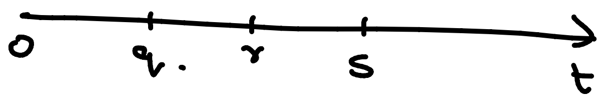
Standard Brownian Motion

Monday, December 6, 2021 8:51 AM

1) It is continuous time and continuous state stochastic process.

<u>Characteristics</u>



A continuous time stochastic process $(B_t)_{t \geq 0}$ is a standard brownian motion if its satisfies following properties

1) $B_0 = 0$.

2) <u>Normal dist</u>:- For $t > 0$, B_t has a normal dist. with mean 0 and Var. t.

3) <u>Stationary increment</u>:- For $s, t > 0$ $B_{t+s} - B_s$ has the same dist. as B_t.

$$P(B_{t+s} - B_s \leq z) = P(B_t \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \, dx$$

for $-\infty < z < \infty$.

4) <u>Independent increment</u>:- If $0 \leq q < r \leq s < t$ then $B_t - B_s$ and $B_r - B_q$ are independent random variables.

5) (continuous path) $\longrightarrow$ The function $t \to B_t$

3) (continuous path) $\longrightarrow$ The function $t \to B_t$ is continuous with prob. 1.

Q1 For $0 < s < t$. Find the dist. of $B_s + B_t$.

Ans Let's write

$$B_s + B_t = 2B_s + (B_t - B_s)$$

By independent increment.

B_s and $B_t - B_s$ are independent random variables.

The sum of independent normal variables are normal.

Thus

$B_s + B_t$ is normally dist. with

mean $E(B_s + B_t) = E(B_s) + E(B_t) = 0$.

and

Variance

$$\begin{aligned}
Var(B_s + B_t) &= Var\left[2B_s + B_t - B_s\right] \\
&= Var(2B_s) + Var(B_t - B_s) \\
&= 4\,Var(B_s) + Var(B_t - B_s) \\
&= 4 \cdot s + (t - s) \\
&= 4s + t - s = 3s + t.
\end{aligned}$$

$\left[Var(ax) = a^2 Var(x) \right]$

$\therefore E(B_s + B_t) = 0$

$V(B_s + B_t) = 3s + t$.

The dist. of $(B_s + B_t)$ is

$B_s + B_t \sim N(0, 3s+t)$.

Q_2 A Particle moves on a line according to Standard brownian motion (BM), $B(t)$. What is it expected position at time $t = 6$.? What is the variance of its position at $t = 6$?

Ans $B(t)$ has a normal dist. with mean $E(B(t)) = 0$ and $Var(B(t)) = t$.

$\therefore$ The expected ~~to~~ position at $t = 6$

$E(t = 6) = 0$.

$V(t = 6) = 6$.

Q_3 :- If the particle is at position 1.7 at time $t = 2$. What is its expected position at time $t = 4$?

Ans $B(4) = B(2) + B(4) - B(2)$. $\quad [\because B(0) = 0]$

$[B(2) - B(0)] + [B(4) - B(2)]$

Where the two increments are independent.

$$E\left(B(4)\,\middle|\,B(2)=1.7\right) = 1.7 + E\left(B(4)-B(2)\,\middle|\,B(2)\right).$$

Conditional expectation

$$= 1.7 + E\left(B(4)-B(2)\right).$$

[independent increment]

$$= 1.7 + 0 = 1.7.$$

Q4 A particle's position is modeled with Brownian motion. If the particle 1 at time $t = 2$. Find the prob. that its position is at most 3 at time $t = 5$.

Ans $P(B_5 \le 3 \mid B_2 = 1) = P(B_5 - B_2 \le 3 - B_2 \mid B_2 = 1)$

$$= \underline{P(B_5 - B_2 \le 2 \mid B_2 = 1)}$$

$$= P(B_3 \le 2)$$

$P(B_3 \le 2) = ? = P(B_1) + P(B_2)$ [From the table of normal dist.]

$$= 0.3413 + 0.4772$$

$$= \underline{\underline{0.8185}}$$

Q5 Find the covariance of B_s and B_t?

Ans $\text{COV}(B_s, B_t) = E(B_s, B_t) - E(B_s)\cdot E(B_t)$

Pro $\text{Cov}(B_s, B_t) = E(B_s, B_t) - E(B_s) \cdot E(B_t)$