Mathematical Logic and Predicate Calculus

- A first and indispensable part of mathematical logic consists of the so-called sentential calculus (calculus of sentences).

- Logic is the discipline that deals with the methods of reasoning. On an elementary level, logic provides rules and techniques for determining whether a given argument is valid or not.

Definition:

A **statement or proposition** is a declarative sentence that is either TRUE or FALSE but not both, which is also meaningful and precise.

The truth values TRUE and FALSE of a statement are denoted by T and F respectively. Also, the truth values TRUE is denoted by 1 and FALSE is denoted by 0.

Examples:

1. The earth is round.
2. $2+8=5$
3. Bhopal is in Sehore.
4. What is your name?
5. $2*x=10$
6. $3+x=y$
7. Collect the papers.
8. Send the message.

Logical Connectives and compound statements

In mathematics, the letters $x, y, z, \ldots$ often denote variables that can be replaced by real numbers, and there variables can be obtained with the familiar operators $+, -, *, /$

In logic, the letters $p, q, r, \ldots$ denote propositional variables, which means variables can be replaced by statements. Statements or propositional variables can be combined by logical operators (operations) to obtain compound statements.

Negation/NOT

If p is a statement, the negation of p is the statement “not p”, denoted by ($\sim p$). Thus $\sim p$ is the statement “it is not the case that p“.

From the definition, it follows that if p is true, then $\sim p$ is false, and if p is false, then $\sim p$ is true.

Truth Table is a table which gives the truth values of a compound statement in terms of its component parts.

NOT is a unary operator

p	$\sim p$
T	F
F	T

Example

Give the negation of the following statements

1. p: $2+3>1$
2. q: It is cold
3. $5<7$
4. Bangalore is in Bihar

Conjunction

If p and q are statements, the conjunction of p and q is the compound statement "*p and q*" or "p meet q", denoted by $p \wedge q$. The connectivity and is denoted by $\wedge$. The compound statement $p \wedge q$ is true when both p and q are true otherwise it is false.

AND is a binary operator.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

If p and q are statements, the disjunction of p and q is the compound statement "*p or q*" or "p join q", denoted by $p \vee q$. The connectivity OR is denoted by $\vee$. The compound statement $p \vee q$ is true if at least one of p or q is true, it is false when both p and q are false.

OR is a binary operator.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example

Form the conjunction and disjunctions of p and q for each of the following:

1	**p: It is snowing**	**q: It is cold**
2	p: $2<3$	q:$-5>-8$
3	p:$2+3=10$	q: Bangalore is in Bihar

Conditional Statement

If p and q are statements, the compound statement "$\boldsymbol{if\ p\ then\ q}$" denoted $\boldsymbol{p \to q}$ is called a conditional statement or implication. A conditional statement $\boldsymbol{p \to q}$ is False only when the hypothesis is True and the conclusion is False.

The statement $\boldsymbol{p}$ is called the **antecedent** or **hypothesis. (Today is a holiday)**

The statement $\boldsymbol{q}$ is called the **consequent** or **conclusion. (The store is closed)**

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q$ can read as

- p only if q
- p implies q
- p is sufficient for q
- q if p

Give the negation of each of the following statements

i. $2 + 7 \leq 11$

ii. 2 is an even integer and 8 is an odd integer

iii. It will rain tomorrow or it will snow tomorrow

2. Determine the truth value of each of the following statements

i. $2 < 3$ and 3 is a positive integer

ii. $2 \geq 3$ or 3 is a positive integer

iii. $2 < 3$ or 3 is not a positive integer

iv. $2 \geq 3$ and 3 is not a positive integer

3. Make a truth table for the following statements

i. $(\sim p \vee q) \wedge r$

ii. $(p \vee q) \vee r$

iii. $p \wedge \left(\sim(q \vee \sim r)\right)$

iv. $(r \wedge \sim q) \vee (p \vee r)$

v. $\sim\left[\left((p \wedge r) \wedge (q \wedge r)\right) \vee \left((\sim p \wedge \sim q) \wedge (\sim q \wedge \sim r)\right)\right]$

Converse, Inverse and Contrapositive

We can arrange some new conditional statements using a conditional statement $p \rightarrow q$. In particular, there exist three inter-related conditional statements.

- The inverse of $p \rightarrow q$ is the proposition $\sim p \rightarrow \sim q$
- The converse of $p \rightarrow q$ is the proposition $q \rightarrow p$
- The contrapositive of $p \rightarrow q$ is the proposition $\sim q \rightarrow \sim p$

Example

Determine the converse, inverse and contrapositive of the conditional statement

"The home team wins whenever it is raining"

1. Write the converse, inverse and contrapositive of the following conditional statements

i. If you buy air-ticket in advance, it is cheaper

ii. If x is an integer, then $x^2 \geq 0$

iii. If it rains, the grass gets wet

iv. If the sprinklers operate, the grass gets wet

2. Find the truth values of the following

i. If -1 is a positive number, then $2 + 2 = 5$

ii. If -1 is a positive integer, then $2 + 2 = 4$

iii. If 1 is a positive number, then $2 + 2 = 5$

iv. If 1 is a positive integer, then $2 + 2 = 4$

v. If $\sin x = 0$, then $x = 0$

Biconditional Statement

A statement of the form "$p\ if\ and\ only\ if\ q$" is called a biconditional statement. It is denoted by $p \leftrightarrow q$.

Biconditional statement is the combination of two conditional statements $p \rightarrow q\ and\ q \rightarrow p$.

A conditional statement $p \leftrightarrow q$ is False only when one of the statement is False.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ex. p: 5+5=10

q: 2*5=10

Which of the following biconditionals are true

1. $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
2. $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
3. $x^2 \geq 0$ if and only if $x \geq 0$

Exclusive or (XOR)

The exclusive or of two statements p and q is denoted by $p \oplus q$.

The exclusive or is true when exactly one of the statement is true other one is false.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \oplus q}$
T	T	F
T	F	T
F	T	T
F	F	F

$$p \oplus q \equiv \sim(p \leftrightarrow q)$$

Example

- The circuit is either on or off but not both.
- Let $ab < 0$, then $a < 0$ or $b < 0$ but not both.

Tautologies, Contradictions and Contingency

In some propositions, if the last column of their truth tables contain only T. i.e., the propositions are true for any truth values of their variables, then such propositions are called tautologies.

A proposition is called a contradiction, if it contains only F in the last column of its truth table.

A proposition is neither a tautology nor a contradiction is called a contingency.

Logical Equivalence

Two propositions p and q are said to be logically equivalent if they have the identical truth tables. It is denoted by $p \equiv q$.

The notion can also be defined as the propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

1. p: I will study Discrete Mathematics

   q: I will go to movie

   r: I am in a good mood

Write English sentences corresponding to the following statements

i. $\left((\sim p) \wedge q\right) \rightarrow r$

ii. $r \rightarrow (p \vee q)$

iii. $(\sim r) \rightarrow \left((\sim q) \vee p\right)$

iv. $\left(q \wedge (\sim p)\right) \leftrightarrow r$

2. P: If the flood destroys my house or the fire destroys my house, then my insurance company will pay me.

   Write the converse and contrapositive of the above statement

1. Construct the truth table for the following statement formulae and determine it is a tautology or contradiction or contingency.

i. $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

ii. $p \wedge (p \vee q)$

iii. $(p \rightarrow q) \rightarrow q$

iv. $\big((p \vee q) \vee (p \rightarrow r)\big) \vee (\sim q \rightarrow \sim r)$

v. $(p \wedge q) \wedge \big(\sim(p \vee q)\big)$

vi. $(p \wedge q) \rightarrow (p \vee q)$

vii. $\big((p \vee q) \wedge \sim p\big) \leftrightarrow (\sim p \wedge q)$

1. Verify the following propositions by using truth tables

i. $(p \rightarrow q) \equiv \big((\sim \mathrm{p}) \vee \mathrm{q}\big)$

ii. $(p \rightarrow q) \equiv (\sim \mathrm{q} \rightarrow \sim \mathrm{p})$

iii. $(p \rightarrow q) \wedge (p \rightarrow r) \equiv \mathrm{p} \rightarrow (\mathrm{q} \wedge \mathrm{r})$

iv. $(p \rightarrow q) \vee (p \rightarrow r) \equiv \mathrm{p} \rightarrow (\mathrm{q} \vee \mathrm{r})$

v. $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (\mathrm{p} \vee \mathrm{q}) \rightarrow \mathrm{r}$

vi. $(p \rightarrow r) \vee (q \rightarrow r) \equiv (\mathrm{p} \wedge \mathrm{q}) \rightarrow \mathrm{r}$

Properties (laws) of propositional operations

Commutative Properties

1. $p \wedge q \equiv q \wedge p$
2. $p \vee q \equiv q \vee p$

Associative Properties

1. $p \vee (q \vee r) \equiv (p \vee q) \vee r$
2. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive Properties

1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Idempotent Properties

1. $p \vee p \equiv p$
2. $p \wedge p \equiv p$

Properties of negation

1. $\sim(\sim p) \equiv p$
2. $p \wedge \sim p \equiv F$
3. $p \vee \sim p \equiv T$

De-Morgan's laws

1. $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
2. $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

Identity Properties

1. $P \wedge T \equiv p$
2. $p \vee F \equiv p$

Domination Properties

1. $P \vee T \equiv T$
2. $p \wedge F \equiv F$

Absorption Properties

1. $p \vee (p \wedge q) \equiv p$
2. $p \wedge (p \vee q) \equiv p$

Without using truth tables, prove the following

1. $\sim(\sim p \wedge q) \equiv p \vee (\sim q)$
2. $(\sim p) \wedge (p \vee q) \equiv \sim p \wedge q$
3. $\sim\big(p \vee (\sim p \wedge q)\big) \equiv \sim p \wedge \sim q$
4. $(p \vee q) \wedge (\sim q) \equiv \sim q \wedge p$
5. $(p \vee q) \wedge \sim(\sim p \wedge q) \equiv p$
6. $\sim\big[\sim\big((p \vee q) \wedge r\big) \vee \sim q\big] \equiv q \wedge r$
7. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology
8. Show that $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Predicates and Quantifiers

Predicates:

Predicate is a declarative sentence whose true or false value depending on one or more variables.

EX:

- $P(x, y)$: $x\ is\ a\ father\ of\ y$
- $P(x)$: $x\ is\ greater\ than\ 3$

x, y are subjects of the statements

$P(x)$ is the propositional function.

Quantifiers

The variable of the predicate is quantified by quantifiers.

There are two types of Quantifiers:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier

The statements within its scope are true for every value of the specific variable.

$$\forall$$

The universal quantifier is of a predicate $P(x)$ is a statement

“for all values of x, $P(x)$ is true”

$$\forall x, P(x)$$

Existential Quantifier

The statements within its scope are true for some value of the specific variable.

$$\exists$$

The Existential quantifier is of a predicate $P(x)$ is a statement

“there is a value of x, $P(x)$ is true”

$$\exists x, P(x)$$

Example

$P(x)$: "$x^2 > 10$"

What is the truth value of $\forall x, P(x)\ and\ \exists x, P(x)$ for each of the following domains

1. The set of real numbers $\mathbb{R}$.
2. The set of positive integers not exceeding 4.
3. The set of real numbers in the interval [10, 39.5]
4. The set of real numbers in the interval $[0,\sqrt{9.6}]$

There is a unique

Uniqueness quantifier is denoted by $\exists!\, or \exists_1$

$\exists!\, x\ P(x)$: "there exists a unique x such that $P(x)$ is true"

Quantifiers with restricted domains

- What do the following statements mean for the domain of real numbers?

$$\forall x < 0, x^2 > 0 \quad \text{same as} \quad \forall x(x < 0 \to x^2 > 0)$$

$$\forall y \neq 0, y^3 \neq 0 \quad \text{same as} \quad \forall y(y \neq 0 \to y^3 \neq 0)$$

$$\exists z > 0, z^2 = 2 \quad \text{same as} \quad \exists z(z > 0 \wedge z^2 = 2)$$

Be careful about → and ∧ in these statements

Precedence of quantifiers

- $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus

$\forall x\, p(x) \vee q(x) \equiv (\forall x \quad p(x)) \vee q(x)$ rather than $\forall x \quad (p(x) \vee q(x))$

Binding variables

- When a quantifier is used on the variable x, this occurrence of variable is **bound.**
- If a variable is not bound, then it is **free.**
- All variables that occur in propositional function of predicate calculus must be bound or set equal to a particular value to turn it into a proposition.
- The part of a logical expression to which a quantifier is applied, is the **scope** of this quantifier.

Example

What are the scope of these expressions? Are all the variables bound?

$$\exists x(x + y = 1)$$

$$\exists x(p(x) \wedge q(x)) \vee \forall x R(x)$$

$$\exists x(p(x) \wedge q(x)) \vee \forall y R(y)$$

The same letter is often used to represent variables bound by different quantifiers with scopes $\exists x \; and \; \forall y$ that do not overlap.

Negation of quantifiers

- Consider the statement:
    1) All students in this class have red hair.
    2) There is a student in this class with red hair.

What is required to show the statement is false?

1) There exists a student in this class that does NOT have red hair.
2) All students in this class do not have red hair.

Negation of quantifiers

1) $\sim\left(\forall x\, P(x)\right) \equiv \exists x\left(\sim P(x)\right)$

2) $\sim\left(\exists x\, P(x)\right) \equiv \forall\, x\left(\sim P(x)\right)$

These are De-Morgan's laws

NOTE:

1. $\forall x\left(P(x) \wedge Q(x)\right) \equiv \left(\forall x\, P(x)\right) \wedge \left(\forall x\, Q(x)\right)$
2. $\forall x\left(P(x) \vee Q(x)\right) \equiv \left(\forall x\, P(x)\right) \vee \left(\forall x\, Q(x)\right)$

Examples:

1. Write the negation of the following statements:

i. There is an honest politician.

ii. All Americans eat cheeseburgers.

iii. $\forall\, x\ (x^2 > x)$

iv. $\exists\, x\ (x^2 = 2)$

2. Show that $\sim\forall x\big(P(x) \to Q(x)\big)$ and $\exists x\ \big(P(x) \wedge \sim Q(x)\big)$ are logically equivalent.

Translating from English into Logical Expressions:

1) Consider “For every student in this class, that student has studied calculus”

- Rephrased: “For every student x in this class, x has studied calculus.”
  - Let C(x) be “x has studied calculus”
  - Let S(x) be “x is a student”
- $\forall x\ C(x)$
  - True if the universe of discourse is all students in this class

- What about if the universe of discourse is all students (or all people?)
  - $\forall x\ (S(x) \wedge C(x))$
    - This is wrong! Why?
  - $\forall x\ (S(x) \rightarrow C(x))$
- Another option for other subjects:
  - Let $Q(x,y)$ be "student x has studied subject y"
  - $\forall x\ (S(x) \rightarrow Q(x, \text{calculus}))$

2) Consider:

- "Some students in this class have visited Mexico"
- "Every student in this class has visited either Canada or Mexico"

- Let:
  - $S(x)$ be "x is a student in this class"
  - $M(x)$ be "x has visited Mexico"
  - $C(x)$ be "x has visited Canada"

- Consider: “Some students in this class have visited Mexico”
  - Rephrasing: “There exists a student x who has visited Mexico”
- $\exists x\, M(x)$
  - True if the universe of discourse is all students
- What about if the universe of discourse is all people?

– $\exists x\, (S(x) \rightarrow M(x))$

- This is wrong! Why?

– $\exists x\, (S(x) \wedge M(x))$

[Caution! Our statement cannot be expressed as $\exists x\, (S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either $F \rightarrow T$ or $F \rightarrow F$.]

- Consider: “Every student in this class has visited Canada or Mexico”
- $\forall x\ (M(x) \vee C(x))$
  - When the universe of discourse is all students
- $\forall x\ (S(x) \rightarrow (M(x) \vee C(x))$
  - When the universe of discourse is all people
- Why isn’t $\forall x\ (S(x) \wedge (M(x) \vee C(x)))$ correct?

- Note that it would be easier to define V(x, y) as “x has visited country y”
  - $\forall x\ (S(x) \wedge V(x, \text{Mexico}))$
  - $\forall x\ (S(x) \rightarrow (V(x, \text{Mexico}) \vee V(x, \text{Canada})))$

Examples:

- Translate the statements:
  - “All hummingbirds are richly colored”
  - “No large birds live on honey”
  - “Birds that do not live on honey, are dull in color”
  - “Hummingbirds are small”
- Assign our propositional functions
  - Let $P(x)$ be “x is a hummingbird”
  - Let $Q(x)$ be “x is large”
  - Let $R(x)$ be “x lives on honey”
  - Let $S(x)$ be “x is richly colored”
- Let our universe of discourse be all birds

- Our propositional functions
  - Let $P(x)$ be “x is a hummingbird”
  - Let $Q(x)$ be “x is large”
  - Let $R(x)$ be “x lives on honey”
  - Let $S(x)$ be “x is richly colored”
- Translate the statements:

  1) “All hummingbirds are richly colored”
     - $\forall x\ (P(x) \rightarrow S(x))$
  2) “No large birds live on honey”

- $\sim\exists x\ (Q(x) \wedge R(x))$
- Alternatively: $\forall x\ (\sim Q(x) \vee \sim R(x))$

3) “Birds that do not live on honey are dull in color”
   - $\forall x\ (\sim R(x) \rightarrow \sim S(x))$
4) “Hummingbirds are small”
   - $\forall x\ (P(x) \rightarrow \sim Q(x))$

Nested Quantifiers

Two quantifiers are nested if one is in the scope of the other

Example:

$\forall x \, \exists \, y \quad (x + y = 0)$

Is same as $\forall x \; Q(x)$ where $Q(x)$ is $\exists y \;\; (x + y = 0)$?

$\forall x \; Q(x)$ where $Q(x)$ is $\exists y \; \mathrm{P}(\mathrm{x}, \mathrm{y})$,

where $\mathrm{P}(\mathrm{x}, \mathrm{y})$ is $(x + y = 0)$

	When it is true?	When it is false?
$\forall x \, \forall y \, P(x, y)$		
$\forall x \, \exists y \, P(x, y)$		
$\exists x \, \forall y \, P(x, y)$		
$\exists x \exists y \, P(x, y)$		

	When it is true?	When it is false?
$\forall x\ \forall y\ P(x, y)$	For every pair of x, y, $P(x, y)$ is TRUE	There is a pair x, y for which $P(x, y)$ is FALSE
$\forall x\ \exists y\ P(x, y)$	For every x, there is a y for which $P(x, y)$ is TRUE	There is an x for every y, $P(x, y)$ is FALSE
$\exists x\ \forall y\ P(x, y)$	There is an x for every y, $P(x, y)$ is TRUE	For every x, there is a y for which $P(x, y)$ is FALSE
$\exists x \exists y\ P(x, y)$	There is a pair x, y for which $P(x, y)$ is TRUE	For every pair of x, y $P(x, y)$ is FALSE

Examples:

1. Translate the following statements into symbolic form
   i. The sum of two positive integers is always positive
   ii. Every real number except zero has a multiplicative inverse
   iii. Every positive integer is the sum of the squares of four integers
2. Translate the following into English

   $C(x)$: x has a computer

   $F(x, y)$: x and y are friends

   Domain: All students in a class

   i. $\forall x \left(c(x) \vee \exists\, y \left(C(y) \wedge F(x, y) \right) \right)$

   ii. $\exists\, x\, \forall y\, \forall z \left(\left(F(x, y) \wedge F(x, z) \wedge (y \neq z) \right) \rightarrow {\sim} F(x, y) \right)$

EX

3. Translate into symbolic form using quantifiers
    i. If a person is female and is a parent, then this person is someone's mother
    ii. Everyone has exactly one best friend
    iii. There is a woman who has taken a flight on every airline of the world
4. Write the negation of the following
    i. $\forall\, x\, \exists y\, (xy = 1)$
    ii. $\forall x\, \exists y\, p(x, y) \lor \forall x\, \exists y\, Q(x, y)$
    iii. $\forall x\, \exists y\, (P(x, y) \land \exists z\, R(x, y, z))$
    iv. Everything is beautiful

Theory of inference for the statement calculus

Mathematical logic is often used for logical proofs. Proofs are valid arguments that determine the truth values of mathematical statements.

- An argument is a sequence of statements that end with a conclusion

- The argument is valid if the conclusion(final statement) follows from the truth values of its preceding statements (premises).

NOTE: the last statement in the argument is called the conclusion and all its preceding statements are called premises (or hypothesis).

Rules of inference are templates for building valid arguments

Theory of inference

Let A and B are two statement formulas, then "B logically follows from A" or "B is a valid conclusion of the premise A" if and only if $A \to B$ is a tautology.

A set of premises, $\{H_1, H_2, H_3, \dots, H_n\}$ derives a conclusion C iff $H_1 \wedge H_2 \wedge H_3 \wedge \cdots \wedge H_n \to C$ is a tautology.

Valid arguments using propositional logic

Consider the following argument (sequence of propositions)

1. If the professor offers chocolate for an answer, you answer the professor's question
2. The professor offers the chocolate for an answer (p)
3. Therefore, you answer the professor's question (q)

The form (symbolic form) of the above argument is

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

The argument is valid since $\big((p \rightarrow q) \wedge p\big) \rightarrow q$ is a tautology.

Rules of inference for propositional logic

Inference Rule	Tautology	Name
$\begin{array}{l} \boldsymbol{p} \\ \underline{\boldsymbol{p \rightarrow q}} \\ \therefore \boldsymbol{q} \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus Ponens (Mode of affirms)
$\begin{array}{l} \boldsymbol{\sim q} \\ \underline{\boldsymbol{p \rightarrow q}} \\ \therefore \boldsymbol{\sim p} \end{array}$	$(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$	Modus tollens (mode that denies)
$\begin{array}{l} \boldsymbol{p \rightarrow q} \\ \underline{\boldsymbol{q \rightarrow r}} \\ \therefore \boldsymbol{p \rightarrow r} \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{l} \boldsymbol{p \vee q} \\ \underline{\boldsymbol{\sim p}} \\ \therefore \boldsymbol{q} \end{array}$	$((p \vee q) \wedge (\sim p)) \rightarrow q$	Disjunctive syllogism

Rules of inference for propositional logic

Inference Rule	Tautology	Name
$\dfrac{\boldsymbol{p}}{\therefore \boldsymbol{p} \vee \boldsymbol{q}}$	$p \rightarrow (p \vee q)$	Addition
$\dfrac{\boldsymbol{p} \wedge \boldsymbol{q}}{\therefore \boldsymbol{p}}$	$(p \wedge q) \rightarrow p$	Simplification
$\dfrac{\begin{array}{c}\boldsymbol{p} \\ \boldsymbol{q}\end{array}}{\therefore \boldsymbol{p} \wedge \boldsymbol{q}}$	$\big((p) \wedge (q)\big) \rightarrow (p \wedge q)$	Conjunction
$\dfrac{\begin{array}{c}\boldsymbol{p} \vee \boldsymbol{q} \\ \sim\boldsymbol{p} \vee \boldsymbol{r}\end{array}}{\therefore \boldsymbol{q} \vee \boldsymbol{r}}$	$\big((p \vee q) \wedge (\sim p \vee r)\big) \rightarrow (q \vee r)$	Resolution

Ex: Which rule of inference is used in each argument below

1. Alice is a Math major. Therefore, Alice is either a Math major or a CSE major
2. Jerry is a Math major and a CSE major. Therefore, Jerry is a Math major
3. If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed
4. If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today
5. If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.
6. I go swimming or eat an Ice-Cream. I did not go swimming. Therefore, I eat an Ice-Cream.

Ex 1: Determine whether the argument is valid and whether the conclusion must be true

"if $\sqrt{\mathbf{2}} > \frac{\mathbf{3}}{\mathbf{2}}$ then $\left(\sqrt{\mathbf{2}}\right)^{\mathbf{2}} > \left(\frac{\mathbf{3}}{\mathbf{2}}\right)^{\mathbf{2}}$. We know that $\sqrt{\mathbf{2}} > \frac{\mathbf{3}}{\mathbf{2}}$. Therefore, $\left(\sqrt{\mathbf{2}}\right)^{\mathbf{2}} = \mathbf{2} > \left(\frac{\mathbf{3}}{\mathbf{2}}\right)^{\mathbf{2}} = \frac{\mathbf{9}}{\mathbf{4}}$."

Sol:

-> is the argument valid?

YES: Modus ponens inference rule

-> does the conclusion must be true

NO: we cannot conclude that the conclusion is true. Since one of its premises, $\sqrt{2} > \frac{3}{2}$ is false.

-> Indeed, in this case, the conclusion is false, since $2 \ngtr \frac{9}{4}$

EX 2: Show that the hypotheses

$\rightarrow$It is not sunny this afternoon and it is colder than yesterday

$\rightarrow$we will go swimming only if it is sunny

$\rightarrow$if we don't go swimming, then we will go shopping

$\rightarrow$if we go shopping, then we will go to a restaurant

lead to the conclusion

$\rightarrow$we will go to a restaurant

Sol: Main steps:

i. Translate the statement into symbolic form(propositional logic)

ii. Write proof using the sequence of steps that states the hypothesis or apply inference rules to previous steps

Ex:

1. Determine that R is a valid inference from the premises
$$"P \to Q, Q \to R, \text{and } P"$$

2. Show that $\sim P$ follows logically from
$$\sim(P \wedge \sim Q), \sim Q \vee R, \sim R$$

3. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises
$$P \vee Q, Q \to R, P \to M, \sim M$$

4. Show that $\sim P \vee Q, \sim Q \vee R, R \to S$ concludes $P \to S$

Rules of inference for quantified statements

Rule of inference	Name
$\dfrac{\forall x\, P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$	Existential generalization

Ex1: Show that the premises

- A student in section A of the course has not read the book
- everyone in section A of the course passed the first exam

imply the conclusion

- Someone who passed the first exam has not read the book