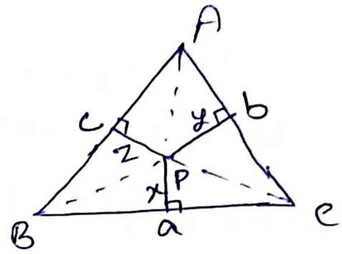
MID TERM EXAM - 22

Course: Calculus and Laplace Transform (MAT1001)

Slot: B21+B22+B23

Ques - 1:

Let x, y, z are the length of perpendicular dropped from point P to the three sides a, b and c of a triangle.



Area(ΔABC)

$= \text{Area}(\Delta PBC) + \text{area}(\Delta PAC) + \text{Area}(\Delta PAB)$

$= \frac{1}{2}x.a + \frac{1}{2}yb + \frac{1}{2}zc$

$A = \frac{1}{2}xa + \frac{1}{2}yb + \frac{1}{2}zc$ ------ ①

$\frac{1}{2}xa + \frac{1}{2}yb + \frac{1}{2}zc - A = 0.$

$\therefore \quad \phi(x,y,z) = \frac{1}{2}xa + \frac{1}{2}yb + \frac{1}{2}zc - A = 0.$

$f(x,y,z) = x^2+y^2+z^2$

Lagrange multiplier method

$(F = f + \lambda\phi)$ ------ ②

$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda\frac{1}{2}a = 0$ ------ ③

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda\frac{1}{2}b = 0$ ------ ④

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z + \lambda\frac{1}{2}c = 0$

multiplying ② by x, ③ by y, ④ by z and adding, we get.

$2(x^2+y^2+z^2) + \lambda(\frac{1}{2}xa + \frac{1}{2}yb + \frac{1}{2}zc) = 0$

$2f + \lambda A = 0 \Rightarrow \lambda = -\frac{2f}{A}$

From ② $2x - \frac{2F}{A}.\frac{1}{2}a = 0 \Rightarrow x = \frac{af}{2A}$

From ③ $2y - \frac{2F}{A}.\frac{1}{2}b = 0 \Rightarrow y = \frac{bF}{2A}$

From ④, $2z - \frac{2F}{A}.\frac{1}{2}c = 0 \Rightarrow z = \frac{cF}{2A}$

putting x, y and z in ① we get.

$A = \frac{1}{2}\frac{aF}{2A}.a + \frac{1}{2}.\frac{bF}{2A}.b + \frac{1}{2}\frac{cF}{2A}.c = \frac{F}{4A}(a^2+b^2+c^2)$

$4A^2 = f(a^2+b^2+c^2) \Rightarrow f = \frac{4A^2}{a^2+b^2+c^2}$

Hence, $x^2+y^2+z^2 = \frac{4A^2}{a^2+b^2+c^2}$

Step-1

2(a) The function is well defined at (0,0)

Step-2
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$
$$= \lim_{x\to 0}\left[\lim_{y\to mx} \frac{x^3-y^3}{x^2+y^2}\right]$$
$$= \lim_{x\to 0} \frac{x^3-m^3x^3}{x^2+x^2m^2}$$
$$= \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2}$$
$$=0$$

Thus, limit exists at (0,0)

Step-3 limit of $f(x)$ at origin = value of the function at origin
$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = f(0,0)=0$$

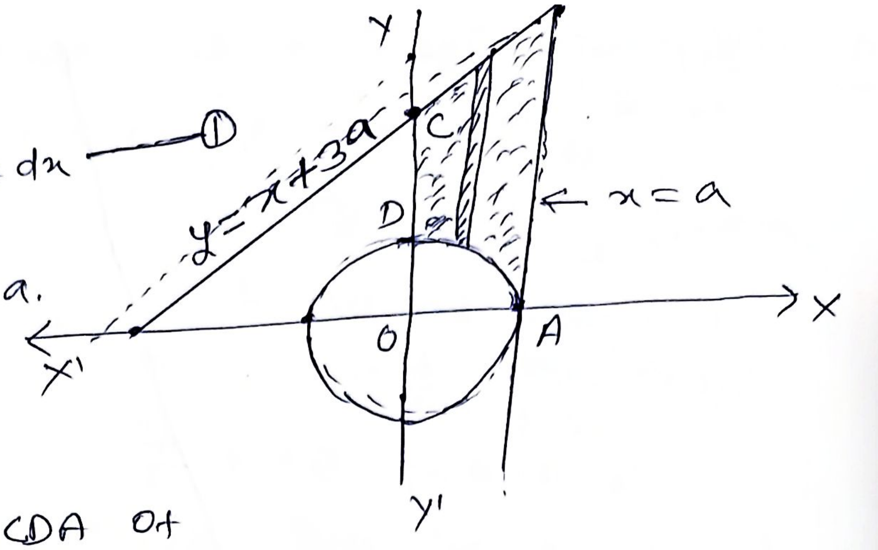
Hence, the function f is continuous at the origin.

2(b)
$$f(x+h, y+k) = \frac{(x+h)(y+k)}{(x+h)+(y+k)}$$
$$f(x,y) = \frac{xy}{x+y}$$
$$\frac{\partial f}{\partial x} = \frac{(x+y)y-xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$
$$\frac{\partial f}{\partial y} = \frac{x^2}{(x+y)^2}$$
$$\frac{\partial^2 f}{\partial x^2} = -\frac{2y^2}{(x+y)^3}$$
$$\frac{\partial^2 f}{\partial y\,\partial x} = \cancel{\frac{2xy}{(x+y)}} \quad \frac{(x+y)^2 2x-2(x+y)x^2}{(x+y)^4} = \frac{(x+y)2x-2x^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{2x^2}{(x+y)^3}$$

Now
$$f(x+h,y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2f_{xx}+2hkf_{xy}+k^2f_{yy}\right)$$
$$= \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3}$$

Ques 3:

$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dy\,dx \quad \text{—— (1)}$$



limit of x are 0 and a.

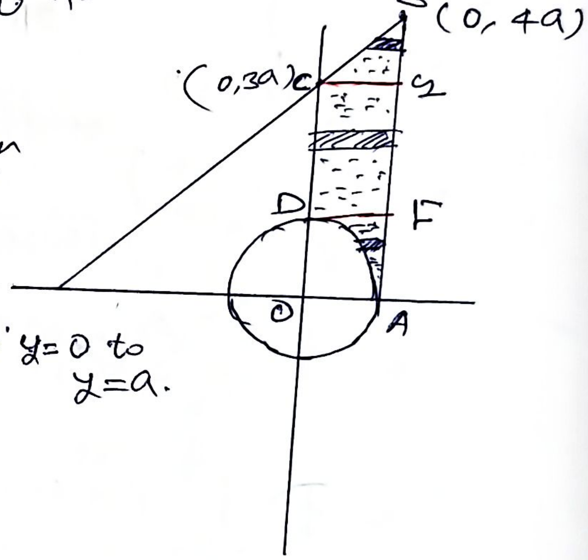
limit of y are

$\sqrt{a^2-x^2}$ and $x+3a$.

The shaded portion ABCDA of the region of the integration is bounded by $y = \sqrt{a^2-x^2}$ and $y = x+3a$.

$x = 0$ and $x = a$.

On changing the order of integration we have to integrate (1) w.r.t. x first and then y.



For this way, we have to divide the region ABCD of integration into three part AFD, DFGC and BCG.

For AFDA

x-limit: $\sqrt{a^2-y^2}$ to $x = a$. y limit: $y = 0$ to $y = a$.

For ~~AFDA~~ FGCD

x-limit: $x = 0$, to $x = a$.

y-limit: $y = a$, $y = 3a$

For BCGB

x-limit: $x = y-3a$, to $x = a$,

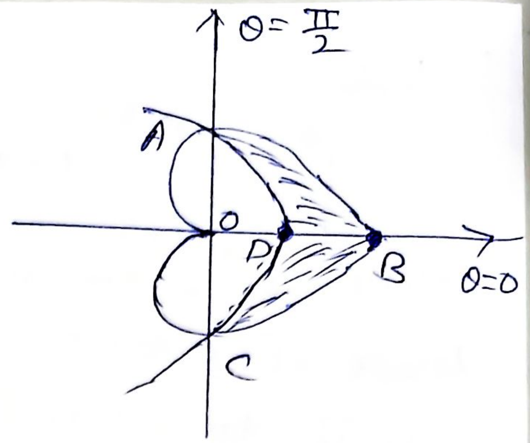
y-limit: $y = 3a$, to $y = 4a$

Hence.
$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dy\,dx = \int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dx\,dy + \int_a^{3a} \int_0^a f(x,y)\,dx\,dy + \int_{3a}^{4a} \int_{y-3a}^{a} f(x,y)\,dx\,dy$$

4

$r = \cancel{a}(1+\cos\theta)$ —— ①

$r = \cancel{a}\,\dfrac{1}{1+\cos\theta}$ —— ②



Solving ① and ②

$(1+\cos\theta)(1+\cos\theta) = 1$

$(1+\cos\theta)^2 = 1$

$1+\cos\theta = 1$

$\cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$

limit of 'r' are $1+\cos\theta$, $\frac{1}{1+\cos\theta}$

limit of 'θ' are, $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Required area = Area ADCBA

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_{\frac{1}{(1+\cos\theta)}}^{1+\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((1+\cos^2\theta + 2\cos\theta) - \frac{1}{(2\cos^2\frac{\theta}{2})^2}\right) d\theta$$

$$= 2\times\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((1+\cos^2\theta + 2\cos\theta) - \frac{1}{4}\sec^4\frac{\theta}{2}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[(1+\cos^2\theta+2\cos\theta) - \frac{1}{4}(1+\tan^2\theta/2)\sec^2\theta/2\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[(1+\cos^2\theta+2\cos\theta) - \frac{1}{4}(1+\tan^2\theta/2)\sec^2\theta/2\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\left(1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right) - \frac{1}{4}(1+\tan^2\theta/2)\sec^2\theta/2\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[1+\frac{1}{2}+\frac{\cos 2\theta}{2}+2\cos\theta - \frac{1}{4}(\sec^2\theta/2 + \tan^2\theta/2\cdot\sec^2\theta/2)\right] d\theta$$

$$= \left[\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta - \frac{1}{4}\left(2\tan\frac{\theta}{2}+\frac{2}{3}\tan^3\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2}+\frac{\pi}{4}+0+2\sin\frac{\pi}{2}-\frac{1}{2}\tan\frac{\pi}{4}-\frac{1}{6}\tan^3\frac{\pi}{4}\right]$$

$$= \frac{3\pi}{4}+2-\frac{1}{2}-\frac{1}{6}$$

$$= \frac{3\pi}{4}+\frac{4}{3}$$

<u>Ques 5</u> Let the given region be R, then R is defines as.

$0 \le z \le 1-x-y$, $0 \le y \le 1-x$, $0 \le x \le 1$.

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2(x+y+z+1)^2} \right]_0^{1-x-y}$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{(x+y+z+1-x-y)^2} - \frac{1}{(x+y+1)^2} \right] dy\, dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy\, dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{y}{4} + \frac{1}{x+y+1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1} \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1} \right] dx$$

$$= -\frac{1}{2} \left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(1+x) \right]_0^1$$

$$= -\frac{1}{2} \left[\frac{1}{2} - \log 2 + \frac{1}{8} \right]$$

$$= -\frac{1}{2} \left[\frac{5}{8} - \log 2 \right]$$

$$= \frac{1}{2} \log 2 - \frac{5}{16}$$