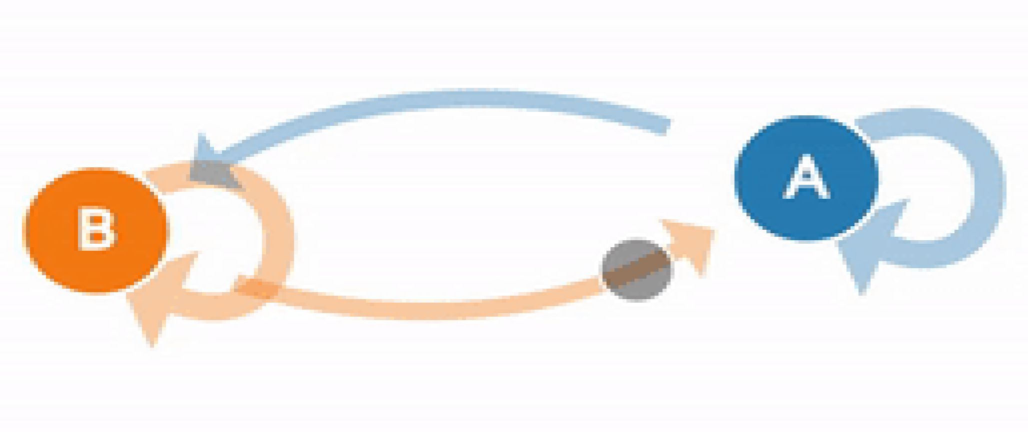
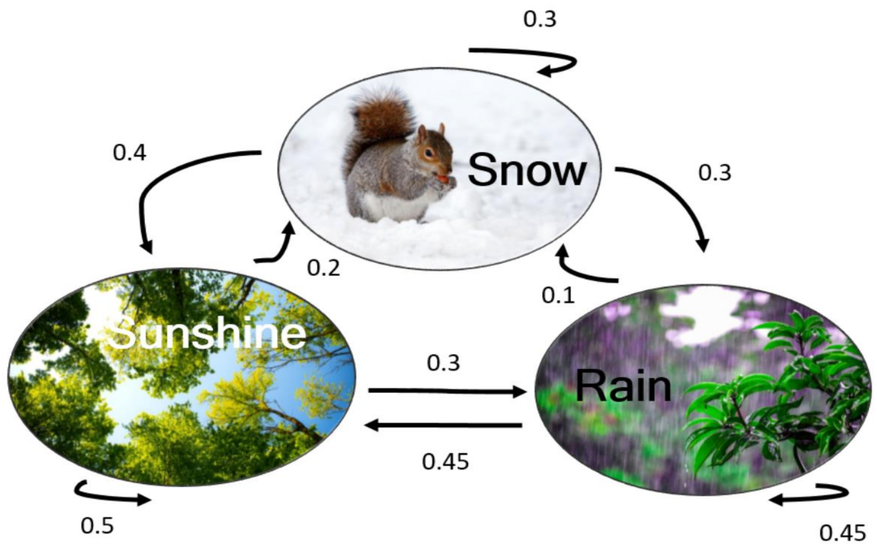


Module-1
Introduction to Stochastic Processes



Presented by:

Dr. Jyoti Badge

Content

- Definitions
- Classification of general stochastic processes
- Markov processes
- Markov chains: definition
- Transition graphs
- Transition probability matrix
- Order of a markov chain
- Chapman-kolmogorov equation

Stochastic Process

A stochastic process is a family of random variables.

"Stochastic process" simply equates to "random process". Image: CSUS.edu

Stochastic Process

- ***Definition:*** *Stochastic process or random process is a collection of random variables ordered by an index set.*

Stochastic Process

Discrete-time Stochastic Process

(Eg: Observing the price of a share at the beginning of each day.)

Continuous- time Stochastic Process

(Eg: No. of people in a supermarket t minutes after the store opens for business may be viewed as continuous.)

Stochastic Process

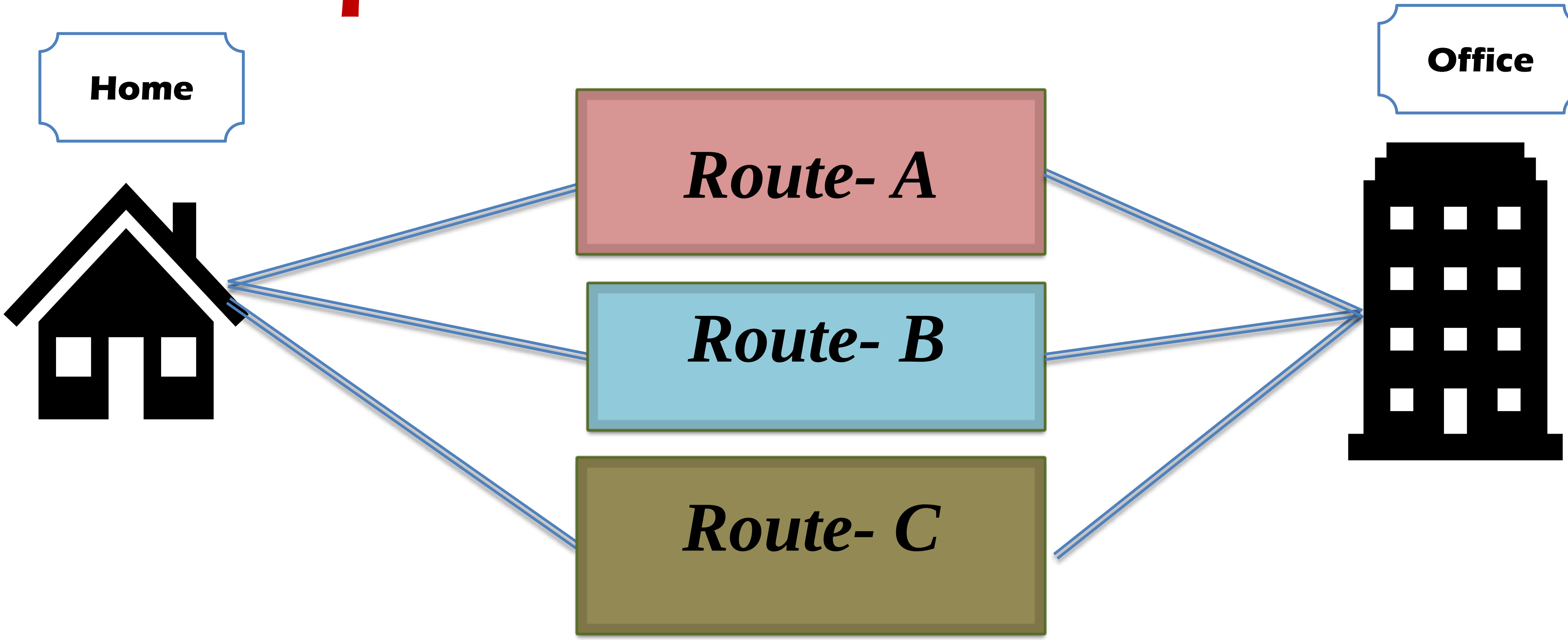
☛ ***Example 1.*** Random variables $X_0, X_1, X_2, \ldots$ form a stochastic process ordered by the *discrete index set* $\{0, 1, 2, \ldots\}$. Notation: $\{X_n : n = 0, 1, 2, \ldots\}$.

☛ ***Example 2.*** Stochastic process $\{Y_t : t \geq 0\}$. with *continuous index set* $\{t : t \geq 0\}$.

The indices n and t are often referred to as "time", so that X_n is a **descrete-time process** and Y_t is a **continuous-time process**.

The range (possible values) of the random variables in a stochastic process is called the state space of the process.

Example of Stochastic Process



(home-->A, A, B, C, A, A, C -->office)

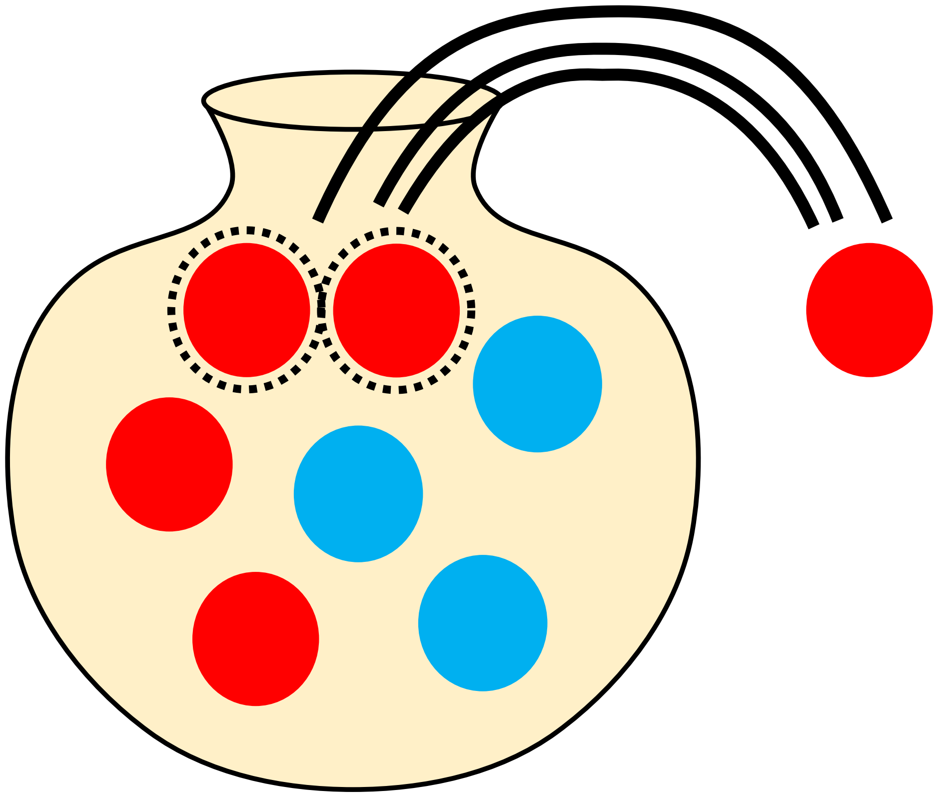
your route sequence is a stochastic process.

Example of Stochastic Process

Markov Chain Model

- Markov Chain Model is one of the most powerful tools for analyzing complex ***stochastic system***.
- A Markov Chain is a mathematical system that experiences transitions from one state to another according to a given set of probabilistic rules.
- Markov chains are stochastic processes, but they differ in that they must lack any "memory".
- Probability of the next state of the system is only dependent on the present state of the system and not on any prior states. This is called the Markov property.(***Memoryless Property of Markov chain***)

Polya's urn model



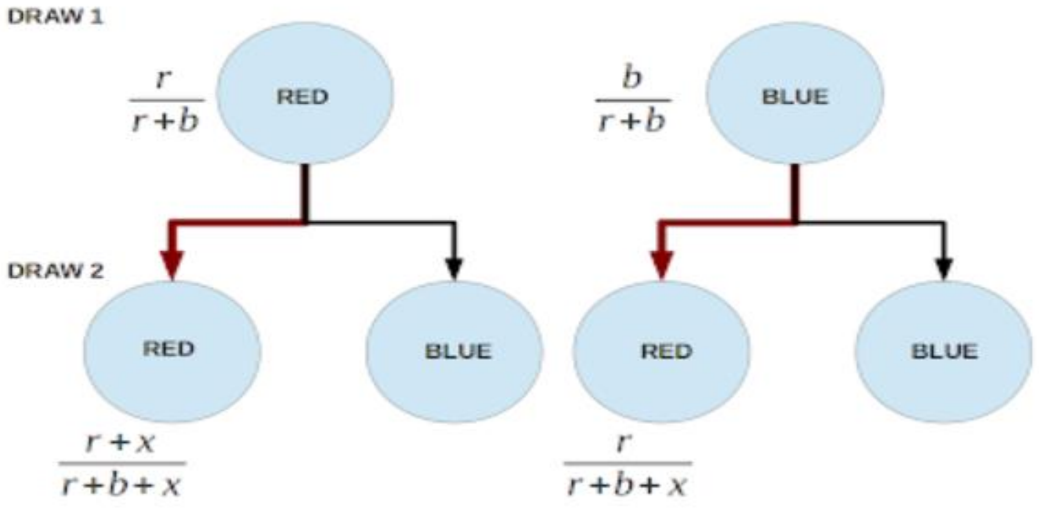
An urn has r red balls and b blue balls. Someone draws a ball at random, its color observed and put back into the urn. You do not know what was observed. However that person puts back x balls of the same color back into the urn. Now, you draw a second ball from this urn. What is the probability that it is red?

Also you have to find the probability of red in the third draw, fourth draw and fifth draw?

Write your observations in this activity.

The probability that a red ball is drawn from the urn in the first draw is $\frac{r}{r+b}$ and for a blue ball would be $\frac{b}{r+b}$. The second draw, if it is a red ball, could be a consequence of either a red ball being drawn the first time or a blue ball.

For the second draw, the probability that a red ball is drawn if a red ball is drawn the first time, would be $\frac{r+x}{r+b+x}$. The probability that a red ball is drawn if a blue ball is drawn the first time, would be $\frac{b}{r+b+x}$. This layout is shown in the figure below.



The probability that a red ball is drawn on the second draw is

$$P(\text{Red: Draw=2}) = \frac{r+x}{r+b+x} \times \frac{r}{r+b} + \frac{r}{r+b+x} \times \frac{b}{r+b}$$

The above simplifies as

$$\frac{(r+b+x)r}{(r+b+x)(r+b)} = \frac{r}{r+b}$$

Markov Chain Definition

Let P *be a* $k \times k$*-matrix with elements* $\{ P_{i,j} \; : i, j = 1, \ldots k \}$ *A random process* $(X_0, X_1, \ldots)$ *with finite state space* $S = \{s_1, s_2, \ldots, s_k\}$ *is said to be a (homogenous) Markov chain with transition matrix* P *if for all* n*, all* $i, j \in \{1, \ldots, k\}$ *and all* $i_0, \ldots, i_{n-1} \in \{1, \ldots, k\}$ *we have*

$$\mathbb{P}(X_{n+1} = s_j | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = \mathbb{P}(X_{n+1} = s_j | X_n = i) = P_{i,j}$$

The Markov property "The future depends on the past through the present."

$$\mathrm{P}(X_{n+1} = s_j | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i) = \mathrm{P}(X_{n+1} = s_j | X_n = i)$$

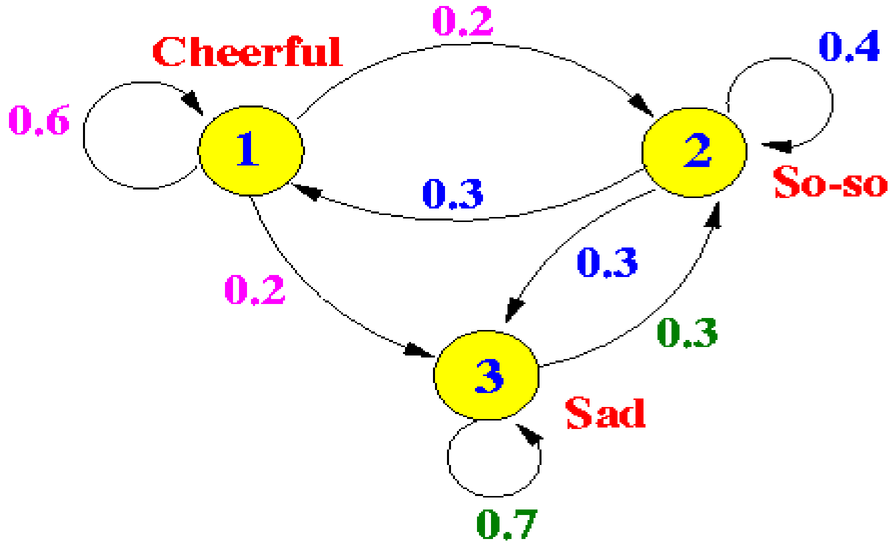
Representation I: transition matrix We can represent the Markov chain by a matrix containing the transition probabilities, or ...

Representation II: transition graph ... we can represent the Markov chain with a transition graph where a positive transition probability is represented by an arrow.

Time homogeneity The property that the transition probabilities doesn't change over time.

State

- A state is a condition or location of an object in the system at particular time.
- The state of a Markov chain at time t is the value of X_t
- For example, if $X_t = 6$, we say the process is in state 6 at time t



- nodes (also often called states), and the arrows connecting them edges. Each edge has a probability associated with it; as specified above, this is the probability of 'following' that edge if you start at that node.

State space

Definition:

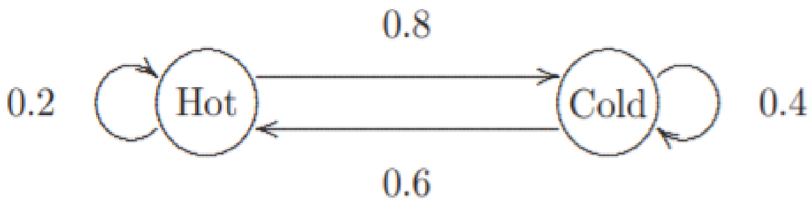
- The state space of a Markov chain, S, is the set of values that each X_t can take. For example, $S = \{1, 2, 3, 4, 5, 6, 7\}$. Let S have size N (possibly infinite).

Transition Probability

- The probabilities associated with various state changes are called transition probabilities.
- Mathematically

$$P(\text{Next state } s_j \text{ at } t=1/\text{Initial state } s_i \text{ at } t=0)$$

Transition Matrix



The matrix describing the Markov chain is called the *transition matrix*.
It is the most important tool for analysing Markov chains.

Transition Matrix

$$\begin{array}{cc} & \overbrace{\longleftarrow \text{ list all states } \longrightarrow}^{X_{t+1}} \\ X_t \left\{ \begin{array}{c} \text{list} \\ \text{all} \\ \text{states} \end{array} \right. \updownarrow & \begin{pmatrix} \texttt{insert} \\ \texttt{probabilities} \\ p_{ij} \end{pmatrix} \begin{array}{l} \longleftarrow \text{rows add to 1} \\ \longleftarrow \text{rows add to 1} \end{array} \end{array}$$

The transition matrix is usually given the symbol $P = (p_{ij})$.

In the transition matrix P:

The transition matrix is usually given the symbol $P = (p_{ij})$.

In the transition matrix P:

- *the ROWS represent NOW, or FROM* (X_t);
- *the COLUMNS represent NEXT, or TO* (X_{t+1});
- *entry* (i, j) *is the CONDITIONAL probability that NEXT* $= j$, *given that NOW* $= i$: *the probability of going FROM state* i *TO state* j.

$$\boxed{p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i).}$$

Notes: 1. The transition matrix P must list *all* possible states in the state space S.

2. P is a *square matrix* $(N \times N)$, because X_{t+1} and X_t both take values in the same state space S (of size N).

3. The **rows** of P should each *sum to 1:*

$$\sum_{j=1}^{N} p_{ij} = \sum_{j=1}^{N} \mathbb{P}(X_{t+1} = j \mid X_t = i) = \sum_{j=1}^{N} \mathbb{P}_{\{X_t = i\}}(X_{t+1} = j) = 1.$$

This simply states that X_{t+1} *must* take one of the listed values.

4. The **columns** of P do **not** in general sum to 1.

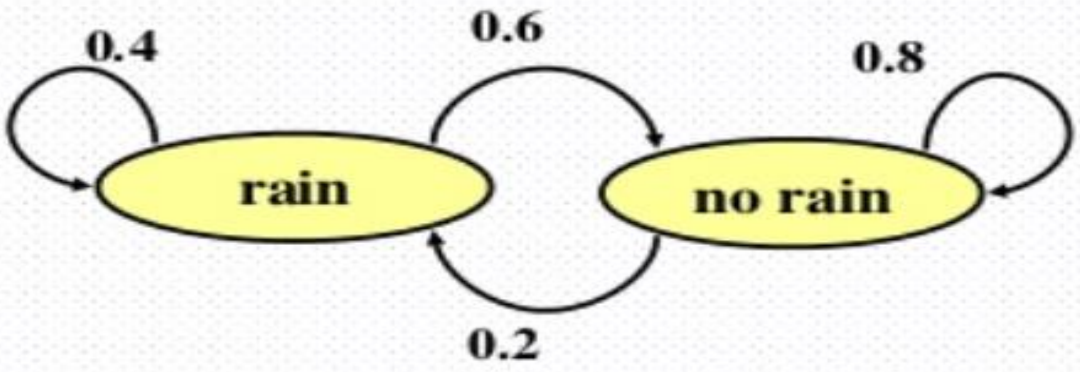
Assumptions of Markov chain

- Finite number of states.
- States are mutually exclusive.
- States are collectively exhaustive.
- Probability of moving from one state to another state is constant over time.

Lets try to understand Markov chain from very simple example

- **Weather:**
- raining today → 40% rain tomorrow
- 60% no rain tomorrow
- not raining today → 20% rain tomorrow
- 80% no rain tomorrow

Stochastic Finite State Machine:



Simple Example

Weather:

- raining today ⇒ 40% rain tomorrow
  ⇒ 60% no rain tomorrow

- not raining today ⇒ 20% rain tomorrow
  ⇒ 80% no rain tomorrow

The transition matrix:

$$
\begin{array}{rc}
 & \begin{array}{cc} \text{Rain} & \text{No rain} \end{array} \\
\begin{array}{r} \text{Rain} \\ \text{No rain} \end{array} & P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}
\end{array}
$$

- Stochastic matrix:
  Rows sum up to 1
- Double stochastic matrix:
  Rows and columns sum up to 1

In a certain market, only two brands of cold drinks A and B are sold. Given that a man last purchased brand A, there is 80% chance that he would buy the same brand in the next purchase, while if a man purchased brand B, there is 90% chance that his next purchase would be brand B, using this information

a. Develop Transition Probability Matrix.

b. Draw Transition diagram.

Multi-step (n-step) Transitions

The **P** matrix is for one step: n to $n + 1$.

How do we calculate the probabilities for transitions involving more than one step?

Consider an Internal Revenue Service IRS auditing example:

Two states: $\mathbf{s}^0 = 0$ (no audit), $\mathbf{s}^1 = 1$ (audit)

Transition matrix $\mathbf{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$

Interpretation: $p_{01} = 0.4$, for example, is conditional probability of an audit next year given no audit this year.

Two-step Transition Probabilities

Let $p_{ij}^{(2)}$ be probability of going from i to j in two transitions.

In matrix form, $\mathbf{P}^{(2)} = \mathbf{P} \times \mathbf{P}$, so for IRS example we have

$$\mathbf{P}^{(2)} = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix}$$

The resultant matrix indicates, for example, that the probability of no audit 2 years from now given that the current year there was no audit is $p_{00}^{(2)} = 0.56$.

n-Step Transition Probabilities

This idea generalizes to an arbitrary number of steps.

For $n = 3$: $P^{(3)} = P^{(2)}\, P = P^2\, P = P^3$

or more generally, $P^{(n)} = P^{(m)}\; P^{(n\text{-}m)}$

The ij th entry of this reduces to

$$p_{ij}^{(n)} = \sum_{k=0}^{m} p_{ik}^{(m)}\, p_{kj}^{(n\text{-}m)} \quad 1 \le m \le n\text{–}1$$

Chapman - Kolmogorov Equations

Interpretation:

RHS is the probability of going from i to k in m steps & then going from k to j in the remaining $n - m$ steps, summed over all possible intermediate states k.

n-Step Transition Matrix for IRS Example

Time, n	Transition matrix, $\mathbf{P}^{(n)}$
1	$\begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$
2	$\begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix}$
3	$\begin{bmatrix} 0.556 & 0.444 \\ 0.555 & 0.445 \end{bmatrix}$
4	$\begin{bmatrix} 0.5556 & 0.4444 \\ 0.5555 & 0.4445 \end{bmatrix}$
5	$\begin{bmatrix} 0.55556 & 0.44444 \\ 0.55555 & 0.44445 \end{bmatrix}$

Application of Markov chains

1) ***How to calculate the probability of the states.***

2) ***How to calculate the probability after n-steps.***

3) ***How to find the probability of the chain.***

Notations

q_0	Initial probability of the states.
q_1	Probabilities of the states after the 1 time period.
q_2	Probabilities of the states after the 2 time period
………	………………
q_n	Probabilities of the states after the n time period
P	TPM after 1 time period.
P_2	TPM after 2 time period.
……...	……..
P_n	TPM after n time period.

1-step transition probability:

$$P_{ij}\text{=}P(X_{n+1} = j|X_n = i)$$

Time

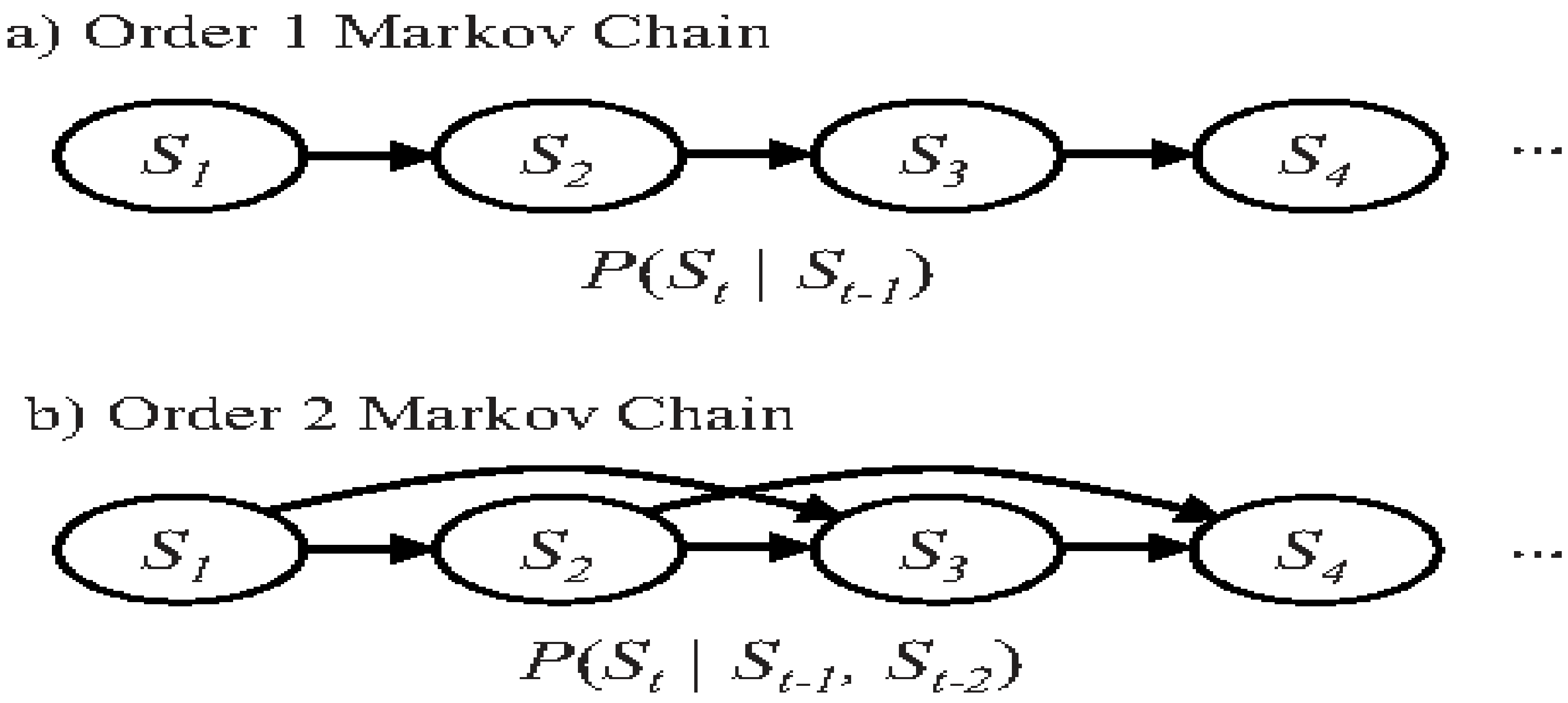
$P_{ij}^{(n)}$

From To

n-step transition probability:

$$P_{ij}^{(n)}\text{=}P(X_{n+1} = j|X_n = i)$$

Order of a Markov Chain



Markov Process

Coke vs. Pepsi Example

- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.

- If a person's last cola purchase was Pepsi, there is an 80% chance that his next cola purchase will also be Pepsi.

Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

Question

- The number of units of an item that are withdrawn from inventory on a day-to-day basis is a markov chain process in which requirement for tomorrow depend on today's requirements. A one day transition matrix is given below:
- $$\begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$
- Develop a two day transition matrix

In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. (i) Find the probability that the grandson of a man from Harvard went to Harvard. (ii) Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.

Solution. We first form a Markov chain with state space $S = \{H, D, Y\}$ and the following transition probability matrix :

$$\mathsf{P} = \begin{pmatrix} .8 & 0 & .2 \\ .2 & .7 & .1 \\ .3 & .3 & .4 \end{pmatrix}.$$

Note that the columns and rows are ordered: first H, then D, then Y. Recall: the ij^{th} entry of the matrix P^n gives the probability that the Markov chain starting in state i will be in state j after n steps. Thus, the probability that the grandson of a man from Harvard went to Harvard is the upper-left element of the matrix

$$\mathsf{P}^2 = \begin{pmatrix} .7 & .06 & .24 \\ .33 & .52 & .15 \\ .42 & .33 & .25 \end{pmatrix}.$$

It is equal to $.7 = .8^2 + .2 \times .3$ and, of course, one does not need to calculate all elements of P^2 to answer this question.
If all sons of men from Harvard went to Harvard, this would give the following matrix for the new Markov chain with the same set of states:

$$\mathsf{P} = \begin{pmatrix} 1 & 0 & 0 \\ .2 & .7 & .1 \\ .3 & .3 & .4 \end{pmatrix}.$$

The upper-left element of P^2 is 1, which is not surprising, because the offspring of Harvard men enter this very institution only.

Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations. Set up the matrix of transition probabilities. Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.

	Professional	Skilled	Unskilled
Professional	.8	.1	.1
Skilled	.2	.6	.2
Unskilled	.25	.25	.5

so that the transition matrix for this chain is

$$\mathsf{P} = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .25 & .25 & .5 \end{pmatrix}$$

with

$$\mathsf{P}^2 = \begin{pmatrix} 0.6850 & 0.1650 & 0.1500 \\ 0.3300 & 0.4300 & 0.2400 \\ 0.3750 & 0.3000 & 0.3250 \end{pmatrix},$$

and thus the probability that a randomly chosen grandson of an unskilled labourer is a professional man is 0.375.

Type -1 : How to calculate the probability of the states.

For example:

$$P(\underbrace{X_2}_{\text{time}} = \underbrace{2}_{\text{State}}), \quad P(\underbrace{X_3}_{\text{time}} = \underbrace{1}_{\text{State}}), \text{ etc}$$

Notation:

1) $X_n = i$

time

state

2) $\boxed{P(X_n = a) = q_n(a)}$

$q_n = q_0 P^n$.
$n = 1, 2, 3 \ldots$

P = TPM
q : initial Prob. of the state.

We compute the Probability of the states by using following Formula

$$\boxed{q_n = q_0 P^n} \quad \text{OR} \quad q_{n+1} = q_n P$$

Example:

$q_3 = q_0 P^3$ which is same as that of $q_3 = q_2 P$

Example 0: A man either uses his car or takes a bus or a train to work each day. The TPM of the Markov chain with these three states 1 (Car), 2 (Bus), 3 (Train) is

$$P = \begin{array}{c|ccc} & C & B & T \\ \hline C & 0.1 & 0.5 & 0.4 \\ B & 0.6 & 0.2 & 0.2 \\ T & 0.3 & 0.4 & 0.3 \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$. Calculate

$P(X_2 = 3)$.

2 step Trans.

$$q_0 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix}$$

$$\begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ \downarrow & \downarrow & \downarrow \\ C & B & T. \end{array}$$

$$P[X_2 = 3]$$

$$\left[q_n = q_0 P^n \right]$$

$$q_2 = q_0 P^2$$

$$P^2 = P \cdot P = \begin{array}{c} C \\ B \\ T \end{array} \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0.385 & 0.336 & 0.279 \end{bmatrix}$$

$$P(X_2 = 3) = 0.279.$$

Example 3: Three boys *A*, *B* and *C*, are throwing a ball to each other. *A* always throws the ball to *B* and *B* always throws the ball to *C*, but *C* is as likely to throw the ball to *B* as to *A*. If the initial probability distribution of three states A, B and C is 0.3, 0.4 and 0.3 respectively. Find

(i) the transition matrix.

(ii) $P(X_2 = B)$.

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$$q_0 = [0.3, 0.4, 0.3]$$

$$q_2 = q_0 P^2.$$

$$q_2 = 0.35$$

$(3-1=2)$

$P(X_3=3 \mid X_1=1) = \boxed{P_{13}^{(2)}}$

Representation:

Type -2: How to calculate the probability after n-steps.

i.e., $P(X_3 = 3|X_1 = 1)$ or $p_{13}^{(2)}$

like $p_{13}^{(2)}$, $p_{23}^{(3)}$ etc

$p_{23}^{(3)} =$

Example: A professor of Statistics not wanting to be predictable, decides on an innovative way of assigning homework based on probabilities. On the first day of the week, he draws a transition diagram as shown in Figure 1. The nodes of the diagram represent full credit (F), half credit (H) and no credit (N) assignments. The transition probabilities for day 1 are as shown in the figure. Construct TPM and compute:

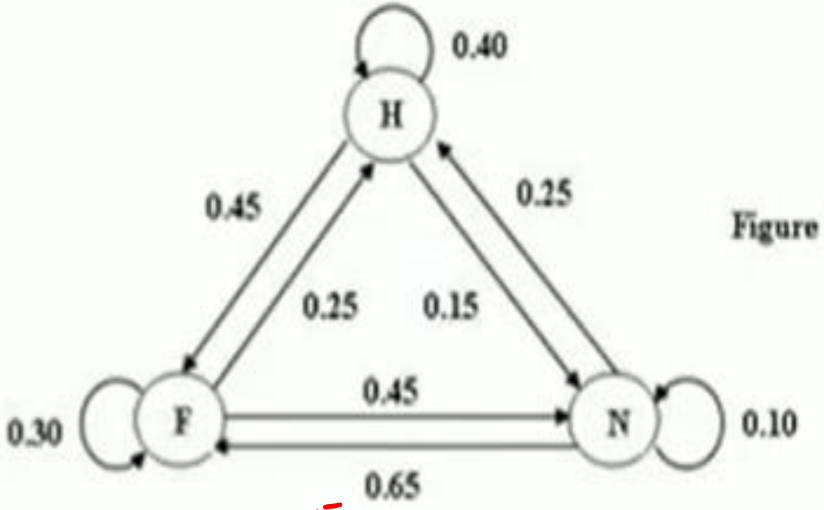


Figure 1

(i) $P(X_3 = F | X_2 = N)$ $= P_{NF}^{(1)} = 0.65$

(ii) $P(X_2 = N | X_1 = H)$ $= P_{HN}^{(1)} = 0.15$

(iii) $P(X_4 = H | X_2 = F)$ $= P_{FH}^{(2)} =$

$$P^2 = P \cdot P = \begin{array}{c|ccc} & F & H & N \\ \hline F & 0.4950 & 0.2876 & 0.275 \\ H & 0.4125 & 0.3100 & 0.2775 \\ N & 0.3725 & 0.2875 & 0.3400 \end{array}$$

Ans $= 0.2876$

How to calculate the probabilities of Markov chain.

$(3-1=2)$ (1)

such as $P(X_3 = 2, X_1 = 3, X_0 = 2)$

Interpretation of $P(X_3 = 2, X_1 = 3, X_0 = 2)$

Its means that you started from State 2

After 1 time period, it move to state 3

Then after 2 time period, it move to state 2 again.

Example 1: The TPM of the Markov chain with three states 1, 2, 3 is

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$. Calculate

(i) $P(X_2 = 1)$ ← Prob. of the state $q_n = q_0 P^n$ $= q_2 = q_0 P^2$

(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array} \qquad q_0 = [0.7 \quad 0.2 \quad 0.1]$$

$$P(x_3 = 2,\ x_2 = 3,\ x_1 = 3,\ x_0 = 2)$$

Inital Prob.

$$\xrightarrow{q_0} 2 \xrightarrow{(1)} 3 \xrightarrow{(1)} 3 \xrightarrow{(1)} 2$$

$$= q_0(2) \times P_{23}^{(1)} \times P_{33}^{(1)} \times P_{32}^{(1)}$$

$$= 0.2 \times 0.2 \times 0.3 \times 0.4$$

$$= \underline{\underline{0.0048}}$$

Example 2: Three boys *A*, *B* and *C*, are throwing a ball to each other. *A* always throws the ball to *B* and *B* always throws the ball to *C*, but *C* is as likely to throw the ball to *B* as to *A*. If the initial probability distribution of three states A, B and C is 0.3, 0.4 and 0.3 respectively. Find

(i) the transition matrix.

(ii) $P(X_2 = B)$,

(iii) $P(X_3 = B, X_2 = C, X_1 = B, X_0 = A)$ and

(iv) the distribution of the balls after two rounds.

Solution:

(i) The required TPM is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array} \quad \& \quad q_0 = [0.3, 0.4, 0.3]$$

$$q_0 \dashrightarrow A \xrightarrow{(1)} B \xrightarrow{(1)} C \xrightarrow{(1)} B$$

$$q_0(A) \times P_{AB}^{(1)} \times P_{BC}^{(1)} \times P_{CB}^{(1)}$$

$$0.3 \times 1 \times 1 \times \tfrac{1}{2} = 0.15$$

(iii)

$$P(X_3 = B, X_2 = C, X_1 = B, X_0 = A)$$

$$\dashrightarrow A \xrightarrow{1} B \xrightarrow{1} C \longrightarrow B$$

$$= q_0(A) \times p_{AB} \times p_{BC} \times p_{CB}$$

$$= 0.3 \times 1 \times 1 \times 1/2$$

$$= 0.15$$

Q

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} \end{array}$$

$$q_0 = \begin{array}{c} \begin{array}{ccc} (0) & (1) & (2) \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix} \end{array}$$

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.27 & 0.39 & 0.34 \\ 0.20 & 0.48 & 0.32 \\ 0.23 & 0.39 & 0.38 \end{bmatrix} \end{array}$$

$$P(x_3 = 2,\ x_1 = 0,\ x_0 = 2)$$

$$\xrightarrow{q_0} 2 \xrightarrow{(1)} 0 \xrightarrow{(2)} 2$$

$$q_0(2) \times P_{20}^{(1)} \times P_{02}^{(2)}$$

$$= 0.2 \times 0.4 \times 0.34$$

$$= 0.0272$$

Q

$S = \{1, 2, 3\}$

{prob. of markov chain}

$P[X_4 = 3 \mid \cancel{X_1 = 3, X_2 = 2}, X_3 = 1]$

(markov's property)

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/4 & 1/4 \\ 2 & 1/3 & 0 & 2/3 \\ 3 & 1/2 & 1/2 & 0 \end{array}$$

$P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$

$q_0 = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right]$

$P[X_4 = 3 \mid X_3 = 1]$

(1)

$P_{13}^{(1)} = ?$

n step

T.P.M

Find

$P(X_1 = 3, X_2 = 2, X_3 = 1)$

$\to q_0(3) \times P_{32}^{(1)} \times P_{21}^{(1)}$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$

A Markov chain has transition matrix

$$
P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix} \end{array}
$$

with initial distribution $\boldsymbol{\alpha} = (0.2, 0.3, 0.5)$. Find the following:

(a) $P(X_7 = 3 | X_6 = 2)$

(b) $P(X_9 = 2 | X_1 = 2, X_5 = 1, X_7 = 3)$

(c) $P(X_0 = 3 | X_1 = 1)$

(d) $E(X_2)$

a) $P(X_7 = 3|X_6 = 2) = P_{2,3} = 0.6.$

b) $P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3) = P(X_9 = 2|X_7 = 3) = P^2_{3,2} = 0.27;$

c) $P(X_0 = 3|X_1 = 1) = P(X_1 = 1|X_0 = 3)P(X_0 = 3)/P(X_1 = 1) = P_{31}\alpha_3/(\alpha P)_1 = (0.3)(0.5)/(0.17) = 15/17 = 0.882;$

d) $E(X_2) = \sum_{k=1}^{3} kP(X_2 = k) = (0.182, 0.273, 0.545) \cdot (1, 2.3) = 2.363.$

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (1/2, 0, 1/2)$. Find the following:

(a) $P(X_2 = 1 | X_1 = 3)$

(b) $P(X_1 = 3, X_2 = 1)$

(c) $P(X_1 = 3 | X_2 = 1)$

(d) $P(X_9 = 1 | X_1 = 3, X_4 = 1, X_7 = 2)$

a) $P(X_2 = 1|X_1 = 3) = P_{3,1} = 1/3$.

b) $P(X_1 = 3, X_2 = 1) = P(X_2 = 1|X_1 = 3)P(X_1 = 3) = P_{3,1}(\alpha P)_3 = (1/3)(5/12) = 5/36$.

c) $P(X_1 = 3|X_2 = 1) = P(X_1 = 3, X_2 = 1)/P(X_2 = 1) = (5/36)/(\alpha P^2)_1 = (5/36)/(5/9) = 1/4$.

d) $P(X_9 = 1|X_1 = 3, X_4 = 1, X_7 = 2) = P(X_9 = 1|X_7 = 2) = P^2_{2,1} = 0$.

- For the general two-state chain with transition matrix

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

and initial distribution $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, find the following:

(a) the two-step transition matrix

(b) the distribution of X_1

4 For the general two-state chain,

a)

$$\boldsymbol{P}^2 = \begin{pmatrix} 1 + p(p+q-2) & p(2-p-q) \\ q(2-p-q) & 1 + q(p+q-2) \end{pmatrix}.$$

b) Distribution of X_1 is $(\alpha_1(1-p) + \alpha_2 q, \alpha_2(1-q) + \alpha_1 p)$.