Differential Equation

An equation involving derivatives or differentials of one or more independent variables is called a differential equation.

Order: Order of the highest order derivatives involved

Degree : Degree of the highest order derivatives involved.

① $$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = e^x, \qquad \text{ODE}$$

Order – 4, degree – 1.

② $$y(y^2+1)\,dx + x(y^2+1)\,dy = 0 \qquad \text{ODE}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y(y^2+1)}{(y^2+1)x} = 0.$$

Order 1, degree 1.

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$$ Order - 3, degree - 2 PDE

Linear differential equation:

A differential eqn is called linear if
i) Every dependent variable and every derivative occur in the first degree only.
ii) No products of dependent variables and/or derivatives occur.

If the differential eqn is not linear, then it is called non linear.

Note → Every linear eqn is of first degree but every first degree eqn may not be linear.

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0.$$
Degree - 1 but nonlinear

Soln of Differential Eqn:

Any relation between the dependent and independent variables (no derivative terms) which satisfy the differential eqn is called a soln of integral of the differential eqn.

Ex. $y = \frac{A}{x} + B$ is a soln of

$y'' + \left(\frac{2}{x}\right)y' = 0$.

$y' = -\frac{A}{x^2} \Rightarrow y'' = \frac{2A}{x^3}$

Here y satisfies the given differential eq^n.

Family of curves: An n parameter family of curves is a set of relations of the form
$\{(x,y): f(x,y,c_1,c_2,\dots c_n) = 0\}$

Ex: 1) Set of all concentric circles $x^2+y^2=c$.
One parameter family if c takes non negative values.

(ii) Set of circles $(x-c_1)^2+(y-c_2)^2=c_3$
Three parameter family if c_1, c_2 takes all real values and c_3 takes all non negative real values.

Sol^n of differential eq^n is family of curves.

Formation of differential eq^{n}s from a given n-paremeters family of curves:

From a given family of curves containing n arbitrary constants, we can obtain an nth order differential eq^n whose sol^n is the given family.

- Differentiate the given eq^n n times to get n additional eq^{n}s containing those arbitrary constants.
- Eliminate n arbitrary constants from the $(n+1)$ eq^{n}s.
- Obtain a differential eq^n of nth order

Ex1.

Obtain the differential eqn satisfied by $xy = ae^{x} + be^{-x} + x^2$, a & b are arbitrary constants.

→ Given family of curves :

$$xy = ae^{x} + be^{-x} + x^2 \quad (1)$$

Differentiating (1) w.r.t x,

$$x\frac{dy}{dx} + y = a e^{x} - b e^{-x} + 2x$$

Differentiating again

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$= xy - x^2 + 2 \quad \text{(From (1))}$$

$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 2 - x^2$$

(Ans).

General and particular soln

Let $F(x, y, y', y'', \ldots y^{n}) = 0$ be an nth order ordinary differential eqn.

1) General soln : Soln containing n-independent arbitrary constant.

II) Particular soln : Soln by giving particular values to one or more of the n independent constants.

Remark : Observe that the number of arbitrary constants in a soln of a differential eqn depends upon the order of differential eqn.

Therefore General soln of nth order differential eqn will contain n arbirary constants.

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Linear first order ODE

An equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{—— (1)}$$

Where P, Q are functions of x only (or constants) is called linear equation of first order in y. The dependent variables as also its derivative in such equations occur in the first degree only, not as higher powers or products.

Let R be an integrating factor of the above ~~left hand side of the eq~~ eq. Then the eq becomes

$$R\frac{dy}{dx} + RPy = RQ$$

$$\text{Now} \quad R\frac{dy}{dx} + RPy = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$$

$$RP = \frac{dR}{dx}.$$

$$\Rightarrow \frac{dR}{R} = P\,dx$$

Integrating $\log R = \int P\,dx$, so $R = e^{\int P\,dx}$

So, $R = e^{\int P\,dx}$ is an integrating factor.

Multiplying both sides of (1) by $e^{\int P\,dx}$ by the integrating factor, we get

$$\frac{dy}{dx}e^{\int P\,dx} + Py\,e^{\int P\,dx} = Qe^{\int P\,dx}$$

$$\text{or } d\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}\,dx$$

Integrating, $ye^{\int P\,dx} = \int Qe^{\int P\,dx} + c$, c is a constant.

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Note → Sometimes ~~eo~~ an eqn may be linear in x, where y is the independent variable. The form of such eqn is

$$\frac{dx}{dy} + P_1 x = Q_1$$

P_1, Q_1 are functions of y only (or constant)

The general soln, in this case, will be

$$x e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + c.$$

Ex 1. $\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{1}{(x^2+1)^3}$

The eqn is ~~not~~ linear form in y.

Here $P = \frac{4x}{x^2+1}$, $Q = \frac{1}{(x^2+1)^3}$

Integrating factor, $e^{\int P dx}$

$$= e^{\int \frac{4x}{x^2+1} dx}$$

$$= e^{2\log(x^2+1)} = e^{\log(x^2+1)^2}$$

$$= (x^2+1)^2.$$

Multiplying both sides of eqn by I.F,

$$(x^2+1)^2 \frac{dy}{dx} + 4x(x^2+1) y = \frac{1}{x^2+1}$$

$$\Rightarrow d\{(x^2+1)^2 y\} = \frac{1}{x^2+1} dx.$$

Integrating both sides,

$$y(x^2+1)^2 = \tan^{-1}x + c,$$

where c is arbitrary constant.

Ex 2. $1+y^2+(x-e^{-\tan^{-1}y})\frac{dy}{dx}=0$

The equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

which is linear in x.

$$P=\frac{1}{1+y^2},\quad Q=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

Integrating factor, $e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}$

$$=e^{\tan^{-1}y}$$

Multiplying both sides by I.F,

$$\frac{dx}{dy}e^{\tan^{-1}y}+\frac{e^{\tan^{-1}y}}{1+y^2}x=\frac{1}{1+y^2}$$

$$\Rightarrow\quad d\left(xe^{\tan^{-1}y}\right)=\frac{1}{1+y^2}dy$$

Integrating both sides, we get

$$\therefore\ xe^{\tan^{-1}y}=\tan^{-1}y+c,$$ c is an arbitrary constant

Solve

1. $\cos x\frac{dy}{dx}+y\sin x=1$
2. $(x^2+1)\frac{dy}{dx}+2xy=4x^2$
3. $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$
4. $\frac{dy}{dx}+2xy=e^{-x^2}$

Equation reducible to linear form

An equation of the form $f'(y)\frac{dy}{dx} + Rf(y) = Q(x)$.

Substituting $f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$.

Equations reduces to $\frac{dv}{dx} + Pv = Q$.

$\hookrightarrow$ linear in v

A special case: Bernoulli's eq^n.

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n.$$

where P and Q are constants or function of x and n is a constant except 0 and 1 is called Bernoulli's differential eq^n.

The above eq^n can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P = Q.$$

Substitute $\frac{1}{y^{n-1}} = v \Rightarrow (1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q.$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n) \quad \text{—— linear in } v.$$

Ex 1. $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0$.

$$\Rightarrow 2xy\frac{dy}{dx} + x^2 - 2x + 2y^2 = 0.$$

$$\Rightarrow 2y\frac{dy}{dx} + \frac{2y^2}{x} = \frac{\cancel{x} \; 2x - x^2}{x} \quad \text{—— (1)}$$

Substitute $y^2 = v \Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$.

$$\frac{dv}{dx} + \frac{2}{x}v = (2-x).$$

$$\text{I.F} = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2.$$

$\therefore \int d(vx^2) = \int x^2(2-x)\,dx + c$, c is an arbitrary constant.

$$\Rightarrow vx^2 = \frac{2}{3}x^3 - \frac{x^4}{4} + c$$

$$\Rightarrow y^2 = \frac{2}{3}x - \frac{x^2}{4} + \frac{c}{x^2}.$$

Ex 2. $$\frac{dy}{dx} - y\tan x = -y^2\sec x.$$

Dividing by y^2

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Substitute $\frac{1}{y} = v$.

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow -\frac{dv}{dx} - v\tan x = -\sec x$$

$$\Rightarrow \frac{dv}{dx} + \tan x \cdot v = \sec x$$

$$\text{I.F} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x.$$

$\therefore \int d(v\sec x) = \int \sec^2 x\,dx + c$, c is an arbitrary constant

$$\Rightarrow v\sec x = \tan x + c$$

$$\Rightarrow \frac{1}{y}\sec x = \tan x + c.$$

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Solve the following problem

1. $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$.
2. $y(2xy + e^x)\,dx - e^x\,dy = 0$.
3. $\frac{dy}{dx} - \frac{1}{1+x}\tan y = (1+x)e^x \sec y$.
4. $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$.
5. $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$.

Linear differential eqn of Higher Order with Constant coeffecients:

• The general form:

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = X \quad (1)$$

where $a_1, a_2, \ldots a_n$ are constants and X is a function of x.

Its general solution will be of the form: General Soln = Complementary fn (C.F) + Particular Integral (P.I).

Complementary fn: It is the general Soln of the corresponding homogeneous eqn

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = 0 \quad (2)$$

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• Particular Integral: If v is any particular solⁿ, of then

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_n v = X$$

$\therefore$ v is the particular integral.

If $y = f(x)$ be the solⁿ of & eqⁿ (2) and $y = \phi(x)$ be particular solⁿ of (1), then the general solⁿ of (1) will take the form

$$y = \underbrace{f(x)}_{C.F} + \underbrace{\phi(x)}_{P.I}$$

• <u>Differential Operator D</u>

We use the symbol D for the differential operator $\frac{d}{dx}$, so that $\frac{d^r y}{dx^r}$; we write $D^r y$.

If m_1 be a constant, $(D - m_1)y \equiv \frac{dy}{dx} - m_1 y$.

The notation $(D-m_1)(D-m_2)y$ is defined to mean that y is operated first with $(D-m_2)$, then the result is operated with $(D-m_1)$

$$\therefore \quad (D-m_1)(D-m_2)y \equiv (D-m_1)\left(\frac{dy}{dx} - m_2 y\right)$$

$$\equiv \frac{d^2y}{dx^2} - m_1\frac{dy}{dx} - m_2\frac{dy}{dx} + m_1 m_2 y$$

$$\equiv \frac{d^2y}{dx^2} - (m_1+m_2)\frac{dy}{dx} + m_1 m_2 y.$$

Also $(D-m_1)(D-m_2)y = (D-m_2)(D-m_1)y$

Solution of Homogeneous linear Equations (Complementary function)

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0$$

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n] y = 0$$

$$\Rightarrow [(D-\alpha_1)(D-\alpha_2) \cdots (D-\alpha_n)] y = 0 .$$

Treating the operator D as a no, the ordinary laws of multiplication works.

Consider $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$.

Writing the eq^n in operator form,

$$(D^2 + a_1 D + a_2) y = 0 .$$

Auxiliary eq^n will be $m^2 + a_1 m + a_2 = 0$ —— (3)

Case 1: Non repeated roots:

Let α_1, α_2 be two non repeated, distinct roots of (3).

$$(D^2 + a_1 D + a_2) y = 0 \Rightarrow (D-\alpha_1)(D-\alpha_2) y = 0$$

A sol^n of above eq^n: $(D-\alpha_2) y = 0$

$$\Rightarrow \frac{dy}{dx} = \alpha_2 y$$

$$\Rightarrow y = e^{\alpha_2 x} .$$

If we consider $(D-\alpha_2)(D-\alpha_1) y = 0$.

A sol^n of $(D-\alpha_1) y = 0 \Rightarrow \frac{dy}{dx} = \alpha_1 y$

$$\Rightarrow y = e^{\alpha_1 x}$$

Thus the general sol^n : $y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$.

Case II : Repeated roots

$(D-\alpha)(D-\alpha)y = 0$

Let $(D-\alpha)y = z$; then $(D-\alpha)z = 0$

$\Rightarrow z = c_1 e^{\alpha x}$

Now solving $(D-\alpha)y = c_1 e^{\alpha x}$

$\Rightarrow \frac{dy}{dx} - \alpha y = c_1 e^{\alpha x}$ → linear in y.

$\therefore$ I.F $= e^{-\alpha x}$

$\therefore y e^{-\alpha x} = \int c_1 e^{\alpha x} e^{-\alpha x} dx + c_2$

$= c_1 x + c_2$

$\Rightarrow y = (c_1 x + c_2) e^{\alpha x}$ will be general sol^n

Case III : Imaginary roots :

Let $\alpha + i\beta$, $\alpha - i\beta$ be two ~~congu~~ conjugate roots.

$\therefore y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$

$\Rightarrow y = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$

$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$= e^{\alpha x}[c_1 (\cos\beta x + i\sin\beta x) + c_2 (\cos\beta x - i\sin\beta x)]$

$= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x]$

$\Rightarrow y = e^{\alpha x}(c_1' \cos\beta x + c_2' \sin\beta x)$.

[Note → Repeated imaginary roots, $\alpha \pm i\beta$, $\alpha + i\beta$

$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$]

Ex 1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

In operator form, $(D^2 - 5D + 6)y = 0$.

Auxiliary eqn $m^2 - 5m + 6 = 0$.

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2, 3$.

The general soln will be $y = c_1 e^{2x} + c_2 e^{3x}$,

c_1, c_2 are arbitrary constants.

Ex 2. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

In operator form, $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$

Auxiliary eqn : $(m^4 - 2m^3 + 5m^2 - 8m + 4) = 0$

Roots are $m = 1, 1, 2i, -2i$.

The general soln will be

$y = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x$.

Solve

① $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = 0$.

② $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$.

③ Solve the eqn $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ and find the particular soln if $y = 3$ when $x = 0$ and $y = 8$ when $x = \log 2$.

④ Solve $\frac{d^2x}{dt^2} + 4t = 0$, satisfying $x = 4$, $\frac{dx}{dt} = 3$ when $t = 0$.

Determination of Particular Integral :

$f(D)y = X$. Particular Integral (P. I)

$$= \frac{1}{f(D)} X .$$

$\frac{1}{f(D)}$ is called the inverse operator.

The operator $f(D)$ can be expressed as

$$(D-\alpha_1)(D-\alpha_2) \ldots (D-\alpha_n)$$

$$\text{Particular Integral (P. I)} = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)} X$$

General method for P. I : $\frac{1}{D-a} X = e^{ax}\int X e^{-ax} dx.$

$$\Rightarrow \quad y = \frac{1}{(D-a)} X$$

$$\Rightarrow \quad (D-a)y = X \Rightarrow \frac{dy}{dx} - ay = X.$$

$$I.F = e^{-\int a\,dx} = e^{-ax}$$

$$\therefore \quad y e^{-ax} = \int X e^{-ax} + c \to 0$$

(Since it is particular Integral)

$$\therefore \quad y = e^{ax}\int X e^{-ax} dx.$$

Ex 1. Solve $(D^2 + a^2)y = \sec ax$

→ Consider corresponding homogeneous eqn.

$$(D^2 + a^2)y = 0.$$

Auxiliary eqn will be $m^2 + a^2 = 0.$

$\Rightarrow m = \pm ia$

Complementary fn will be $c_1 \cos ax + c_2 \sin ax.$

$$P.I = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec a$$

Consider $\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\, e^{-iax}$

$= e^{iax}\left[x + \frac{i}{a}\ln|\cos ax|\right]$

Similarly $\frac{1}{D+ia}\sec ax = e^{-iax}\left[x - \frac{i}{a}\ln|\cos\right.$

$\therefore P.I = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\ln|\cos ax|\right\} - e^{-iax}\left\{x - \frac{i}{a}\ln|\cos ax|\right\}\right]$

$= \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cos ax.$

General sol^n :

$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\ln|\cos ax|\cos ax.$

Particular Integral for some special forms of X (e^{ax}, $\cos(ax)$, $\sin(ax)$) :

- If a is a constant, then

$$f(D)\, e^{ax} = f(a)\, e^{ax}.$$

$$D e^{ax} = a e^{ax}$$

$$\Rightarrow \quad D^2 e^{ax} = a^2 e^{ax}.$$

$$\therefore \quad f(D)\, e^{ax} = f(a)\, e^{ax}. \quad \text{—— (1)}$$

- When X is of the form e^{ax},

$$f(D)\, y = e^{ax}.$$

$$\Rightarrow \quad y = \frac{1}{f(D)} e^{ax}.$$

Operating $\frac{1}{f(D)}$ on both sides of (1).

$$e^{ax} = \frac{1}{f(D)} f(a) e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0$$

If $f(a) = 0$, then $(D-a)$ is a factor of $f(D)$

Consider $f(D) = (D-a)\, g(D)$.

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)} \frac{1}{g(D)} e^{ax}$$

$$= \frac{1}{(D-a)} \frac{1}{g(a)} e^{ax}, \text{ provided } g(a) \neq 0$$

$$= \frac{1}{g(a)} \frac{1}{(D-a)} e^{ax}$$

$$= \frac{1}{g(a)} x e^{ax}$$

In summary: When $X = e^{ax}$.

1. $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, when $f(a) \neq 0$.
2. If $f(a) = 0$, then $f(D)$ must have a factor of type $(D-a)^r$. Then $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$.

Ex 1. Find the general sol^n of $(D^2 - 3D + 2)y = e^{3x}$ (1).

→ Complementary f^n:

The auxiliary eq^n of (1) is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m = 2, 1.$$

$\therefore$ C.F, $y_c = c_1 e^{2x} + c_2 e^{x}$, c_1, c_2 being arbitrary constants

Particular integral: $y_p = \frac{1}{D^2-3D+2} e^{3x}$

$= \frac{1}{3^2-3\cdot3+2} e^{3x}$

$= \frac{1}{2} e^{3x}$.

The general sol^n is $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$.

2. Find Particular sol^n of $(D^3-D^2-D+1)y = e^x$

$$P.I = \frac{1}{D^3-D^2-D+1} e^{x}$$

$$= \frac{1}{(D-1)^2(D+1)} e^{x}$$

$$= \frac{1}{2} \frac{1}{(D-1)^2} e^{x}$$

$$= \frac{1}{2} \frac{x^2}{2!} e^{x} = \frac{1}{4} x^2 e^{x}.$$

- If X is of the form $\cos\alpha x$ or $\sin\alpha x$

If α, β are constants, then

$$\phi(D^2) \sin(\alpha x+\beta) = \phi(-\alpha^2) \sin(\alpha x+\beta)$$

and $$\phi(D^2) \cos(\alpha x+\beta) = \phi(-\alpha^2) \cos(\alpha x+\beta).$$

$\therefore$ If $\phi(-\alpha^2) \neq 0$,

$$\frac{1}{f(D)} \sin(\alpha x+\beta) = \frac{1}{\phi(D^2)} \sin(\alpha x+\beta) = \frac{1}{\phi(-\alpha^2)} \sin(\alpha x+\beta).$$

$$\frac{1}{f(D)} \cos(\alpha x+\beta) = \frac{1}{\phi(D^2)} \cos(\alpha x+\beta) = \frac{1}{\phi(-\alpha^2)} \cos(\alpha x+\beta).$$

Ex 1. Evaluate $\frac{1}{D^4+D^2+1}\cos 2x$.

$$\frac{1}{D^4+D^2+1}\cos 2x = \frac{1}{(-2^2)^2+(-2^2)+1}\cos 2x$$
$$= \frac{1}{13}\cos 2x.$$

Ex 2. Evaluate $\frac{1}{D^2-2D+1}\cos 3x$

$$= \frac{1}{-3^2-2D+1}\cos 3x$$
$$= \frac{1}{-2(D+4)}\cos 3x.$$
$$= -\frac{1}{2}\cdot\frac{(D-4)}{(D+4)(D-4)}\cos 3x$$
$$= -\frac{1}{2}\frac{D-4}{D^2-16}\cos 3x$$
$$= -\frac{1}{2}\frac{(D-4)}{-3^2-16}\cos 3x$$
$$= \frac{1}{50}(D-4)\cos 3x$$
$$= \frac{1}{50}(3\sin 3x - 4\cos 3x)$$
$$= -\frac{1}{50}(3\sin 3x + 4\cos 3x).$$

If $\phi(-\alpha^2) = 0$. $\frac{1}{D^2+\alpha^2}\sin\alpha x$

Consider P.I $= \frac{1}{D^2+\alpha^2}\cos\alpha x + i\frac{1}{D^2+\alpha^2}\sin\alpha x\}$

$$= \text{Im}\left\{\frac{1}{D^2+\alpha^2}e^{i\alpha x}\right\}$$
$$= \text{Im}\left\{\frac{1}{D^2+\alpha^2}e^{i\alpha x}\right.$$

Consider $\frac{1}{D^2+\alpha^2}e^{i\alpha x} = \frac{1}{(D-i\alpha)(D+i\alpha)}e^{i\alpha x}$

$$= \frac{1}{2i\alpha}\left(\frac{1}{(D-i\alpha)}e^{i\alpha x}\right)$$
$$= \frac{x}{2i\alpha}e^{i\alpha x}$$

$$\frac{1}{D^2+\alpha^2}\sin\alpha x = im\left\{\frac{x}{2\alpha}\sin\alpha x - i\frac{x}{2\alpha}\cos\alpha x\right\}$$

$$= -\frac{x}{2\alpha}\cos\alpha x$$

$$\frac{1}{D^2+\alpha^2}\cos\alpha x = Re\left\{\frac{x}{2\alpha}\sin\alpha x - i\frac{x}{2\alpha}\cos\alpha x\right\}$$

$$= \frac{x}{2\alpha}\sin\alpha x.$$

Ex. Find the general sol^n of $(D^2+4)y = \sin^2 x$.

$C.F = c_1\cos 2x + c_2\sin 2x.$

$$P.I = \frac{1}{D^2+4}\sin^2 x = \frac{1}{D^2+4}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{8} - \frac{1}{2}\frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{8} - \frac{1}{2}\cdot\frac{x}{\cancel{2\alpha}\,2\cdot 2}\sin 2x$$

$$= \frac{1}{8} - \frac{x}{8}\sin 2x$$

$\therefore$ General sol^n $y = CF + PI$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{8}(1-x\sin 2x)$$

I. X is x^m or a polynomial of degree m

Take out the lowest degree term from $f(D)$, so as ~~it~~ to reduce it in the form $[1 \pm F(D)]^\alpha$.

Take it to numerator and expand it.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Ex. Evaluate $\frac{1}{D^3 - D^2 - 6D}(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1)$$

$$= -\frac{1}{6D}\left(1+\frac{D}{6}-\frac{D^2}{6}\right)^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2 - \dots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}+\dots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{2x}{6}+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

- If X is of the form $e^{ax}V$, where V is any function of x.

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$$

Ex. Evaluate $\frac{1}{D^2+3D+2}e^{2x}\sin x$

$$= e^{2x}\cdot\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

$$= e^{2x}\cdot\frac{1}{D^2+7D+12}\sin x$$

$$= e^{2x}\cdot\frac{1}{-1+7D+12}\sin x$$

$$= e^{2x}\cdot\frac{1}{7D+11}\sin x$$

$$= e^{2x}\frac{7D-11}{49D^2-121}\sin x$$

$$= \frac{e^{2x}}{170}(11\sin x - 7\cos x).$$

- If X is of the form xV, where V is any function of x, $\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V$

Ex. Evaluate $\frac{1}{D^2-2D+1}x\sin x$.

$$= x\cdot\frac{1}{D^2-2D+1}\sin x - \frac{2D-2}{(D^2-2D+1)^2}\sin x$$

$$= x\cdot\frac{1}{-1-2D+1}\sin x - \frac{2D-2}{(-1-2D+1)^2}\sin x$$

$$= x\cdot\frac{1}{-2D}\sin x - \frac{2D-2}{4D^2}\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

Solve

① $(D^2-4D+4)y = x^3$

② $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$.

③ $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2e^{3x}$

④ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos^2 x$

Method of Variation of Parameter

Let us consider general linear eqn of second order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad \text{——(1)}$$

where P, Q and X are functions of X.

Let $y = Au + Bv$ (2) be complementary fn of corresponding homogeneous eqn (1), where A, B are constants and u, v are independent soln of fns of x and are independent soln of homogeneous eqn corresponding to (1).

Thus

$$\left.\begin{aligned} \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \\ \frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \end{aligned}\right\} \quad (3)$$

Let us assume that $y = \phi u + \psi v$ (4) be the complete premitive of (1). We take ϕ, ψ as a fn of x in place of A and B.

Differentiating (4) w.r.t x

$$\frac{dy}{dx} = \phi\frac{du}{dx} + u\frac{d\phi}{dx} + \psi\frac{dv}{dx} + v\frac{d\psi}{dx}.$$

We choose ϕ, ψ so that

$$u\frac{d\phi}{dx} + v\frac{d\psi}{dx} = 0 \quad \text{——(5)}$$

So that $\frac{dy}{dx} = \phi\frac{du}{dx} + \psi\frac{dv}{dx}$.

Differentiating this once again

$$\frac{d^2y}{dx^2} = \phi\frac{d^2u}{dx^2} + \frac{d\phi}{dx}\frac{du}{dx} + \frac{d\psi}{dx}\frac{dv}{dx} + \psi\frac{d^2v}{dx^2}$$

Substituting these values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1), we get

$$\phi\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+\psi\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv\right)$$
$$+\frac{d\phi}{dx}\frac{du}{dx}+\frac{d\psi}{dx}\frac{dv}{dx}=X$$

$$\therefore\ \frac{d\phi}{dx}\frac{du}{dx}+\frac{d\psi}{dx}\frac{dv}{dx}=X \quad \text{---(6)}$$

$$\therefore\ \frac{d\phi}{dx}=\frac{vX}{v\frac{du}{dx}-u\frac{dv}{dx}},\quad \frac{d\psi}{dx}=-\frac{uX}{v\frac{du}{dx}-u\frac{dv}{dx}}$$

Integrating,

$$\phi=\int\frac{vX\,dx}{v\frac{du}{dx}-u\frac{dv}{dx}}+c_1,\ =c_1-\int\frac{vX\,dx}{W}$$

$$\psi=-\int\frac{uX\,dx}{v\frac{du}{dx}-u\frac{dv}{dx}}+c_2=c_2+\int\frac{uX}{W}dx$$

<u>Wronskian</u> : ~~If u,~~ The Wronskian of two functions $u(x)$ and $v(x)$ are defined as

$$W(u,v)=\begin{vmatrix} u(x) & v(x) \\ \frac{du}{dx} & \frac{dv}{dx} \end{vmatrix}$$

Independent solⁿ : Two solⁿs u and v are called independent if $W(u,v)\neq 0$.

i.e. $uv'-u'v\neq 0$.

Ex Solve by the method of variation of parameter:

$$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$$

→ The auxiliary eqn is $m^2 + 4 = 0$

$$\Rightarrow m = \pm 2i$$

∴ The complementary function will be

$$C.F = A\cos 2x + B\sin 2x,$$ A, B are arbitrary constants

$$W(\cos 2x, \sin 2x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0$$

∴ $\cos 2x$, $\sin 2x$ are independent soln.

Assume the complete soln of the eqn will be $y = \phi(x)\cos 2x + \psi(x)\sin 2x$.

Using method of variation of parameter

$$\frac{d\phi}{dx} = -\frac{\sin 2x . 4\tan 2x}{2W}$$

$$\Rightarrow \phi(x) = -\int \frac{\sin 2x . \frac{4\sin 2x}{\cos 2x}}{2} dx + C_1$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x} dx + C_1$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= \sin 2x - \log(\sec 2x + \tan 2x) + C_1$$

$$\frac{d\psi}{dx} = \frac{\cos 2x \cdot 4\tan 2x}{2}$$

$$\psi(x) = 2\int \sin 2x \, dx + c_2$$

$$= -\cos 2x + c_2$$

The complete solⁿ will be

$$y = \phi(x)\cos 2x + \psi(x)\sin 2x$$

$$c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$$

Solve by method of variation of parameter

① $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

② $\frac{d^2y}{dx^2} + a^2y = \sec ax$

③ $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

④ $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$

⑤ $\frac{d^2y}{dx^2} + y = \sec x \tan x$

Cauchy - Euler Equation

A linear differential eqⁿ of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X$$

Denoting $\frac{d^n}{dx^n} \equiv D$

$$\left(x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_n\right) y = X.$$

Cauchy Euler eqn :

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = X$$

Choose $x = e^z$ or $z = \ln x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$.

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\Rightarrow x D y = D_1 y \quad \text{where } D_1 \equiv \frac{d}{dz}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2D^2 \equiv D_1^2 - D_1 = D_1(D_1 - 1).$$

So,

$$xD \equiv D_1$$

$$x^2D^2 \equiv D_1(D_1-1)$$

$$x^3D^3 \equiv D_1(D_1-1)(D_1-2).$$

$$x^nD^n \equiv D_1(D_1-1)(D_1-2)\cdots(D_1-(n-1)).$$

Ex 1. Find the general soln of

$$(x^2D^2 - xD + 2)y = x\ln x.$$

Let $x = e^z \Rightarrow z = \ln x.$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x}.$$

$$xD \equiv D_1, \quad \& \quad x^2D^2 \equiv D_1(D_1-1).$$

$$\therefore [D_1(D_1-1) - D_1 + 2]y = \cancel{z} \, ze^z.$$

$$\Rightarrow (D_1^2 - 2D_1 + 2)y = ze^z. \quad \text{—— (1)}$$

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Auxiliary eqn will be $m^2 - 2m + 2 = 0$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ C.F. $y_c = e^z(c_1 \cos z + c_2 \sin z)$.

$= x(c_1 \cos(\log x) + c_2 \sin(\log x))$.

$\therefore$ P.I. $y_p = \frac{1}{D_1^2 - 2D_1 + 2} z e^z$

$\cancel{= \frac{e^z}{1} \cdot \frac{1}{D_1^2 - 2D_1 + 2}} z$

$$= e^z \frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 2} z$$

$$= e^z \frac{1}{D_1^2 + 1} z$$

$$= e^z (1 + D_1^2)^{-1} z$$

$$= e^z (1 - D_1^2 + \cdots) z$$

$$= z e^z = x \ln x$$

$\therefore$ General soln, $y = x(c_1 \cos(\ln x) + c_2 \sin(\ln x)) + x \ln x$.

Solve

(1) $(x^2 D^2 + 3xD + 1)y = \frac{1}{(1-x)^2}$, where $D \equiv \frac{d}{dx}$

(2) $(x^2 D^2 - 3xD + 5)y = x^2 \sin(\log x)$

(3) $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^4$.

(4) $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$.

Laplace Transform

The Laplace transform of the function $F(t)$, which will be denoted by either $f(p)$ or $L\{F(t)\}$, is defined as

$$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\,dt, \quad p>0. \quad \text{(1)}$$

p is called the transform parameter.

Let us now find the Laplace transforms of some elementary fn from def^n.

(i) If $F(t) = c$, then (c is constant)

$$L\{c\} = \int_0^\infty c e^{-pt}\,dt = \left[-\frac{c e^{-pt}}{p}\right]_0^\infty = \frac{c}{p}, \quad p>0.$$

(ii) If $F(t) = t^2$, then

$$L\{t^2\} = \int_0^\infty e^{-pt} t^2\,dt.$$

$$= \left[-\frac{t^2 e^{-pt}}{p}\right]_0^\infty + \int_0^\infty 2t\,\frac{e^{-pt}}{p}\,dt$$

$$= \frac{2}{p}\left[-\frac{e^{-pt}}{p}t\right]_0^\infty + \frac{2}{p}\int_0^\infty \frac{e^{-pt}}{p}\,dt \qquad \because t^2 e^{-pt} \to 0 \text{ as } t\to\infty$$

$$= \frac{2}{p^2}\left[-\frac{e^{-pt}}{p}\right]_0^\infty = \frac{2}{p^3}, \quad p>0.$$

$$\therefore \quad L\{t^n\} = \frac{n!}{p^{n+1}}, \quad n \text{ is +ve integer.}$$

(iii) If $F(t) = e^{\alpha t}$, then

$$L\{e^{\alpha t}\} = \int_0^\infty e^{-pt} e^{\alpha t}\,dt = \left[-\frac{e^{-(p-\alpha)t}}{p-\alpha}\right]_0^\infty$$

$$= \frac{1}{p-\alpha}, \quad p>\alpha.$$

(iv) $$L\{\cosh at\} = L\left\{\tfrac{1}{2}(e^{at}+e^{-at})\right\}$$

$$= \frac{1}{2}\left(\frac{1}{p-a}+\frac{1}{p+a}\right) = \frac{p}{p^2-a^2}, \quad p>|a|$$

Similarly $L\{\sinh at\} = \dfrac{a}{p^2 - a^2}, \; p > |a|$

(v) If $F(t) = \cos at$ then

$$L\{\cos at\} = \int_0^\infty e^{-pt} \cos at \, dt$$

$$= \left[\frac{e^{-pt}}{p^2 + a^2} (-p \cos at + a \sin at) \right]_0^\infty$$

$$= \frac{p}{p^2 + a^2}, \quad p > 0$$

Similarly $L\{\sin at\} = \dfrac{a}{p^2 + a^2}$

Example 1 Find the Laplace transform of $f(t)$ defined as

$$f(t) = \begin{cases} t/k, & \text{when } 0 < t < k \\ 1, & \text{when } t > k. \end{cases}$$

$$L\{f(t)\} = \int_0^k \frac{t}{k} e^{-pt} \, dt + \int_k^\infty 1 \cdot e^{-pt} \, dt$$

$$= \frac{1}{k} \left[\left(t \frac{e^{-pt}}{-p} \right)_0^k - \int_0^k \frac{e^{-pt}}{-p} \, dt \right] + \left[\frac{e^{-pt}}{-p} \right]_k^\infty$$

$$= \frac{1}{k} \left[\frac{k e^{-pk}}{-p} - \left(\frac{e^{-pt}}{p^2} \right)_0^k \right] + \frac{e^{-kp}}{p}$$

$$= \frac{1}{kp^2} \left[e^{-kp} + 1 \right].$$

Inverse of the Laplace transform will be defined by $L^{-1}\{f(p)\} = F(t)$, if $f(p)$ be the Laplace transform of $F(t)$.

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According to the definition of inverse transform, we can state

(a) $L^{-1}\left(\frac{1}{p}\right) = 1$, since $L\{1\} = \frac{1}{p}$.

(b) $L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{n!}$ since $L\{t^n\} = \frac{n!}{p^{n+1}}$

(c) $L^{-1}\left(\frac{1}{p-\alpha}\right) = e^{\alpha t}$, since $L\{e^{\alpha t}\} = \frac{1}{p-\alpha}$

(d) $L^{-1}\left(\frac{1}{p^2+\alpha^2}\right) = \frac{1}{\alpha}\sin\alpha t$, since $L\{\sin\alpha t\} = \frac{\alpha}{p^2+\alpha^2}$.

(e) $L^{-1}\left(\frac{p}{p^2+\alpha^2}\right) = \cos\alpha t$, since $L\{\cos\alpha t\} = \frac{p}{p^2+\alpha^2}$

(f) $L^{-1}\left(\frac{1}{p^2-\alpha^2}\right) = \frac{1}{a}\sinh\alpha t$, since $L\{\sinh\alpha t\} = \frac{\alpha}{p^2-\alpha^2}$

(g) $L^{-1}\left(\frac{p}{p^2-\alpha^2}\right) = \cosh\alpha t$ since $L\{\cosh\alpha t\} = \frac{p}{p^2-\alpha^2}$