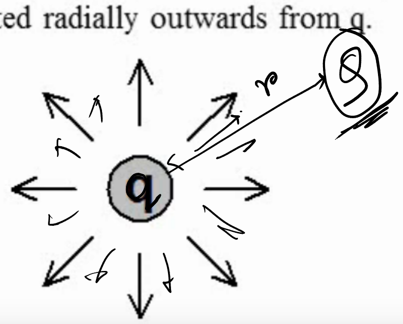
Maxwell's Equation

1. Statement
2. Integral form
3. Use Gauss divergence / Stoke's line integral
4. Differential form
5. Significance

1. $\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$
2. $\vec{\nabla} \cdot \vec{B} = 0$
3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\vec{\nabla} \times \vec{B} = ?$

Coulomb's Law

The static charge q develops a spherical electric field surrounding it. The line of force of the field are directed radially outwards from q.

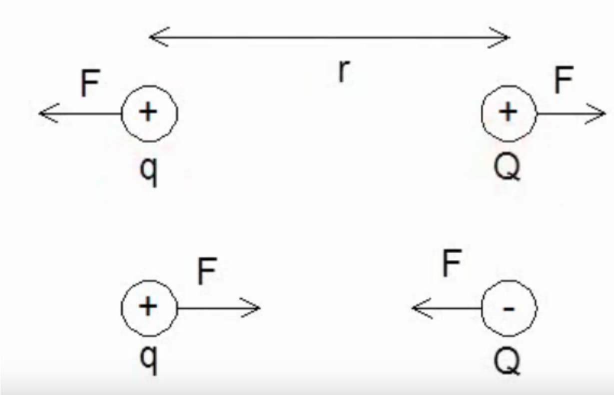


The force exerted by this field on a test charge Q is given by Coulomb's law

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r}$$

$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{C^2}{Nm^2}$$

is called permittivity of free space



Repulsion

Attraction

Electric Field Intensity

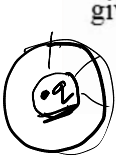
Electric Field Intensity is defined as the force per unit charge at any point. In the field region. It is given by -

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$Q = 1$

$\vec{E} = \frac{D}{\epsilon_0}$

$\vec{D} = \epsilon_0 \vec{E}$



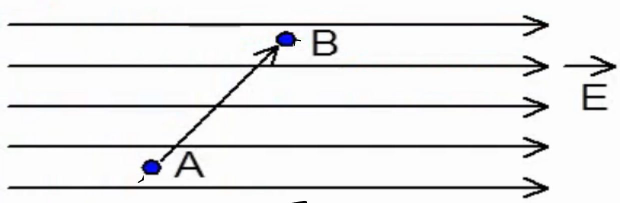
Electric flux density

Electric flux density is defined as the electric charge over unit area of the spherical surface having its centre at charge q. Thus, electric flux density is same as electric charge density and is given by -

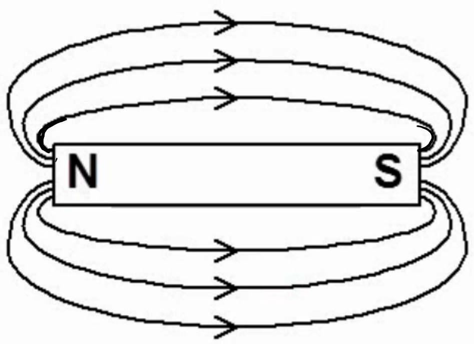
$$\vec{D} = \frac{q}{4\pi r^2}\hat{r} = \epsilon_0\vec{E}$$

Electric potential

$$V_{AB} = \frac{W}{q} = -\int_A^B \vec{E}\, dl$$



Magnetic Field



An isolated magnetic pole cannot exist.

$\vec{D} = \epsilon_0 \vec{E}$

$\vec{H} \rightarrow$ Magnetic field intensity

$\vec{B} \rightarrow$ Magnetic flux density

$\boxed{\vec{B} = \mu \vec{H}}$

$\mu = \mu_0 \mu_r$

$\hookrightarrow$ permeability of free space

$$\vec{B} = \mu\vec{H}$$

Where μ is permeability,

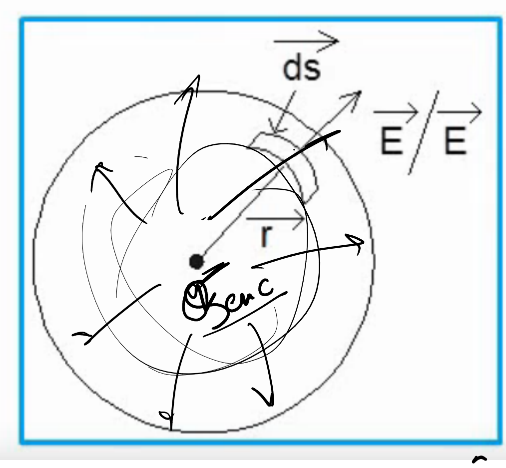
$\mu = \mu_0\mu_r$ where $\mu_0 = 4\pi \times 10^{-7} N/A^2$ is permeability of free space and μ_r is relative permeability

First Maxwell's Equation (Gauss Law for electric field)

Gauss law for electric field : The total electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$\phi = \oint_S \vec{D}\, ds = \oint_S \epsilon_0 \vec{E}\, ds = Q_{enc}$$

$$= \int_V \rho_v \, dV$$



$\rho_v \rightarrow$ charge density

$$\boxed{\oint D\, ds = \epsilon_0 \oint_S E\, ds = \int_V \rho_v \, dV}$$

integral form

Gauss's Divergence theorem

$$\epsilon_0 \int_V \vec{\nabla} \cdot \vec{E}\, dV = \int_V \rho_v \, dV$$

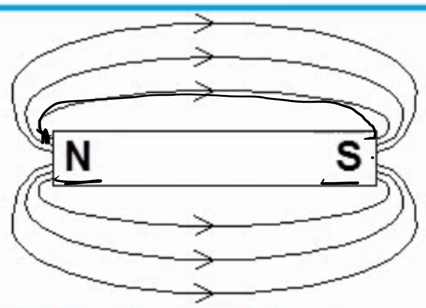
$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}}$$

Differential form

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Second Maxwell's Equation
(Gauss Law for Static Magnetic Field)

Gauss Law for Static Magnetic Field states that – "In a magnetic field, the magnetic lines of force are closed on themselves as shown -



$$\oint_S \vec{B}\, ds = 0 \quad (1)$$

This is called Gauss Law for magnetic field in integral form

Using divergence theorem, we can write -

$$\int_V \vec{\nabla}\cdot\vec{B}\; dv = \oint_S \vec{B}\; ds = 0 \quad (2)$$

Thus, $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

This is second Maxwell's Equation in differential form

Third Maxwell's Equation (Faraday's Law)

A changing magnetic field induces an electric field.

Faraday's law states that "Whenever there is a change in magnetic flux linked with the circuit, an emf is induced in that circuit. The magnitude of induced emf is equal to the rate of change of the flux."

Therefore, the work done in moving a test charge from one point to the other point is given by –

$$\oint_P \vec{E}\ dl = -\frac{d\phi}{dt} = -\oint_S \frac{\partial \vec{B}}{\partial t}\ ds$$

This is integral form of the Faraday's law.

$\frac{\partial \vec{B}}{\partial t} \neq 0$

Using Stoke's theorem, we can write –

$$\oint_P \vec{E}\ dl = \int_S (\vec{\nabla} \times \vec{E})\ ds = -\oint_S \frac{\partial \vec{B}}{\partial t}\ ds$$

$$\boxed{\text{Therefore, } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

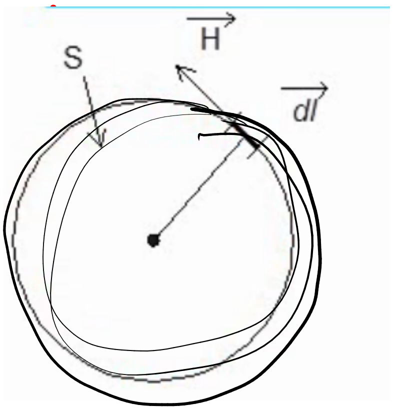
This is third Maxwell's equation in differential form

$$\oint_P \vec{E}\,dl = -\frac{\partial \phi}{\partial t} = \oint_S \frac{\partial \vec{B}}{\partial t}\,ds$$

$$\int_S (\vec{\nabla} \times \vec{E})\,ds = -\oint_S \frac{\partial \vec{B}}{\partial t}\,ds$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

Fourth Maxwell's Equation (Ampere's Circuital law for Static Magnetic field)



Ampere's circuital law states that- "The line integral of magnetic field intensity $\vec{H}$ around a closed path is equal to the current enclosed by that path."

$$\boxed{\oint_P H\,dl = I = \int_S J\,ds}$$

$J \rightarrow$ current density

$$\oint_S \vec{\nabla} \times \vec{H}\,ds = \oint_S J\,ds$$

$$\vec{\nabla} \times \vec{H} = J$$

$$\vec{B} = \mu_0 H \qquad \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}} \rightarrow \text{static field}$$

for time varying field

↳ Displacement current density $\vec{D}$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

↳ time varying field

Therefore, the fourth Maxwell equation for time varying fields becomes -

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

Law	Integral	Differential
1. Gauss law for electric field	$\epsilon_0 \oint E\, ds = \int_V \rho_v \, dV$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}$
2. Gauss law for magnetic field	$\oint \vec{B} ds = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
3. Faraday's law	$\oint \vec{E}\, dl = \oint_S \frac{\partial B}{\partial t} ds$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (time varying) $\vec{\nabla} \times \vec{E} = 0$ (static)
4. Ampere's Circuital Law	$\oint H\, dl = I = \oint_S \vec{J} ds$	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (static) $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$ (time varying field)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Copy of Equation 16.24)}$$

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) \qquad \text{(Equation 18.1)}$$

Next, we will start with Ampere's Law and will take the dot product of **E** with both sides:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \qquad \text{(Copy of Equation 17.13)}$$

$$\mathbf{E} \bullet (\nabla \times \mathbf{H}) = \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) \qquad \text{(Equation 18.2)}$$

Now, let's subtract both sides of Equation 18.2 from both sides of Equation 18.1:

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) \text{(Equation 18.3)}$$

We can now apply the following mathematical identity to the left side of Equation 18.3:

$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B}) \text{ (Equation 18.4)}$$

This substitution yields:

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) \qquad \text{(Equation 18.5)}$$

Distributing the $\mathbf{E}$ across the right side gives:

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \bullet \mathbf{J}$$

find:

$$\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) = \mathbf{H} \bullet \mu \left(\frac{\partial \mathbf{H}}{\partial t}\right)$$

Let's consider for a moment what the derivative of $\mathbf{H}^2$ would yield:

$$\frac{\partial \mathbf{H}^2}{\partial t} = \frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = 2\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t}$$

Dividing this equation by 2, we find:

$$\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = \frac{1}{2}\frac{\partial (H^2)}{\partial t}$$

A similar analysis of the second term on the right side yields:

$$\mathbf{E} \bullet \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial (\mathbf{E} \bullet \mathbf{E})}{\partial t} = \frac{1}{2}\frac{\partial (E^2)}{\partial t} \qquad \text{(Equation 18.10)}$$

Substituting Equations 18.9 and 18.10 into Equation 18.6 and applying two constitutive relations, we obtain:

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\frac{1}{2}\frac{\partial}{\partial t}\left(\mu H^2 + \epsilon E^2\right) - \mathbf{E} \bullet \mathbf{J} \qquad \text{(Equation 18.11)}$$

Taking a volume integral of both sides, we find:

$$\int_{\Delta V} \nabla \bullet (\mathbf{E} \times \mathbf{H}) dV = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\Delta V}\left(\mu H^2 + \epsilon E^2\right) dV - \int_{\Delta V} \mathbf{E} \bullet \mathbf{J} dV \qquad \text{(Equation 18.12)}$$

Applying Ohm's Law (**J**=σ**E**) to the right side, we find:

$$\int_{\Delta V} \nabla \bullet (\mathbf{E} \times \mathbf{H}) dV = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\Delta V}\left(\mu H^2 + \epsilon E^2\right) dV - \int_{\Delta V} \sigma E^2 dV \qquad \text{(Equation 18.13)}$$

Applying the divergence theorem to the left side gives:

$$\boxed{\oint_{\Delta s} (\mathbf{E} \times \mathbf{H}) \bullet \mathbf{dS} = -\frac{\partial}{\partial t}\int_{\Delta V}\frac{1}{2}\left(\mu H^2 + \epsilon E^2\right) dV - \int_{\Delta V} \sigma E^2 dV} \qquad \text{(Equation 18.14)}$$