Elementary Renewal theorem:-

Statement :- For a Poisson Process which is a renewal Process with exponential inter-arrival times x_n, we know that

$E[x_n] = \frac{1}{a}$.

$$M(t) = at$$

(Mean of Exp dist.)

$a = \frac{1}{E[x_n]}$

and therefore

$$\frac{M(t)}{t} = a$$

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In general case the result is

$$\boxed{\frac{M(t)}{t} = \frac{1}{\mu}}$$

holds as $t \to \infty$

Theorems:- with Prob. 1.

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty$$

when $\mu = E[x_n] < \infty$

[S_n = n^{th} time of renewal]

Proof:- By definition of $N(t)$

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we have the interval $[0,t]$.

$$S_{N(t)} \leq t \leq S_{N(t)+1} \quad — ①$$

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)} \quad — ②$$

[$\because$ since $\frac{S_n}{n} \to \mu$ with Prob. 1 as $n \to \infty$

and since $N(t) \to \infty$ with Prob. 1 as $t \to \infty$].

$$\frac{S_{N(t)}}{N(t)} \longrightarrow \mu \quad \text{as} \quad t \to \infty \quad \text{with Prob. 1.} \quad — ③$$

Now

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}.$$

$$\frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \longrightarrow \mu \quad \text{as } t \to \infty.$$

$$\because \quad \frac{N(t)+1}{N(t)} = 1 \quad — ④$$

Taking limits in eq (2) as $t \to \infty$.

it then follows in views of ③ and ④

with Prob. 1.

$$\frac{t}{N(t)} \to \mu \quad \text{i.e} \quad \frac{N(t)}{t} \to \frac{1}{\mu}$$