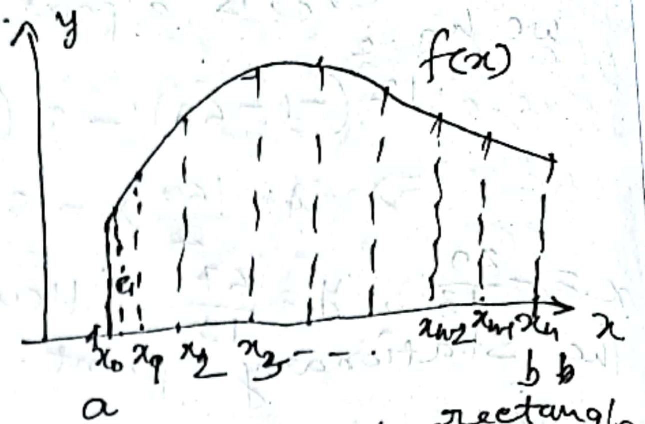
Double Integral

Integrals of Functions of Single Variable :

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$



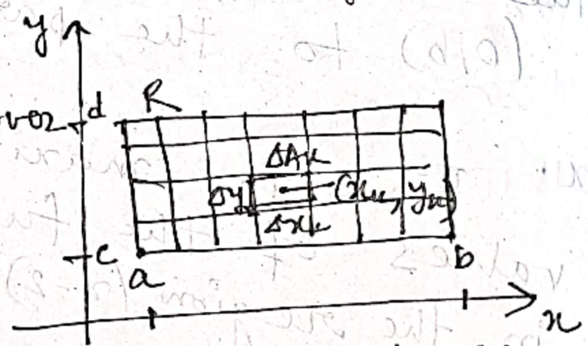
Area of each rectangle $= f(c_k).\Delta x_k$, $\Delta x_k = x_k - x_{k-1}$

In limiting case, when these interval $([a, x_1], [x_1, x_2] - - [x_{n-1}, x_n])$ tends to zero, the limit will approach to the area under the curve.

$\int_a^b f(x)\,dx$ denotes the area under this curve $y=f(x)$ within $[a,b]$.

Double Integral over Rectangular region:

Let $f(x,y)$ be defined over R. $R: a \le x \le b$, $c \le y \le d$



$\Delta A_k = \Delta x_k \Delta y_k$

$$\iint_R f(x,y)\,dA \text{ or } \iint_R f(x,y)\,dx\,dy \text{ or } \iint_R f(x,y)\,dy\,dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k,$$ if the limit exist. (x_k, y_k) be some point of ΔA_k

Fubini's Theorem (First form)

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

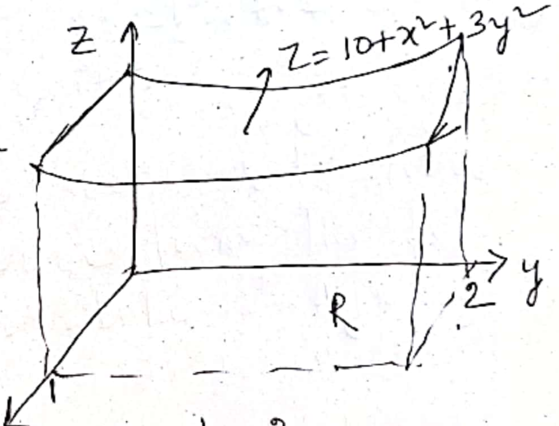
Geometrical Interpretation of Double Integral:

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\,\Delta x\,\Delta y = \iint_R f(x,y)\,dx\,dy$$

represents volume. or area of R if $f(x,y)=1$

Example: Find the volume of the region bounded ~~&~~ above ~~the~~ by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1,\ 0 \le y \le 2$.

→ The Volume is given by the double integral



$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_{x=0}^{1} \int_{y=0}^{2} (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[10y + x^2y + \frac{3y^3}{3}\right]_{y=0}^{2} dx$$

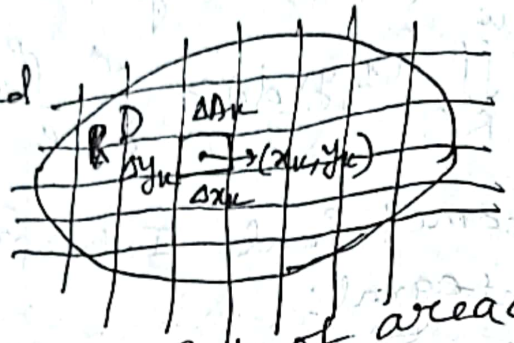
$$= \int_{x=0}^{1} (20 + 2x^2 + 8)\,dx$$

$$= \left[28x + \frac{2x^3}{3}\right]_0^1$$

$$= 28 + \frac{2}{3} = \frac{86}{3}$$

Double Integrals over bounded, non rectangular region (General region)

Let $f(x,y)$ be defined in a closed region D of the xy plane.



Let (x_k, y_k) be some point of area ΔA_k.

Consider $S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \iint_D f(x,y)\, dA$$

Fubini's theorem (Stronger form)

Let $f(x,y)$ be continuous on a region R.

1. R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

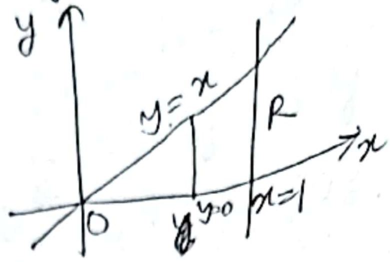
2. If R is defined by $c \le y \le d$, ~~$h_1(y) \le x \le$~~ $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R f(x,y)\, dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example: Find the volume of the prism whose base is the triangle in the xy plane bounded by the x-axis and lines $y=x$ and $x=1$ and whose top line lies in the plane

$$z = f(x,y) = 3-x-y.$$

$$V = \int_{x=0}^{1}\int_{y=0}^{x} z\,dy\,dx$$



$$= \int_{x=0}^{1}\int_{y=0}^{x} (3-x-y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

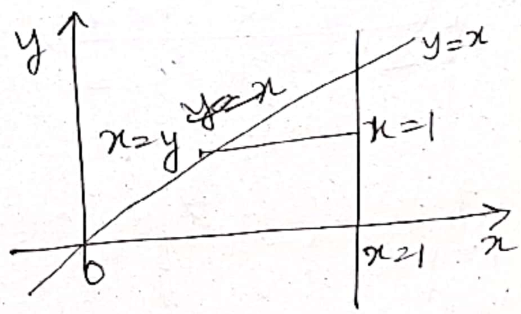
$$= \int_{x=0}^{1}\left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^1 - \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2} = 1.$$

Alternative

When x varies from y to 1, y varies 0 to 1.



$$V = \int_{y=0}^{1}\int_{x=y}^{1} (3-x-y)\,dx\,dy$$

$$= \int_{y=0}^{1}\left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{1}$$

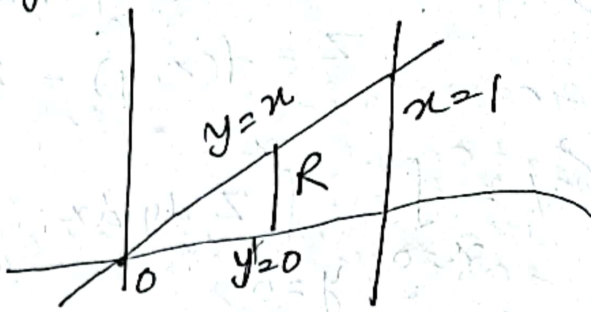
$$= \int_{y=0}^{1}\left(3 - 3y - \frac{1}{2} + \frac{y^2}{2} - y + y^2\right) dy$$

$$= \frac{5}{2}[y]_0^1 - 4\left[\frac{y^2}{2}\right]_0^1 + \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{5}{2} - 2 + \frac{1}{2} = 1.$$

2. Evaluate $\iint_R \frac{\sin x}{x} \, dA$

Where R is the region in the xy plane bounded by x-axis, the lines $y=x$ and $x=1$



$$\iint_R \frac{\sin x}{x} \, dA$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{x} \frac{\sin x}{x} \, dy \right) dx$$

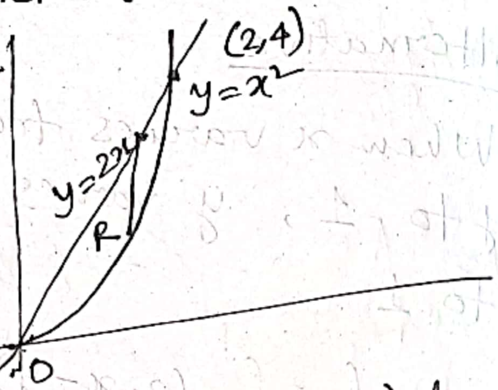
$$= \int_{x=0}^{1} \left[y \frac{\sin x}{x} \right]_0^x dx$$

$$= \int_{x=0}^{1} \sin x \, dx = (-\cos x)\Big|_0^1$$

$$= 1 - \cos(1)$$

$$\simeq 0.46.$$

3. Evaluate $\int_0^2 \int_{x^2}^{2x} (4x+2) \, dy \, dx$ and write equivalent integral with reverse order.



$\rightarrow$

$$\int_{x=0}^{2} \int_{y=x^2}^{2x} (4x+2) \, dy \, dx$$

$$= \int_{x=0}^{2} (4x+2)(2x-x^2) \, dx$$

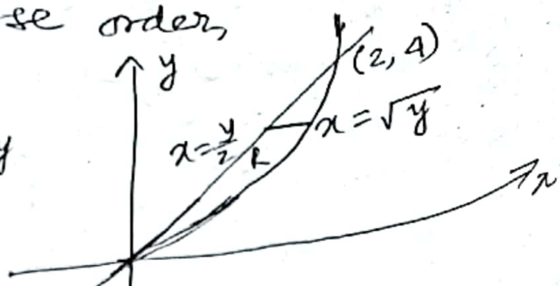
$$= \int_{x=0}^{2} (8x^2 - 4x^3 + 4x - 2x^2) \, dx$$

$$= 6\left[\frac{x^3}{3}\right]_0^2 - 4\left[\frac{x^4}{4}\right]_0^2 + 4\left[\frac{x^2}{2}\right]_0^2$$

$$= 16 - 16 + 2 \cdot 4 = 8$$

Changing the reverse order,

$$\int_{y=0}^{4}\int_{x=\cancel{\sqrt{y}}\frac{y}{2}}^{x=\sqrt{y}} (4x+2)\,dx\,dy$$



$$= \int_{y=0}^{4}\left[4\left(y-\frac{y^2}{4}\right) + 2\left(\sqrt{y}-\frac{y}{2}\right)\right] dy$$

$$= 8.$$

Properties of Double Integral:

If $f(x,y)$, $g(x,y)$ are continuous on the bounded region R, then the following properties hold.

1. Constant Multiple: $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA.$

2. Sum and difference:
$$\iint_R \left(f(x,y) \pm g(x,y)\right) \overset{dA}{\cancel{dxd}} = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA.$$

3. Domination:
(a) $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R
(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on

4. Additivity: If R is the union of two non overlapping regions R_1 and R_2, then
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

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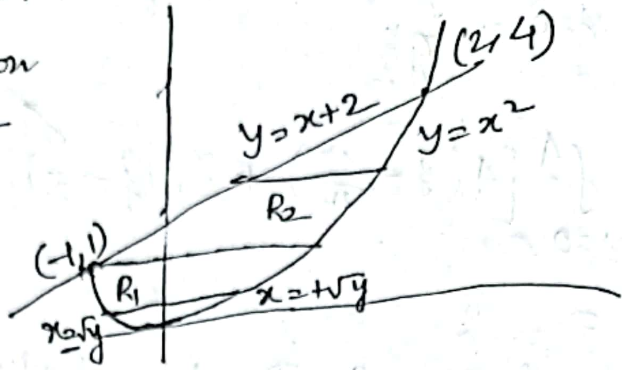
Example: Find the area of region R enclosed by parabola $y=x^2$ and the line $y=x+2$.



→ Find intersection points of $y=x^2$ and $y=x+2$

$x^2=x+2$

$\Rightarrow x^2-x-2=0$

$\Rightarrow (x-2)(x+1)=0$

$\Rightarrow x=2,-1.$

The points are $(-1,1)$ and $(2,4)$.

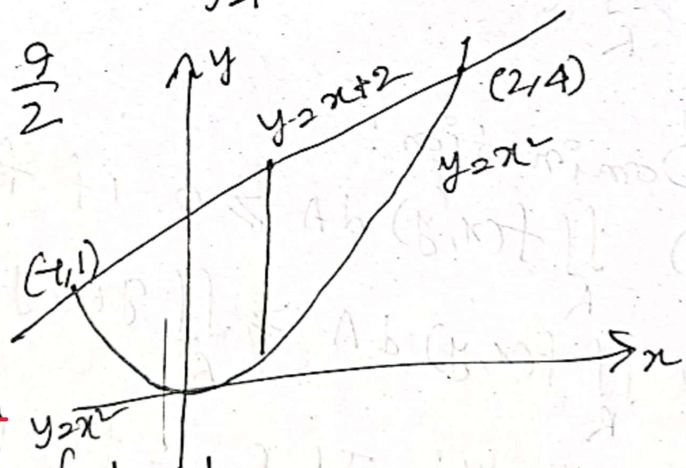
$$A=\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$

$$=\int_{y=0}^{1}\int_{x=-\sqrt{y}}^{+\sqrt{y}} dx\,dy + \int_{y=1}^{4}\int_{x=y-2}^{x=\sqrt{y}} dx\,dy$$

$$=\int_{y=0}^{1} 2\sqrt{y}\,dy + \int_{y=1}^{4}(\sqrt{y}-y+2)\,dy$$

$$=\frac{9}{2}$$

<u>Alternative</u>



$$A=\iint_R dA=\int_{x=-1}^{2\,\cancel{4}}\int_{y=x+2}^{y=x^2} dy\,dx$$

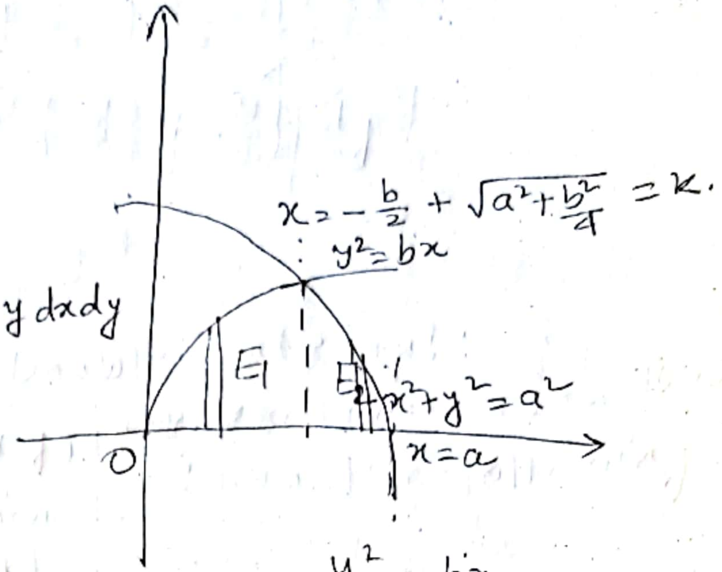
$$=\int_{x=-1}^{2\,\cancel{4}}(x^2-x-2)\,dx = \frac{9}{2}$$

Example: Evaluate $\iint_E y\,dx\,dy$ over the region E in the first quadrant bounded by the x-axis, the curves $x^2+y^2=a^2$, $y^2=bx$.

$\longrightarrow$

$$\iint_E y\,dx\,dy$$

$$= \iint_{E_1} y\,dx\,dy + \iint_{E_2} y\,dx\,dy$$



$$= \int_{x=0}^{k}\int_{y=0}^{\sqrt{bx}} y\,dy\,dx + \int_{x=k}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$y^2 = bx$

$y^2 = a^2-x^2$

$x^2+bx-a^2=0$

$\Rightarrow x = \frac{-b \pm \sqrt{b^2+4a^2}}{2}$

$= -\frac{b}{2} \pm \sqrt{a^2+\frac{b^2}{4}}$

$$= \int_{x=0}^{k} \left[\frac{y^2}{2}\right]_0^{\sqrt{bx}} dx + \int_{x=k}^{a} \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

Let $k = -\frac{b}{2} + \sqrt{a^2+\frac{b^2}{4}}$

$$= \frac{b}{2}\left[\frac{x^2}{2}\right]_0^k + \frac{1}{2}\left[a^2x - \frac{x^3}{3}\right]_k^a$$

$$= \frac{b}{4}k^2 + \frac{1}{2}a^2(a-k) - \frac{1}{6}(a^3-k^3)$$

$$= \frac{1}{3}a^3 - \frac{1}{2}a^2k + \frac{k^3}{6} + \frac{b}{4}k^2$$

where $k = -\frac{b}{2} + \sqrt{a^2+\frac{b^2}{4}}$

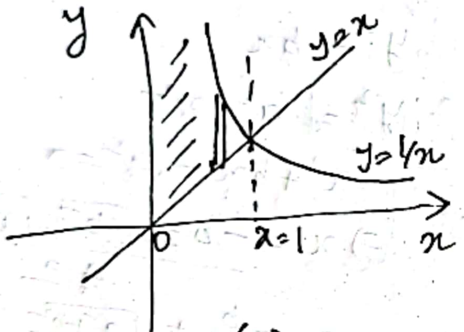
Ex Show that $\iint_E (x^2+y^2)\,dx\,dy = \frac{6}{35}$, where E is bounded by $y=x^2$ and $y^2=x$.

Change of order in Integration

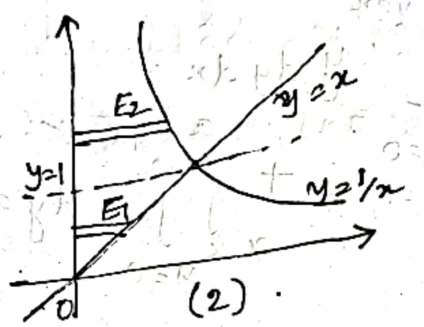
When the limits of integrations are not constants, we may change the order of integration provided the limits of integration are changed so that the entire region of integration is covered.

Example: By changing the order of integration prove that

$$\int_0^1 dx \int_x^{1/x} \frac{y^2\,dy}{(x+y)^2\sqrt{1+y^2}} = \frac{1}{2}\left(2\sqrt{2}-1\right).$$



(1)



(2).

$$\int_0^1 dx \int_x^{1/x} \frac{y^2\,dy}{(x+y)^2\sqrt{1+y^2}} \quad (\text{See fig (1)})$$

$$= \iint_{E_1} \frac{y^2\,dx\,dy}{(x+y)^2\sqrt{1+y^2}} + \iint_{E_2} \frac{y^2\,dx\,dy}{(x+y)^2\sqrt{1+y^2}}$$

$$= \int_{y=0}^{1}\int_{x=0}^{y} \frac{y^2\,dx\,dy}{(x+y)^2\sqrt{1+y^2}} + \int_{y=1}^{\infty}\int_{x=0}^{1/y} \frac{y^2}{\sqrt{1+y^2}}\cdot\frac{dx}{(x+y)^2}\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{y} \frac{y^2}{\sqrt{1+y^2}}\cdot\frac{dx}{(x+y)^2}\,dy + \int_{y=1}^{\infty}\int_{x=0}^{1/y} \frac{y^2}{\sqrt{1+y^2}}\,\frac{dx\,dy}{(x+y)^2}$$

$$= \int_{y=0}^{1} \frac{y^2}{\sqrt{1+y^2}}\left[-\frac{1}{x+y}\right]_0^y dy + \int_{y=1}^{\infty} \frac{y^2}{\sqrt{1+y^2}}\left[-\frac{1}{x+y}\right]_0^{1/y} dy$$

$$= \int_{y=0}^{1} \frac{y^2}{\sqrt{1+y^2}}\left[\frac{1}{y} - \frac{1}{2y}\right] dy + \int_{y=1}^{\infty} \frac{y^2}{\sqrt{1+y^2}}\left(\frac{1}{y} - \frac{y}{1+y^2}\right) dy$$

$$= \int_{y=0}^{1} \frac{1}{2} \frac{y}{\sqrt{1+y^2}} dy + \int_{y=1}^{\infty} \frac{y \, dy}{(1+y^2)^{3/2}}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \int_{z=1}^{2} \frac{dz}{\sqrt{z}} + \lim_{M\to\infty} \int_{y=1}^{M} \frac{y\, dy}{(1+y^2)^{3/2}} \qquad 1+y^2 = z, \; 2y\,dy = dz$$

$$= \frac{1}{4} \cdot \left[\frac{z^{1/2}}{1/2}\right]_1^2 + \lim_{M\to\infty} \int_{z=2}^{1+M^2} \frac{1}{2} \frac{dz}{z^{3/2}}$$

$$= \frac{1}{2}\left[\sqrt{2} - 1\right] + \lim_{M\to\infty} \frac{1}{2} \cdot \left(\frac{-1}{\frac{1}{2}}\right)\left[\frac{1}{z^{1/2}}\right]_2^{1+M^2}$$

$$= \frac{1}{2}(\sqrt{2} - 1) - \lim_{M\to\infty} \left[\frac{1}{(1+M^2)^{1/2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2}(\sqrt{2} - 1) + \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{2} = \frac{1}{2}(2\sqrt{2} - 1).$$

[proved]

Example: By changing the order of integration, prove that $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}} = \frac{\pi}{2} \log\left(\frac{2e}{1+e}\right)$

Hint

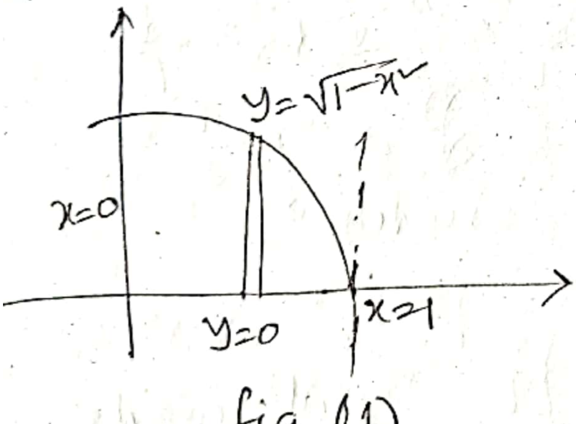
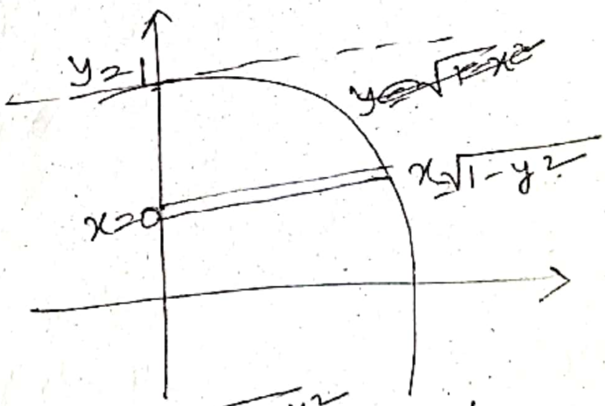


fig (1)



Solve: $\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{1}{(1+e^y)} \frac{dx}{\sqrt{(1-x^2-y^2)}} dy$

$$= \frac{\pi}{2} \log\left(\frac{2e}{1+e}\right)$$

Change of variable:

Jacobian: If $x_i = \phi_i(u_1, u_2, \ldots, u_n)$, $i = 1, 2, \ldots n$ are differentiable functions of n variables $u_1, u_2, \ldots, u_n$, then the Jacobian of functions $x_1, x_2, \ldots, x_n$ with respect to variables $u_1, u_2, \ldots u_n$ is denoted by $\frac{\partial(x_1, x_2, \ldots x_n)}{\partial(u_1, u_2, \ldots u_n)}$ or J which is defined by the determinant

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

Notes: 1. If $x = r\cos\theta$, $y = r\sin\theta$, then

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

2. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then the J is

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta.$$

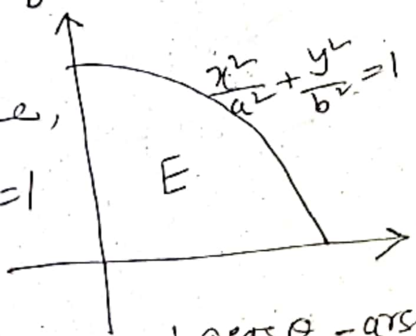
Theorem: Let E be a closed and bounded region in R^n $(n>1)$ and $f: E \to R$ be a continuous function of n variables $x_1, x_2, \dots x_n$ where $(x_1, x_2, \dots x_n) \in E$.

Suppose the transformation $x_i = \phi_i(u_1, u_2 \dots u_n)$ $(i=1,2,\dots n)$ map the domain E into E' and the boundary C of E into C' of E' in such a way that there is one to one correspondence between E and E' and so also between C and C' and the functions ϕ_i $(i=1,2,\dots n)$ possess continous first order partial derivatives at every point of E' then

$$\iint \dots \int_E f(x_1, x_2, \dots, x_n)\, dx_1 \dots dx_n$$
$$= \iint \dots \int_{E'} F(u_1, u_2 \dots u_n)\, |J|\, du_1 du_2 \dots du_n$$

where $F(u_1, u_2, \dots u_n) = f(\phi_1, \phi_2, \dots \phi_n)$.

Example: Evaluate $\iint_E \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy$ where E is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\to$ E is the closed and bounded region in xy plane, bounded by $x=0$, $y=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We use the transformation

$x = ar\cos\theta$, $y = br\sin\theta$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$$
$$= abr > 0.$$

$0 \le r \le 1$. $0 \le \theta \le \frac{\pi}{2}$

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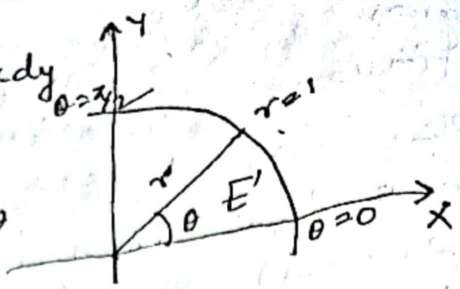
$$\iint_E \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$$



$$= \iint_{E'} (1-r^2)\,|J|\,dr\,d\theta$$

$$= \iint_{E'} ab\,r\,(1-r^2)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} ab\,r\,(1-r^2)\,dr\,d\theta$$

$$= \frac{ab}{2}\int_{\theta=0}^{\pi/2} \int_{z=0}^{1} (1-z)\,dz\,d\theta$$

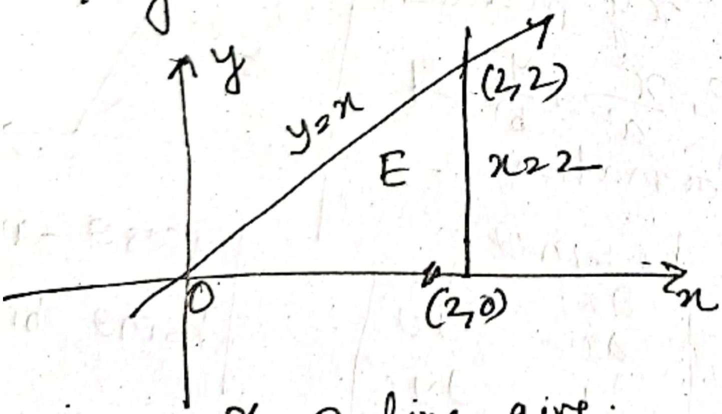
$r^2 = z$

$\Rightarrow 2r\,dr = dz$

$$= \frac{ab}{2}\cdot\frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{\pi ab}{8}.$$

2. Show that $\displaystyle\iint_E \frac{dx\,dy}{\sqrt{(x+y+1)^2-4xy}} = \frac{1}{2}\log_e\left(\frac{16}{e}\right)$ by using the transformation $x = u(1+v)$, $y = v(1+u)$, where E is the triangle with vertices $(0,0)$, $(2,0)$, $(2,2)$.



$x = u(1+v)$

$y = v(1+u)$

$y=x$ line gives

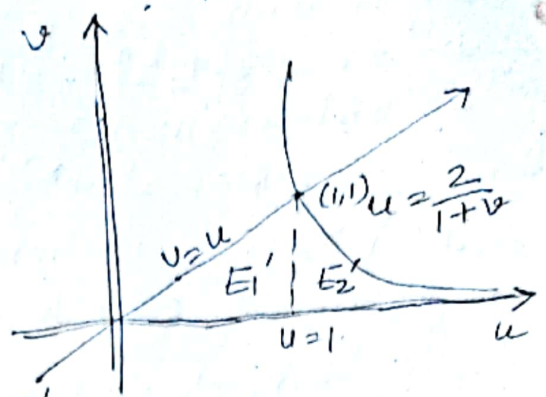
$u(1+v) = v(1+u)$

$\Rightarrow u = v$

$x=2$ line give.

$u(1+v) = 2 \Rightarrow u = \dfrac{2}{1+v}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$



$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= 1+u+v > 0 \text{ on } E'$$

$E_1' : 0 \le v \le 1,\ v \le u \le 1, \cancel{0 \le v \le 1}$

$E_2' : 0 \le v \le 1,\ 1 \le u \le \frac{2}{1+v}.$

$$(x+y+1)^2 - 4xy$$
$$= (x+y)^2 + 2(x+y) + 1 - 4xy$$
$$= (x-y)^2 + 2(x+y) + 1$$
$$= (u+uv-v-uv)^2 + 2(u+v+2uv) + 1$$
$$= (u-v)^2 + 2(u+v+2uv) + 1$$
$$= (u+v)^2 + 2(u+v) + 1$$

$$\iint_E \frac{dx\,dy}{\sqrt{(x+y+1)^2 - 4xy}} = \iint_{E'} \frac{(1+u+v)\,du\,dv}{\sqrt{(u+v)^2 + 2(u+v)+1}}$$

$$= \iint_{E'} \frac{1+u+v}{1+u+v}\,du\,dv$$

$$= \int_{v=0}^{1}\int_{u=v}^{1} du\,dv + \int_{v=0}^{1}\int_{u=1}^{2/1+v} du\,dv$$

$$= \int_{v=0}^{1} (1-v)\,dv + \int_{v=0}^{1} \left(\frac{2}{1+v} - 1\right) dv$$

$$= \left[-\frac{(1-v)^2}{2}\right]_0^1 + 2\left[\log_e(1+v)\right]_0^1 - 1 = \log_e 4 - \frac{1}{2}$$

$$= \frac{1}{2} + 2\log_e 2 - 1 = \frac{1}{2}\log_e\left(\frac{16}{e}\right).$$

Double Integral in Polar Co-ordinate

Cartesian Integral to Polar form:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

[G denotes the same region of integration but described in polar co-ordinate]

Ex 1. Transform the integral into cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta \, dr\,d\theta$

$\rightarrow$ $\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3 \sin\theta\cos\theta\, dr\, d\theta$

Consider $x = r\cos\theta$, $y = r\sin\theta$ and

$dx\,dy = r\,dr\,d\theta$.

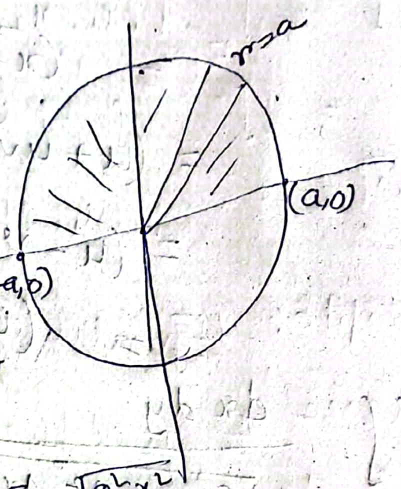
~~If $r=0$, $x=0$, $y=0$~~

If $r=0$, i.e. x axis

$y=0$

~~$x=a$~~

$\theta = \pi$ means x axis in the second quadrant



$x^2+y^2 = r^2 = a^2$

$\Rightarrow y^2 = a^2 - x^2 \Rightarrow y = \sqrt{a^2-x^2}$

The above integral transforms to

$$\int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} x\,y\,dy\,dx$$

$$= \int_{x=-a}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=-a}^{a} \frac{x}{2}(a^2-x^2)\,dx$$

$$= \frac{1}{2}\int_{x=-a}^{a} x(a^2-x^2)\,dx$$

$= 0$ [$\because f$ is odd fn]

2. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$.

Limits of y is $y = \sqrt{2x-x^2} \Rightarrow x^2+y^2-2x=0$

$\Rightarrow (x-1)^2+y^2=1$

which represents a circle with centre $(1,0)$ and radius 1.

Put $x = r\cos\theta$, $y = r\sin\theta$.

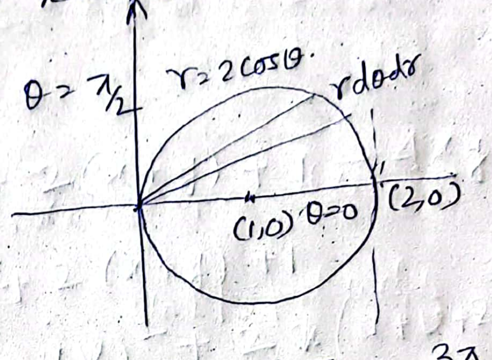
$x^2+y^2=r^2$

$r^2 - 2r\cos\theta = 0 \Rightarrow r(r-2\cos\theta)=0$

$\Rightarrow r=0$ or $r = 2\cos\theta$.

Lower limit of $y=0 \Rightarrow x$ axis

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot r\, dr\, d\theta$$



$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta\, d\theta = 4 \times \frac{3\times 1\times\pi}{4\times 2\times 2} = \frac{3\pi}{4}$$

Ex. Evaluate: $\int_{x=0}^{2} \int_{y=0}^{\sqrt{2x-x^2}} \frac{x\, dy\, dx}{\sqrt{x^2+y^2}}$ by changing polar co-ordinate.

Problem Set

① Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ by changing order of integration.

② Evaluate the integral $\int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx$ by ~~th~~ changing the order of integration.

③ Change the order of integration in the double integral $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dx\, dy$.

Triple Integral

Evaluation: Divide the region V into n sub regions of respective volumes $\delta V_1, \delta V_2, \ldots \delta V_n$.

Let (x_j, y_j, z_j) be an arbitrary point in the jth sub region.

Consider the sum $\displaystyle\sum_{j=1}^{n} f(x_j, y_j, z_j)\,\delta V_j$.

If $n \to \infty$ and $\delta V_j \to 0$ then

$$\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j, z_j)\,\delta V_j$$

$$\iiint_V f(x,y,z)\,dV = \int_{z=a}^{b} \underbrace{\left\{ \int_{y=\psi_1(z)}^{\psi_2(z)} \underbrace{\left\{ \int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\,dx \right\}}_{g(y,z)} dy \right\}}_{h(z)} dz$$

Note: Similar to double integrals; the order of integration is immaterial if the limits of integrations are constants.

$$\int_a^b \int_c^d \int_e^f f(x,y,z)\,dx\,dy\,dz = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dz\,dy\,dx$$

$$= \int_c^d \int_e^f \int_a^b f(x,y,z)\,dz\,dx\,dy$$

Example 1: Evaluate $I = \displaystyle\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$$\rightarrow I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dy\,dx$$

$$= \int_0^a \int_0^x \left[e^{2(x+y)} \cancel{dy\,dx} - e^{(x+y)}\right] dy\,dx$$

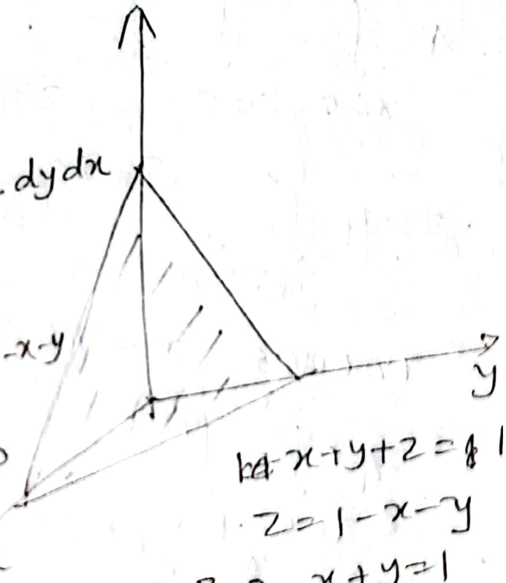
$$= \int_0^a \frac{e^{2(x+y)}}{2}\Big|_0^x dx - \int_0^a e^{x+y}\Big|_0^x dx$$

$$= \frac{1}{2}\int_0^a (e^{4x} - e^{2x})\,dx - \int_0^a (e^{2x} - e^{x})$$

$$= \frac{1}{2}\int_0^a (e^{4x} - 3e^{2x} + 2e^{x})\,dx$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + 2e^{a} - \frac{3}{8}.$$

Ex 2: Evaluate $I = \iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, R is the region bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=1$.



$x+y+z=1$

$z=1-x-y$

If $z=0$, $x+y=1$

$y=1-x$

If $y=0$, $x=1$

$$\rightarrow I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left(-\frac{1}{2}\right)\cdot\left[2^{-2} - (1+x+y)^{-2}\right] dy\,dx$$

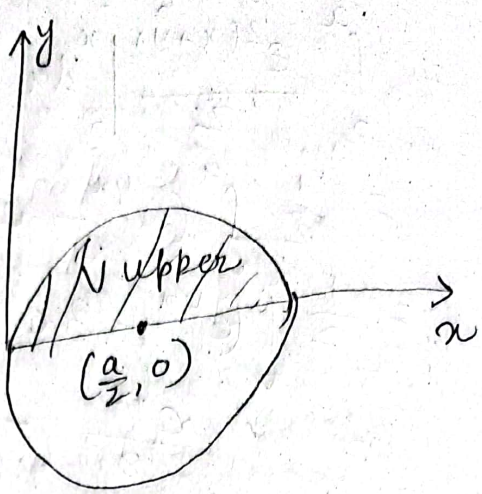
$$= -\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left(\frac{1}{4} - \frac{1}{(1+x+y)^2}\right) dy\,dx$$

$$= -\frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{4}(1-x) - \frac{(-2)}{1+x+y}\Big|_0^{1-x}\right] dx$$

$$+ \frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{2} - \frac{1}{1+x}\right] dx$$

$$= -\frac{1}{8}\int_0^1 (1-x)\,dx + \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

Ex 3: Using the triple integral find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.



$$V = \iiint_V dx\,dy\,dz$$

$$= \iiint_V dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \int_{z=-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 4\int_{x=0}^{a} \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 4\int_{0}^{a} \int_{y=0}^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

~~$r=a\cos\theta$~~, $x = r\cos\theta$, $y = r\sin\theta$

$$= \cancel{4\int_{r=0}^{a}\int_{\theta=0}^{\theta_2}}$$

$$= 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2}\,.\,r\,dr\,d\theta$$

$$= 4\cdot\left(-\frac{1}{2}\cdot\frac{2}{3}\right)\cdot\int_{0}^{\pi/2} \left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{4}{3}a^3\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$$

$[4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$= \frac{2}{3}a^3\left(\pi - \frac{4}{3}\right).$$

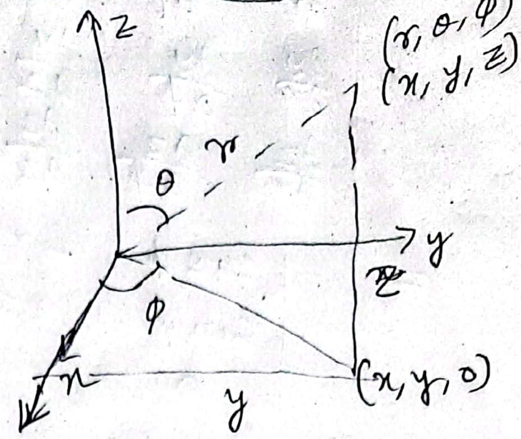
Change of variables in Triple Integral:

Cartesian to spherical polar co-ordinate

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



Note that

$x^2 + y^2 + z^2 = r^2$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\,|J|\,dr\,d\theta\,d\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \cancel{\sin\theta\cos\phi \quad r\cos\theta\cos\phi}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

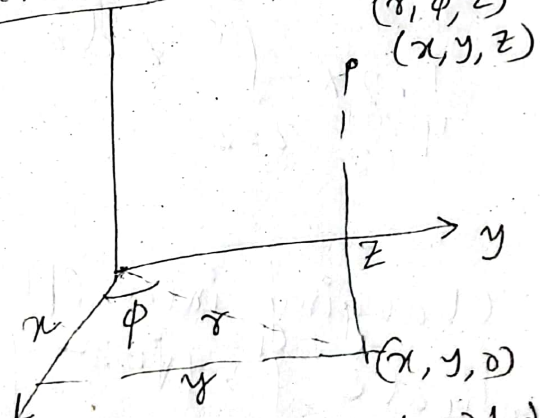
$$= r^2\sin\theta$$

Cartesian to Cylindrical co-ordinate

$x = r\cos\phi$

$y = r\sin\phi$

$z = z$



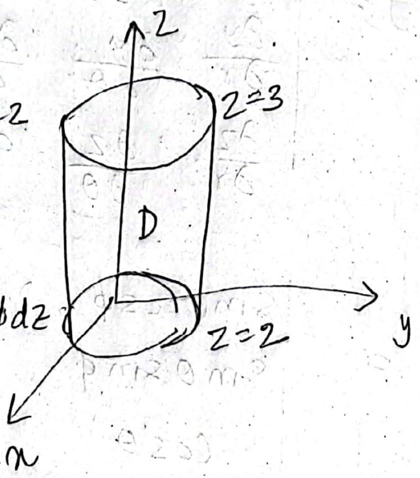
$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D f(r\cos\phi,\ r\sin\phi,\ z)\,|J|\,dr\,d\phi\,dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Ex 1 Changing to cylindrical co-ordinate, evaluate

$$\iiint_D z(x^2+y^2)\,dx\,dy\,dz, \quad D: x^2+y^2 \le 1, \; 2 \le z \le 3.$$



$x = r\cos\phi,\; y = r\sin\phi,\; z = z$

$x^2+y^2 = r^2, \quad J = r$

$$I = \int_{z=2}^{3} \int_{\phi=0}^{2\pi} \int_{r=0}^{1} z \cdot r^2 \cdot r \, dr\, d\phi\, dz$$

$$= \int_{z=2}^{3} \int_{\phi=0}^{2\pi} z \cdot \left[\frac{r^4}{4}\right]_0^1 d\phi\, dz$$

$$= \frac{1}{4} \cdot 2\pi \int_{z=2}^{3} z\, dz = \frac{\pi}{2}\left[\frac{z^2}{2}\right]_2^3$$

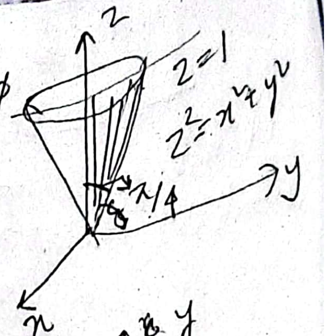
$$= \frac{\pi}{4} \cdot (3^2 - 2^2) = \frac{5\pi}{4}.$$

2. Changing into spherical polar co-ordinate

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$

$\rightarrow$ $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi$

$z = r\cos\theta$

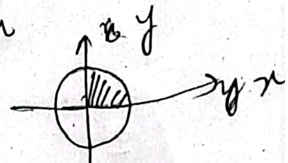


$z = 1 \Rightarrow r\cos\theta = 1$

$r = 1/\cos\theta$

θ varies : 0 to $\pi/4$

ϕ varies : 0 to $\pi/2$



$$= \int_0^{\pi/2}\int_0^{\pi/4}\int_{r=0}^{\sec\theta} \frac{1}{r}\cdot r^2\sin\theta \, dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/4} \frac{1}{2}\sec^2\theta\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec\theta\Big|_0^{\pi/4} d\phi$$

$$= \frac{(\sqrt{2}-1)\pi}{4}.$$

Ex3: Changing spherical co-ordinate, evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-(x^2+y^2+z^2)}}\,dz\,dy\,dx$

$\rightarrow$ $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$J = r^2\sin\theta; \quad x^2+y^2+z^2 = r^2$

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,dr\,d\phi\,d\theta$$

First evaluate $\displaystyle\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr = \int_0^{\pi/2} \sin^2 t\,dt$ [$r = \sin t$]

$$= \frac{1}{2}\int_0^{\pi/2}(1-\cos 2t)\,dt$$

$$= \pi/4.$$

$$= \frac{\pi}{4}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\,d\phi$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}\cdot\left[-\cos\theta\right]_0^{\pi/2} = \frac{\pi^2}{8}.$$