Laplace Transform

The Laplace transform of the function $F(t)$, which will be denoted by either $f(p)$ or $L\{F(t)\}$, is defined as

$$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt, \quad p > 0. \quad (1)$$

p is called the transform parameter.

Let us now find the Laplace transforms of some elementary fn from def^n.

(i) If $F(t) = c$, then (c is constant)

$$L\{c\} = \int_0^\infty c e^{-pt} dt = \left[-c\frac{e^{-pt}}{p}\right]_0^\infty$$

$$= \frac{c}{p}, \quad p > 0.$$

(ii) If $F(t) = t^2$, then

$$L\{t^2\} = \int_0^\infty e^{-pt} t^2\, dt.$$

$$= \left[-\frac{t^2 e^{-pt}}{p}\right]_0^\infty + \int_0^\infty 2t \frac{e^{-pt}}{p} dt$$

$$= \frac{2}{p}\left[-\frac{e^{-pt}}{p} t\right]_0^\infty + \frac{2}{p}\int_0^\infty \frac{e^{-pt}}{p} dt \quad \because t^2 e^{-pt} \to 0 \text{ as } t \to \infty$$

$$= \frac{2}{p^2}\left[-\frac{e^{-pt}}{p}\right]_0^\infty = \frac{2}{p^3}, \quad p > 0.$$

$$\therefore L\{t^n\} = \frac{n!}{p^{n+1}}, \quad n \text{ is +ve integer.}$$

(iii) If $F(t) = e^{\alpha t}$, then

$$L\{e^{\alpha t}\} = \int_0^\infty e^{-pt} e^{\alpha t} dt = \left[-\frac{e^{-(p-\alpha)t}}{p-\alpha}\right]_0^\infty$$

$$= \frac{1}{p-\alpha}, \quad p > \alpha.$$

(iv) $$L\{\cosh at\} = L\left\{\frac{1}{2}(e^{at} + e^{-at})\right\}$$

$$= \frac{1}{2}\left(\frac{1}{p-a} + \frac{1}{p+a}\right) = \frac{p}{p^2 - a^2}, \quad p > |a|$$

Similarly $L\{\sinh at\} = \frac{a}{p^2 - a^2}$, $p > |a|$

(v) If $F(t) = \cos at$ then

$$L\{\cos at\} = \int_0^\infty e^{-pt} \cos at \, dt$$

$$= \left[\frac{e^{-pt}}{p^2 + a^2} (-p \cos at + a \sin at) \right]_0^\infty$$

$$= \frac{p}{p^2 + a^2}, \quad p > 0.$$

Similarly $L\{\sin at\} = \frac{a}{p^2 + a^2}$

Example 1 Find the Laplace transform of $f(t)$ defined as

$$f(t) = \begin{cases} t/k, & \text{when } 0 < t < k \\ 1, & \text{when } t > k. \end{cases}$$

$$L\{f(t)\} = \int_0^k \frac{t}{k} e^{-pt} dt + \int_k^\infty 1 \cdot e^{-pt} dt$$

$$= \frac{1}{k} \left[\left(t \frac{e^{-pt}}{-p} \right)_0^k - \int_0^k \frac{e^{-pt}}{-p} dt \right] + \left[\frac{e^{-pt}}{-p} \right]_k^\infty$$

$$= \frac{1}{k} \left[\frac{k e^{-pk}}{-p} - \left(\frac{e^{-pt}}{p^2} \right)_0^k \right] + \frac{e^{-kp}}{p}$$

$$= \frac{1}{kp^2} \left[e^{-kp} + 1 \right].$$

Inverse of the Laplace transform will be defined by $L^{-1}\{f(p)\} = F(t)$, if $f(p)$ be the Laplace transform of $F(t)$.

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According to the definition of inverse transform, we can state

(a) $L^{-1}\left(\frac{1}{p}\right) = 1$, since $L\{1\} = \frac{1}{p}$.

(b) $L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{n!}$ since $L\{t^n\} = \frac{n!}{p^{n+1}}$

(c) $L^{-1}\left(\frac{1}{p-\alpha}\right) = e^{\alpha t}$, since $L\{e^{\alpha t}\} = \frac{1}{p-\alpha}$

(d) $L^{-1}\left(\frac{1}{p^2+\alpha^2}\right) = \frac{1}{\alpha}\sin\alpha t$, since $L\{\sin\alpha t\} = \frac{\alpha}{p^2+\alpha^2}$.

(e) $L^{-1}\left(\frac{p}{p^2+\alpha^2}\right) = \cos\alpha t$, since $L\{\cos\alpha t\} = \frac{p}{p^2+\alpha^2}$

(f) $L^{-1}\left(\frac{1}{p^2-\alpha^2}\right) = \frac{1}{a}\sinh\alpha t$, since $L\{\sinh\alpha t\} = \frac{\alpha}{p^2-\alpha^2}$

(g) $L^{-1}\left(\frac{p}{p^2-\alpha^2}\right) = \cosh\alpha t$ since $L\{\cosh\alpha t\} = \frac{p}{p^2-\alpha^2}$

Existence of Laplace Transform :

<u>Defn</u> A function F(t) is said to be piecewise continuous on a closed interval $a \le t \le b$ if it be defined on that interval and be & such that the interval can be sub-divided into a finite no. of intervals, in each of which F(t) is continuous.

<u>Defn</u> A function F(t) is said to be of exponential order as $t \to \infty$, if there exists a constant $a > 0$ s.t. $e^{-at}|F(t)|$ is bounded for all $t > T$; i.e. for some constant $M > 0$,

$$|F(t)| \le M e^{at}.$$

Theorem : Let F be a piecewise continuous in the interval [0, T] for every positive T and let F be of exponential order as $t \to \infty$ for some $a > 0$. Then the Laplace transform of F exists for $p > a$.

Some property :

(a) Linearity property :

~~If~~ We have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$,

~~then~~ If a_1, a_2 be constants, then

$$L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\}$$

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(b) First shifting property:

(i) If $L\{F(t)\} = f(p)$, then $L\{e^{at}F(t)\} = f(p-a)$

(ii) If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(p-a)\} = e^{at}F(t)$$
$$= e^{at}L^{-1}\{f(p)\}.$$

(c) Change of scale property:

(i) If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)$

(ii) Similarly if $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right).$$

Laplace transform of derivative:

(i) $L\{F'(t)\} = p\,L\{F(t)\} - F(0)$.

(ii) $L\{F''(t)\} = p^2 L\{F(t)\} - pF(0) - F'(0)$.

(iii) $L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - \cdots$
$$- pF^{n-2}(0) - F^{n-1}(0).$$

In case of inverse transform,

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^n(p)\} = L^{-1}\left\{\frac{d^n}{dp^n}(f(p))\right\}.$$
$$= (-1)^n t^n F(t),\ n = 1, 2, \ldots$$

Laplace transform of integrals

$$L\left\{\int_0^t F(\tau)\,d\tau\right\} = \frac{f(p)}{p} = \frac{1}{p}L\{F(t)\}$$

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\int_0^\infty f(x)\,dx\right\} = \frac{F(t)}{t}.$$

Convolution theorem

Let the two functions $F(t)$ and $G(t)$ be two functions. Then $F * G$ be defined by $F * G = \int_0^t F(x)\, G(t-x)\, dx$ is called the convolution of the functions $F(t)$ and $G(t)$.

<u>Theorem</u> If $f(p)$ and $g(p)$ be the Laplace transforms of $F(t)$ and $G(t)$ respectively then the Laplace transform of the product $f(p)\, g(p)$ is the Convolution $F * G$.

In other words, $L^{-1}\{f(p)\, g(p)\} = F * G$.

<u>Example</u>

① Find $L\{\sin^2 at\}$

$$L\{\sin^2 at\} = L\{\tfrac{1}{2}(1-\cos 2at)\}$$

$$= \tfrac{1}{2} L\{1\} - \tfrac{1}{2} L\{\cos 2at\}$$

$$= \tfrac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4a^2}\right), \quad p>0.$$

$$= \frac{2a^2}{p(p^2+4a^2)}$$

② Evaluate $L\{F(t)\}$, where $F(t) = \begin{cases} (t-1)^2, & t \geq 1 \\ 0, & 0 \leq t \leq 1 \end{cases}$

$$L\{F(t)\} = \int_0^1 0 \cdot e^{-pt}\, dt + \int_1^\infty (t-1)^2 e^{-pt}\, dt$$

$$= \left[-\frac{(t-1)^2}{p} e^{-pt}\right]_1^\infty + \frac{2}{p}\int_1^\infty (t-1) e^{-pt}\, dt$$

$$= \frac{2}{p}\left[-\frac{(t-1)}{p} e^{-pt} - \frac{e^{-pt}}{p^2}\right]_1^\infty$$

$$= \frac{2e^{-p}}{p^3}.$$

(3) If $L\{F(t)\} = \dfrac{p^2 - p + 1}{(2p+1)^2 (p-1)}$, find $L\{F(2t)\}$.

applying the change of scale property.

If $\quad L\{F(t)\} = f(p)$

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right).$$

$$= \frac{1}{2} \cdot \frac{\left(\frac{p}{2}\right)^2 - \frac{p}{2} + 1}{\left(2 \cdot \frac{p}{2} + 1\right)^2 \left(\frac{p}{2} - 1\right)}$$

$$= \frac{p^2 - 2p + 4}{4\,(p+1)^2\,(p-2)}$$

(4) Find $L\{t \cos at\}$

We know $L\{\cos at\} = \dfrac{p}{p^2 + a^2}$, $p > 0$.

$$L\{t \cos at\} = -\frac{d}{dp} L\{\cos at\}$$

$$= -\frac{d}{dp}\left(\frac{p}{p^2 + a^2}\right) = \frac{p^2 - a^2}{(p^2 + a^2)^2}$$

(5) Evaluate $L^{-1}\left(\dfrac{3p - 2}{p^2 - 4p + 20}\right)$

$$= L^{-1}\left\{\frac{3p - 2}{p^2 - 4p + 4 + 16}\right\}$$

$$= L^{-1}\left\{\frac{3(p-2) + 4}{(p-2)^2 + 4^2}\right\}$$

$$= 3L^{-1}\left\{\frac{(p-2)}{(p-2)^2 + 4^2}\right\} + 4L^{-1}\left\{\frac{1}{(p-2)^2 + 4^2}\right\}$$

$$= 3e^{2t} L^{-1}\left\{\frac{p}{p^2 + 4^2}\right\} + 4e^{2t} L^{-1}\left\{\frac{1}{p^2 + 4^2}\right\}$$

$$= 3e^{2t}\cos 4t + e^{2t}\sin 4t.$$

(6) Use convolution theorem to find

$$L^{-1}\left\{\frac{1}{(p-1)(p-2)}\right\}.$$

Since $L^{-1}\left(\frac{1}{p-1}\right) = e^{t}$, $L^{-1}\left(\frac{1}{p-2}\right) = e^{2t}$

So by convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(p-1)(p-2)}\right\} = L^{-1}\left(\frac{1}{(p-1)}\cdot\frac{1}{(p-2)}\right)$$

$$= \int_0^t e^{x} e^{2(t-x)}\,dx$$

$$= \int_0^t e^{2t}\cdot e^{-x}\,dx$$

$$= e^{2t}\left[-e^{-x}\right]_0^t = e^{2t} - e^{t}$$

Solve

(1) Show that $L\{2t^3 - 6t + 8\} = \frac{12}{p^4} - \frac{6}{p^2} + \frac{8}{p}$, $p > 0$

(2) A function $F(t)$ is defined by

$$F(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3, & t > 2 \end{cases}$$

Show that $L\{F(t)\} = (1-e^{-2p})\,p^{-2} + p^{-1}$.

$L\{F'(t)\} = (1-e^{-2p})\,p^{-1}$, $p > 0$

(3) Use shifting ~~them~~ theorem to show that

$$L\{e^t \sin^2 t\} = \frac{1}{2(p-1)} - \frac{p-1}{2(p^2-2p+5)}$$

(4) Evaluate $L^{-1}\left(\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right)$.

(5) Use convolution theorem to show that

$$L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\} = \frac{1}{16}\left(e^{2t} - 4te^{-2t} - e^{-2t}\right)$$