$$= \left(\frac{\quad}{\lambda + \mu} \right) \cdot \frac{\quad}{\lambda + \mu} \qquad [\text{since } \Gamma(r+1) = r!]$$

23.12-1. Inter-Relationship Between L_s, L_q, W_s, W_q

It can be proved under (rather) general conditions of arrival, departure, and service discipline that the formulae,

$$L_s = \lambda W_s, \qquad \text{...(23.69)}$$

and

$$L_q = \lambda W_q, \qquad \text{...(23.70)}$$

will hold in general. These formulae act as key points in establishing the strong relationships between W_s, W_q, L_s and L_q which can be found as follows.

By definition,

$$W_q = W_s - 1/\mu. \qquad \text{...(23.71)}$$

Thus, multiplying both sides by λ and substituting the values from (23.69) and (23.70),

$$L_q = L_s - \lambda/\mu. \qquad \text{...(23.72)}$$

This means that one of the four expected values (together with λ and μ) should immediately yield the remaining three values.

Q. In $(M \mid M \mid 1) : (\infty \mid FCFS)$ model obtain p.d.f. of waiting time (excluding service time) and hence obtain $E(W_q)$, $E(W_s)$, $E(L_q)$, $E(L_s)$. **[Garhwal M.Sc. (Stat.) 93]**

23.12-2. Illustrative Examples on Model I

Example 1. *A TV repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?*

[JNTU 2002; Agra 98, 93; Karnataka B.E. (CSE) 93; Meerut (Maths.) 91]

Solution. Here, $\mu = 1/30$, $\lambda = 10/(8 \times 60) = 1/48$. Therefore, expected number of jobs are

$$L_s = \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda} = \frac{1/48}{1/30 - 1/48} = 1\tfrac{2}{3} \text{ jobs.} \qquad \textbf{Ans.}$$

Since the fraction of the time the repairman is busy (*i.e.* traffic intensity) is equal to λ/μ, the number of hours for which the repairman remains busy in a 8-hour day is

$$= 8 \,.\, (\lambda/\mu) = 8 \times 30/48 = 5 \text{ hours.}$$

Therefore, the time for which the repairman remains idle in 8-hour day = (8 − 5) hours = 3 hours. **Ans.**

Example 2. *At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes ? It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.* **[Meerut (Maths.) 39, 96]**

Solution. Here, $\lambda = 15/60 = 1/4$ customer/minute, $\mu = ?$ Prob. [waiting time ≥ 12] $= 1 - 0.90 = 0.10$.

Therefore,
$$\int_{12}^{\infty} \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda)w}\, dw = 0.10 \quad \text{or} \quad \lambda \left(1 - \frac{\lambda}{\mu}\right) \left[\frac{e^{-(\mu - \lambda)w}}{-(\mu - \lambda)} \right]_{12}^{\infty} = 0.10$$

or
$$e^{(3 - 12\mu)} = 0.4\,\mu \quad \text{or} \quad 1/\mu = 2.48 \text{ minute per service.} \qquad \textbf{Ans.}$$

Example 3. *Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call assumed to be distributed exponentially with mean 3 minutes. Then,* (*a*) *What is the probability that a person arriving at the booth will have to wait ?*

[Meerut 2002; I.A.S. (Main) 91]

(*b*) *What is the average length of the queues that form from time to time ?* [Meerut 2002; Kanpur 2000]

(*c*) *The telephone department will install a second booth when convinced that an arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth ?*

[Meerut 2002; I.A.S. (Maths) 91]

Solution. Here, $\lambda = 1/10$ and $\mu = 1/3$

(*a*) Prob. $(W > 0) = 1 - P_0 = \lambda/\mu$ [from eqn. (23.60)] $= \frac{1}{10} \times \frac{3}{1} = \frac{3}{10} = 0.3$ **Ans.**

(*b*) $(L \mid L > 0) = \mu/(\mu - \lambda) \quad = ⅓/(⅓ - \frac{1}{10}) = 1.43$ persons [from eqn. (23.68)] **Ans.**

(*c*) $W_q = \lambda/\mu(\mu - \lambda)$ [from eqn. (23.65)]

Since $W_q = 3$, $\mu = ⅓$, $\lambda = \lambda'$ (say) for second booth, therefore

$$3 = \frac{\lambda'}{⅓\,(⅓ - \lambda')}, \quad \text{giving } \lambda' = 0.16.$$

Hence, increase in the arrival rate $= 0.16 - 0.10 = 0.06$ arrivals per minute. **Ans.**

Example 4. *As in Example 3, a telephone booth with Poisson arrivals spaced 10 minutes apart on the average, and exponential call lengths averaging 3 minutes.*

(*a*) *What is the probability that an arrival will have to wait more than 10 minutes before the phone is free ?*

(*b*) *What is the probability that it will take him more than 10 minutes altogether to wait for phone and complete his call ?*

(*c*) *Estimate the fraction of a day that the phone will be in use.*

(*d*) *Find the average number of units in the system.*

Solution. Here $\lambda = 0.1$ arrival per minute, $\mu = 0.33$ service per minute.

(*a*) Prob. [waiting time ≥ 10] $= \int_{10}^{\infty} \Psi(w)\, dw = \int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu - \lambda)w}\, dw$

$$= -\frac{\lambda}{\mu}\left[e^{-(\mu-\lambda)w}\right]_{10}^{\infty} = 0.3e^{-2.3} = 0.03 \text{ [see Exponential Tables]}$$ **Ans.**

(*b*) Prob. [waiting time in the system ≥ 10]

$$= \int_{10}^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)w}\, dw = e^{-10(\mu-\lambda)} = e^{-2.3} = 0.10 \text{ [see Exp. Table]}$$ **Ans.**

(*c*) The fraction of a day that the phone will be busy = traffic intensity $\rho = \lambda/\mu = 0.3$.

(*d*) Average number of units in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{10}}{⅓ - \frac{1}{10}} = \frac{3}{7} = 0.43 \text{ customer.}$$ **Ans.**

Example 5. (a) *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average 36 minutes. Calculate the following :*

(*i*) *The average number of trains in the queue.*

(*ii*) *The probability that the queue size exceeds 10.* [JNTU 99, 98; Raj. Univ. (M. Phil.) 93, 90]

If the input of trains increases to an average 33 per day, what will be change in (i) and (ii) ? Establish the formula you use in your calculations.

Solution. Here [JNTU 2002; Agra 98; I.A.S. (Main) 90]

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute; } \mu = \frac{1}{36} \text{ trains/minute, and } \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75.$$

(*i*) $$L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1 - 0.75} = 3 \text{ trains}.$$

(*ii*) Prob [*queue size* ≥ 10] $= \rho^{10} = (0.75)^{10} = 0.06.$ **Ans.**

When the input increases to 33 trains per day $\lambda = \frac{1}{43}$, $\mu = \frac{1}{36}$.

Therefore, $\rho = \lambda/\mu = 36/43 = 0.84$. Hence,

(*i*) $L_s = 0.84/0.16 = 5$ trains (*ii*) Prob (queue size $\geq 10) = (0.84)^{10} = 0.2$ (approx.)

(b) *Trains arrive at the yard every 20 minutes and the service time is 40 minutes. If the line capacity of the yard is limited to 6, find*

(i) the probability the yard is empty.

(ii) the average number of trains in the system. **[JNTU (B. Tech.) 2003]**

Solution. Proceed as in part (a).

Example 6. *In the above problem calculate the following :*

(*i*) *Expected waiting time in the queue.*

(*ii*) *The probability that number of trains in the system exceeds 10.* **[JNTU (B. Tech.) 98]**

(*iii*) Average number of trains in the queue.

Solution. Here $\lambda = 1/48$, $\mu = 1/36$ and $\rho = 0.75$.

(*i*) Expected waiting time in the queue is

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1/48}{1/36\,(1/36 - 1/48)} = 108 \text{ minutes } \textit{or 1 hr 48 mts.}$$

(*ii*) $P(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06.$

(*iii*) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{(1/48)^2}{1/36\,(1/36 - 1/48)} = \dfrac{108}{48} = 2.25$ or nearly 2 trains.

Example 7. *Consider an example from a maintenance shop. The inter-arrival times at toolcrib are exponential with an average time of 10 minutes. The length of the service time (amount of time taken by the toolcrib operator to meet the needs of the maintenance man) is assumed to be exponentially distributed, with mean 6 minutes. Find :*

(*i*) *The probability that a person arriving at the booth will have to wait.*

(*ii*) *Average length for the queue that forms and the average time that an operator spends in the Q-system.*

(*iii*) *The manager of the shop will install a second booth when an arrival would have to wait 10 minutes or more for the service. By how much must the rate of arrival be increased in order to justify a second booth.*

(*iv*) *The probability that an arrival will have to wait for more than 12 minutes for service and to obtain his tools.*

(*v*) *Estimate the fraction of the day that toolcrib operator will be idle.*

(*vi*) *The probability that there will be six or more operators waiting for the service.*

Solution. Here $\lambda = 60/10 = 6$ per hour, $\mu = 60/6 = 10$ per hour **[Virbhadrah 2000]**

(*i*) *A person will have to wait if the service facility is not idle.*

Probability that the service facility is idle = Probability of no customer in the system (P_0)

$\therefore$ Probability of waiting $= 1 - P_0 = 1 - (1 - \rho) = \rho = \lambda/\mu = 6/10 = 0.6$ **Ans.**

(*ii*) $L_q = \rho^2/(1 - \rho) = (0.6)^2/(1 - 0.6) = 0.9$

$L_s = L_q + \lambda/\mu = 0.9 + 0.6 = 1.5$. $\therefore$ $W_s = L_s/\lambda = 1.5/6 = 1/4$ hours

(*iii*) $W_q = L_q/\mu = 0.9/6$ hrs. $= 9$ minutes.

Let λ be the arrival rate when a second booth is justified, *i.e.*, $W_q \geq 10$ minutes.

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{60}.$$

$\therefore$ $6\lambda = 10(10 - \lambda)$ or $16\lambda = 100$ or $\lambda = 6.25$.

Hence if the arrival rate exceeds 6.25 per hour, the second booth will be justified.

(*iv*) Probability of waiting for 12 minutes or more is given by

$$\text{Prob. } (W \geq 12) = \int_{12}^{\infty} \rho(\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw = -\rho\left[e^{-(\mu - \lambda)w}\right]_{12/60}^{\infty}$$

$$= \rho e^{-(\mu - \lambda)12/60} = 0.6\, e^{-(10-6).12/60} = 0.6\, e^{-4/5} = 0.27.$$ **Ans.**

(*v*) $P_0 = 1 - \rho = 0.4$, 40% of the time of toolcrib operator is idle.

(*vi*) Probability of six or more operators waiting for the service $= \rho^6 = (0.6)^6$.

Example 8. *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient.* **[JNTU (B. Tech.) 2004; I.A.S. (Main) 93]**

Solution. Here $\lambda = \dfrac{96}{24 \times 60} = \dfrac{1}{15}$ patient/minute, $\mu = \dfrac{1}{10}$ patient/minute

Expected number of patients in the waiting line

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(1/15)^2}{1/10\,(1/10 - 1/15)} = 1\tfrac{1}{3} \text{ patients.}$$

But, $L_q = 1\frac{1}{3}$ is reduced to $L_q' = \frac{1}{2}$.

Therefore, substituting $L_q' = \frac{1}{2}$, $\lambda' = \lambda = 1/15$ in the formula $L_q' = \dfrac{\lambda'^2}{\mu'(\mu' - \lambda')}$, we get

$$\tfrac{1}{2} = \frac{(1/15)^2}{\mu'(\mu' - 1/15)}$$

which gives $\mu' = 2/15$ patient/minute.

Hence the average rate of treatment required is $1/\mu' = 7.5$ minutes.

Consequently, the decrease in the average rate of treatment $= 10 - 15/2 = 5/2$ minutes; and the budget per patient $= 100 + 5/2 \times 10 =$ Rs. 125. So in order to get the required size of the queue, the budget should be increased from Rs. 100 to Rs. 125 per patient.

Example 9. *The mean rate of arrival of planes at an airport during the peak period is 20 hour, but the actual number of arrivals in any hour follows a Poisson distribution with the respective averages. When there is congestion, the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.*

(*i*) *How many planes would be flying over the field in the stack on an average in good weather and in bad weather ?*

(*ii*) *How long a plane would be in the stack and in the process of landing in good and in bad weather ?*

(*iii*) *How much stack and landing time to allow so that priority to land out of order would have to be requested only one time in twenty ?* **[Agra 98]**

Solution. Here

$$\lambda = 20 \text{ planes/hour}$$

and

$$\mu = \begin{cases} 60 \text{ planes/hour in good weather} \\ 30 \text{ planes/hour in bad weather} \end{cases}$$

(*i*) $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \begin{cases} 20^2/60(60 - 20) = 1/6 \text{ in good weather} \\ 20^2/30(30 - 20) = 1/3 \text{ in bad weather} \end{cases}$ **Ans.**

(*ii*) $W_s = \dfrac{1}{\mu - \lambda} = \begin{cases} 1/(60 - 20) = 1/40 \text{ hrs. in good weather} \\ 1/(30 - 20) = 1/10 \text{ hrs. in bad weather} \end{cases}$ **Ans.**

(*iii*) The waiting time 'w' taken by the plane in landing and in the stack is given by

$$\int_0^w (\mu - \lambda)\, e^{-(\mu - \lambda)w}\, dw = 0.95$$

or

$$(\mu - \lambda)\left[\frac{e^{-(\mu - \lambda)w}}{\lambda - \mu}\right]_0^w = 0.95$$

or

$$1 - e^{-(\mu - \lambda)w} = 0.95$$

or

$$e^{-(\mu - \lambda)w} = 0.05 \qquad \text{(A)}$$

Now the value of w can be determined in both the cases.

Case I. *When weather is good* :

Substituting $\lambda = 20$, $\mu = 60$ in eqn. (A), $e^{-40w} = 0.05$.

Taking the logarithm of both sides to the base e (instead of 10),

$$-40w = \log_e (0.05) = -2.9957 \qquad [\because \log_e e = 1, \therefore \log_e (0.05) = -2.9957]$$

or
$$w = \frac{2.9957}{40} = 0.075 \text{ hour} = 4.5 \text{ minutes } \textbf{Ans.}$$

Case 2. *When weather is bad :*

Substitution $\lambda = 20$, $\mu = 30$ in eqn. (A),

$$e^{-10w} = 0.05$$

Solving this equation as in case I above, we get

$$w = 0.3 \text{ hour} = 18 \text{ minutes.}$$ **Ans.**

Example 10. *A refinery distributes its products by trucks, loaded at the loading dock. Both company trucks and independent distributor's trucks are loaded. The independent firms complained that sometimes they must wait in line and thus lose money paying for a truck and driver, that is only waiting. They have asked the refinery either to put in a second loading dock or to discount prices equivalent to the waiting time. Extra loading dock cost Rs. 100/- per day whereas the waiting time for the independent firms cost Rs. 25/- per hour. The following data have been accumulated. Average arrival rate of all trucks is 2 per hour and average service rate is 3 per hour. Thirty per cent of all trucks are independent. Assuming that these rates are random according to the Poisson distrbutions, determine :*

(*a*) *the probability that a truck has to wait*

(*b*) *the waiting time of a truck that waits, and*

(*c*) *the expected cost of waiting time of independent trucks per day.*

Is it advantageous to decide in favour of a second loading dock to ward off the complaints ?

Solution. We are given that $\lambda = 2$ per hour and $\mu = 3$ per hour.

(*a*) The probability that a truck has to wait for service is the utilization factor,

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} = 0.66.$$

(*b*) The waiting time of a truck that waits is

$$(W \mid W > 0) = \frac{W_s}{\text{Prob } (W > 0)} = \left[\frac{\lambda}{\mu(\mu - \lambda)} / (\lambda / \mu)\right]$$

$$= \frac{1}{\mu - \lambda} = \frac{1}{3 - 2} = 1 \text{ hour.}$$

(*c*) The total expected waiting time of independent trucks per day is given by :

Expected waiting time = Trucks per day × % Independent truck × Expected waiting time per truck

$$= (2 \times 8)\,(0.3W_q) = 16 \times 0.3 \times \frac{\lambda}{\mu(\mu - \lambda)} = 4.8 \times \frac{2}{3(3 - 2)} = 3.2 \text{ hour per day.}$$

$\therefore$ Expected Cost = Rs. (3.2 × 25) = Rs. 80.

Example 11. (a) *Barber A takes 15 minutes to complete one hair cut. Customers arrive in his shop at an average rate of one every 30 minutes and the arrival process is Poisson. Barber B takes 25 minutes to complete one hair-cut and customers arrive in his shop at an average rate of one every 50 minutes, the arrival process being Poisson during steady state.*

(*i*) *Where would you expect the bigger queue ?*

(*ii*) *Where would you require more times waiting included, to complete a hair-cut.*

(b) *In a hair dressing salon with one barber the customer arrival follows Poission distribution at an average rate of one every 45 minutes. The service time is exponentially distributed with a mean of 30 minutes. Find (i) Average number of customers in the salon.*

(*ii*) *Average waiting time of a customer before service.*

(*iii*) *Average idle time of the barber.* **[VTU (BE Mech.) 2003]**

Solution. Proceed as in above solved examples.

Example 12. (a) *An airlines organisation has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of* *customers arriving during any given period is Poisson distributed with an arrival rate of eight per hour and that the reservation clerk can serve a customer in six minutes on an average, with an exponentially distributed service time.*

(i) What is the probability that the system is busy ?

(ii) What is the average time a customer spends in the system ?

(iii) What is the average length of the queue and what is the number of customers in the system ? **[C.A., Nov. 96]**

(b) *If for a period of 2 hours in a day (8–10 AM) planes arrive at the aerodrome for every 20 minutes but the service time continues to remain 32 minutes then calculate for this period :*

(i) the probability that the aerodrome is empty (ii) average queue length on the assumption that the time capacity of the aerodrome is limited to 6 planes. **[JNTU (Mech. & Prod.) May 2004]**

Solution. (a) According to the given information :

Mean arrival rate, $\lambda = 8$ customers per hour

Mean service rate, $\mu = \dfrac{60}{6} = 10$ customers per hour

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{8}{10} \quad \text{or} \quad \frac{4}{5}$$

(i) The probability that the system is busy is given by :

$$1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = 0{\cdot}8\text{, } i.e., 80\% \text{ of the time system is busy.}$$

(ii) The average time a customer spends in the system is given by :

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{10 - 8} = \frac{1}{2} \text{ hours or 30 minutes}$$

(iii) The average length of the queue is given by :

$$L_q = \frac{\lambda}{\mu} \times \frac{\lambda}{\mu - \lambda} = \frac{8}{10} \times \frac{8}{10 - 8} = 3{\cdot}2 \text{ customers}$$

The average number of customers in the system is given by :

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{8}{10 - 8} \text{ or 4 customers.}$$

(b) [**Hint :** Proceed as (a)]

Example 13. *Customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.* **[Poona (M.B.A.) 98]**

Solution. Here we are given :

$$\lambda = 20 \text{ per hour}, \mu = \frac{60 \times 60}{100} = 36 \text{ per hour}$$

The average waiting time of a customer in the queue is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{36(36 - 20)} = \frac{5}{36 \times 4} \text{ hours or } \frac{5 \times 3600}{36 \times 4}\text{, } i.e., \text{ 125 seconds.}$$

The average waiting time of a customer in the system is given by :

$$W_s = \frac{1}{(\mu - \lambda)} = \frac{1}{(36 - 20)} \text{ or } \frac{1}{16} \text{ hour } i.e., \text{ 225 seconds.}$$

Example 14. *Customers arrive at a one-window drive according to a Poisson distribution with mean of 10 minutes and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have to wait outside this space. Calculate :*

(i) Probability that an arriving customer can drive directly to the space in front of the window.

(ii) Probability that an arriving customer will have to wait outside the directed space.

(iii) How long an arriving customer is expected to wait before getting the service ?

[JNTU (B. Tech.) 2003; SJMIT (BE Mech.) 2002; C.A. (May) 98]

Solution. From the given information, we find that :

Mean arrival rate, $\lambda = 6$ customers per hour

and mean service rate, $\mu = 10$ customers per hour

(i) Probability that an arriving customer can drive directly to the space in front of the window is given by :

$$P_0 + P_1 + P_2 = \left(1 - \frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right]$$

$$= \left(1 - \frac{6}{10}\right)\left[1 + \frac{6}{10} + \left(\frac{6}{10}\right)^2\right] = \frac{98}{1225} \text{ or } 0{\cdot}784$$

(ii) Probability that an arriving customer will have to wait outside the directed space is given by :

$$1 - (P_0 + P_1 + P_2) = 1 - 0{\cdot}784 = 0{\cdot}216 \text{ or } 21{\cdot}6\%$$

(iii) Expected waiting time of a customer being getting the service is given by :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{6}{10(10-6)} = \frac{3}{20} \text{ hr. or 9 minutes.}$$

Example 15. *The rate of arrival of customers at a public telephone booth follows Poisson distribution, with an average time of 10 minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution, with mean time of 3 minutes.*

(i) What is the probability that a person arriving at the booth will have to wait ?

(ii) What is the average length of the non-empty queues that form from time to time ?

(iii) The Mahanagar Telephone Nigam Ltd. will install a second booth when it is convinced that the customers would expect waiting for at least 3 minutes for their turn to make a call. By how much time should the flow of customers increase in order to justify a second booth ?

(iv) Estimate the fraction of a day that the phone will be in use. **[C.A., (May) 1999; Delhi (M. Com.) 99]**

Solution. Here we are given :

$$\lambda = \frac{1}{10} \times 60 \text{ or 6 per hour and } \mu = \frac{1}{3} \times 60 \text{ or 20 per hour}$$

(i) Probability that a person arriving at the booth will have to wait

$$= 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{6}{20} \text{ or } 0{\cdot}3.$$

(ii) Average length of non-empty queues

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 6} = 1{\cdot}42.$$

(iii) The installation of a second booth will be justified if the arrival rate is greater than the waiting time. Now, if λ' denotes the increased arrival rate, expected waiting time :

$$W_q' = \frac{\lambda'}{\mu(\mu - \lambda')} \Rightarrow \frac{3}{60} = \frac{\lambda'}{20(20 - \lambda')} \text{ or } \lambda' = 10.$$

(ii) P_0 = Prob. of no customer in the system $= 1 - \frac{\lambda}{\mu} = 0{\cdot}5$

Thus 50% of time an arrival will not have to wait.

(iii) Average time spent by a customer $= \frac{1}{\mu - \lambda} = \frac{1}{5}$ hour or 12 minutes

(iv) Average queue length $= \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{5 \times 5}{10(10-5)} = 0{\cdot}5$

(v) The management will deploy the person exclusively for Xeroxing when the average time spent by a customer exceeds 15 minutes. We wish to calculate the arrival rate that will lead to such situation. Let this arrival rate be λ'. Then

$$\frac{1}{\mu - \lambda'} > \frac{15}{60} \text{ or } \frac{1}{10 - \lambda'} > \frac{1}{4}, \textit{i.e.}, \lambda' > 6.$$

Hence, if the arrival rate of customers is greater than 6 customers per hour, the average time spent by a customer will exceed 15 minutes.

Example 16. *Telephone users arrive at a booth following a Poisson distribution with an average time of 5*