Graph Theory

MODULE [illegible]: Graph Theory

- Formation of graph
- Basic terminologies of directed and undirected graphs
- Matrix representation of graphs (Adjacency Matrix and Incidence Matrix)
- Isomorphism
- Walk, Path, Circuit, Euler Path and Circuit, Hamilton Path and Circuit
- Shortest path problem, Dijkstras Algorithm.

Some Applications :

• We can determine whether two computers are connected by a communications link using graph models of computer networks.

• Graphs are used to model telephone calls between telephone numbers, and links between websites.

• To model roadmaps and the assignment of jobs to employees of an organization.

• To solve problems such as finding the shortest path between two cities in a transportation network.

• Graph models are useful tools in the design of software.

Graph

A graph $G = (V, E)$ consists of V, a nonempty set of vertices or nodes and E, a set of edges.

Each edge has either one or two vertices associated with it, called its endpoints.

An edge is said to connect its endpoints.

Adjacent nodes or vertices

Any pair of nodes (vertices) that are connected by edge in a graph is called adjacent nodes.

Isolated nodes

A vertex that is not adjacent to another vertex in a graph is called isolated vertex.

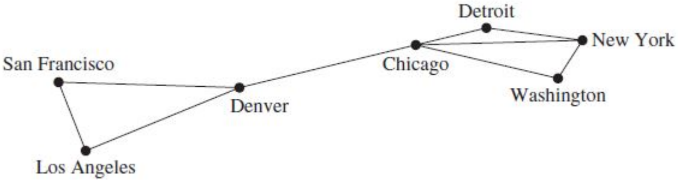
Finite and Infinite Graph

The set of vertices V of a graph G may be infinite.

- A graph with an infinite number of edges is called an infinite graph.
- a graph with a finite vertex set and a finite edge set is called a finite graph.

e.g. Suppose that a network is made up of data centers and communication links between computers.

We can represent the location of each data center by a point and each communications link by a line segment.



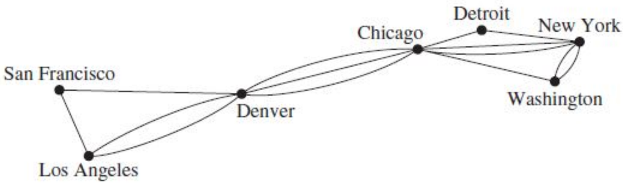
Simple Graph

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

Multi Graph

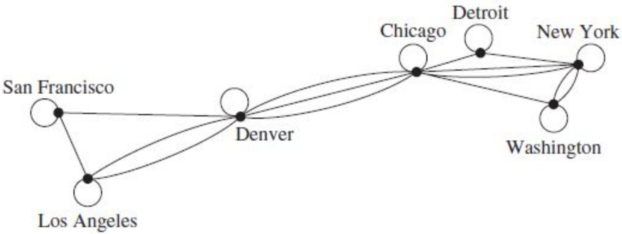
Graphs that may have multiple edges connecting the same vertices are called multigraphs.

When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we say that $\{u, v\}$ is an edge of multiplicity m.



Pseudo Graph

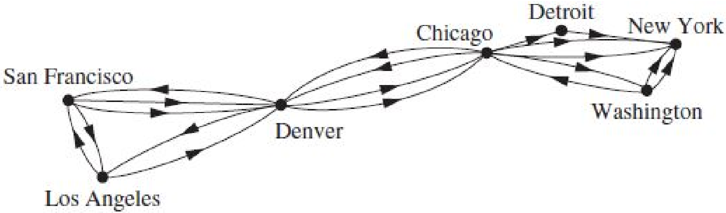
A graph in which loops and multiple edges are allowed is called Pseudo graph.



Directed Graph

A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges E.

Each directed edge is associated with an ordered pair of vertices.



Suppose $e = (u, v)$ is a directed edge in a digraph, then

- u is called initial vertex of e and v is called terminal vertex of e.
- e is said to incident from u and to be incident to v.
- u is adjacent to v and v is adjacent to u.

Degree of a Vertex

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

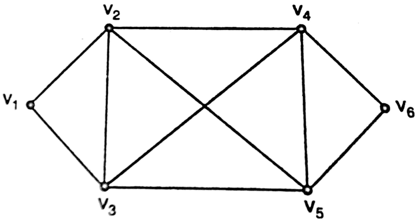
The degree of the vertex v is denoted by $deg(v)$.

A vertex of degree zero is called isolated vertex.

A vertex is called pendant vertex if and only if it has degree one.

Example :

Find the degree of each vertex of following graph



Solution :

$$deg(v_1) = 2 \quad deg(v_2) = 4 \quad deg(v_3) = 4$$
$$deg(v_4) = 4 \quad deg(v_5) = 4 \quad deg(v_6) = 2.$$

Theorem

- The sum of degrees of the vertices in an undirected graph G is even i.e.

$$\sum_{v \in V} deg(v) = 2e$$

  where e denotes the edges in G.
- The total number of odd vertices in a graph is even in an undirected graph.

- The degree of a vertex of a simple graph G on n vertices cannot exceed $n-1$.
- The maximum number of edges in a simple graph with n vertices is

$$\frac{n(n-1)}{2}$$

.

In-degree, Out-degree and Total degree

- In a graph with directed edges the in-degree of a vertex v, denoted by

$$deg^{-}(v),$$

  is the number of edges with v as their terminal vertex.
- The out-degree of v, denoted by

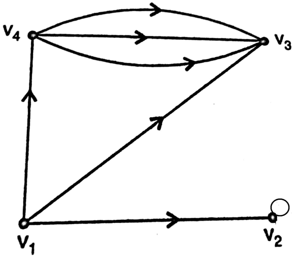
$$deg^{+}(v),$$

  is the number of edges with v as their initial vertex.

  Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.
- The sum of in-degree and out-degree of a vertex is called total degree of vertex.
- A vertex with zero in-degree is called source.
- A vertex with zero out-degree is called sink.

Example :

(1) Find in degree, out degree and total degree of each vertex of following graph



Solution :

$$deg^{-}(v_1) = 0 \quad deg^{-}(v_2) = 2 \quad deg^{-}(v_3) = 4 \quad deg^{-}(v_4) = 1$$

$$deg^{+}(v_1) = 3 \quad deg^{+}(v_2) = 1 \quad deg^{+}(v_3) = 0 \quad deg^{+}(v_4) = 3$$

$$deg(v_1) = 3 \quad deg(v_2) = 3 \quad deg(v_3) = 4 \quad deg(v_4) = 4$$

(2) Does there exists a simple graph corresponding to following degree sequence ? (a) $(1,1,2,3)$ (b) $(2,2,4,6)$.

Solution : (a) The sum of degrees

$$1+1+2+3=7$$

is odd.

Hence there is no graph corresponding to the given degree sequence.

(b) The number of vertices in a given graph sequence is four and maximum degree of vertex in a simple graph with 4 vertices is 3.

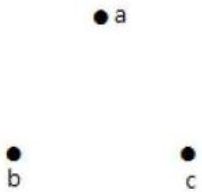
Hence there is no graph corresponding to the given degree sequence.

Null Graph

A graph which contains only isolated vertex is called Null graph.

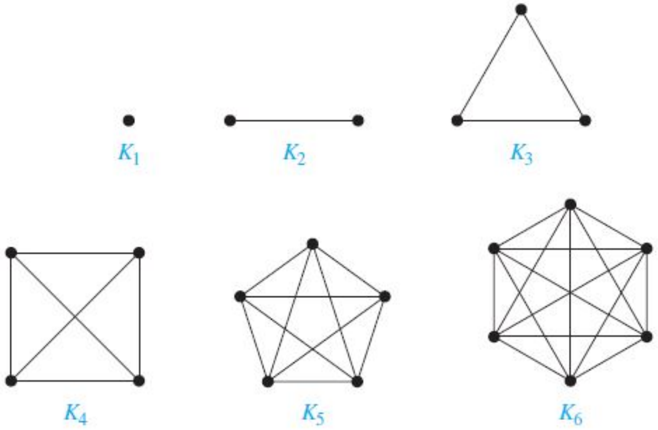
Null graph on n vertices is denoted by N_n.

N_3 =Null graph with 3 vertices



Complete Graph

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices



Regular Graph

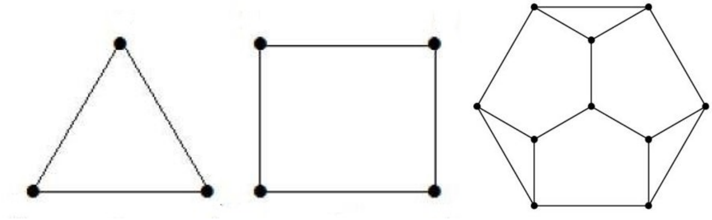
A graph in which all vertices are of same degree is called a regular graph.

If degree of each vertex is r, then graph is called regular graph of degree r.

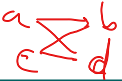
Note :

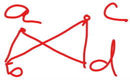
- Every null graph is a regular graph of degree 0.
- Every complete graph is regular graph of degree $n-1$.
- If G has n vertices and is regular of degree r then G has $\frac{1}{2}nr$ edges.

Two and Three regular graph



$V_1 = \{a, c\}$
$V_2 = \{b, d\}$





Cycle

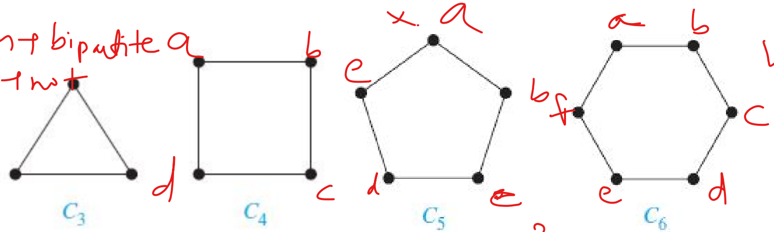
A cycle C_n, $n \geq 3$, consists of n vertices

$$v_1, v_2, \cdots, v_n$$

and edges

$$\{v_1, v_2\},\ \{v_2, v_3\},\ \cdots\ \{v_{n-1}, v_n\}\ \text{and}\ \{v_n, v_1\}.$$

even → bipartite
odd → not



X ✓

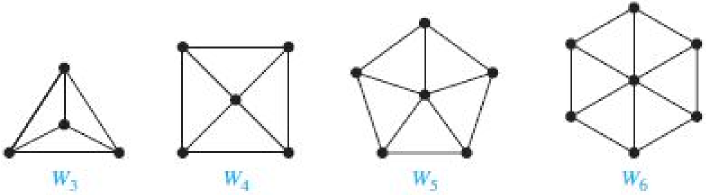
$V_1 = \{a, c\}$
$V_2 = \{b, \cancel{e}\, d\}$

$V_1 = \{a, c, e\}$
$V_2 = \{b, f, d\}$

$W_n \to$ not

Wheel

A wheel W_n is obtained when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



$Q_n \to V$

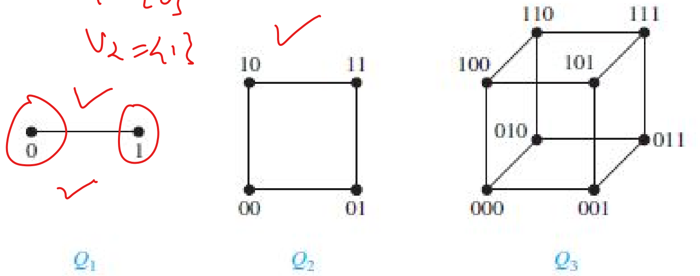
N-cube

An n-dimensional hypercube, or n-cube, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n.

Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

$V_1 = \{0\}$

$V_2 = \{1\}$



Bipartite Graph

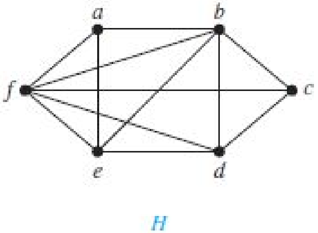
A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Note :

- Q_n is a bipartite graph.
- A graph which contains triangle cannot be bipartite.

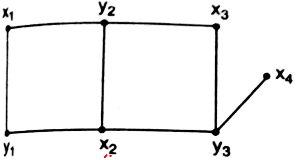
e.g. Graph H is not bipartite because it contain triangle.



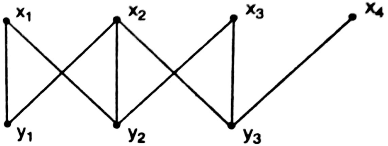
$V_1 = \{x_1, x_2, x_3, x_4\}$

$V_2 = \{y_1, y_2, y_3\}$

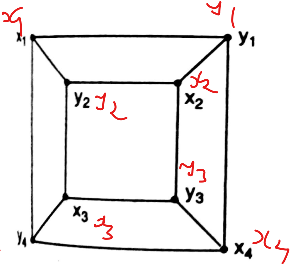
G



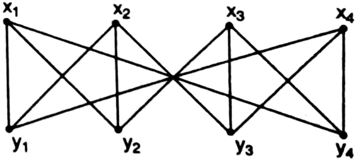
which redrawn as:



H



which redrawn as:



V_1

V_2

$V_1 = \{x_1, x_2, x_3, x_4\}$

$V_2 = \{y_1, y_2, y_4, y_3\}$

Example : Show that C_6 is a bipartite graph.

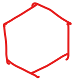
Solution : C_6 is bipartite, because its vertex set can be partitioned into the two sets

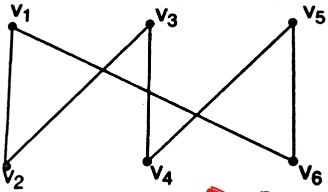
$$V_1 = \{v_1, v_3, v_5\}$$

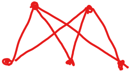
and

$$V_2 = \{v_2, v_4, v_6\}$$

and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.







$K_{2,3}$

$K_{2,4}$

$K_{2,6}$

$K_{m,n}$

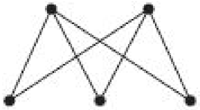
Complete Bipartite Graph

A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 and V_2 of m and n vertices respectively in which there is an edge between each pair of vertices v_1 and v_2 where v_1 is in V_1 and v_2 is in V_2.

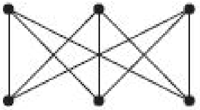
$|E| = rs$
$|V| = r+s$

$\downarrow$ V_1 $\downarrow$ V_2

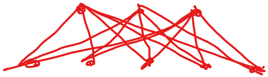
$K_{r,s}$ has $r+s$ vertices and rs edges.

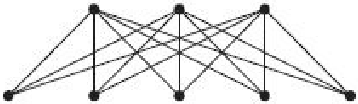


$K_{2,3}$

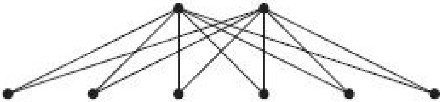


$K_{3,3}$





$K_{3,5}$



$K_{2,6}$

$K_{1,2}$

Note :

- $K_{1,n}$ is called star graph.
- $\cancel{K_5}$ and $K_{3,3}$ are called Kuaratowski graph.

$K_{5,5}$

Adjacency matrix

- Undirected Graph $n \times n$

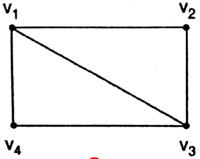
The adjacency matrix A of G is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } v_i \text{ and } v_j \text{ vertices} \\ 0 & \text{if there is no edge between } v_i \text{ and } v_j \text{ vertices} \end{cases}$$

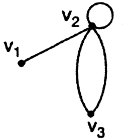
v_1 v_2 v_3

$A(G) =$

Example : Use adjacency matrix to represent the following graphs



G



H

Solution :(1) We order the vertices as v_1, v_2, v_3 and v_4. Since there are four vertices the adjacency matrix will be square matrix of order four.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(2) We order the vertices as v_1, v_2 and v_3. Since there are three vertices the adjacency matrix will be square matrix of order three.

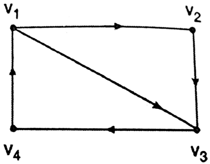
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

- Directed Graph

The adjacency matrix A of a directed graph G is the $n \times n$ zero-one matrix where

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j \end{cases}$$

Example : Use adjacency matrix to represent the graph



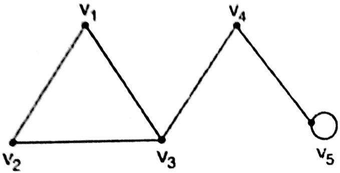
Solution : We order the vertices as $v_1, v_2.v_3$ and v_4. Since there are four vertices the adjacency matrix will be square matrix of order four.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Example : Draw the undirected graph represented adjacency matrix given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

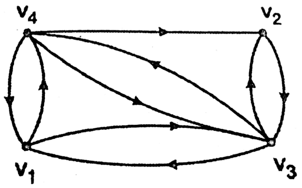
Solution : Since the matrix given is of order five, the graph has five vertices $v_1, v_2.v_3, v_4$ and v_5. The required graph is



Example : Draw the directed graph represented adjacency matrix given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution : Since the matrix given is of order four, the graph has four vertices $v_1, v_2.v_3$ and v_4. The required graph is



Example : Draw the undirected graph representing adjacency matrix given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrix

- Undirected Graph

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \cdots, v_n$ are the vertices and $e_1, e_2, \cdots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

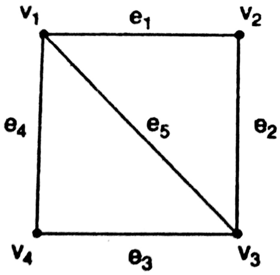
$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

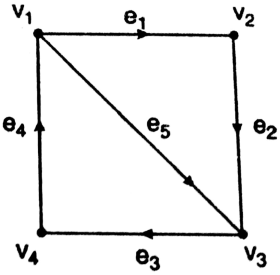
- Directed Graph

The incidence matrix here is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is directed away from } v_i \\ -1 & \text{when edge } e_j \text{ is towards } v_i \\ 0 & \text{otherwise} \end{cases}$$

Example : Find the incidence matrix to represent the following graphs





Solution : The incidence matrix for undirected graph is an 4×5 matrix

$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The incidence matrix for directed graph is an 4×5 matrix

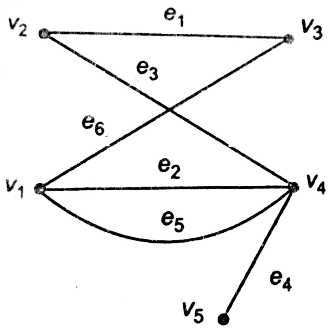
$$I(D) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Example :Draw the graph whose incidence matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution :

Here there are 5 rows and 6 columns in a matrix, therefore the graph has 5 vertices and 6 edges. The graph of given incidence matrix is

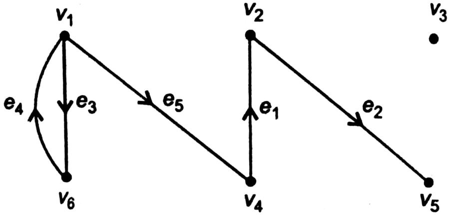


Example :Draw the graph whose incidence matrix is

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

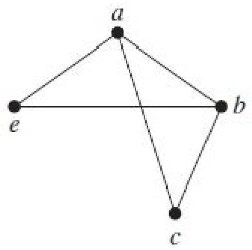
Solution :

Here there are 6 rows and 5 columns in a matrix, therefore the graph has 6 vertices and 5 edges. Since -1 is there in matrix it is digraph. The graph of given incidence matrix is

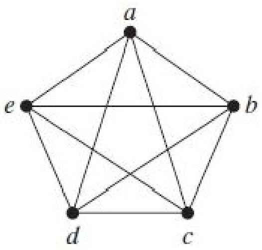


Subgraph

- A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$.
- A subgraph H of G is a proper subgraph of G if $H \neq G$.



is a subgraph of



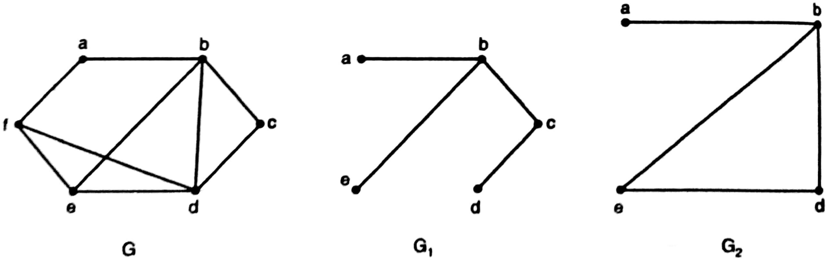
Graph Theory

If G is graph with m vertices and n edges then the number of sub-graphs of $G = (2^m - 1) \times 2^n$

Induced Subgraph

Let $G = (V, E)$ be any graph, and let $S \subset V$ be any subset of vertices of G. Then the induced subgraph is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

In below figure G_1 is not an Induced subgraph of G but G_2 is an Induced subgraph of G

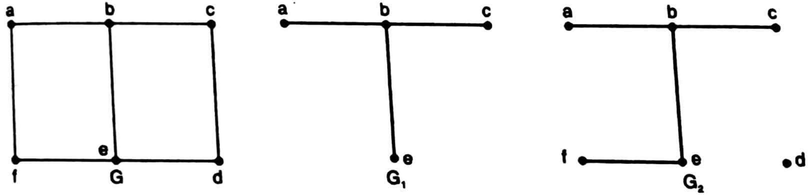


Spanning Subgraph

A subgraph H of G is called spanning subgraph of G if and only if $V(H) = V(G)$.

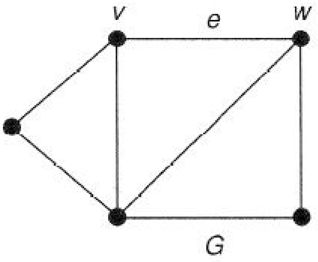
In below figure G_1 is not spanning subgraph of G but G_2 is spanning subgraph of G

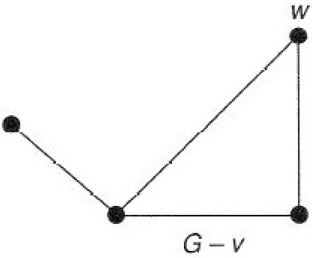
G_1 is an induced subgraph of G but G_2 is not an induced subgraph of G



Vertex deleted Subgraph

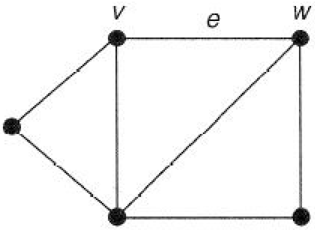
- When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$.
- Similarly, if V' is a subset of V , then the graph $G - V$ is the subgraph $(V - V', E')$, where E' is the set of edges of G not incident to a vertex in V.



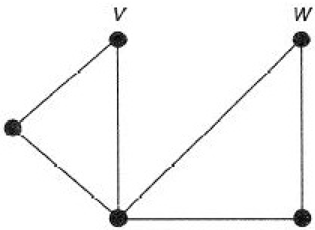


Edge deleted Subgraph

- If e is an edge in G, we denote by $G - e$ the graph obtained from G by deleting the edge e.
- Similarly, if F is a subset of E , then the graph $G - F$ is the subgraph obtained from G by deleting edges in F



G



G − e

Isomorphic Graphs

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists $f : V_1 \to V_2$ such that

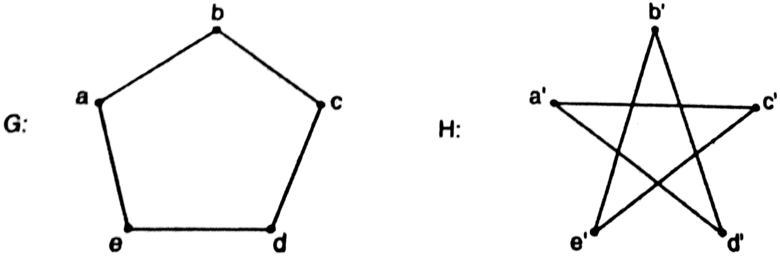
- f is one-one and onto function
- $\{a, b\}$ is an edge in E_1 if and only if $\{f(a), f(b)\}$ is an edge in E_2

Such a function f is called an isomorphism.

Two simple graphs that are not isomorphic are called nonisomorphic.

- Isomorphic simple graphs must have the same number of vertices.
- Isomorphic simple graphs also must have the same number of edges.
- The degrees of the vertices in isomorphic simple graphs must be the same. That is, a vertex v of degree d in G must correspond to a vertex $f(v)$ of degree d in H.

Example : Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.



Solution : Here $V(G) = \{a, b, c, d, e\}$ and $V(H) = \{a', b', c', d', e'\}$ so $|V(G)| = 5 = |V(H)|$. Also $|E(G)| = 5 = |E(H)|$

The vertices of degree 2 in G are $\{a, b, c, d, e\}$ and vertices of degree 2 in H are $\{a', b', c', d', e'\}$

Define a function $f : V(G) \to V(H)$ as

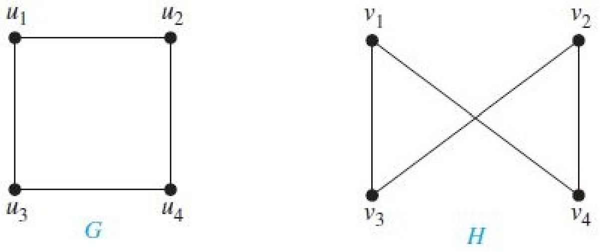
$$f(a) = a', \quad f(b) = c', \quad f(c) = e', \quad f(d) = b', \quad f(e) = d'$$

then f is one-one and onto. Further

$$\begin{aligned}
\{a, b\} \in E(G) \quad &\text{and} \quad \{f(a), f(b)\} = \{a', c'\} \in E(H) \\
\{a, e\} \in E(G) \quad &\text{and} \quad \{f(a), f(e)\} = \{a', d'\} \in E(H) \\
\{b, c\} \in E(G) \quad &\text{and} \quad \{f(b), f(c)\} = \{c', e'\} \in E(H) \\
\{c, d\} \in E(G) \quad &\text{and} \quad \{f(c), f(d)\} = \{e', b'\} \in E(H) \\
\{d, e\} \in E(G) \quad &\text{and} \quad \{f(d), f(e)\} = \{b', d'\} \in E(H)
\end{aligned}$$

Therefore, G and H are isomorphic.

Example : Show that the graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic.



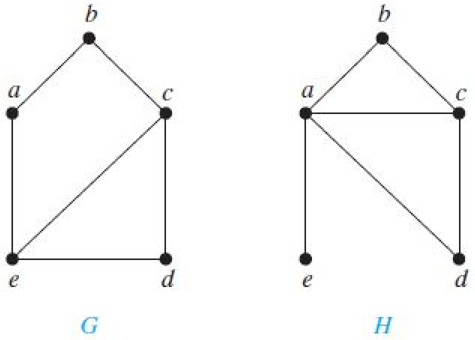
Solution : The function f with

$$f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3, \text{ and } f(u_4) = v_2$$

is a one-one correspondence between V and W.

Hence $G = (V, E)$ and $H = (W, F)$ are isomorphic.

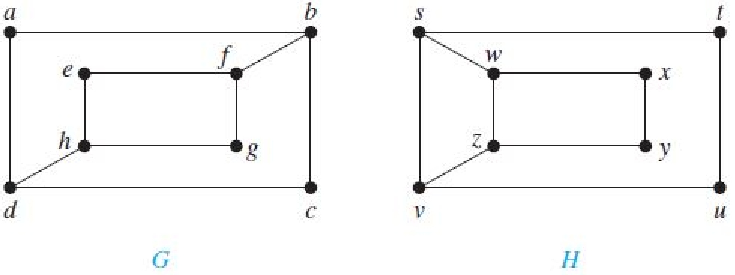
Example : Show that the following graphs are not isomorphic.



Solution : Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one.

So G and H are not isomorphic.

Example : Determine whether the following graphs are isomorphic.

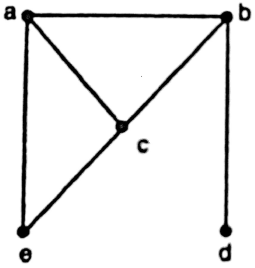


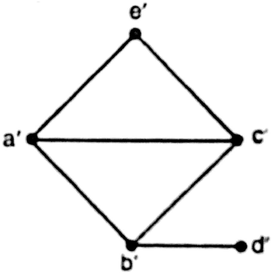
Solution : The graphs G and H both have eight vertices and 10 edges. They also both have four vertices of degree two and four of degree three.

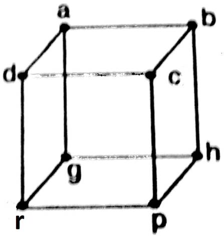
As $deg(a) = 2$ in G, a must correspond to either $t, u, x,$ or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

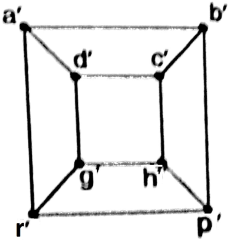
So G and H are not isomorphic.

Example : Determine whether following graphs are isomorphic









Walk

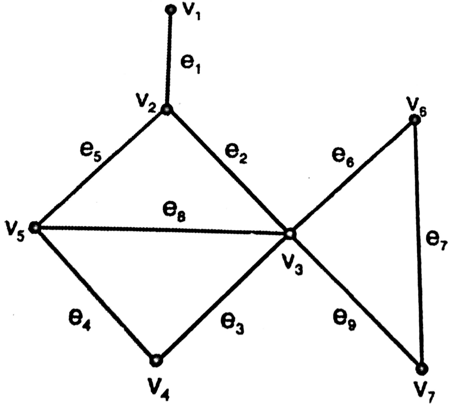
- A walk in a graph G is a finite alternating sequence

$$v_0 - e_1 - v_1 - e_2 - v_2 - e_3 - \cdots - v_{n-1} - e_n - v_n$$

  of vertices and edges of the graph such that each edge e_i in the sequence joins vertices v_{i-1} and v_i, $1 \leq i \leq n$.
- The end vertices v_0 and v_n are called terminal vertices of the walk.
- The vertices $v_1, v_2, \cdots, v_{n-1}$ are called internal vertices.
- The number of edges in a walk is called length of the walk.
- A walk is called open when the terminal vertices are distinct.
- A walk is called closed when the terminal vertices are same.

Trail, Circuit and Path

- A walk is called trail if all edges are distinct.
- A trail is called open when the terminal vertices are distinct.
- A trail is called closed when the terminal vertices are same.
- A closed trail is called Circuit.
- A walk is called a path if all vertices and edges are distinct.
- A path in which only repeated vertex is first vertex is called cycle.
- A path or circuit is simple if it does not contain the same edge more than once.



v1
e1
v2
v6
e5
e2
e6
e8
v5
e7
v3
e4
e3
e9
v4
v7

- The sequence

$$v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$$

  is a walk of length 8.
- The sequence

$$v_1 - e_1 - v_2 - e_5 - v_5 - e_3 - v_3 - e_3 - v_4 - e_4 - v_5$$

  is a trail.
- The sequence

$$v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$$

  is a path.
- The sequence

$$v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$$

  is a cycle.

Konigsberg Bridge Problem

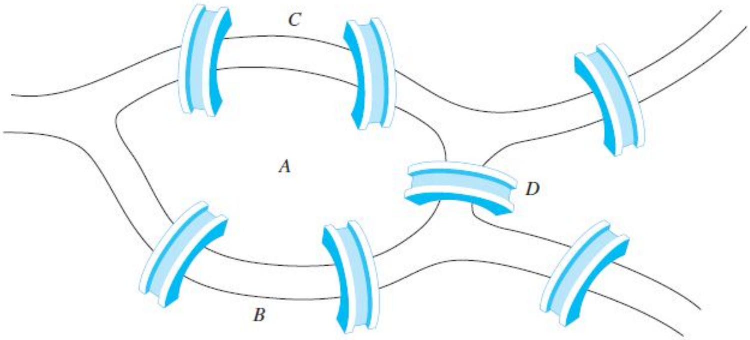


FIGURE 1 **The Seven Bridges of Königsberg.**

Whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.

The Swiss mathematician Leonhard Euler solved this problem. Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure 2.

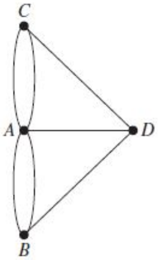
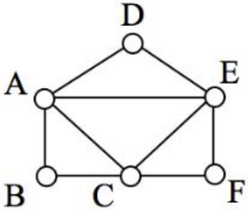
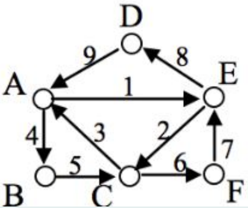


FIGURE 2 **Multigraph Model of the Town of Königsberg.**

Euler path and Circuit

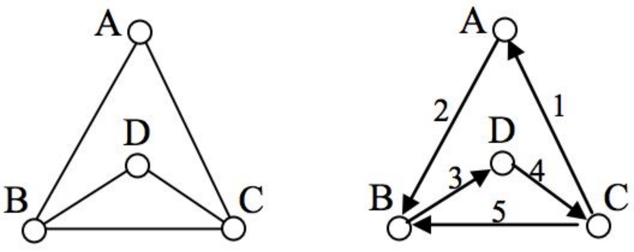
- A circuit in a connected graph is an Euler circuit if it contains every edge of graph exactly once.





- A connected graph with with an Euler circuit is called an Euler graph or Eulerian graph.

- If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, then trail is called Euler trail.
- An Euler path in G is a simple path containing every edge of G.



Theorem

A nonempty connected graph G is Eulerian if and only if all its vertices are of even degree.

Solution to Konigsberg Bridge Problem

In Konigsberg Bridge Problem, the degree of every vertex is odd, so graph is not Eulerian.

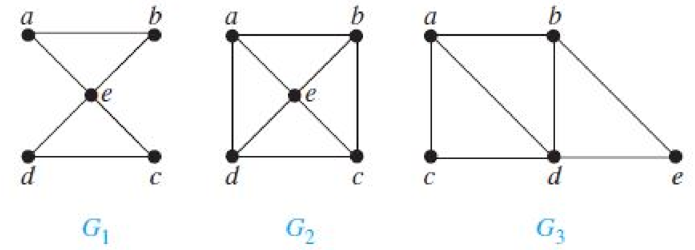
Hence it is impossible to walk through the town crossing each bridge exactly once and return to starting point

Determination of Euler Circuit and Euler path in a graph

- List the degree of all vertices in a graph.
- If any value is zero it cannot have Euler path or Euler circuit.
- If all degrees are even, then G has both Euler path and Euler Circuit.
- If exactly two vertices are of odd degree, then G has Euler path but no Euler circuit.

Note :
(i) K_n is Eulerian if and only if n is odd.
(ii) $K_{m,n}$ if and only if m and n both are even.

Example : Which of the undirected graphs below have an Euler circuit? Of those that do not, which have an Euler path?



Solution : For G_1

$$deg(a) = 2, \;\; deg(b) = 2, \;\; deg(c) = 2, \;\; deg(d) = 2, \;\; deg(e) = 4.$$

Since degree of all vertices is even, G_1 has both Euler path and Euler circuit.

The graph G_1 has an Euler circuit, for example, a, e, c, d, e, b, a.

For G_2

$$deg(a) = 3, \;\; deg(b) = 3, \;\; deg(c) = 3, \;\; deg(d) = 3, \;\; deg(e) = 4.$$

Since degree of all vertices is not even, G_2 has no Euler circuit.

Also since there are more than two vertices of odd degree G_2 has no Euler path.

For G_3

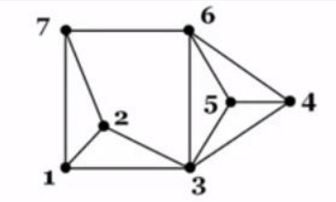
$$deg(a) = 3, \;\; deg(b) = 3, \;\; deg(c) = 2, \;\; deg(d) = 4, \;\; deg(e) = 2.$$

Since degree of all vertices is not even, G_3 has no Euler circuit.

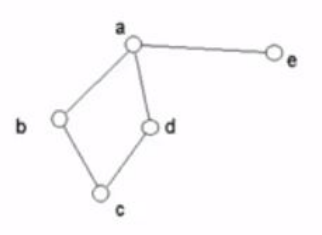
As there are exactly two vertices a and d of odd degree, G_3 has Euler path namely, a, c, d, e, b, d, a, b.

Hamiltonian Path and Circuit

A Hamiltonian circuit is a circuit that visits every vertex once with no repeats, except it must start and end at the same vertex.

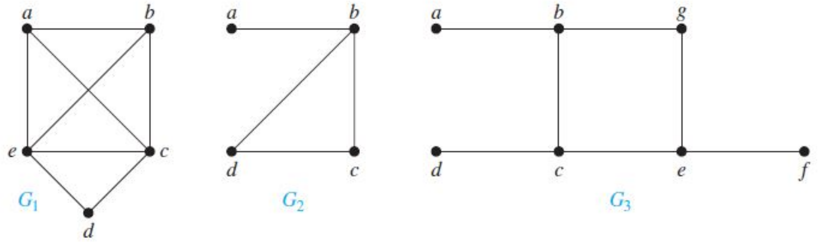


A Hamiltonian path contains every vertex once with no repeats, but does not have to start and end at the same vertex.



A connected graph with with an Hamiltonian circuit is called Hamiltonian graph.

Example : Which of the simple graphs below have a Hamilton circuit or, if not, a Hamilton path?



Solution : G_1 has a Hamilton circuit:

$$a, b, c, d, e, a.$$

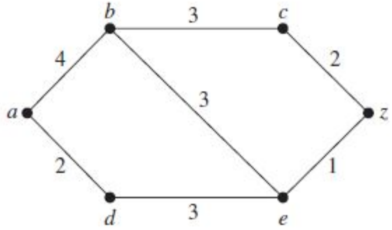
There is no Hamilton circuit in G_2 since any circuit containing every vertex must contain the edge $\{a, b\}$ twice but G_2 does have a Hamilton path

$$a, b, c, d.$$

G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}, \{e, f\}$, and $\{c, d\}$ more than once.

Shortest Path Problem

(1) Weighted Graph



Shortest path from to a to z is

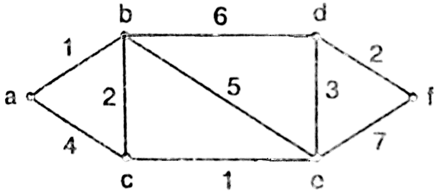
$$a, d, e, z$$

and length of shortest path is 6.

Dijkstras algorithm

To find the length of a shortest path between two vertices in a connected simple undirected weighted graph.

Example : Apply Dijkstra's algorithm to the graph give below and find the shortest path from a to f



Solution :

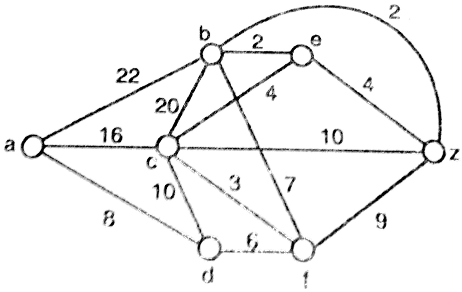
v	a	b	c	d	e	f
a	0_a	1_a	4_a	∞	∞	∞
b		1_a	3_b	7_b	6_b	∞
c			3_b	7_b	4_c	∞
e				7_e	4_c	11_e
d				7_e		9_d

Hence the shortest path from a to f is

$$a\ b\ c\ e\ d\ f$$

and its length is equal to 9.

Example : Apply Dijkstra's algorithm to the graph give below and find the shortest path from a to z



Solution :

v	a	b	c	d	e	f	z
a	0_a	22_a	16_a	8_a	∞	∞	∞
d		22_a	16_a	8_a	∞	14_d	∞
f		21_f	16_a		∞	14_d	23_f
c		21_f	16_a		20_c		23_f
e		21_f			20_c		23_f
b		21_f					23_b

Hence the shortest path from a to z is

$$a\ d\ f\ b\ z$$

and its length is equal to 23.

Graph without weight

- Breadth First Search Algorithm

To find the shortest path

(i) Label vertex s with 0, set $i = 0$.

(ii) Find all unlabelled vertices in graph which are adjacent to vertices labelled i. If there are no such vertices then i is not connected to s.

(iii) If t is labelled go to next step.

(iv) The length of shortest path from s to t is then $i + 1$.

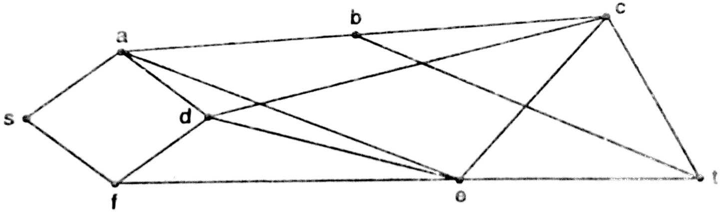
- Back Tracking Algorithm

(i) Set $i = \lambda(t)$ and assign $v_i = t$

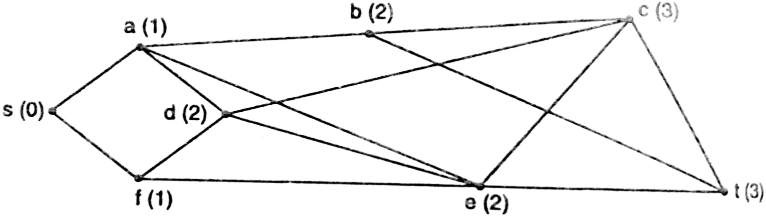
(ii) Find a vertex u adjacent to v_i and with $\lambda(u) = i - 1$. Assign $v_{i-1} = u$.

(iii) If $i = 1 - 1$ then stop. Otherwise decrease i to $i - 1$ and follow step (ii).

Example : Find the shortest path from vertex s to t and its length from the graph given below



Solution : First we apply Breadth First Search Algorithm



Next we apply Back Tracking Algorithm to find the path

Since $\lambda(t) = 3$, assign $v_3 = t$

We choose e or b adjacent to $v_3 = t$ with $\lambda(e) = 2$, assign $v_2 = e$

Next choose f adjacent to $v_2 = e$ with $\lambda(f) = 1$, assign $v_1 = f$

At last we take s adjacent to $v_1 = f$ with $\lambda(s) = 0$, assign $v_0 = s$

Thus we have the shortest path $v_0\ v_1\ v_2\ v_3 = s\ f\ e\ t$ from s to t.