Module-2

Classification of States and Chains

Content

Transient, persistent and ergodic states, limiting behaviour of n-step transition probabilities, stationary distribution, stationary distribution of the chain, types of states, periodicity of a state, Types of Markov chain – irreducible, regular, ergodic, absorbing, mean time spent in a transient state, average number of visits to a state before absorption, probability of absorption, Markov chains with finite and countable state space, reducible chains, continuous time Markov processes, random walk, gambler ruins problem.

Accessibility and Communicating for Markov Chain

i and j

P_{ij}

Row column.

Accessibility

$j \leftarrow i$ Row column

- State j is <u>accessible</u> from state *i* if $p_{ij}^{(n)} > 0$ for some n>= 0, meaning that starting at state i, there is a positive probability of transitioning to state j in **some** number of steps.
- This is written $\boxed{j \leftarrow i}$
- State j is accessible from state i ≠j **if and only if** there is a **directed path** from i to j in the state transition diagram.
- Note that every state is accessible from itself because we allow n=0 in the above definition and $p^{(0)}_{ii}=P(X_0=i \mid X_0=i)=1>0$.

Accessibility and Communicating for Markov Chain

Communicability

$j \longleftrightarrow i$

- States i and j <u>communicate</u> if state j is accessible from state i, and state i is accessible from state j (denote $j \leftrightarrow i$)
- Communicability is
  - <u>Reflexive</u>: Any state communicates with itself
  - <u>Symmetric</u>: If state i communicates with state j, then state j communicates with state i
  - <u>Transitive</u>: If state i communicates with state j, and state j communicates with state k, then state i communicates with state k

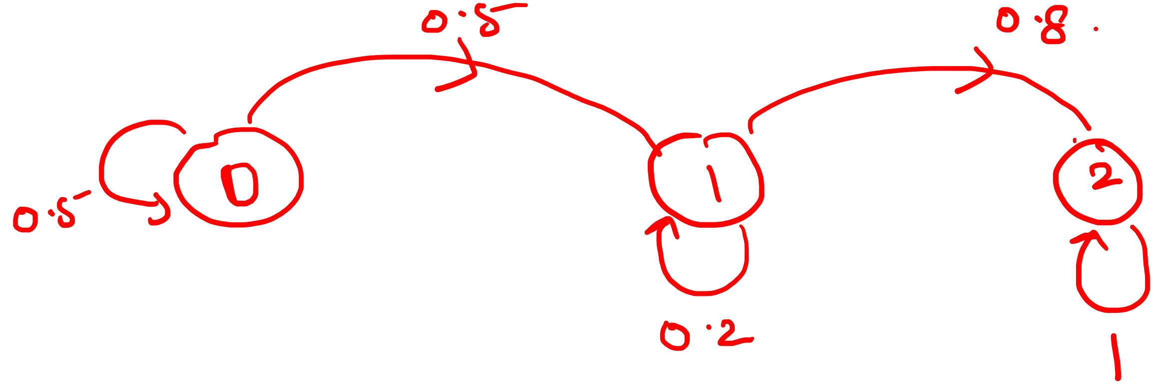
State Classes

- Two states are said to be in the same class if the two states communicate with each other, that is $i \leftrightarrow j$, then i and j are in same class.

  $\{i\}$ $\{j\}$

- Thus, all states in a Markov chain can be partitioned into disjoint classes
  - If states i and j are in the same class, then $i \leftrightarrow j$.
  - If a state i is in one class and state j is in another class, then i and j do not communicate.

Q



States classes

$0 \leftrightarrow 0$

$1 \leftrightarrow 1$

$2 \leftrightarrow 2$.

(3 classes)

$\{0\}$

$\{1\}$

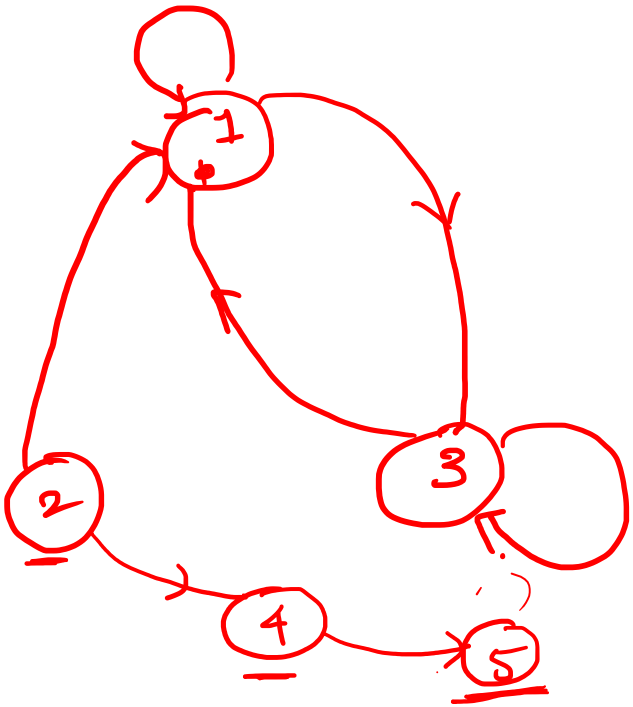
$\{2\}$

$\boxed{\text{3 classes}}$

State ~~clases~~ classes

$\{1,3\}, \{2\}, \{4\}, \{5\}$

$= 4$ classes



State classes

$1 \longleftrightarrow 3$, $\{1,3\}$

~~$\{2\}$~~

$\{2\}$ ~~classes~~

(~~4 classes~~)

$\{4\}$

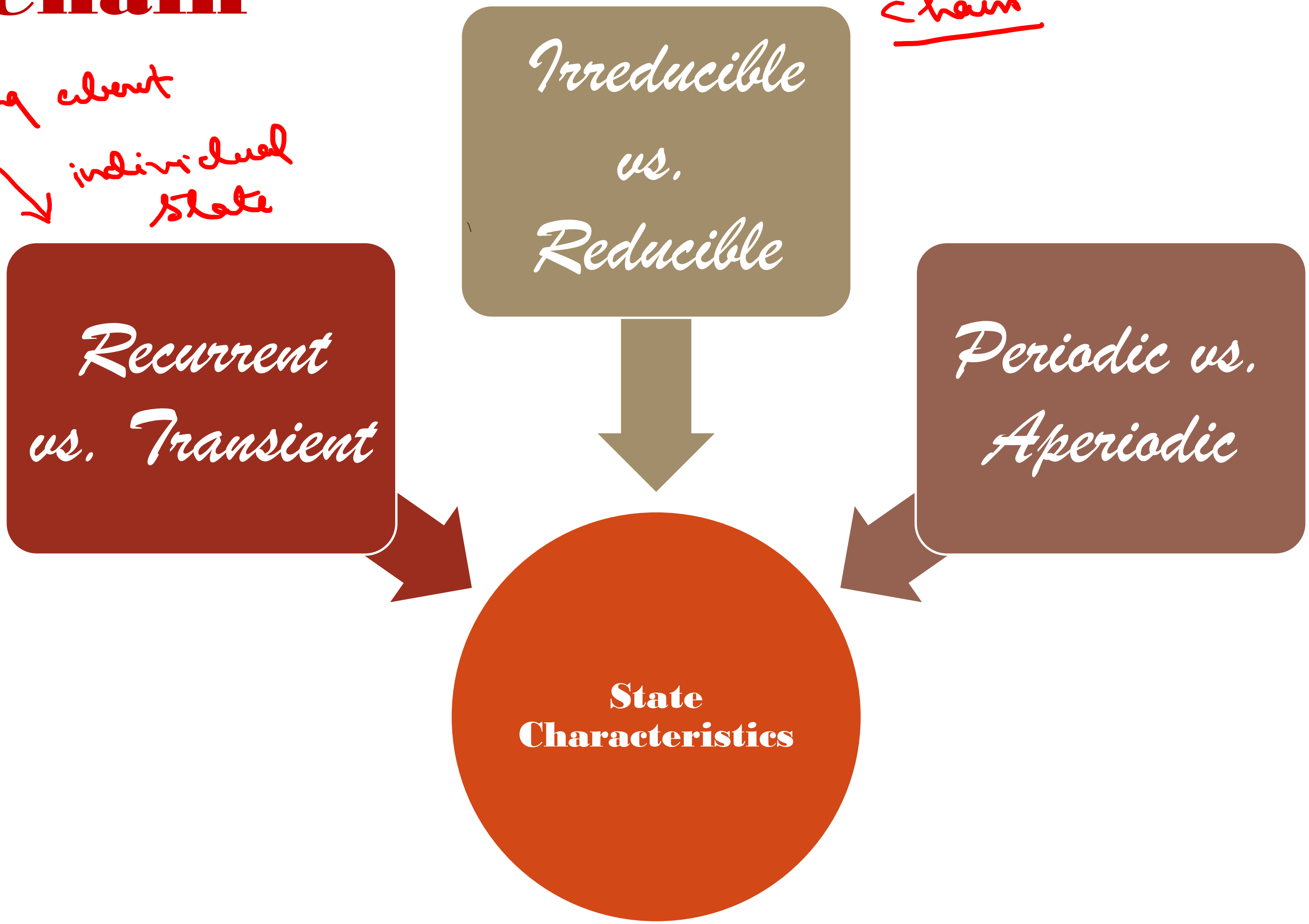
$\{5\}$

(4 classes)

Characteristic of Markov Chain

Talking about individual state

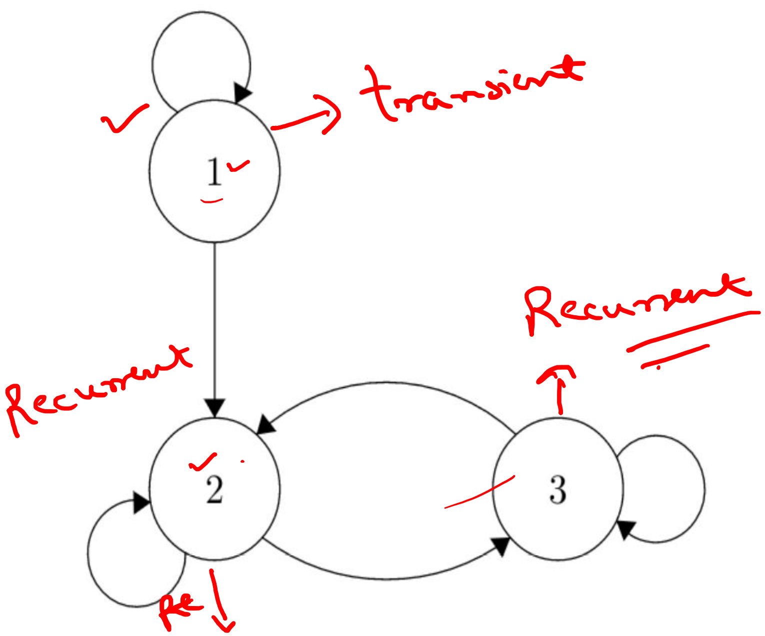
Entire chain



Recurrent vs. Transient (state characteristic)

a. A state j is called a *transient* (or *nonrecurrent*) state if there is a positive probability that the process will never return to j again after it leaves j.

b. A state j is called a *recurrent* (or *persistent*) state if, with probability 1, the process will eventually return to j after it leaves j. A set of recurrent states forms a *single chain* if every member of the set communicates with all other members of the set.



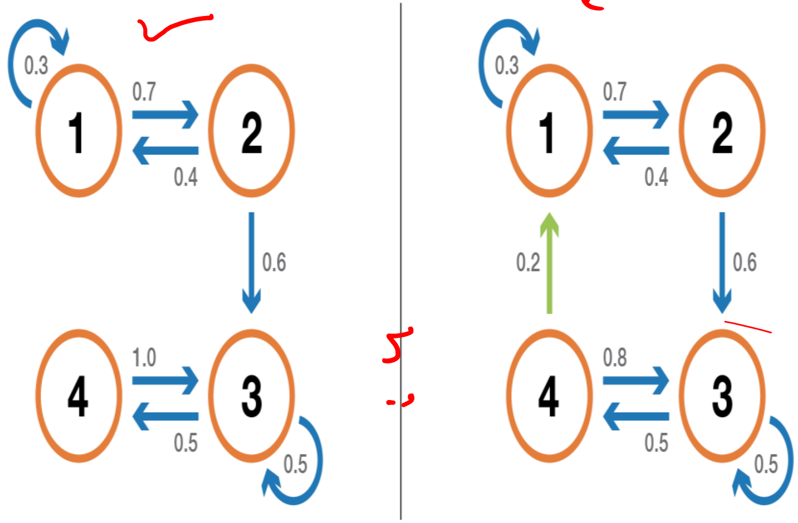
Here, State 1 is transient; if we start in State 1, we might circle back to State 1 for a while, but eventually we'll go to states 2 and 3 and we will bounce around there for all of time, never returning to 1. States 2 and 3 are recurrent; if you start in either one, you know you'll get back eventually.

Irreducible vs. Reducible(State Characteristic)

- A Markov chain is **irreducible** if it is possible to reach any state from any other state (not necessarily in a single time step). If the state space is finite and the chain can be represented by a graph, then we can say that the graph of an irreducible Markov chain is strongly connected (graph theory).

Reducible

irreducible



The chain on the left is not irreducible: from 3 or 4 we can't reach 1 or 2. The chain on the right (one edge has been added) is irreducible: each state can be reached from any other state.

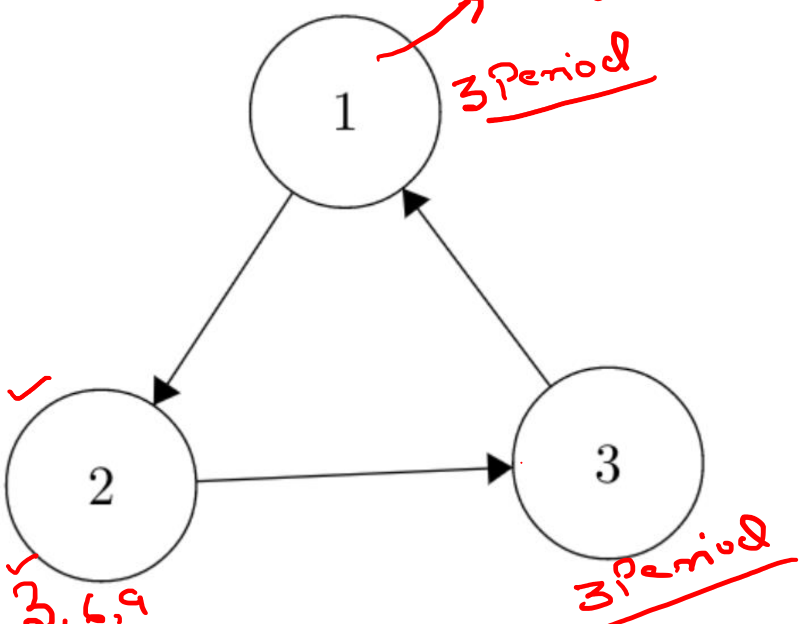
Period (state characteristic)

- Each state in a Markov Chain has a period. The period is defined as the greatest common denominator of the length of return trips (i.e., number of steps it takes to return), given that you start in that state.

$\boxed{\text{3 Periodic}}$

(G.C.D)

3, 6, 9, - - - -



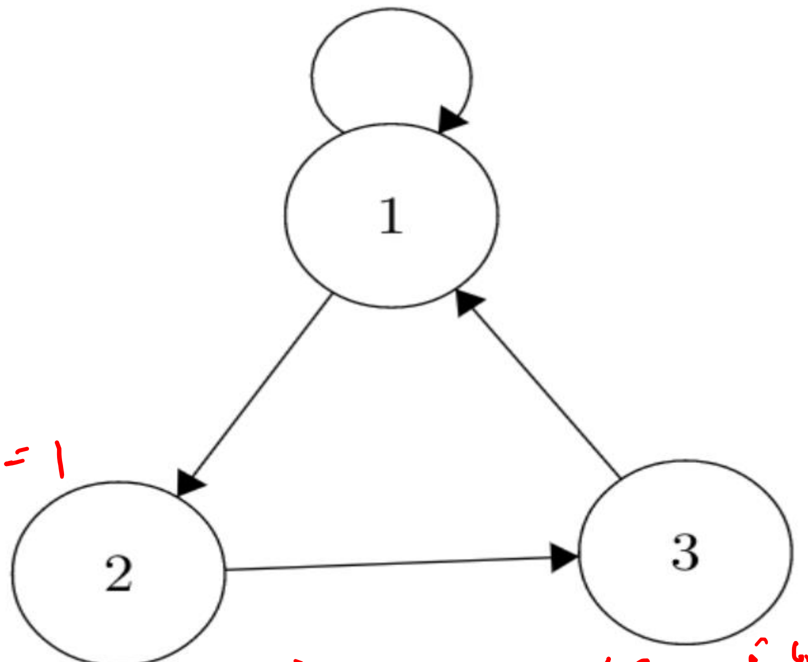
Imagine first that we start in State 1. We know with certainty that in the next step we go to State 2, and then State 3, and then back to State 1. In fact, we know how long every possible path of return to State 1 is: 3 steps, 6 steps, 9 steps, etc. The greatest common denominator of these path lengths is 3, so State 1 (and the other two states) have period 3.

Period (state characteristic)

- Consider this chain

G.C.D

(1, 3, 4, 5, 6 - - -)



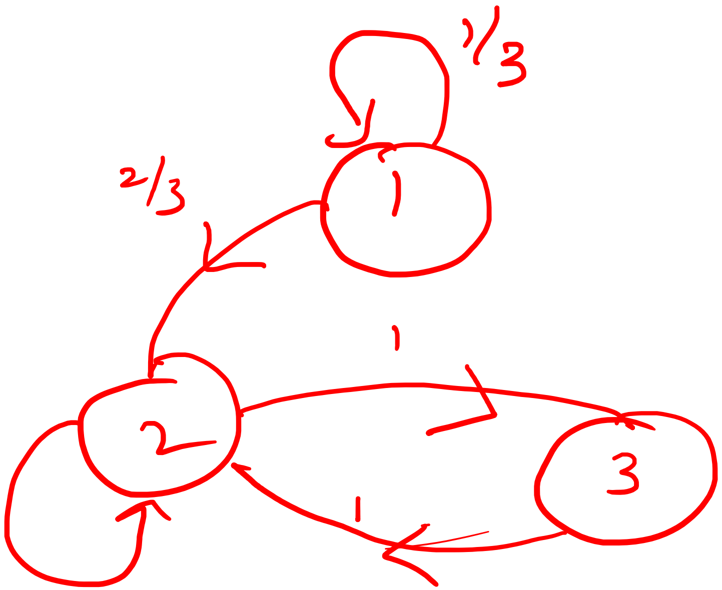
GCD = 1

(3, 4, 5, 6, 7, - -)

(3, 4, 5, 6, 7)

GCD = 1

If we start in State 1, we could automatically just loop back after one period, but we could also go through the whole chain (to State 2, then 3, then 1) in 3 steps. So two possible return lengths are 1 and 3 (there are more, of course), and already the greatest common denominator of these path lengths is 1 So State 1 has period one. What about State 2? If we start at State 2, we know we will go to State 3 and then State 1. However, there we could stay in State 1 for any period of time before returning to State 2. We could return in 3, 4, 5, etc. number of steps. Again, the greatest common denominator here is 1, so State 2 also has period 1. You can check, but the same holds for State 3.



$$
\begin{array}{c|ccc}
 & 1 & 2 & 3 \\
\hline
1 & \frac{1}{3} & & \\
2 & & & \\
3 & & &
\end{array}
$$

GCD

1 State $\longrightarrow$ 1 Period

2 State $\longrightarrow$ 1 Period. (1, 2, 3, 4, - - -)

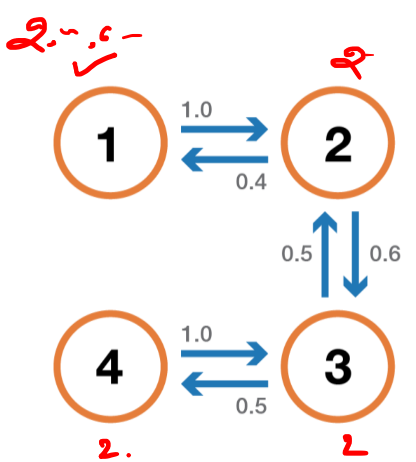
3 State $\longrightarrow$ 1 Period. (2, 3, 4, 5, - - -)

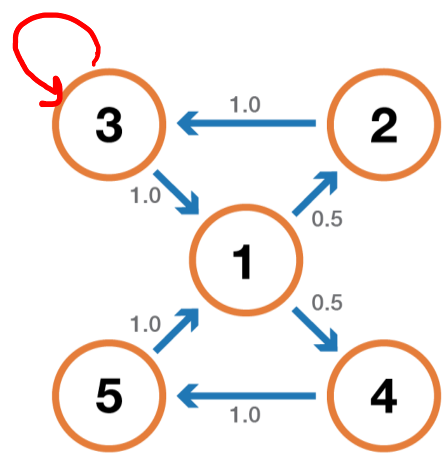
$\downarrow$

1

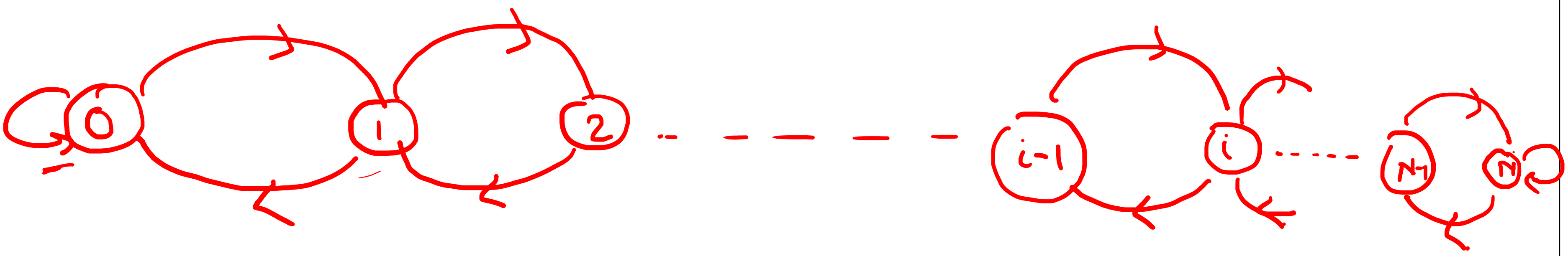
Periodic vs. Aperiodic (chain characteristic)

- If all states in a chain have period 1, that chain is aperiodic.
- A state has period k if, when leaving it, any return to that state requires a multiple of k time steps (k is the greatest common divisor of all the possible return path length). If k = 1, then the state is said to be aperiodic and a whole Markov chain is **aperiodic** if all its states are aperiodic. For an irreducible Markov chain, we can also mention the fact that if one state is aperiodic then all states are aperiodic.





The chain on the left is 2-periodic: when leaving any state, it always takes a multiple of 2 steps to come back to it. The chain on the right is 3-periodic.



The Period of the chain is 1

aperiodic

Regular Transition Matrix

- **A transition matrix is regular, if some power of the matrix contains all positive entries. A Markov chain is a regular Markov Chain if its transition matrix is regular.**

$$\begin{bmatrix} & a_2 \\ a_2 & a_2 \end{bmatrix}_{2\times 2}$$

Decide whether the following transition matrices are ~~following.~~ Regular transition matrix.

(a) $$A = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$A^2, A^3, - - - ..$

(b) $$B = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) $A = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$

Solution Square A.

$$A^2 = \begin{bmatrix} 0.5625 & 0.3125 & 0.125 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$

Since all entries in A^2 are positive, matrix A is regular.

(b) $B = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution Find various powers of B.

$$B^2 = \begin{bmatrix} 0.25 & 0 & 0.75 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ B^3 = \begin{bmatrix} 0.125 & 0 & 0.875 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ B^4 = \begin{bmatrix} 0.0625 & 0 & 0.9375 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Further powers of B will still give the same zero entries, so no power of matrix B contains all positive entries. For this reason, B is not regular.

NOTE If a transition matrix P has some zero entries, and P^2 does as well, you may wonder how far you must compute P^k to be certain that the matrix is not regular. The answer is that if zeros occur in the identical places in both P^k and P^{k+1} for any k, they will appear in those places for all higher powers of P, so P is not regular.

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Regular Matrix: ?

The given matrix is not regular

Equilibrium Vector of Markov Chain or (steady State Condition of Makov Chain)

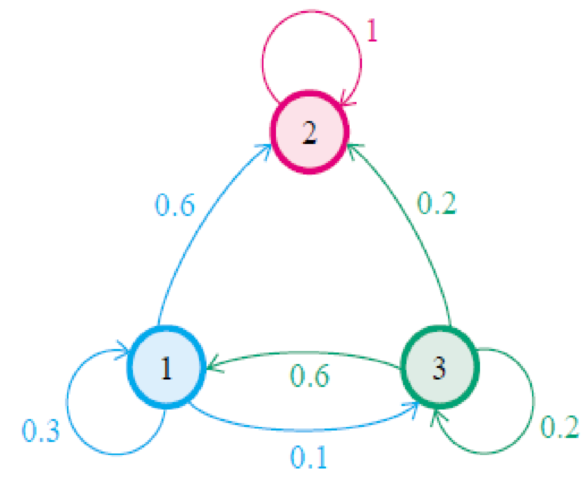
If a Markov chain with transition matrix P is regular, then there is a unique vector V such that, for any probability vector v and for large values of n,

$$v \cdot P^n \approx V.$$

Vector V is called the **equilibrium vector** or the **fixed vector** of the Markov chain.

Absorbing Markov Chain

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0 & 1 & 0 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} = P. \end{array}$$



ABSORBING STATE

State i of a Markov chain is an **absorbing state** if $p_{ii} = 1$.

ABSORBING MARKOV CHAIN

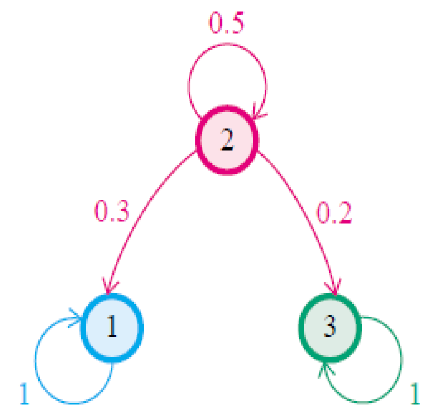
A Markov chain is an **absorbing chain** if and only if the following two conditions are satisfied:

1. the chain has at least one absorbing state; and
2. it is possible to go from any nonabsorbing state to an absorbing state (perhaps in more than one step).

Identify all absorbing states in the Markov chains having the following matrices. Decide whether the Markov chain is absorbing.

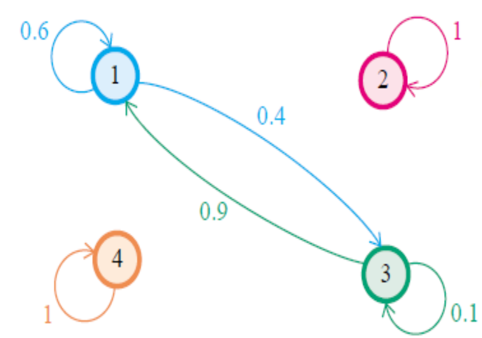
(a)

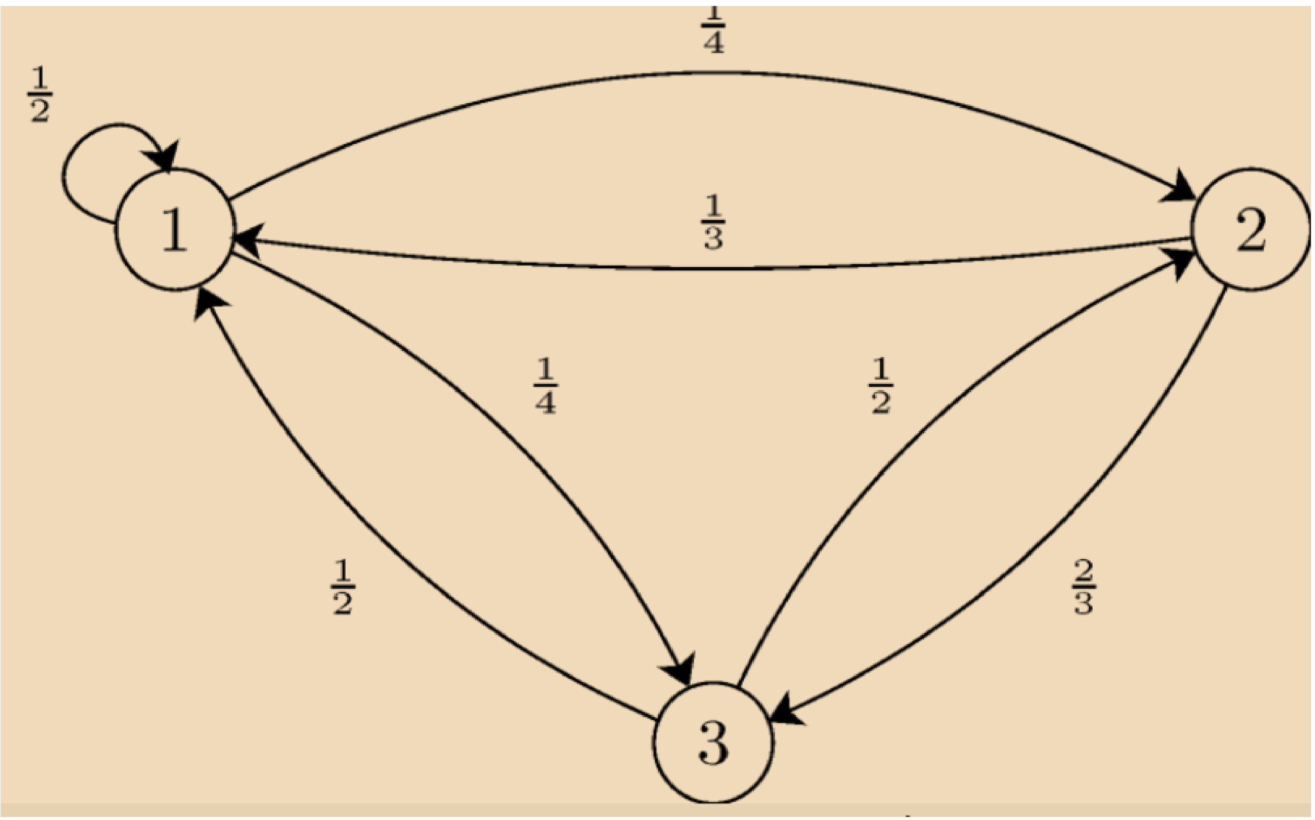
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



(b)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$





a. Is this chain irreducible?

b. Is this chain aperiodic?

c. Find the stationary distribution for this chain.

d. Is the stationary distribution a limiting distribution for the chain?

a. The chain is irreducible since we can go from any state to any other states in a finite number of steps.

b. The chain is aperiodic since there is a self-transition, i.e., $p_{11} > 0$.

c. To find the stationary distribution, we need to solve

$$\pi_1 = \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3,$$
$$\pi_2 = \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3,$$
$$\pi_3 = \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2,$$
$$\pi_1 + \pi_2 + \pi_3 = 1.$$

We find

$$\pi_1 \approx 0.457,\ \pi_2 \approx 0.257,\ \pi_3 \approx 0.286$$

d. The above stationary distribution is a limiting distribution for the chain because the chain is irreducible and aperiodic.