Gradient

Let us consider a scalar function of three variables T(x,y,z).

A theorem on partial derivative states that –

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

This gives how T changes when we alter all three variables by infinitesimal amounts dx, dy and dz.

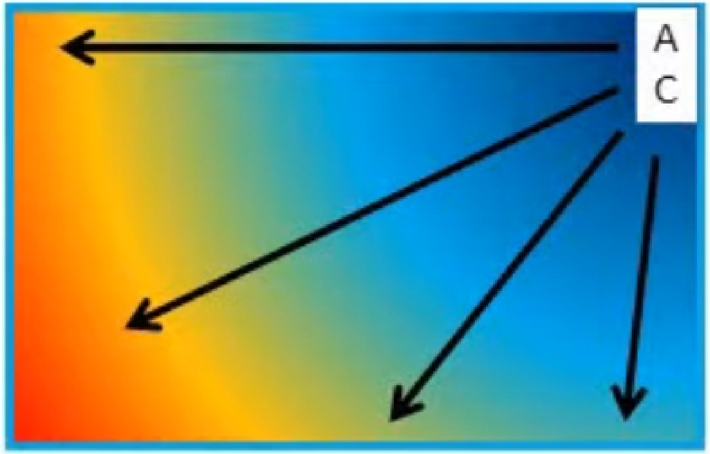
Above equation is reminiscent of a dot product –

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \quad = \quad \overrightarrow{\nabla T} \cdot \overrightarrow{dl}$$

Where $\overrightarrow{\nabla T} = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$ is a gradient of T.

$\overrightarrow{\nabla T}$ is a vector quantity with three components.

Significance of Gradient



- Like any vector, gradient has a magnitude and direction.
- $dT = \vec{\nabla T} \cdot \vec{dl} = |\vec{\nabla T}|\,|\vec{dl}|\, cos\theta$

  Where θ is angle between $\vec{\nabla T}$ and $\vec{dl}$
- If we fix the magnitude $|\vec{dl}|$ and search in various directions for different values of θ, maximum change in T will occur for $cos\theta = 1$ $i.e.$ for $\theta = 0$.
- It means that for a fixed distance $|\vec{dl}|$, change in T i.e. dT is maximum in the same direction of $\vec{\nabla T}$.

Thus, the gradient $\vec{\nabla T}$ points in the direction of maximum increase of the function T.

Also, the magnitude $|\vec{\nabla T}|$ gives us the slope (i.e. rate of increase) along this direction.

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (magnitude of a position vector)

$$\vec{\nabla r} = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z}$$

$$= \frac{\partial\left(\sqrt{x^2+y^2+z^2}\right)}{\partial x}\hat{x} + \frac{\partial\left(\sqrt{x^2+y^2+z^2}\right)}{\partial y}\hat{y} + \frac{\partial\left(\sqrt{x^2+y^2+z^2}\right)}{\partial z}\hat{z}$$

$$= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$

$$= \frac{x\,\hat{x} + y\,\hat{y} + z\,\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r} = \hat{r}$$

Significance : The distance from origin increases most rapidly along the direction of $\vec{r}$.

Find the gradient of $\phi(x, y, z) = 3x^2y - y^3z^2$ at point (1,-2,-1).

$$\frac{\partial\phi}{\partial x} = \frac{\partial(3x^2y - y^3z^2)}{\partial x} = 6xy$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial(3x^2y - y^3z^2)}{\partial y} = 3x^2 - 3y^2z^2$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial(3x^2y - y^3z^2)}{\partial z} = -2y^3z$$

$$\overrightarrow{\nabla\phi} = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$$

$$= (6xy)\,\hat{x} + (3x^2 - 3y^2z^2)\,\hat{y} - (2y^3z)\,\hat{z}$$

$$\therefore\ \overrightarrow{\nabla\phi}\Big|_{(1,-2,-1)} = 6(1)(-2)\,\hat{x} + (3(1)^2 - 3(-2)^2(-1)^2)\,\hat{y} - 2(-2)^3(-1)\,\hat{z}$$

$$= -12\,\hat{x} - 9\,\hat{y} - 16\,\hat{z}$$

The operator ∇ (del operator)

The gradient has a formal appearance of a operator $\vec{\nabla}$ acting on a scalar T.

$$\overrightarrow{\nabla T} = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \quad \text{i.e.}\ \overrightarrow{\nabla T} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)T$$

Here, $\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$ is called as ‘del’.

- This ‘del’ is not a vector.
- It does not multiply a function; rather it is an instruction to differentiate what follows.
- It does not have any meaning until we provide it with a function to act upon.
- To be precise, $\vec{\nabla}$ is not a vector that multiplies T. $\vec{\nabla}$ is a vector operator that acts on T.

The operator $\vec{\nabla}$ can act in three ways –

- On a scalar function T –

  i.e. $\overrightarrow{\nabla T}$ (the gradient)

- On a vector function $\vec{v}$ via dot product –

  $\vec{\nabla} \cdot \vec{v}$ (the divergence)

- On a vector function $\vec{v}$ via cross product -

  $\vec{\nabla} \times \vec{v}$ (the curl)

The Divergence

- When the operator $\vec{\nabla}$ act on a vector function $\vec{\mathrm{v}}$, via dot product, we get divergence of a vector function $\vec{\mathrm{v}}$.

$$\vec{\nabla} \cdot \vec{\mathrm{v}} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\mathrm{v}_x\hat{x} + \mathrm{v}_y\hat{y} + \mathrm{v}_z\hat{z}\right)$$

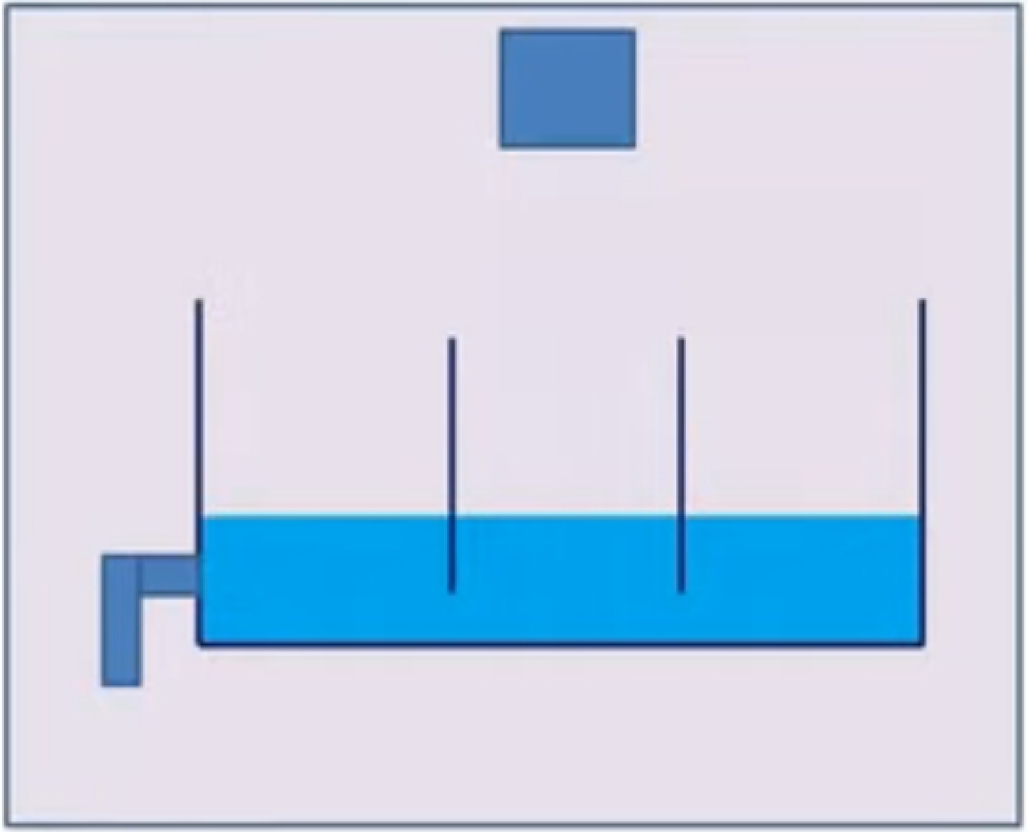
$$\vec{\nabla} \cdot \vec{\mathrm{v}} = \left(\frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} + \frac{\partial \mathrm{v}_z}{\partial z}\right)$$

- The divergence of a vector function is a scalar. The divergence of a scalar function can not be written and it is meaningless.

Significance of divergence :

Divergence $\vec{\nabla} \cdot \vec{\mathrm{v}}$ is a measure of how much the vector $\vec{\mathrm{v}}$ spreads out (diverges) from the given point.

The Divergence



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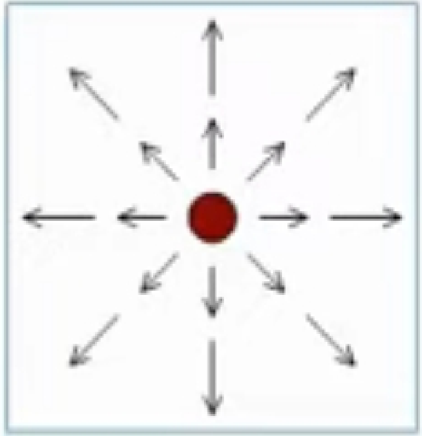


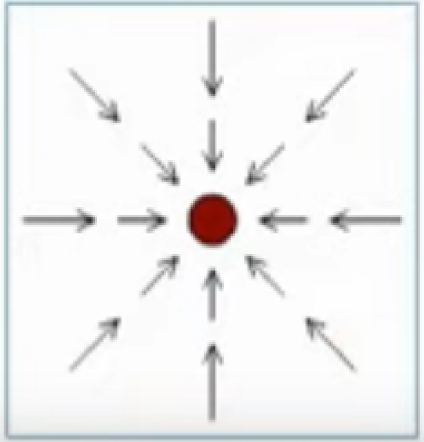
$$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

This vector function has a large positive divergence.

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(x\hat{x} + y\hat{y} + z\hat{z})$$

$$\vec{\nabla}\cdot\vec{v} = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1 + 1 + 1$$

$$= 3$$



$$\vec{v} = -x\hat{x} - y\hat{y} - z\hat{z}$$

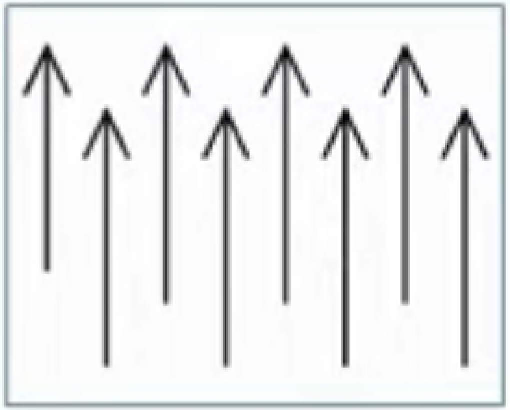
This vector function has large negative divergence

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(-x\hat{x} - y\hat{y} - z\hat{z})$$

$$\vec{\nabla}\cdot\vec{v} = \left(\frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial(-z)}{\partial z}\right)$$

$$= -1 - 1 - 1 = -3$$

The Divergence



$\vec{\mathrm{v}} = K\,\hat{z}$ K is constant. This vector has zero divergence	$\vec{\nabla}\cdot\vec{\mathrm{v}} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(0\hat{x} + 0\hat{y} + K\hat{z})$ $\vec{\nabla}\cdot\vec{\mathrm{v}} = \left(\frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial K}{\partial z}\right)$ $= 0 + 0 + 0 = 0$
$\vec{\mathrm{v}} = y\,\hat{y}$ This vector has positive divergence.	$\vec{\nabla}\cdot\vec{\mathrm{v}} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(0\hat{x} + y\hat{y} + 0\hat{z})$ $\vec{\nabla}\cdot\vec{\mathrm{v}} = \left(\frac{\partial 0}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial 0}{\partial z}\right)$ $= 0 + 1 + 0 = 1$

If $\overrightarrow{\mathbf{v_A}} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ and $\overrightarrow{\mathbf{v_B}} = y\,\hat{y}$, calculate their divergence.

$$\vec{\nabla} \cdot \overrightarrow{v_A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (x\hat{x} + y\hat{y} + z\hat{z})$$

$$\vec{\nabla} \cdot \overrightarrow{v_A} = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1 + 1 + 1 = 3$$

$$\vec{\nabla} \cdot \overrightarrow{v_B} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (y\hat{y})$$

$$\vec{\nabla} \cdot \overrightarrow{v_B} = \left(\frac{\partial 0}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial 0}{\partial z}\right) = 0 + 1 + 0 = 1$$

If $\vec{\mathbf{v}} = \mathbf{1}\,\hat{x} + \mathbf{2}\,\hat{y} + \mathbf{3}\,\hat{\mathbf{z}}$, find divergence of $\vec{\mathbf{v}}$.

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (1\hat{x} + 2\hat{y} + 3\hat{z})$$

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \left(\frac{\partial(1)}{\partial x} + \frac{\partial(2)}{\partial y} + \frac{\partial(3)}{\partial z}\right)$$

$$= 0 + 0 + 0 = 0$$

If $\vec{A} = x^2 z\,\hat{x} - 2y^2 z^2\,\hat{y} + xy^2 z\,\hat{z}$, Find $\vec{\nabla} \cdot \vec{A}$ at point (1,-1,1)

$$\vec{\nabla} \cdot \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(x^2 z\,\hat{x} - 2y^2 z^2\,\hat{y} + xy^2 z\,\hat{z}\right)$$

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial(x^2 z)}{\partial x} + \frac{\partial(-2y^2 z^2)}{\partial y} + \frac{\partial(xy^2 z)}{\partial z}\right)$$

$$= 2xz - 4yz^2 + xy^2$$

$$\vec{\nabla} \cdot \vec{A}\Big|_{(1,-1,1)} = 2(1)(1) - 4(-1)(1)^2 + (1)(-1)^2$$

$$= 2 + 4 + 1 = 7$$

The Curl

When the operator $\vec{\nabla}$ act on a vector function $\vec{v}$ via cross product, we get curl of a vector function $\vec{v}$.

$$\vec{\nabla}\times\vec{v}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\left(v_x\hat{x}+v_y\hat{y}+v_z\hat{z}\right)$$

$$\vec{\nabla}\times\vec{v}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

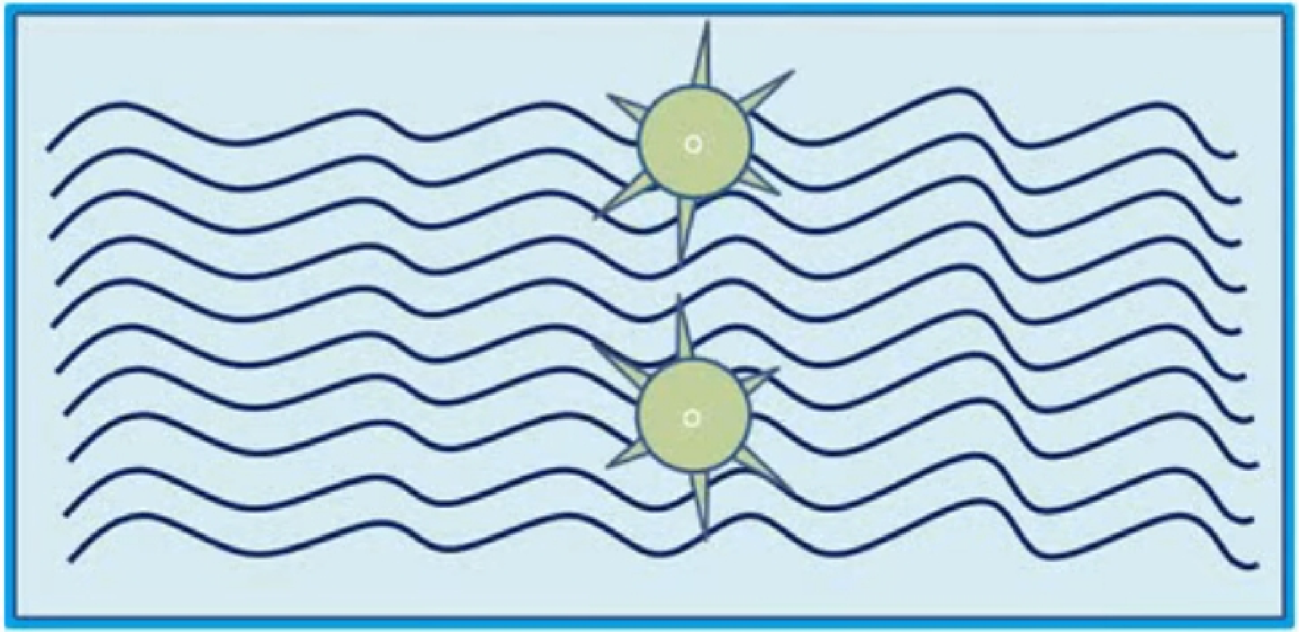
$$\vec{\nabla}\times\vec{v}=\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

Significance of the curl

$\vec{\nabla}\times\vec{v}$ is a measure of how much the vector $\vec{v}$ curls around the given point. Zero curl means there is no rotation.

The curl of a vector function is a vector.

The Curl



Significance of the curl

$\vec{\nabla} \times \vec{v}$ is a measure of how much the vector $\vec{v}$ curls around the given point. Zero curl means there is no rotation.

If $\vec{\mathbf{v}} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, calculate its curl.

$$\vec{\nabla} \times \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times (x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{v} = \hat{x}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{y}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{z}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0 + 0 + 0 = 0$$

Calculate curl of $\vec{\mathbf{v}}$ if $\vec{\mathbf{v}} = -y\hat{x} + x\hat{y}$.

$$\vec{\nabla} \times \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times (-y\,\hat{x} + x\,\hat{y} + 0\,\hat{z})$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{v} = \hat{x}\left(\frac{\partial(0)}{\partial y} - \frac{\partial(x)}{\partial z}\right) + \hat{y}\left(\frac{\partial(-y)}{\partial z} - \frac{\partial(0)}{\partial x}\right) + \hat{z}\left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{v} = \hat{x}(0) + \hat{y}(0) + \hat{z}(1 - (-1)) = 2\hat{z}$$

Show that divergence of a curl is zero.

Let us consider $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$.

Curl of $\vec{v}$ **is given by -** $\quad \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$

$$\vec{\nabla} \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Divergence of Curl of $\vec{v}$ **is given by -**

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \left(\frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z}\right) + \left(\frac{\partial^2 v_x}{\partial y \partial z} - \frac{\partial^2 v_z}{\partial y \partial x}\right) + \left(\frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y}\right) = 0$$

Thus divergence of a curl is zero.

Gauss Theorem (Divergence theorem)

Gauss Theorem States that – If V is the volume bound by the surface S, volume integral of divergence of a function $\vec{v}$ over volume V is equal to surface integral of the function $\vec{v}$ over the surface S that surrounds the given volume.

$$\int_V \vec{\nabla} \cdot \vec{v} \; dv = \oint_S \vec{v} \cdot \overrightarrow{ds}$$

Stoke's Theorem (Fundamental Theorem for Curl)

Stokes Theorem States that – If S is the surface area bound by the boundary P, surface integral of curl of a vector function $\vec{v}$ over surface area S is equal to line integral of the vector function $\vec{v}$ over the closed curve P binding that surface.

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot ds = \oint_P \vec{v} \cdot dl$$