Module-5

Renewal Theory:-

[It is a generalization of Poisson Process].

Introduction

Let x_n , $n = 1, 2, ---$ be the non-negative

i.i.d rv's with $S_n = x_1 + x_2 + --- + x_n$, $n \geq 1$.

$S_0 = 0$. F is the d.F. of X and assume

$P(x_n = 0) < 1$.

[The supremum of a set is its least upper bound].

Define $N(t) = \sup\{n \mid S_n \leq 1\}$

The $\{N(t) \geq 0\}$ is called the renewal Process.

x_i = lifetime of the machines being replaced.

The first machine is installed at time

$t = 0$. and is replaced instantaneously at

time $t = x_1$.

The replaced machine is again

replaced at time $t = x_1 + x_2$ and so on.

If $S_n = x_1 + x_2 + --- + x_n$

The Partial sum S_n can be interpreted to be the time at which the nth replacement is made.

$N(t)$ is the largest Value of n for which $S_n \le t$.

In other words $N(t)$ is the no. of renewals that would have occured at time t.

The renewal Theory is a Special case of a random walk with absorbing barrier we are sampling the X_i untill S_n shoots the barrier at time t and $N(t)+1$ is the sample size when we stop.

Hence the renewal theory is also linked with sequential Analysis in Statistics

<u>Theorem 1</u>:- The dist. of $N(t)$ is given by

$$P_n(t) = P[N(t)=n] = F_n(t) - F_{n+1}(t)$$

<u>Proof</u>:- $P[N(t)=n] = P[N(t)\geq n] - P[N(t)\geq n+1]$.

$$= P[S_n \le t] - P[S_{n+1} \le t].$$

$$\boxed{P[N(t)=n] = F_n(t) - F_{n+1}(t)}$$ (Hence Proved).

$S_0 = 0$
$S_1 = x_1$
$S_2 = x_1 + x_2$
$\vdots$

__Question__

Find $P[N(t)=n]$ given F.

__Soln:-__ $N(t)$ = No. of renewal at time t.

$$P[S_2 \le t] = \int_0^\infty F(t-u)\,dF(u) \quad \text{[Recursive integral eq.]}.$$

$$= \int_0^t F(t-u)\,dF(u) \quad \text{[convolution]}.$$

$$= F * F(t).$$

$$= F^{(2)}(t).$$

$$P[S_n \le t] = F^{(n)}(t)$$

$$= \int_0^t F^{(n-1)}(t-u)\,dF(u) \quad , n \ge 1.$$

Define $F^{(0)}(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \ge 0. \end{cases}$

Now

$$P[N(t)=n] = P[S_1 \le t, S_2 \le t, \cdots S_n \le t, S_{n+1} > t].$$

$$= P[S_n \le t, S_{n+1} > t] \quad \text{[by non-negativeness of } x_i].$$

Since

$$S_n = x_1 + x_2 + \cdots + x_n.$$

$$S_{n+1} = \underbrace{\boxed{x_1 + x_2 + \cdots - + x_n}}_{S_n} + x_{n+1}$$

$$S_{n+1} = S_n + x_{n+1}$$

$$P[N(t)=n] = P[S_n \le t,\ S_n + x_{n+1} > t].$$

$$= P[t - x_{n+1} < S_n \le t].$$

$$= \int_0^\infty P\{t - x_{n+1} < S_n \le t \mid u < x_{n+1} \le u + du]\, dF(u).$$

$$= \int_0^\infty P[t - u < S_n \le t \mid u < x_{n+1} \le u + du]\, dF(u)$$

Since S_n is independent of x_{n+1}

$$\int_0^\infty P[t-u < S_n \le t]\, dF(u).$$

$$= \int_0^\infty \{F^{(n)}(t) - F^{(n)}(t-u)\}\, dF(u).$$

$$= \int_0^\infty F^{(n)}\, dF(u) - \int_0^\infty F^{(n)}(t-u)\, dF(u)$$

$$\boxed{P[N(t)=n] = F^{(n)}(t) - \int_0^\infty F^{(n)}(t-u)\, dF(u)}$$

Theorem-2

$$H(t) = \sum_{n=1}^\infty F^{(n)}(t).$$

Here $H(t)$ = Renewal Function.

Proof:- $H(t) = E[N(t)]$ $\quad [E(x) = x \cdot P(x)]$

$$H(t) = \sum_{n=0}^{\infty} n \cdot P[N(t)=n].$$

$$= 0 + 1 \cdot P[N(t)=1] + 2 \cdot P[N(t)=2] + 3 \cdot P[N(t)=3] + \cdots$$

$$= [F^{(1)}(t) - F^{(2)}(t)] + 2[F^{(2)}(t) - F^{(3)}(t)] + 3[F^{(3)}(t) - F^{(4)}(t)] + \cdots$$

Renewal Function

$$= F^{(1)}(t) + F^{(2)}(t) + F^{(3)}(t) + \cdots$$

$$\boxed{H(t) = \sum_{n=1}^{\infty} F^{(n)}(t)}$$ (Hence Proved).

Theorem 3

$$H(t) = F(t) + \int_0^t H(t-u)\, dF(u) \quad [\text{Renewal Equation}].$$

Proof:-

$$H(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$$

$$= F^{(1)}(t) + \sum_{n=2}^{\infty} F^{(n)}(t).$$

$$= F(t) + \sum_{n=1}^{\infty} F^{(n+1)}(t).$$

$$= F(t) + \sum_{n=1}^{\infty} \int_0^t F^{(n)}(t-u)\, dF(u)$$

$$= F(t) + \int_0^t \sum_{n=1}^{\infty} F^{(n)}(t-u)\, dF(u)$$

Renewal Equation $\quad$ [By Fubini theorem]

Renew... [By Fubini Theorem]

$$\boxed{H(t) = F(t) + \int_0^t H(t-u)\, dF(u)}$$

Theorem 4

$\{N(t),\ t \in [0, \infty)\}$ is completely determined by $H(t)$.

Proof:- $H(t) = F(t) + \int_0^t H(t-u)\, dF(u)$. → convolution

$$H(t) = F(t) + H * F(u).$$

where $*$ is the convolution operator.

Taking Laplace transform on both sides.

$$L(s) = F(s) + L(s) \cdot F(s)$$

$$L(s) = F(s)\,[1 + L(s)].$$

$$\frac{L(s)}{1+L(s)} = F(s)$$

Again $L(s) - L(s) \cdot F(s) = F(s)$

$$L(s)\,[1 - F(s)] = F(s)$$

$$L(s) = \frac{F(s)}{1-F(s)}.$$

This shows that $H(t)$ and $F(t)$ can be determined uniquely one from the other.

Since Laplace transform determines a non-decreasing function uniquely. Hence $N(t)$ is completely determined by $H(t)$.