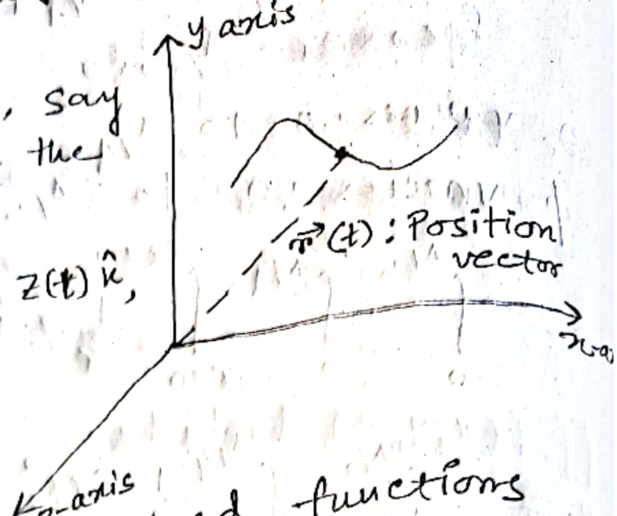
Vector valued function

Vector valued functions are functions which map a real number to a vector.

A vector function, say $\vec{r}(t)$, is written in the form

$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

$a \le t \le b$



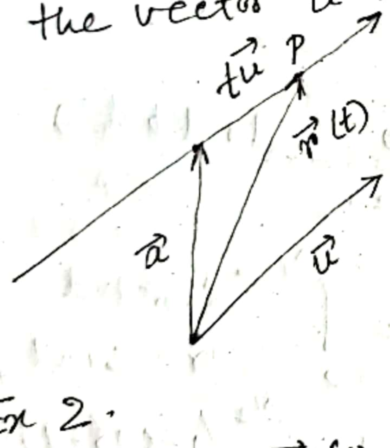
Here x, y, z are real valued functions of the parameter t and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y and z-axes respectively.

In 2D-plane, $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$; $a \le t \le b$.

Vector functions of one variable :

Ex 1. Equation of a straight line passing through A with position vector $\vec{a}$ parallel to the vector $\vec{u}$.

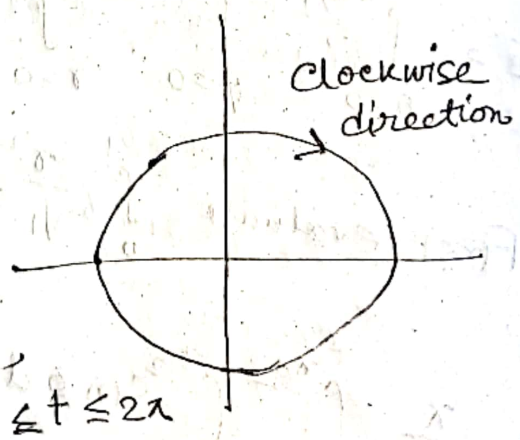
$\vec{r}(t) = \vec{a} + t\vec{u}$, $t \in R$



Ex 2.

Consider $\vec{r}(t) = 3\cos t\,\hat{i} - 2\sin t\,\hat{j}$,

$0 \le t \le 2\pi$



Ex 3. $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + t\hat{k}$, $0 \le t \le 2\pi$

Limit and Continuity of Vector functions:

- Limit: $\lim_{t\to a} \vec{r}(t) = [\lim_{t\to a} x(t)]\hat{i} + [\lim_{t\to a} y(t)]\hat{j} + [\lim_{t\to a} z(t)]\hat{k}$.

provided $x(t)$, $y(t)$ and $z(t)$ have limits as $t \to a$.

- Continuity: A vector-valued function $\vec{r}(t)$ is continuous at $t=a$ if and only if each of its component functions is continuous at $t=a$.

- Example: Discuss continuity of $\vec{r}(t) = t\hat{i} + \hat{j} + (2-t^2)\hat{k}$

→ Since each component of $\vec{r}(t)$ is continuous for all $t \in \mathbb{R}$, the given vector function of one variable is continuous for all $t \in \mathbb{R}$.

- Example: Discuss continuity of
$$\vec{r}(t) = \frac{1}{t-2}\hat{i} + t\hat{j} + \ln(t)\hat{k}$$

→ The first component $\frac{1}{t-2}$ is ~~continuous~~ defined except $t \neq 2$.

Also, $\ln(t)$ is defined only for $t > 0$.

So, $\vec{r}(t)$ is continuous for $t > 0$ and $t \neq 2$.

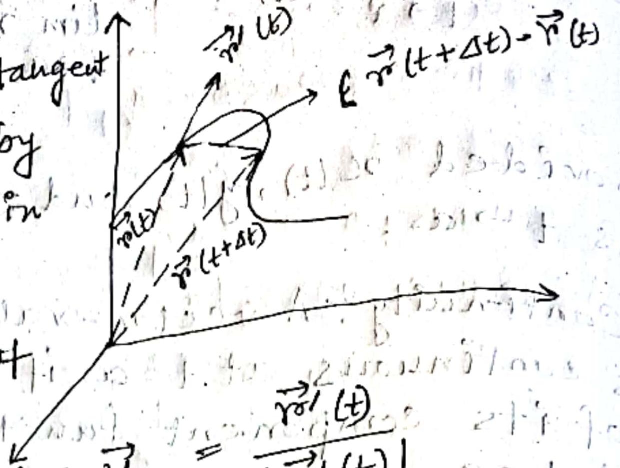
Differentiability of vector functions

- Differentiability: $\vec{r}(t)$ is said to be differentiable if $\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$ exist

Similar to limit evaluation, differentiation of vector valued functions can be done on a component wise. as

$$\frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}.$$

• Geometrical Interpretation:



$\vec{r}'(t)$ is a vector tangent to the curve given by $\vec{r}(t)$ and pointing in the direction of increasing values of t.

Unit tangent vector: $\vec{u} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

• Arc length of a curve:

Let a curve be given by the vector function $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$

Parametric equations of curve:

$x = x(t)$, $y = y(t)$, $z = z(t)$.

$$\text{Length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

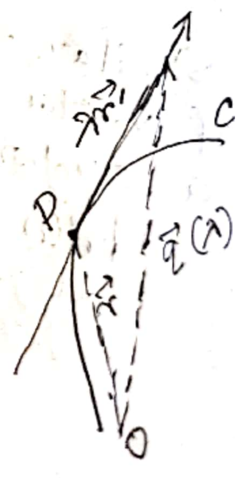
Note that $|\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

(length of the tangent vector)

∴ Length of arc in terms of position vector, $\vec{r}(t) = \int_a^b |\vec{r}'(t)|\, dt$

• Equation of tangent to a curve C at point P:

$\vec{q}(\lambda) = \vec{r} + \lambda \vec{r}'$, $\lambda \in R$



Example: Consider $\vec{r} = t\hat{i} + (t^2+1)\hat{j}$

Tangent vector, $\vec{r}\,' = \hat{i} + 2t\hat{j}$

Equation of tangent at $t=2$:

$$\vec{q}(\lambda) = (2\hat{i}+5\hat{j}) + \lambda(\hat{i}+4\hat{j})$$
$$= (2+\lambda)\hat{i} + (5+4\lambda)\hat{j}.$$

• Gradient of a Scalar function

Let $f(x,y,z)$ be a function of x, y and z such that f_x, f_y and f_z exist.

The gradient of f, denoted by grad f, is the vector

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

↳ which is vector function.

• Nabla or Del Operator:

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

$$\Rightarrow \text{grad } f = \vec{\nabla} f.$$

• Tangent plane and normal line to a surface:

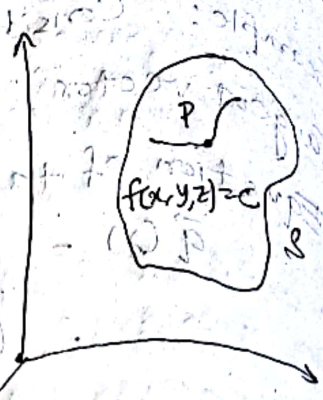
Let a surface S be given by $z = g(x,y)$

Define the function $f(x,y,z) = g(x,y) - z$

Then the given surface $z = g(x,y)$ can be treated as the level surface of $f(x,y,z)$ be given by $f(x,y,z) = 0$.

Note that level surface of a function $f(x,y,z)$ be given by $f(x,y,z) = c$.

Let $P(x_0, y_0, z_0)$ be a point on S and let C be a curve ~~through~~ through P that is defined by the vector valued function



$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Since the curve lies on the surface, we have $f(x(t), y(t), z(t)) = c \quad \forall t$

$$\Rightarrow \frac{d}{dt}\left(f(x(t), y(t), z(t))\right) = 0$$

$$\Rightarrow f_x(x,y,z)\,x' + f_y(x,y,z)\,y' + f_z(x,y,z)\,z' = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}\right) = 0$$

At (x_0, y_0, z_0), we have $\nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(t) = 0$

The gradient at P is orthogonal to the tangent vector of every curve on S through P.

Unit normal to vector to a surface $f(x,y,z)$: $= \; c \dfrac{\nabla f}{|\nabla f|}$

The plane through $P(x_0, y_0, z_0)$ that is normal to $\nabla f(x_0, y_0, z_0)$ is called the tangent plane to S at P.

Let $Q(x, y, z)$ be an arbitrary point in the tangent plane.

Then the vector $(x - x_0)\,\hat{i} + (y - y_0)\,\hat{j} + (z - z_0)\,\hat{k}$ lies in the tangent plane.

$$((x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}) \cdot (f_x(P_0)\hat{i} + f_y(P_0)\hat{j} + f_z(P_0)\hat{k}) = 0$$

$$\boxed{(x-x_0) f_x(x_0, y_0, z_0) + (y-y_0) f_y(x_0, y_0, z_0) + (z-z_0) f_z(x_0, y_0, z_0) = 0.}$$

$\hookrightarrow$ eqn of tangent plane at point P

Example: Find unit normal to the surface $x^2+y^2-z = 0$ at the point $(1, 1, 2)$.

Define $f = x^2+y^2-z$.

$$\Rightarrow \nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla f(1,1,2) = 2\hat{i} + 2\hat{j} - \hat{k}.$$

Unit normal vector $\hat{n} = \dfrac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{2^2+2^2+1}}$

$$= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}.$$

The other unit normal vector is

$$-\hat{n} = -\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}.$$

Vector field: Function that maps a point in space | plane to a vector.

A vector field over a solid region (or a plane) R is a function that assigns a vector $\vec{F}(x,y,z)$ or $(\vec{F}(x,y))$ to each point in R:

$$\vec{F}(x,y,z) = f(x,y,z)\,\hat{i} + g(x,y,z)\,\hat{j} + h(x,y,z)\,\hat{k}.$$

Example: Velocity of air inside a room is defined by a vector field.

Example: Gradient of a function is an example of a vector field.

Suppose $f(x,y) = 3x^2y + 2xy^3$.

$\text{grad} f = \nabla f(x,y) = (6xy + 2y^3)\hat{i} + (3x^2+6xy$

Scalar field: Function that maps a point in space | plane to a scalar.

A scalar field over a solid region (or a plane) R is a function that assigns a scalar to each point in R.

$f(x,y,z) = 3x^2 + 2y^2 + z^2$.

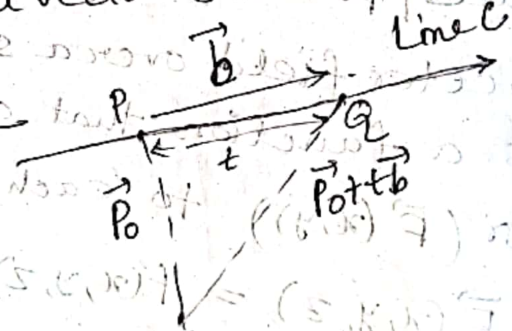
Temparature inside a room is defined by a scalar field.

In the context of vector, a real valued function of several variables is called a scalar field.

Example: Consider $F(x,y) = 2x^2 + y^2 = c$

Scalar field may be visualized using level curves of $F(x,y)$ (level surface in case of 3D)

Directional derivative of a scalar field $f(x,y,z)$ at $P(x_0, y_0, z_0)$ along a vector $\vec{b}$:



Let $|\vec{b}| = 1$. Let C be the line passing through P and parallel to $\vec{b}$

Position vector of the line C is:

$$\vec{r}(t) = \vec{P_0} + t\vec{b} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Rate of change of f in the direction $\vec{b}$ is given as

$$\lim_{t\to 0} \frac{f(Q)-f(P)}{t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$= \vec{\nabla f}\cdot\frac{d\vec{r}}{dt} = \vec{\nabla} f(x_0, y_0, z_0)\cdot\vec{b}.$$

At any point P, the directional derivative of f represents the rate of change in f along $\vec{b}$ at the point P, it is denoted by $D_b f = \vec{\nabla f}|_P \cdot \vec{b}$.

Ex. Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $\hat{i} + \hat{j} + \hat{k}$ and evaluate this at the ~~origin point~~ origin.

$\vec{\nabla f} = 2\hat{i} + \hat{j} + 2z\hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$. Unit vector $\vec{u}$ along $\vec{b} = \frac{\vec{b}}{|\vec{b}|}$

$$= \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$~~\vec{\nabla f}\cdot\vec{b}~~$

$$D_{(0,0,0)}\vec{b} = (2\hat{i} + \hat{j} + 2z\hat{k})\Big|_{(0,0,0)} \cdot \left(\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right)$$

$$= \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2z}{\sqrt{3}}\right)\Big|_{(0,0,0)}$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

Ex. Find the directional derivative of ~~f(x,y)~~ $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1,2)$ in the direction $u = \hat{i} + \sqrt{3}\hat{j}$

$\vec{\nabla f} = -2x\hat{i} - \frac{1}{2}y\hat{j}$

$\vec{\nabla f}|_{(1,2)} = -2\hat{i} - \hat{j}$

Unit vector along $\hat{u} = \frac{\hat{i}}{2} + \frac{\sqrt{3}}{2}\hat{j}$

$$D_u f = (-2\hat{i}-\hat{j})\cdot\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right)$$

$$= -1 - \frac{\sqrt{3}}{2}.$$

Maximum Rate of Change of a scalar field :

Rate of change of f in the direction of a unit vector $\vec{b}$:

$$D_b f = \vec{\nabla f}\cdot\vec{b} = |\vec{\nabla f}|\,|\vec{b}|\cos\theta$$

$$= |\vec{\nabla f}|\cos\theta \quad [\because |\vec{b}|=1]$$

⇒ Rate of change is maximum when $\theta = 0$ i.e. in the direction of $\vec{\nabla f}$.

⇒ Rate of change is minimum when $\theta = \pi$ i.e. in the opposite direction of $\vec{\nabla f}$

⇒ Gradient vector ~~points~~ ∇f points in the direction in which f increases rapidly and $-\nabla f$ points in the direction in which f decreases rapidly.

Example: Let $f(x,y,z) = x^2+y^2 - 2z$. Find the direction of maximum increase of f at $(2,1,-1)$

Gradient of f: $2x\hat{i} + 2y\hat{j} - 2\hat{k}$.

Direction of maximum increase : $\vec{\nabla f}\big|_{(2,1,-1)}$

$$= 4\hat{i} + 2\hat{j} + 2\hat{k}.$$

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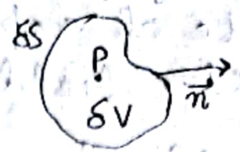
Divergence of a Vector Field:

The divergence of a vector field $\vec{v}$ at a point P is defined as



$$\text{div}\,\vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \vec{v}\cdot\vec{n}\, d\sigma$$

Flux of the vector field $\vec{v}$ out of a small closed surface where δV is a small volume enclosing P with surface δS and $\vec{n}$ is the outward pointing normal to δS.

Computation of Divergence:

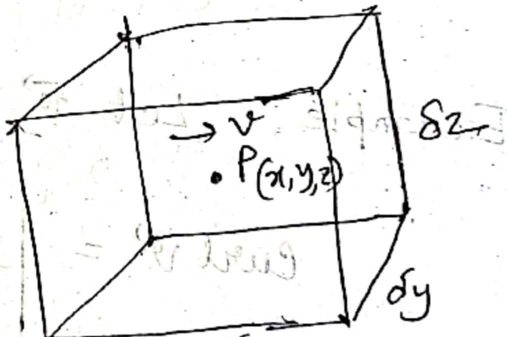
The divergence of a vector field $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is the scalar field given by

$$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

Physical Interpretation of Divergence of a vector field:

Suppose $\vec{v}(x, y, z)$ is the velocity of a fluid at a point $P(x, y, z)$.



Measure the rate per unit volume at which fluid flows out of this box across its faces:

$$\text{div}\,\vec{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{S} \vec{v}\cdot\vec{n}\, d\sigma$$

$$= \lim_{\substack{\delta x \to 0\\ \delta y \to 0\\ \delta z \to 0}} \frac{1}{\delta x\,\delta y\,\delta z} \left(\sum_{i=1}^{6} \iint_{S_i} \vec{v}\cdot\vec{n}\, d\sigma\right)$$

$$\approx \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \vec{\nabla}\cdot\vec{v}$$

Divergence can be interpreted as the rate of expression or compression of the vector field.

- If div $\vec{v} > 0$, the ~~fluid~~ field is expanding and ~~de~~ if div $\vec{v} < 0$, the field is contracting.

- If $\vec{\nabla}\cdot\vec{v} = 0$ for a vector ~~v~~ field $\vec{v}$, ~~it~~ then $\vec{v}$ is said to solenoidal.

Curl of a vector field: Curl of a vector $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is given by

$$\text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

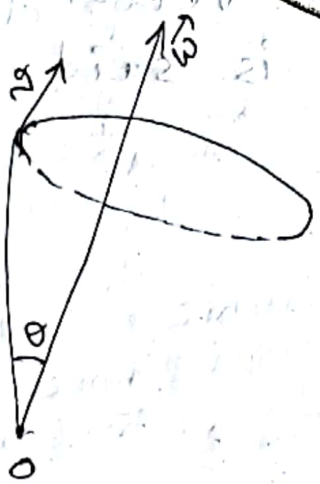
$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}.$$

Example: Let $\vec{v} = y\hat{i} + 2xz\hat{j} + ze^x\hat{k}$

$$\text{curl}\,\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2xz & ze^x \end{vmatrix}$$

$$= -2x\hat{i} - ze^x\hat{j} + (2z-1)\hat{k}.$$

Physical interpretation of Curl of a vector field :



Suppose an object rotates with uniform angular velocity $\vec{\omega}$.

tangential speed = angular speed × radius

$$|\vec{v}| = |\vec{\omega}||\vec{r}|\sin\theta$$
$$= |\vec{\omega}\times\vec{r}|$$

Note that the direction of $\vec{v}$ is perpendicular to both $\vec{r}$ and $\vec{\omega}$.

Since $\vec{v}$ and $\vec{r}\times\vec{\omega}$ both have same direction and same magnitude, we conclude

$$\vec{v} = \vec{\omega}\times\vec{r}$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{\omega} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\cancel{\vec{v} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \vec{a} & \vec{b} & \vec{c} \\ x & y & z \end{vmatrix}}$$

$$\vec{v} = \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \vec{v} = (bz - cy)\hat{i} + (cx - az)\hat{j} + (ay - bx)\hat{k}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz-cy & cx-az & ay-bx \end{vmatrix}$$

$$= 2a\hat{i} + 2b\hat{j} + 2c\hat{k} = 2\vec{\omega}$$

Curl $\vec{v}$ signifies the tendency of rotation. It is directed along the axis of rotation with magnitude twice the angular speed

A vector field $\vec{v}$ for $\vec{\nabla} \times \vec{v} = 0$ everywhere is said to irrotational.

Line Integral

Smooth Curves : Let $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$, $t \in [a, b]$

denote a curve in space.

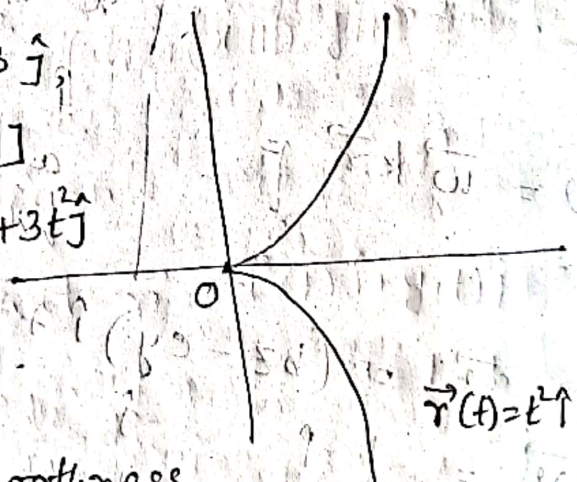
If $\vec{r}(t)$ posses a continuous first order derivative (nowhere zero) for the given values of t, then the curve is known as smooth.

In other words, the space curve $\vec{r}(t)$ is smooth when $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are continuous on $[a, b]$ and not simultaneously zero on (a, b).

Note that ~~the~~ condition nowhere zero ensures that the curve has no ~~stor~~ sharp corners.

Consider $\vec{r}(t) = t^2\hat{i} + t^3\hat{j}$,

$t \in [-1, 1]$.



$\vec{r}(t) = t^2\hat{i} + t^3$.

Compute $\frac{d\vec{r}(t)}{dt} = 2t\hat{i} + 3t^2\hat{j}$

$\Rightarrow \frac{d\vec{r}(t)}{dt} = \vec{0}$ for $t = 0$.

This indicate nonsmoothness for $t = 0$

Note that $\frac{d\vec{r}(t)}{dt} = 0$ does not necessarily imply nonsmoothness.

However $\frac{d\vec{r}(t)}{dt} \neq 0$ ~~do~~ always implies smoothness.

Consider $\vec{r}(t) = t^3\hat{i} + t^6\hat{j}$; $t \in [-1, 1]$

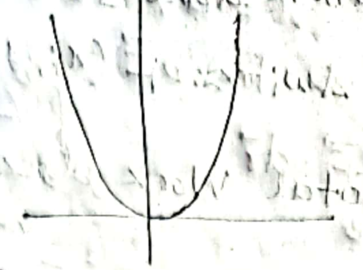
$\Rightarrow \frac{d\vec{r}(t)}{dt} = 0$ for $t = 0$.

But the curve is smooth.

$x(t) = t^3 \quad y(t) = t^6$.

$\Rightarrow y = x^2$.

$\vec{r}(t) = t^3\hat{i} + t^6\hat{j}$



Alternative parametrization,

$$\vec{r}(t) = t\hat{i} + t^2\hat{j}, \; t \in [-1, 1].$$

$$\frac{d\vec{r}(t)}{dt} \neq 0 \quad \forall t.$$

Piece-wise smooth curve: If it is made up to a finite number of smooth curves.

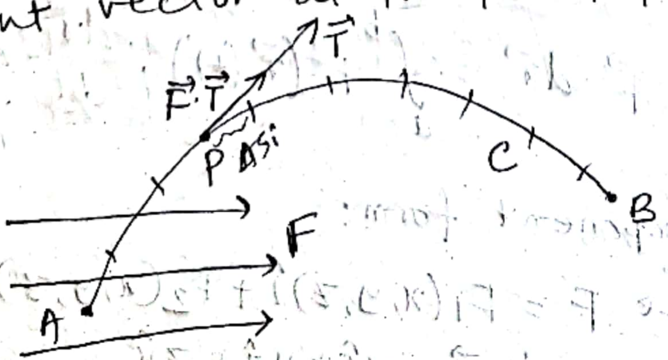
Simple closed curve: A curve which does not intersect itself anywhere and initial and end points are same is known as simple closed curve.

Simple closed curve

Simple not closed

Closed but not simple.

Line integrals: Let a force $\vec{F}$ ~~act on~~ act upon a particle which is displaced along a given curve C in space. Let $\vec{T}$ be a unit tangent vector at the point $P(x_i, y_i, z_i)$



On a small subarc of length Δs_i the work done is

$$\Delta w_i \approx \vec{F}(x_i, y_i, z_i) \cdot \vec{T}(x_i, y_i, z_i)\, \Delta s_i$$

Total Work done: $W = \lim_{N \to \infty} \sum_{i=1}^{N} \Delta w_i$

$$= \int_C \vec{F} \cdot \vec{T}\, ds$$

Let the curve C be given by

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}.$$

Note that $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and $ds = |\vec{r}'(t)|\,dt$

$$= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

$$\vec{F} \cdot \vec{T}\, ds = \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|}\, |\vec{r}'(t)|\, dt$$

$$= \vec{F} \cdot \vec{r}'(t)\, dt = \vec{F} \cdot d\vec{r}$$

$$W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t)\, dt$$

Evaluation of line integrals $\int_C \vec{F} \cdot d\vec{r}$

In vector form, note that $\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

$$a \le t \le b$$

and $d\vec{r} = \dfrac{d\vec{r}}{dt}\, dt$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

In component form:

Suppose $\vec{F} = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

Ex 1: Find the work done by $\vec{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$ over the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

Sol:- $\dfrac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$. Put $x = t$, $y = t^2$, $z = t^3$

$$F(\vec{r}(t)) = (t^2-t^2)\hat{i} + (t^3-t^4)\hat{j} + (t-t^6)\hat{k}$$

$$= (t^3-t^4)\hat{j} + (t-t^6)\hat{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = 2t(t^3-t^4) + 3t^2(t-t^6)$$

$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

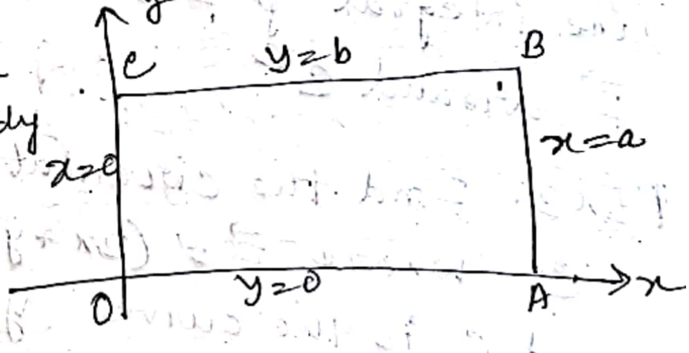
$$\int \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \frac{29}{60}.$$

Ex 2. Evaluate : $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$

C : rectangle in the xy plane bounded by $y=0$, $x=a$; $y=b$, $x=0$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y^2)\,dx - 2xy\,dy$$



~~Alog~~ Along OA :

$y=0$, $dy=0$ and x varies from 0 to a.

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{a} x^2\,dx = \frac{a^3}{3}.$$

Along AB:

$x = a$, $dx = 0$ and y varies from 0 to b.

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=0}^{b} (-2ay)\,dy$$

$$= -ab^2$$

Along BC: ~~$x = a$, $dx = 0$~~ $y = b \Rightarrow dy = 0$ and x varies from a to 0.

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{x=a}^{0} (x^2+b^2)\,dx = -\left(\frac{a^3}{3} + ab^2\right)$$

Along CO: $x = 0 \Rightarrow dx = 0$ and y varies from b to 0.

$$\int_{CO} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{C} \vec{F}\cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

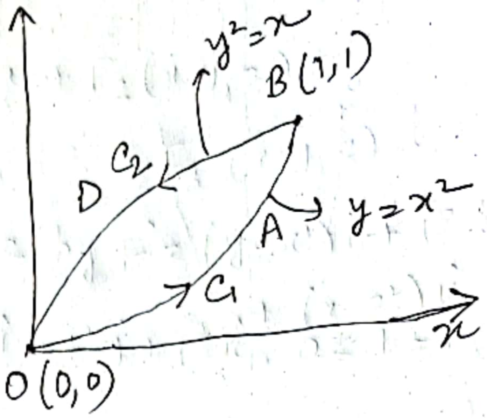
$$= -2ab^2.$$

Line integral as circulation: Let C be an oriented closed curve. We call the line integral $\oint_C \vec{F}\cdot d\vec{r}$ the circulation of $\vec{F}$ around C.

Ex 3. Find the circulation of $\vec{F}$ around C where $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C is the curve $y = x^2$ from (0,0) to (1,1) and the curve $y^2 = x$ from (1,1) to (0,0).

Sol^{n}: $\vec{F}\cdot d\vec{r} = (2x+y^2)\,dx + (3y-4x)\,dy$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$



Along OAB: $x^2 = y \Rightarrow 2x\,dx = dy$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} (2x+x^4)\,dx + \int_{x=0}^{1} (3x^2-4x)\,2x\,dx$$

$$= \frac{1}{30}.$$

Along BDO: $x = y^2 \Rightarrow dx = 2y\,dy$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_0^1 (2y^2+y^2)\,2y\,dy - \int_0^1 (3y-4y^2)\,dy$$

$$= -\frac{6}{4} - \frac{3}{2} + \frac{4}{3} = -\frac{5}{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}.$$

Ex 4. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, $\vec{F} = y\hat{i} - 2x\hat{j}$,
$C: x^2+y^2=9$.

Sol^{n}: Parametric eq^{n} of circle: $x = 3\cos t$
$y = 3\sin t$,
$0 \le t \le 2\pi$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{t=0}^{2\pi} (-9\sin^2 t - 18\cos^2 t)\,dt$$

$$\left[\because \vec{F}\cdot\frac{d\vec{r}}{dt} = (3\sin t\,\hat{i} - 2\cdot 3\cos t\,\hat{j})\cdot(3\sin t\,\hat{i} + 3\cos t\,\hat{j}) = -9\sin^2 t - 18\cos^2 t\right]$$

$$= -9\oint_0^{2\pi} (1+\cos^2 t)\,dt = -9\oint_0^{2\pi}\left(1+\frac{1}{2}(1+\cos 2t)\right)dt$$

$$= -9\left(\frac{3}{2}2\pi + 0\right) = -27\pi.$$

Conservative Vector field

A vector field $\vec{V}$ is said to be conservative if the vector function can be written as the gradient of a scalar function. i.e.

$$\vec{V} = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

The vector function f is called a potential function or potential of $\vec{V}$.

Ex1. ~~Show that~~ If the vector field $\vec{F} = (2x+y)\hat{i} + x\hat{j} + 2z\hat{k}$ is conservative, find the potential function.

$\vec{F}$ is conservative if it can be written as

$$\vec{F} = \vec{\nabla} f$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial x} = 2x+y}_{(1)}, \quad \underbrace{\frac{\partial f}{\partial y} = x}_{(2)}, \quad \underbrace{\frac{\partial f}{\partial z} = 2z}_{(3)}$$

$$\frac{\partial f}{\partial x} = 2x+y \Rightarrow f = x^2 + xy + h(y,z). \quad (4)$$

$$\therefore \frac{\partial f}{\partial y} = x + \frac{\partial h}{\partial y} = x \cdot [\text{From } (2)]$$

$$\Rightarrow \frac{\partial h}{\partial y} = 0$$

$\Rightarrow h$ is independent of y.

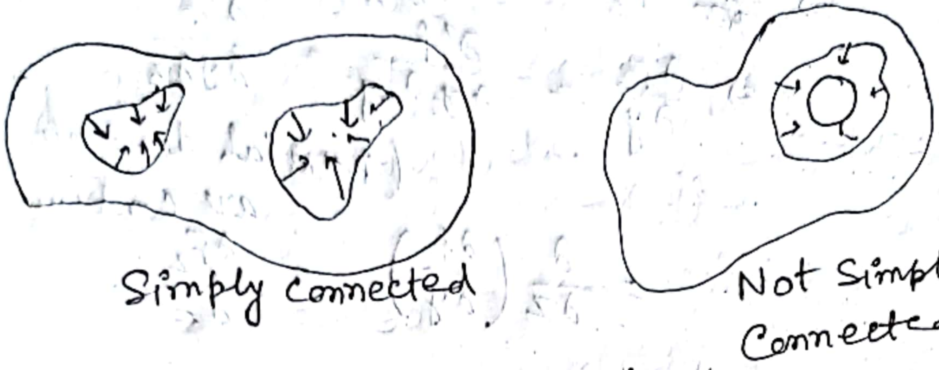
$$\therefore h \equiv h(z)$$

From (4), $\frac{\partial f}{\partial z} = \frac{dh}{dz} \Rightarrow \frac{dh}{dz} = 2z$ [from (3)]

$$\Rightarrow h = z^2 + c.$$

$$\therefore f = x^2 + xy + z^2 + c$$

Simply Connected domain: A domain D (in R^2 or R^3) is simply connected if it consists of a single connected piece and if every simple, closed curve C in D can be continuously shrunk to a point while remaining in D throughout the deformation.



Test for conservative field

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field whose components have continuous first order partial derivatives in a simply connected domain D.

F is conservative if and only if $\boxed{\vec{\nabla} \times \vec{F} = \vec{0}}$ at all points of D.

Equivalently $\vec{F}$ is conservative if and only if $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}$, $\dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$ and $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ $\hookrightarrow$ (A)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

$\vec{\nabla} \times \vec{F} = \vec{0} \Leftrightarrow$ (A) holds.

Proof (F is conservative $\Rightarrow \vec{\nabla} \times \vec{F} = \vec{0}$)

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$ (assuming that $\vec{F}$ is conservative)

$$\Rightarrow F_3 = \frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial y \partial z}$$

(partial derivatives are continuous)

$$= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial F_2}{\partial z}$$

$$\therefore \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

Similarly other relations can be proved.

Ex. Show that $\vec{F} = (e^x \cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy + z)\hat{k}$

is conservative.

Sol^n. $F_1 = (e^x \cos y + yz)$, $F_2 = (xz - e^x \sin y)$,

$F_3 = (xy + z)$.

To show that $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$, $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$,

$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$

$$\frac{\partial F_3}{\partial y} = x = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial F_2}{\partial x} = z - e^x \sin y$$

$$\frac{\partial F_1}{\partial y} = -e^x \sin y + z$$

$$\therefore \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = y = \frac{\partial F_3}{\partial x}.$$

Hence $\vec{F}$ is conservative.

Path Independence

Let $\vec{F}$ be a vector field defined on a simply connected domain D and suppose that for any two points A and B in D, the integral

$$\int_A^B \vec{F}\cdot d\vec{r}$$

is same over all paths ~~for~~ from A to B in the domain D.

Independence of path and conservative vector fields

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field whose components are continuous throughout a simply connected region D in space. Then there exists a differentiable fn f such that

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

($\vec{F}$ is conservative in D)

if and only if the integral $\int_A^B \vec{F}\cdot d\vec{r}$ is independent of the path in D.

i.e. $\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A)$.

f=x^2 y+xz^3+constant

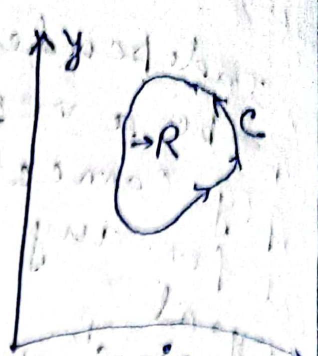
Ex 1. (a) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Hints: $[(\vec{\nabla}\times\vec{F} = 0)$

(b) Find the scalar potential. $(\vec{F} = \nabla f)\,]$

(c) Find the work done moving from (1, -2, 1) to (3, 1, 4) $[(f(B) - f(A))]$ =202 unit

Green's theorem:

Let R be a region in R^2 whose boundary is a simple closed curve C which is piecewise smooth (oriented counted clockwise – when traversed C in the region R always lies left).



Let $\vec{F} = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$ be smooth vector field (F_1, F_2 are continuously differentiable fn) on both R and C.

then
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Note that the above can also be written as
$$\oint_C \vec{F}\,d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot\hat{k}\;dx\,dy$$

$$\text{curl}\,F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl}\,\vec{F}\cdot\hat{k} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\cdot\hat{k}$$
$$= 1\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

Problem 1. Verify Green's theorem for the vector field $\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$. The region R is bounded by the circle $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j},\ 0 \le t \le 2\pi$.

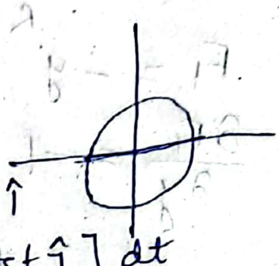
Sol^{n}. $F_1 = x - y \Rightarrow \dfrac{\partial F_1}{\partial y} = -1$.

$F_2 = x \Rightarrow \dfrac{\partial F_2}{\partial x} = 1$.

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = 2\iint_R dx\,dy$$

$$= 2\pi \cdot 1^2 = 2\pi$$

$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \cancel{\cos t\,\hat{i} + \sin t\,\hat{j} + \sin}$$



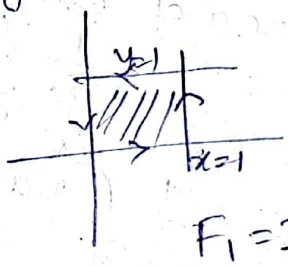
$$= \int_0^{2\pi} [(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}]\cdot[-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$$

$$= \int_0^{2\pi} [\sin t\,(\sin t - \cos t) + \cos^2 t]\,dt$$

$$= 2\pi - \frac{1}{2}\int_0^{2\pi} \sin 2t\,dt$$

$$= 2\pi - \frac{1}{2}\cdot 0 = 2\pi.$$

2. Evaluate the integral $\oint_C xy\,dy - y^2\,dx$ using Green's theorem. Here C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.



Sol^{n}. $$\oint_C (xy\,dy - y^2\,dx)$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$F_1 = y^2$

$\dfrac{\partial F_1}{\partial y} = 2y$

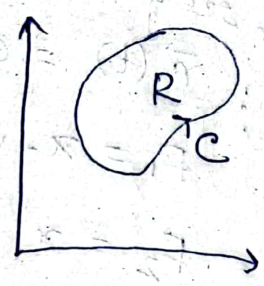
$F_2 = xy$

$\dfrac{\partial F_2}{\partial x} = y$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (y + 2y)\,dx\,dy$$

$$= 3\left[\frac{y^2}{2}\right]_0^1 = \frac{3}{2}$$

3. Show that the area bounded by a simple closed ~~curc~~ curve C is given by $\frac{1}{2}\oint_C (xdy - ydx)$



Sol^n Green's theorem

$$\frac{1}{2}\oint_C (xdy - ydx)$$

$$= \frac{1}{2}\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$F_1 = -y$ $\quad$ $F_2 = x$.

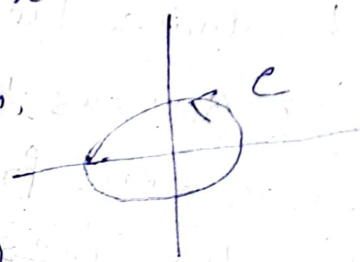
$\frac{\partial F_1}{\partial y} = -1$ $\quad$ $\frac{\partial F_2}{\partial x} = 1$.

$$= \frac{1}{2}\iint_R [1 - (-1)]\, dxdy$$

$$= \frac{1}{2} \cdot 2 \iint_R dxdy$$

$$= \iint_R dxdy = \text{Area of } R.$$

4. Using Green's theorem, find the area of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.



Sol^n. Using Green's theorem,

Area of ellipse

$$= \frac{1}{2}\oint_C (xdy - ydx)$$

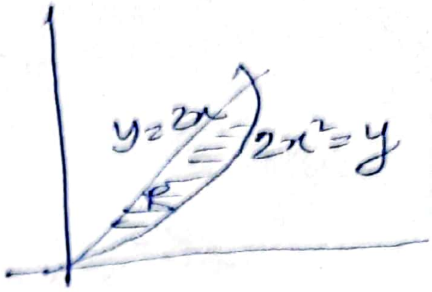
$$= \frac{1}{2}\int_0^{2\pi} [a\cos\theta . b\cos\theta - b\sin\theta(-a\sin\theta)]\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)\,d\theta$$

$$= \pi ab.$$

6. Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$, C is the boundary of the region $R = \{(x,y) : 0 \le x \le 1,\ 2x^2 \le y \le 2x\}$

Sol: Using Green's theorem,



$$\oint_C (x^2+y^2)\,dx + 2xy\,dy$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$F_1 = x^2+y^2$ $\qquad$ $F_2 = 2xy$

$\frac{\partial F_1}{\partial y} = 2y$ $\qquad$ $\frac{\partial F_2}{\partial x} = 2y.$

$$= \iint_R (2y-2y)\,dx\,dy = 0.$$

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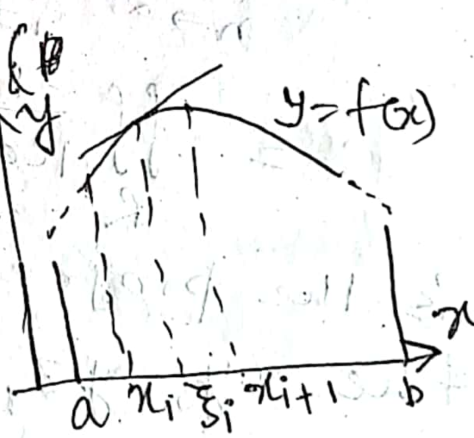
Surface Integral

Smooth Surface: A curve is called smooth if it has a continuous tangent.
Similarly a surface is smooth if it has a continuous normal vector.
A surface is called piecewise smooth if it consists of finite number of smooth surfaces.

Example: Surface of a sphere $\Rightarrow$ a smooth surface.
surface of a cube $\rightarrow$ a piecewise smooth surface.

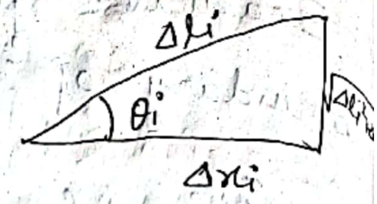
Does not have a normal vector along any of its edges.

Evaluation of Arc length:



Let θ be the angle of the tangent at ξ_i with +ve x axis.

$$\frac{\Delta x_i}{\Delta l_i} = |\cos\theta_i|$$



$$\Rightarrow \Delta l_i = \frac{1}{|\cos\theta_i|}\,\Delta x_i$$

Alternatively

$$f'(\xi_i) = \frac{\sqrt{\Delta l_i^2 + \Delta x_i^2}}{\Delta x_i}$$

$\tan\theta_i \rightarrow$

$$\Rightarrow \Delta l_i = \sqrt{1+(f'(\xi_i))^2}\,\Delta x_i$$

Arc Length $L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta l_i$

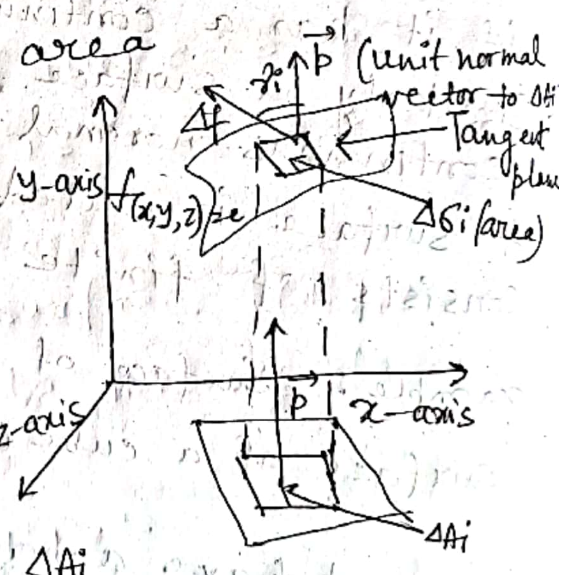
$$= \int_C dl = \int_a^b \frac{1}{|\cos\theta_i|}\,dx$$

$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Arc length differential $dl = \frac{1}{|\cos\theta|}\,dx$

$$= \sqrt{1+f'^2}\,dx.$$

Evaluation of surface area



$$\frac{\Delta A_i}{\Delta \sigma_i} = |\cos\gamma_i|$$

$$\Rightarrow \Delta\sigma_i = \frac{1}{|\cos\gamma_i|}\,\Delta A_i$$

Surface area:

$$S = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta\sigma_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{|\cos\gamma_i|}\,\Delta A_i$$

$$= \iint_R \frac{1}{|\cos\gamma_i|}\,dA$$

R is the projection of the surface on the xy, yz, or zx plane.

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Note that $|\vec{\nabla f}\cdot\vec{p}| = |\vec{\nabla f}|\,|\vec{p}|\,|\cos\gamma|$

$$\Rightarrow \frac{1}{\cos\gamma} = \frac{|\vec{\nabla f}|}{|\vec{\nabla f}\cdot\vec{p}|}$$

The area of the surface $f(x,y,z)=c$ over a closed and bounded plane R:

$$S = \iint_S d\sigma = \iint_R \frac{|\nabla f|}{|\vec{\nabla f}\cdot\vec{p}|}\, dA,$$

R is the projection of S on the xy, yz or zx plane. $\vec{p}$ is the unit normal to R and $\nabla f\cdot\vec{p}\neq 0$

Remark: Let $z = g(x,y)$ be the equation of a surface. Then the surface area:

(~~Jnt~~) $$S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy$$

where R is the projection of the surface in the xy plane.

In the vector form, the same can be calculated using $S = \iint_R \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\, dA$.

Let $f = z - g(x,y) = 0 \Rightarrow \nabla f = (-g_x, -g_y, 1)$

$\Rightarrow |\nabla f| = \sqrt{1+z_x^2+z_y^2}$, $|\vec{\nabla f}\cdot\vec{p}| = 1$ (considering $\vec{p}$ as the unit normal to xy plane)

Surface Integral $\iint_S g(x,y,z)\, d\sigma$

Integrating a fn over surface using the idea just developed for calculating surface area.

Suppose we have electric charge distribution over $f(x, y, z) = c$.
Let the fn $g(x, y, z)$ gives the charge per unit area (charge density) at each point on S.

Total charge $S = \iint_S g(x, y, z)\, d\sigma$

$$= \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA,$$

surface integral of g over S.

Note: If g gives the mass density of a thin shell of material S, the integral gives the mass of shell.

If $g = 1$, then the integral will simply give the total area of surface.

Ex 1. Find the area of the cap cut from the hemisphere $x^2+y^2+z^2 = 2$, $z \geq 0$ by the cylinder $x^2+y^2 = 1$.

Soln: Projection of the surface $f(x,y,z) = c$ onto the xy plane:
i.e. $x^2+y^2+z^2 = 2$
$x^2+y^2 \leq 1$.

Now $f(x, y, z) = x^2+y^2+z^2$

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$.

$|\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2}$

The vector $\vec{p} = \hat{k}$ is normal to the xy plane $\Rightarrow |\nabla f \cdot \vec{p}| = |2z| = 2z$

$z \geq 0$.

Surface area, $S = \iint\limits_R \frac{\nabla f}{|\nabla f \cdot \vec{p}|}\, dA$

$$= \iint\limits_R \cancel{2\sqrt{2}}\ \frac{2\sqrt{2}}{2z}\, dA$$

$$= \sqrt{2} \iint\limits_R \frac{dA}{z}$$

$x^2+y^2+z^2 = 2, \quad z \geq 0$

$|\nabla f| = 2\sqrt{2}, \quad |\nabla f \cdot \vec{p}| = 2z$

$$\therefore S = \sqrt{2} \iint\limits_R \frac{dA}{\sqrt{2-(x^2+y^2)}}$$

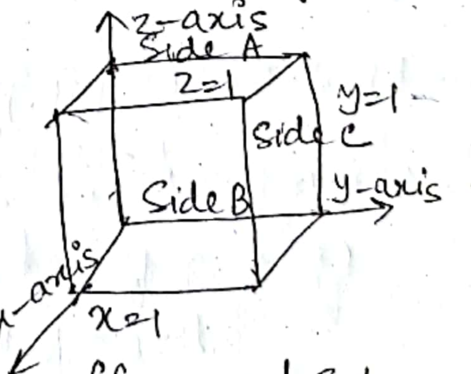
$$2\sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r\,dr\,d\theta}{\sqrt{2-r^2}} \qquad \% \ x^2+y^2 = r^2$$

$$= \frac{\sqrt{2}}{\cancel{2}} \int_{\theta=0}^{2\pi} \left[-\sqrt{2-r^2}\right]_{r=0}^{1} d\theta$$

$$= \sqrt{2} \int_0^{2\pi} (\sqrt{2}-1)\, d\theta$$

$$= 2\pi(2-\sqrt{2}).$$

2. Integrate $g(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$ and $z=1$.



The integral over the the surface of the cube reduces to

$$\iint\limits_{\substack{\text{Cube}\\ \text{surface}}} xyz\,d\sigma = \iint\limits_{\substack{\text{Side}\\ A}} xyz\,d\sigma + \iint\limits_{\substack{\text{Side}\\ B}} xyz\,d\sigma + \iint\limits_{\substack{\text{Side}\\ C}} xyz\,d\sigma.$$

Side A is the surface $f(x,y,z) = z = 1$
over the region
R_{xy} : $0 \le x \le 1$, $0 \le y \le 1$ in the xy plane

For this surface (side A) and region R_{xy}

$\hat{p} = \hat{k}$ $\quad \vec{\nabla f} = \hat{k}$, $|\vec{\nabla f}| = 1$.

$|\vec{\nabla f} \cdot \vec{p}| = 1$. $\Rightarrow d\sigma = \dfrac{|\vec{\nabla f}|}{|\vec{\nabla f} \cdot \vec{p}|} dA$

$= dx\,dy$.

$$\iint_{\text{Side A}} xyz\, d\sigma = \int_{x=0}^{1} \int_{y=0}^{1} xy \cdot 1 \, dx\,dy \qquad d\sigma = dx\,dy.$$

$$= \frac{1}{4}$$

Similarly $\displaystyle\iint_{\text{Side B}} xyz\, d\sigma = \frac{1}{4}$. $\quad \displaystyle\iint_{\text{Side C}} xyz\, d\sigma = \frac{1}{4}$.

$$\iint_{\substack{\text{Cube} \\ \text{surface}}} xyz\, d\sigma = \frac{3}{4}$$

Flux of a vector field $\vec{F}$ through a surface S.

The flux of a vector field $\vec{F}$ across an orientable surface S in the of $\vec{n}$ (unit normal to S) is given by the integral

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, d\sigma .$$

Geometrically a flux integral is the surface integral over S of normal components of $\vec{F}$.

If $\vec{F}$ is the continuous velocity field of a fluid and $\rho(x,y,z)$ is the density of the fluid at (x,y,z), then the flux integral

$$\iint_S \rho\vec{F}\cdot\vec{n}\,d\sigma$$

represents the mass of fluid flowing across S per unit of time.

Evaluation of flux integral $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$

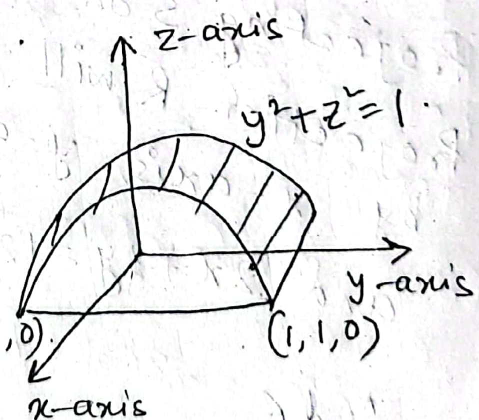
Suppose S is a part of a level surface $f(x,y,z)=c$, then $\vec{n}$ may be taken either of the two unit vectors

$$\vec{n} = \pm\frac{\nabla f}{|\nabla f|}$$

$$\text{Flux} = \pm\iint_R \vec{F}\cdot\left(\frac{\nabla f}{|\nabla f|}\right)\frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA$$

$$= \pm\iint_R \vec{F}\cdot\frac{\nabla f}{|\nabla f\cdot\vec{p}|}\,dA.$$

Ex 1. Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2+z^2=1$, $z>0$ by the planes $x=0$ and $x=1$.



Sol^{n}. Surface $f(x,y,z)=c$

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4(y^2+z^2)}}$$

$$= y\hat{j}+z\hat{k}\,;\quad \vec{p}=\hat{k}$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA = \frac{|\nabla f|}{|\nabla f\cdot\hat{k}|}\,dA = \frac{2}{|2z|}\,dA$$

$$= \frac{1}{z}\,dA.$$

Also $\vec{F}\cdot\vec{n} = y^2 z + z^3 = z$

Flux through S. 6: $\iint_S \vec{F}\cdot\vec{n}\, d6 = \iint_{R_{xy}} z\cdot\frac{1}{z}\, dA$

$= \iint_{R_{xy}} dA = 2.$

2. Evaluate the integral $\iint \vec{F}\cdot\vec{n}\, d\sigma$ where $\vec{F} = 6z\hat{i} + 6\hat{j} + 3y\hat{k}$ and S is the portion of the plane $2x + 3y + 4z = 12$ which is in the first octant.

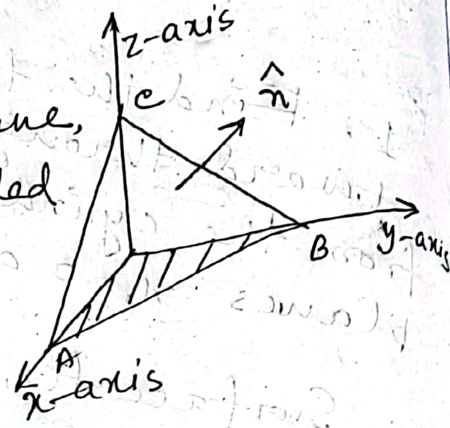
Let $f(x,y,z) = 2x + 3y + 4z$

$\Rightarrow \vec{\nabla} f = 2\hat{i} + 3\hat{j} + 4\hat{k}$

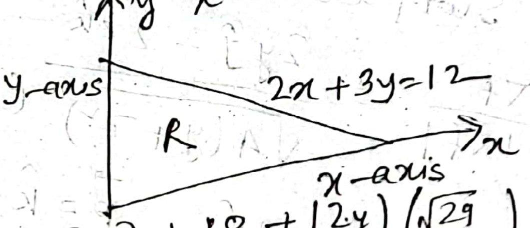
$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{29}}(2\hat{i} + 3\hat{j} + 4\hat{k})$

$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{29}}(12z + 18 + 12y)$

$d6 = \frac{|\nabla f|}{|\nabla f\cdot\hat{k}|}\, dA = \frac{\sqrt{29}}{4}\, dA \quad (\vec{p} = \hat{k})$



Projecting S on the xy-plane,
Projection R will be bounded by x-axis, y-axis and $2x + 3y = 12$



$\iint_S \vec{F}\cdot\hat{n}\, d6$

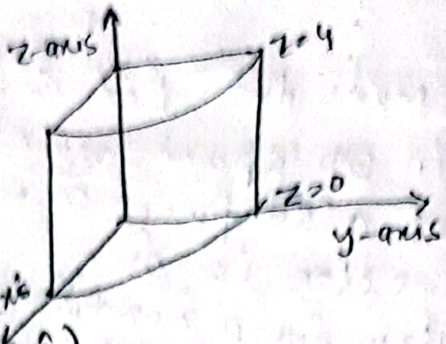
$= \iint_R \frac{1}{\sqrt{29}}\left(12\frac{(12-2x-3y)}{4} + 18 + 12y\right)\left(\frac{\sqrt{29}}{4}\right) dA$

$= \frac{1}{4}\iint_R (54 - 6x + 3y)\, dA$

$= \frac{1}{4}\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (54 - 6x + 3y)\, dy\, dx$

$= 138$

3. Evaluate the surface integral $\iint_S \vec{F}\cdot\hat{n}\, d\sigma$ where $\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$ and S is the portion of the surface of the cylinder $x^2+y^2=36$, $0 \le z \le 4$ included in the first octant.



Let $f(x,y,z) = x^2+y^2-36$

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$|\nabla f| = \sqrt{4\times 36} = 12$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{12}(2x\hat{i} + 2y\hat{j})$$

$$= \frac{1}{6}(x\hat{i} + y\hat{j})$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\, dA \qquad \vec{p} = \hat{i} \text{ (if projection is on } yz\text{-plane)}$$

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$|\nabla f| = 12$

$|\nabla f\cdot\vec{p}| = 2x \qquad d\sigma = \frac{12}{|2x|}\, dA.$

$\vec{F}\cdot\hat{n} = \frac{1}{6}(z^2 x + xy^2)$

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iint_{R_{yz}} \frac{1}{6}(xz^2 + xy^2)\,\frac{6}{x}\, dA$$

$$= \int_{y=0}^{6}\int_{z=0}^{4} (z^2+y^2)\, dy\, dz$$

$$= \int_0^4 \left(\frac{y^3}{3} + z^2 y\right)_0^6 dz$$

$$= \int_0^4 (72 + 6z^2)\, dz = 72\times 4 + \frac{6\times 64}{3}$$

$= 416.$

Stoke's theorem

If we recall, Green's theorem in plane

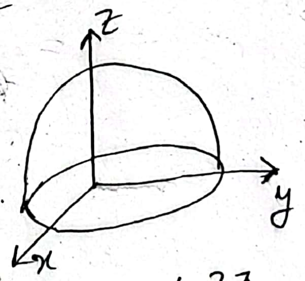
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D (\nabla\times\vec{F})\cdot\hat{k}\,dx\,dy$$

Stoke's th: Let C be a closed curve in 3D space which forms the boundary of a surface S whose unit normal vector is $\hat{n}$.
Then for a continuous differentiable fn vector field $\vec{F}$, we have

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$, where the direction of the line integral around C and the normal $\hat{n}$ are oriented in a right handed sense.

If $\nabla\times\vec{F} = 0$ ($\vec{F}$ is irrotational or $\vec{F}$ is conservative) then Stoke's theorem tells us that $\oint_C \vec{F}\cdot d\vec{r} = 0$

Ex 1. Verify Stoke's theorem for the hemisphere $S: x^2+y^2+z^2=9,\ z\geq 0$, its boundary $C: x^2+y^2=9,\ z=0$ and the field $\vec{F} = y\hat{i} - x\hat{j}$.



Sol^n Parametric eq^n of curve

$$\vec{r}(\theta) = 3\cos\theta\,\hat{i} + 3\sin\theta\,\hat{j},\quad 0\leq\theta\leq 2\pi$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = -3\sin\theta\,\hat{i} + 3\cos\theta\,\hat{j}$$

$$\vec{F} = 3\sin\theta\,\hat{i} - 3\cos\theta\,\hat{j}$$

$$\vec{F}\cdot\frac{d\vec{r}}{d\theta} = -9\sin^2\theta - 9\cos^2\theta$$

$$= -9.$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} \vec{F}\cdot\frac{d\vec{r}}{d\theta}\,d\theta = \int_0^{2\pi} -9\,d\theta$$

$$= -18\pi.$$

S: $x^2+y^2+z^2=9$, $f = x^2+y^2+z^2$, $\vec{F} = y\hat{i} - x\hat{j}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \hat{i}\cdot(0) + \hat{j}\cdot(0) + \hat{k}(-1-1)$$

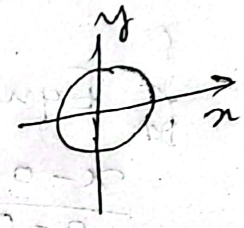
$$= -2\hat{k}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4\times 9}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \iint_{x^2+y^2\le 9} -\frac{2z}{3}\frac{|\nabla(x^2+y^2+z^2)|}{|\nabla(x^2+y^2+z^2)\cdot\hat{k}|}\,dx\,dy$$

$$= \iint_{x^2+y^2\le 9} -\frac{2z}{3}\cdot\frac{6}{2z}\,dx\,dy$$

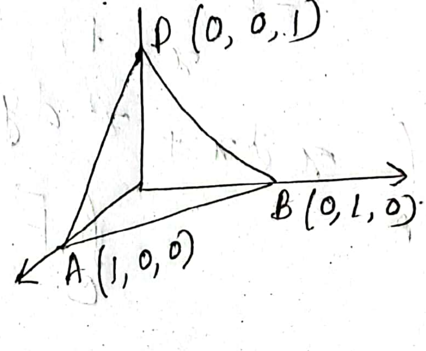


$$= -2\iint_{x^2+y^2\le 9} dx\,dy$$

Ex 2. Verify Stoke's theorem for function $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface $x+y+z=1$ lying in the first quadrant.

Stoke's theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$

S: triangle ABD, C: lines AB, BD, DA.



$$\oint_C \vec{F}\cdot d\vec{r}$$

$$= \oint_C (x\hat{i} + z^2\hat{j} + y^2\hat{k})\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_{AB} (x\,dx + z^2 dy + y^2 dz) + \int_{BD} (x\,dx + z^2 dy + y^2 dz)$$
$$+ \int_{DA} (x\,dx + z^2 dy + y^2 dz)$$

Equating to the line AB :

$$\frac{x-1}{0-1} = \frac{y-0}{1-0} = \frac{z-0}{0-0} = t \qquad 0 \le t \le 1.$$

$$\Rightarrow x = 1-t, \quad y = t, \quad z = 0,$$

$$\int_{AB} (x\,dx + z^2 dy + y^2 dz) = \left[\frac{(1-t)^2}{2}\right]_0^1 = -\frac{1}{2}$$
$$= \int_{t=0}^{1} (1-t)(-dt)$$

Equating to line BD :

$$\frac{x-0}{0-0} = \frac{y-1}{0-1} = \frac{z-0}{1-0} = t$$

$$x = 0, \quad y = 1-t, \quad z = t$$

$$\int_{BD} (x\,dx + z^2 dy + y^2 dz) = \int_{t=0}^{1} t^2(-dt) + (1-t)^2 dt$$
$$= \int_{t=0}^{1} (1-2t)\,dt = 0$$

Equating to line DA ;

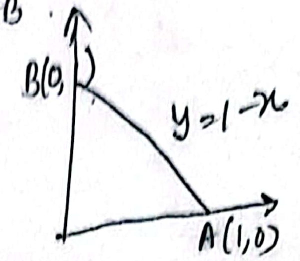
$$\frac{x-0}{1-0} = \frac{y-0}{0-0} = \frac{z-1}{0-1} = t$$

$$x = t, \quad y = 0, \quad z = 1-t,$$

$$\int_{DA} x\,dx + z^2 dy + y^2 dz = \int_0^1 t\,dt = \frac{1}{2}$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Projecting S on the x-y plane, let R be its projection. R is bounded by x-axis, y-axis and straight line AB.



Given surface

$f = x + y + z = 1 \Rightarrow$

$$\nabla f = \hat{i} + \hat{j} + \hat{k}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

$$\frac{|\nabla f|}{|\nabla f \cdot \hat{k}|} = \frac{\sqrt{3}}{|1|} = \sqrt{3}$$

$$\text{curl } \vec{F} \cdot \hat{n} = (2(y-z)\,\hat{i}) \cdot \left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}(y-z) = \frac{2}{\sqrt{3}}(2y+x-1)$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_{R_{xy}} \frac{2}{\sqrt{3}}(2y+x-1)\sqrt{3}\, dx\, dy$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x} (2y+x-1)\, dy\, dx$$

$$= 2\int_0^1 (1-x)^2 + (x-1)(1-x)\, dx$$

$$= 0.$$

Ex 3. Let $\vec{F} = -y\hat{i} + x\hat{j} - xyz\hat{k}$ and let S be the part of the cone $z = \sqrt{x^2+y^2}$ for $x^2+y^2 \le 9$. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ or $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$, whichever appear easier. Here $\hat{n}$ is the inner normal vector.

[Hint : $C: x^2+y^2 = 9,\ z = 3$

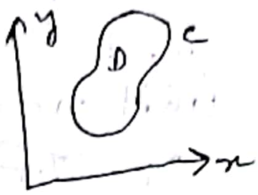
$x = 3\cos t,\ y = 3\sin t,\ z = 3.$ $\oint \vec{F} \cdot d\vec{r}$, $t = 0$ to 2π

Ans $= 18\pi$]

Divergence theorem

(Volume integrals $\leftrightarrow$ surface integrals)

recall Green's theorem,



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot\hat{k}\, dA$$

its generalization in space

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds \quad \text{(Stoke's theorem)}$$

Divergence theorem

Define a vector field $\vec{F} = F_2(x,y)\hat{i} - F_1(x,y)\hat{j}$

$$\text{Div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Differential element along tangent to C:

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}, \qquad \vec{r} = x\hat{i} + y\hat{j}$$

$$|d\vec{r}| = \sqrt{dx^2+dy^2} = ds$$

Unit tangent vector to C:

$$\left(\frac{d\vec{r}}{ds}\right) = \hat{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}.$$

Unit normal to C: $\hat{n} = \dfrac{dy}{ds}\hat{i} - \dfrac{dx}{ds}\hat{j}$

$$F_1\,dx + F_2\,dy = \left(F_1\frac{dx}{ds} + F_2\frac{dy}{ds}\right) ds$$

$$= \vec{F}\cdot\hat{n}\, ds$$

Green's theorem: $\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \nabla\cdot\vec{F}\, dA}$.

$$\left(\oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA\right)$$

$\hookrightarrow$ Green's Theorem)

Divergence theorem (Generalization of Green's theorem)

Replace the closed curve C $\to$ a closed surface S in 3D

Replace the bounding domain D $\to$ the bounding volume M.

Vector field $\vec{F}(x,y) \to$ vector field $\vec{F}(x,y,z)$

$$\iint_S F\cdot\hat{n}\, d\sigma = \iiint_M \nabla\cdot\vec{F}\, dV$$

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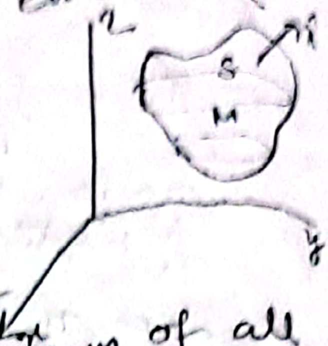
Divergence theorem: The flux of a vector field $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ across a closed surface S, in the direction of the surface's outward unit normal field $\hat{n}$ equals the integral of $\nabla\cdot\vec{F}$ over the region M enclosed by the surface



$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_M \nabla\cdot\vec{F}\,dV$$

Intuitavely it states that sum of all Sum of all sources minus sum of all sinks gives the net flow of the region.

Ex 1. Verify Divergence theorem for the field $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the sphere $x^2+y^2+z^2 = 9$.

Solⁿ. $\hat{n} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$ $\because \nabla\vec{F} = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$|\nabla\vec{F}| = \sqrt{4x^2+4y^2+4z^2}$

$= \sqrt{4\times 9} = 6.$

$\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{6} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$

$\Rightarrow \vec{F}\cdot\hat{n} = \frac{1}{3}(x^2+y^2+z^2) = \frac{1}{3}\cdot 9 = 3.$

$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S 3\,d\sigma = 3\iint_S d\sigma$

$= 3\cdot(4\pi\cdot 3^2) = 108\pi.$

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 3.$

$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 3\,dV = 3\times\frac{4}{3}\pi\cdot 3^3$

$= 108\pi$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V \nabla\cdot\vec{F}\,dV.$

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Ex 2 Find the flux of $\vec{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ outward through the surface of the cube from the first octant by the planes $x = 2, y = 2, z = 2$.

Sol^n $\vec{\nabla}\cdot\vec{F} = y + z + x$.

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV \quad \text{(Divergence theorem)}$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2}(x+y+z)\,dz\,dy\,dx$$

$$= 24.$$

Ex 3 If V is the volume enclosed by a closed surface S and $\vec{F} = 3x\hat{i} + 2y\hat{j} + z\hat{k}$, show that $\iint_S \vec{F}\cdot\hat{n}\,ds = 6V$.

Sol^n $\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z)$

$$= 3 + 2 + 1 = 6$$

By Gauss divergence theorem, $\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV$

$$= 6\iiint_D dV$$

$$= 6V.$$

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Ex 4. Evaluate $\iint_S \left((x^3 - yz)\hat{i} - 2x^2y\,\hat{j} + 2\hat{k}\right)\hat{n}\cdot d\sigma$ where S denotes the surface of the cube bounded by the planes $x=0$, $x=3$, $y=0$, $y=3$, $z=0$, $z=3$.

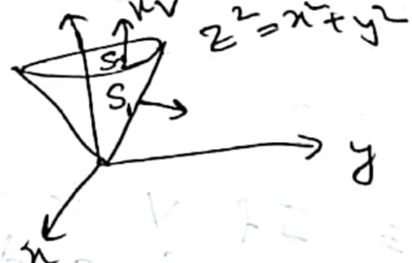
$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(2z)$$
$$= 3x^2 - 2x^2 + 0 = x^2$$

By Gauss divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \iiint_D x^2\,dx\,dy\,dz$$
$$= \int_{x=0}^{3}\int_{y=0}^{3}\int_{z=0}^{3} x^2\,dz\,dy\,dx$$
$$= 9\cdot\frac{x^3}{3}\Big|_0^3 = 81.$$

Ex 5. Let S be given by the cone $z=\sqrt{x^2+y^2}$ for $x^2+y^2 \le 1$ together with the disk $x^2+y^2\le 1$, $z=1$. For $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, verify the divergence theorem.

~~sol~~ Divergence th : $\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$



Sol^n. Let S_2 : $x^2+y^2\le 1$, $z=1$

Surface integral :

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{S_1}\vec{F}\cdot\hat{n}\,d\sigma + \iint_{S_2}\vec{F}\cdot\hat{n}\,d\sigma$$

For S_1, $\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}$

$$\vec{F}\cdot\hat{n} = \frac{1}{\sqrt{2}z}(x^2+y^2-z^2) = \frac{1}{\sqrt{2}z}\cdot 0 = 0$$

For S_2, $x^2+y^2 \le 1$; $z=1$. $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{n} = \hat{k}$. $\vec{F}\cdot\hat{n} = z$.

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = 0 + \iint_{S_2} z\, d\sigma = \iint_{S_2} 1 \cdot d\sigma = \pi \cdot 1^2 = \pi.$$

Volume integral $\iiint_M \nabla\cdot\vec{F}\, dV = 3\iiint_M dV$

(Volume of a cone with height h and radius r $= \pi r^2 \cdot \frac{h}{3}$)

Here $h=1$, $r=1$. $= 3\cdot\pi\cdot 1\cdot\frac{1}{3} = \pi$.