Ex $f(x,y) = |x| + |y|$

Show that $f_x(0,0)$ and $f_y(0,0)$ donot exist but $f(x,y)$ has minimum value at $(0,0)$.

$\rightarrow$ $\lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ does not exist

$= 1$ if $h \to 0^+$

-1 if $h \to 0^-$

$f_x(0,0)$ and $f_y(0,0)$ donot exist.

$f(x,y) - f(0,0) = |x| + |y| \geq 0$ for every point $(x,y) \in N(0,0)$.

$\Rightarrow f(x,y) \geq f(0,0) \quad \forall (x,y) \in N(0,0)$.

So f has a minimum value at $(0,0)$.

Def$^{\underline{n}}$ Stationary point $\rightarrow$ $f : S (\subseteq \mathbb{R}^2) \rightarrow \mathbb{R}$, (a,b) be an interior point of S. (a,b) is called stationary point of f in S if both $f_x(a,b)$ and $f_y(a,b)$ exist and $f_x(a,b) = 0 = f_y(a,b)$.

Ex Find all the stationary points of

$f(x,y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$.

$f_x = 8x - y + 3x^2y + y^3 \longrightarrow (1)$

$f_y = -x + 8y + x^3 + 3xy^2 \longrightarrow (2)$

$f_x = 0 = f_y \Rightarrow$ $(2)-(1)$ $(x-y)^3 - 9(x-y) = 0$.

$\Rightarrow (x-y)\{(x-y)^2 - 9\} = 0$

Either $x - y = 0 \Rightarrow x = y$.

or $(x-y+3)(x-y-3) = 0$

$\Rightarrow x - y = -3$ or $x - y = 3$

If $x = y$, $f_x = 8x - x + 3x^3 + x^3 = 0$

$\Rightarrow 4x^3 + 7x = 0$

$\Rightarrow x(4x^2+7) = 0 \Rightarrow x = 0, y = 0$

(0, 0) is a stationary point

If $x = y-3$,

$f_x = 8(y-3) - y + 3(y-3)^2 y + y^3 = 0$

$\Rightarrow 4y^3 - 18y^2 + 34y - 24 = 0$

$y = 3/2 \Rightarrow x = -3/2$.

If $x = y+3$

$4y^3 + 18y^2 + 34y + 24 = 0$

$\Rightarrow y = -3/2 \quad x = 3/2$.

$(0,0)$, $(-3/2, 3/2)$, $(3/2, -3/2)$ are stationary points of function.

Def^n An interior point (a,b) of fn $f(x,y)$ is said to saddle point of f if it is a stationary point of f i.e. $\left.\begin{array}{l} f_x(a,b)=0 \\ f_y(a,b)=0 \end{array}\right\}$ but f has neither minima nor maxima value.

Ex Show that $(0,0)$ is a saddle point of

$f(x,y) = \underline{x^6 + (x-y)^3}$.

$f_x(x,y) = 6x^5 + 3(x-y)^2$

$f_y(x,y) = -3(x-y)^2$

$f_x(0,0) = 0 = f_y(0,0)$.

In the $N(0,0)$, there are points where $x > y$ or $x < y \Rightarrow (x-y) > 0$ for some points in the $N(0,0)$

or $(x-y) < 0$ for some points in the $N(0,0)$.

$\underline{(x-y)^3} > 0$ in some points of $N(0,0)$

or < 0 in some points of $N(0,0)$

$x^6 + (x-y)^3 > 0$ in some point $N(0,0)$
or < 0 in " " "
$f(x,y) - f(0,0)$ changes its sign in $N(0,0)$.
neither minima or neither maxima will exist at $(0,0)$.

Ex $f(x,y) = (x-y)^3 + (2-x)^2$
Show that function has a saddle point at $(2,2)$.

$f_x(2,2) = 0 = f_y(2,2)$

$f(2+h, 2+k) - f(2,2) = (h-k)^3 + h^2$

$h=0,\ k>0, \quad < 0.$

$h>0,\ k=0 \quad > 0$

Working procedure to find the extremums.

① Find the stationary points of $f(x,y)$

$f_x(a,b) = 0, \quad f_y(a,b) = 0.$

② Find $f_{xx}(a,b)$, $f_{xy}(a,b)$, $f_{yy}(a,b)$

Consider $H = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}$

$\det H = \underset{A}{f_{xx}(a,b)}\ \underset{C}{f_{yy}(a,b)} - [\underset{B}{f_{xy}(a,b)}]^2$

$= AC - B^2$

③ If $AC-B^2 < 0$, then f has neither minima nor maxima

If $AC-B^2 = 0$, No conclusion

If $AC-B^2 > 0$, Extrema will exist

If $A > 0$, minima will exist

$A < 0$, maxima will exist.

Ex. $f(x,y) = x^4 + y^4 - 2x^2$. Show that f has a local minima at $(-1,0)$ and $(1,0)$. $(0,0)$ is a saddle point.

$f_x = 4x^3 - 4x \Rightarrow f_x = 0 \Rightarrow 4x(x^2-1)=0$
$\Rightarrow x = 0$ or $x = \pm 1$.

$f_y = 4y^3 \Rightarrow f_y = 0$
$\Rightarrow y = 0$.

$(0,0)$, $\underline{(1,0)}$, $\underline{(-1,0)}$ all are stationary points.

$f_{xx} = 12x^2 - 4 \qquad f_{yy} = 12y^2$

$f_{xy} = 0$

At $(0,0)$

$\underbrace{f_{xx}(0,0)}_{A} = -4 \qquad \underbrace{f_{yy}(0,0)}_{C} = 0$.

$\underbrace{f_{xy}(0,0)}_{B} = 0$.

$H = (-4)\cdot 0 - 0 = 0 \Rightarrow$ No conclusion can be drawn

$f(h,k) - f(0,0) = h^4 + k^4 - 2h^2$
$= h^2(h^2-2) + k^4$.

In $N(0,0)$ when $0 < |h| < \sqrt{2}$, $k = 0$, $f(h,k) - f(0,0) < 0$.

For $h = 0$, $k \neq 0$, $f(h,k) - f(0,0) > 0$

$f(h,k) - f(0,0)$ changes its sign in $N(0,0)$.

So, f has no extreme value at $(0,0)$.

$(0,0)$ is saddle point

At $(1,0)$

$A = f_{xx}(1,0) = 12\cdot 1^2 - 4 = 8 > 0$.

$B = f_{xy}(1,0) = 0 \qquad H = AC - B^2 = 8\cdot 0 - 0 = 0$

$C = f_{yy}(1,0) = 0$

$f(1+h,k) - f(1,0) = (1+h)^4 + k^2 - 2(1+h)^2 + 1$
$= [(1+h)^2 - 1]^2 + k^2 > 0$

f has minimum value at $(1, 0)$.

At $(-1, 0)$

$A = f_{xx}(-1, 0) = 12 \cdot (-1)^2 - 4 = 8$

$B = f_{xy}(-1, 0) = 0$

$C = f_{yy}(-1, 0) = 0$.

$H = AC - B^2 = 0$.

To prove $f(-1+h, 0+k) - f(-1, 0) > 0$ in $N(-1, 0)$

Conservative field

Ex 1. If the vector field $\vec{F} = (2x+y)\hat{i} + x\hat{j} + 2z\hat{k}$ is conservative, find the potential function.

$\rightarrow$ $\vec{F}$ is conservative if it can be written as $\vec{F} = \vec{\nabla f} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

$(2x+y)\hat{i} + x\hat{j} + 2z\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

$\frac{\partial f}{\partial x} = 2x+y$ —(1), $\quad \frac{\partial f}{\partial y} = x$, —(2) $\quad \frac{\partial f}{\partial z} = 2z$ —(3)

From (1)

Integrate (1) w.r.t x,

$f = \int (2x+y)\,dx + \phi(y, z)$ —(4)

$= x^2 + xy + \phi(y, z)$

$\frac{\partial f}{\partial y} = x + \frac{\partial \phi}{\partial y}$

$x = x + \frac{\partial \phi}{\partial y}$

$\Rightarrow \frac{\partial \phi}{\partial y} = 0 \Rightarrow \phi$ is independent of y

$\Rightarrow \phi \equiv \phi(z)$.

From (4)

$f = x^2 + xy + \phi(z)$ —(5)

$\frac{\partial f}{\partial z} = \phi'(z) \quad \Rightarrow \phi'(z) = 2z$

$\Rightarrow \phi(z) = z^2 + c$

$$f(x,y,z) = x^2 + xy + z^2 + C.$$

Green's theorem

Ex Verify Green's theorem for the vector field $\vec{F}(x,y) = (x-y)\hat{\imath} + x\hat{\jmath}$. The region R is bounded by the circle $C: \vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath}$, $0 \le t \le 2\pi$

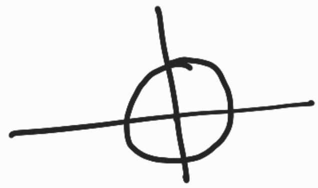
$\rightarrow$ $F_1 = (x-y) \Rightarrow \dfrac{\partial F_1}{\partial y} = -1$

$F_2 = x \Rightarrow \dfrac{\partial F_2}{\partial x} = 1.$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = 2\iint_R dx\,dy$$

$$= 2\cdot\pi\cdot 1^2$$

$$= 2\pi.$$



$$=$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} [(\cos t - \sin t)\hat{\imath} + \cos t\,\hat{\jmath}]\cdot[-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}]\,dt$$

$$= \int_0^{2\pi} [-\sin t(\cos t - \sin t) + \cos^2 t]\,dt$$

$$= \int_0^{2\pi} (-\sin t\cos t + 1)\,dt$$

$$= 2\pi - \frac{1}{2}\int_0^{2\pi} \sin 2t\,dt$$

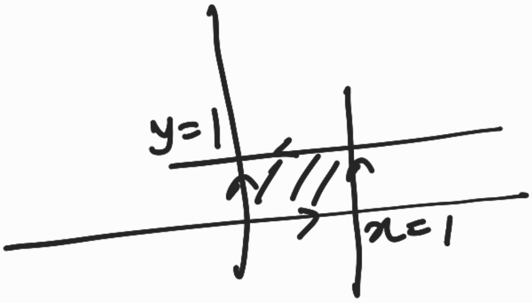
$$= 2\pi - \frac{1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= 2\pi.$$

2. Evaluate the integral $\oint_C (xy\,dy - y^2\,dx)$ using Green's theorem. Here C is the square cut from the first quadrant by the lines $x=1$ and $y=1$.

$$\oint_C (xy\,dy - y^2\,dx)$$

$$= \oint_C (F_1\,dx + F_2\,dy)$$



$F_1 = -y^2, \quad F_2 = xy.$

$$\frac{\partial F_1}{\partial y} = -2y \qquad \frac{\partial F_2}{\partial x} = y.$$

Using Green's theorem

$$\oint (xy\,dy - y^2\,dx) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1} (y + 2y)\,dx\,dy$$

$$= \int_{y=0}^{1} 3y\,dy = \frac{3}{2}\left[y^2\right]_0^1$$

$$= \frac{3}{2}.$$

Surface Integral

Evaluation of arc length of a curve $y = f(x)$.

$$\text{Arc length, } L = \int_a^b \sqrt{1 + f'(x)^2}\,dx.$$

Evaluation of surface area

$f(x, y, z) = c$ be eqn of surface

The area of the surface $f(x, y, z) = c$ over a bounded and closed plane R

$$\boxed{S = \iint_R \frac{|\nabla \vec{f}|}{|\nabla \vec{f} \cdot \vec{p}|}\,dA}$$

, R is the projection of S on the xy, yz or zx plane.

$\vec{p}$ is the unit normal to R and $\nabla \vec{f} \cdot \vec{p} \neq 0$.

Remark : Let $z = g(x,y)$ be the equation of a surface. Then the surface area :

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx dy.$$

where R is the projection of the surface in x-y plane.

Ex Find the area of the cap cut from the hemisphere $x^2+y^2+z^2=2$, $z \geq 0$ by the cylinder $x^2+y^2=1$.

Sol^n Projection of the surface $f(x,y,z)=c$ i.e. $x^2+y^2+z^2=2$ onto the xy plane: $x^2+y^2 \leq 1$

Now $f(x,y,z) = x^2+y^2+z^2$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

$$|\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2}$$
$$= 2\sqrt{2}.$$

The vector $\vec{p} = \hat{k}$ is normal to the xy-plane

$$|\nabla \vec{f} \cdot \vec{p}| = |(2x\hat{i} + 2y\hat{j} + 2z\hat{k}) \cdot \hat{k}|$$
$$= |2z| = 2z, \quad z \geq 0.$$

$$S = \iint_R \frac{\nabla \vec{f}}{|\nabla \vec{f} \cdot \vec{p}|}\, dA = \iint_R \frac{2\sqrt{2}}{2z}\, dA.$$
$$= \sqrt{2} \iint_R \frac{1}{z}\, dA.$$

$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{1}{\sqrt{2-r^2}}\, r dr d\theta$$

$x^2+y^2=r^2$

$dA = r\, dr d\theta.$

$r=0$ to $r=1$

θ varies from 0 to 2π

$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{r}{\sqrt{2-r^2}}\, dr\, d\theta$$

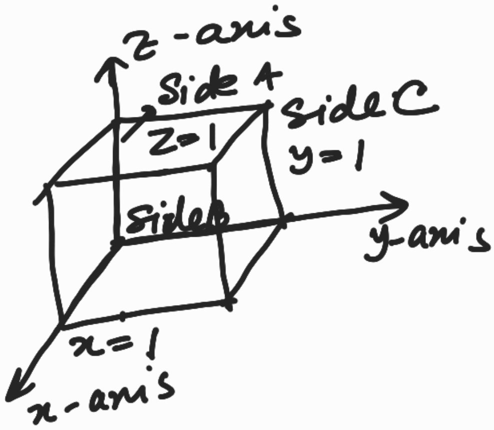
$$= \sqrt{2} \int_{\theta=0}^{2\pi} \left[-\sqrt{2-r^2}\right]_0^1 d\theta$$

$$= \sqrt{2} \int_0^{2\pi} (\sqrt{2}-1)\, d\theta$$

$$= 2\sqrt{2}\pi(\sqrt{2}-1) = \underline{2\pi(2-\sqrt{2})} \text{ Ans}$$

2. Integrate $g(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$ and $z=1$.

$\rightarrow$ The integral over the surface of the cube reduces to



$$\iint_{\substack{\text{cube}\\\text{surface}}} xyz\, d\sigma = \iint_{\text{Side A}} xyz\, d\sigma + \iint_{\text{Side B}} xyz\, d\sigma + \iint_{\text{side C}} xyz\, d\sigma.$$

Side A is the surface $f(x,y,z) = z-1$ over the region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$ in the xy plane.

$\hat{p} = \hat{k}$. $\vec{\nabla f} = \hat{k}$. $|\vec{\nabla f}| = 1$.

$$d\sigma = \frac{|\vec{\nabla f}|}{|\vec{\nabla f} \cdot \vec{p}|} dA$$

$$\iint_{\text{Side A}} xyz\, d\sigma = \int_{x=0}^{1} \int_{y=0}^{1} xy \cdot 1 \cdot 1 \cdot dx\, dy \qquad d\sigma = dx\,dy$$

$$= \frac{1}{4}.$$

Calculate $\iint_{\text{Side B}} xyz\, d\sigma$, $\iint_{\text{Side C}} xyz\, d\sigma$.

$$\iint_{\substack{\text{cubic}\\\text{surface}}} xyz\, d\sigma = 3/4$$

Flux of a vector field $\vec{F}$ through a surface S.

The flux of a vector field $\vec{F}$ across an orientable surface S in the direction of $\hat{n}$ (unit normal vector to surface S) is given by

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, d\sigma.$$

Geometrically a flux integral is the surface integral over S of normal components of $\vec{F}$.

Evaluation of Flux integral $\iint_S \vec{F}\cdot\hat{n}\, d\sigma$.

Suppose S is a part of a level surface $f(x,y,z)=c$, $\hat{n}$ may be taken by

$$\hat{n} = \pm\frac{\vec{\nabla f}}{|\vec{\nabla f}|}$$

$$\text{Flux} = \pm\iint_R \vec{F}\cdot\left(\frac{\vec{\nabla f}}{|\vec{\nabla f}|}\right)\frac{|\vec{\nabla f}|}{|\vec{\nabla f}\cdot\vec{p}|}\, dA$$

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \pm\iint_R \vec{F}\cdot\frac{\vec{\nabla f}}{|\vec{\nabla f}\cdot\vec{p}|}\, dA.}$$

__Ex__ Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2+z^2=1$, $z>0$ by the planes $x=0$ and $x=1$.

$\rightarrow$ Surface $f(x,y,z)=c$.

$y^2+z^2=1$, $z>0$

$$\hat{n} = \frac{\vec{\nabla f}}{|\vec{\nabla f}|} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \frac{2y\hat{j}+2z\hat{k}}{\sqrt{4\cdot 1}}$$

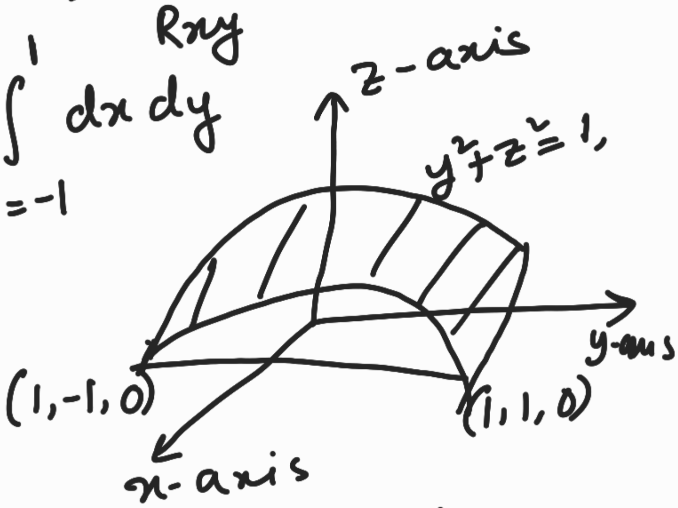
$$= y\hat{j}+z\hat{k}.$$

$\hat{p} = \hat{k}$

$$d\sigma = \frac{|\nabla f|}{|\nabla \vec{f} \cdot \vec{p}\,|}\, dA = \frac{2}{|2z|}\, dA$$
$$= \frac{1}{z}\, dA$$

Also $\vec{F} \cdot \hat{n} = (yz\hat{j} + z^2\hat{k}) \cdot (y\hat{j} + z\hat{k})$
$$= y^2 z + z^3 = z(y^2+z^2) = z$$

Flux through $S = \iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iint_{R_{xy}} z \cdot \frac{1}{z}\, dA$
$$= \iint_{R_{xy}} dA$$
$$= \int_{x=0}^{1} \int_{y=-1}^{1} dx\, dy$$
$$= 2 \text{ unit}.$$



2. Evaluate the integral $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$ where $\vec{F} = 6z\hat{i} + 6\hat{j} + 3y\hat{k}$ and S is the portion of the plane $2x+3y+4z=12$ which is in the first octant.

→ $f(x,y,z) := 2x+3y+4z = 12 \Rightarrow z = \dfrac{12-2x-3y}{4}$

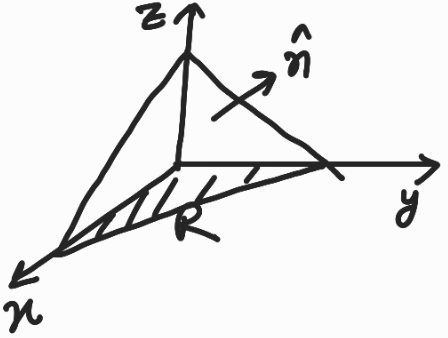
$\vec{\nabla f} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

$$\hat{n} = \frac{\vec{\nabla f}}{|\vec{\nabla f}|} = \frac{2\hat{i} + 3\hat{j} + 4\hat{k}}{\sqrt{2^2+3^2+4^2}} = \frac{1}{\sqrt{29}}(2\hat{i}+3\hat{j}+4\hat{k})$$

$$\vec{F} \cdot \hat{n} = \frac{1}{\sqrt{29}}(6z\hat{i} + 6\hat{j} + 3y\hat{k}) \cdot (2\hat{i}+3\hat{j}+4\hat{k})$$
$$= \frac{1}{\sqrt{29}}(12z + 18 + 12y). \qquad \hat{p} = \hat{k}$$

$$d\sigma = \frac{|\vec{\nabla f}|}{|\vec{\nabla f} \cdot \hat{k}|}\, dA = \frac{\sqrt{29}}{4}\, dA.$$

Projecting S in the xy-plane
Projection R will be bounded by
x-axis, y-axis and
$$2x + 3y = 12$$

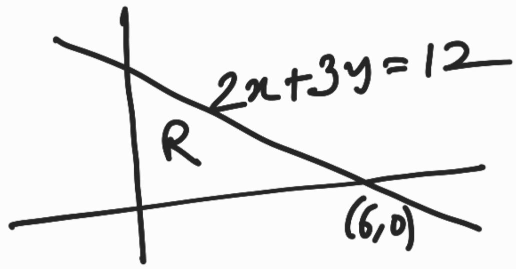


$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iint_{R_{xy}} \frac{1}{\sqrt{29}} \left(\frac{12(12-2x-3y)}{4} + 18 + 12y \right) \cdot \frac{\sqrt{29}}{4}\, dA$$

$$= \frac{1}{4} \iint_{R_{xy}} (54 - 6x + 3y)\, dA$$

$$= \frac{1}{4} \int_{x=0}^{6} \int_{y=0}^{(12-2x)/3} (54 - 6x + 3y)\, dy\, dx$$

$$= 138 \text{ unit}.$$



Ex Evaluate the surface integral $\iint_S \vec{F}\cdot\hat{n}\, d\sigma$
where $\vec{F} = z^2\,\hat{i} + xy\,\hat{j} - y^2\,\hat{k}$ and S is
the portion of the surface of the cylinder
$x^2 + y^2 = 36$, $0 \le z \le 4$ included in first
octant. (Hint: Take projection in y-z plane)

Stoke's theorem $\rightarrow$ Let C be a closed curve
in 3D space which forms the boundary
of a surface S whose unit normal vector
is $\hat{n}$.
Then for a continuous differentiable vector
field $\vec{F}$, we have
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$
where

the direction of the line integral is around C and the normal $\hat{n}$ are oriented in right handed sense.

If $\vec{\nabla} \times \vec{F} = \vec{0}$, then Stoke's theorem says

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

<u>Ex</u> Verify Stoke's theorem for the hemisphere $S: x^2+y^2+z^2 = 9,\ z \geq 0$, its boundary $C: x^2+y^2 = 9,\ z = 0$ and the field $\vec{F} = y\hat{i} - x\hat{j}$.

$\rightarrow$ $C: x^2+y^2 = 9,\ z = 0$. $\quad x = r\cos\theta$. $\quad y = r\sin\theta$.

$$\vec{r}(\theta) = 3\cos\theta\,\hat{i} + 3\sin\theta\,\hat{j}$$

$$d\vec{r} = (-3\sin\theta\,\hat{i} + 3\cos\theta\,\hat{j})\,d\theta$$

$$\vec{F} = 3\sin\theta\,\hat{i} - 3\cos\theta\,\hat{j}.$$

$$\vec{F} \cdot d\vec{r} = -(9\sin^2\theta + 9\sin^2\theta)\,d\theta$$

$$= -9\,d\theta.$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} -9\,d\theta = -18\pi.$$

$S: x^2+y^2+z^2 = 9,\ z \geq 0.$ $\quad f(x,y,z) = x^2+y^2+z^2$

$\vec{F} = y\hat{i} - x\hat{j}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}$$

$$= \hat{i}.0 + \hat{j}.0 + \hat{k}(-1-1)$$

$$= -2\hat{k}.$$

$$\hat{n} = \frac{\vec{\nabla f}}{|\nabla f|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4 \times 9}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{3}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds$$

$$= \iint_{x^2+y^2 \le 9} -2\hat{k} \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{3} \frac{|\nabla \vec{f}|}{|\nabla \vec{f} \cdot \vec{p}|}\, dx\,dy$$

$\vec{p} = \hat{k}$

$$= \iint_{x^2+y^2 \le 9} \left(-\frac{2z}{3}\right) \frac{6}{2z}\, dx\,dy$$

$$= -2 \iint_{x^2+y^2 \le 9} dx\,dy \quad = -2 \cdot \pi \cdot 3^2$$

$$= -18\pi.$$

<u>Ex</u> Verify Stoke's theorem for function $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface $x + y + z = 1$ lying in the first quadrant.

Module-5

Laplace Transform

The Laplace transform of the function $F(t)$, which will be denoted by $L\{F(t)\}$ or $f(p)$ is defined as

$$L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt, \quad p > 0$$

p is called the transform parameter.

Let us now find the Laplace transforms of some elementary function from definition.

(i) If $F(t) = c$, then

$$L\{c\} = \int_0^\infty c \cdot e^{-pt}\,dt = \lim_{M\to\infty}\left[-\frac{ce^{-pt}}{p}\right]_0^M$$

$$= \frac{c}{p}, \quad p > 0.$$

(ii) If $F(t) = t^2$, then

$$L\{t^2\} = \int_0^\infty e^{-pt}\,t^2\,dt$$

$$= \lim_{M\to\infty}\left[-\frac{e^{-pt}\cdot t^2}{p}\right]_0^M + \int_0^\infty 2t \cdot \frac{e^{-pt}}{p}\,dt$$

$$= \frac{2}{p}\lim_{M_1\to\infty}\left[-\frac{e^{-pt}}{p}\,t\right]_0^{M_1} + \frac{2}{p}\int_0^\infty \frac{e^{-pt}}{p}\,dt$$

$$= \frac{2}{p^2}\lim_{M_2\to\infty}\left[\frac{e^{-pt}}{p}\right]_0^{M_2} = \frac{2}{p^3}, \quad p > 0.$$

$$L\{t^2\} = \frac{2}{p^3}, \quad p > 0.$$

$$\boxed{L\{t^n\} = \frac{n!}{p^{n+1}}}, \quad n \text{ is a +ve integer.}$$

(iii) If $F(t) = e^{\alpha t}$, then

$$L\{e^{\alpha t}\} = \int_0^\infty e^{-pt}e^{\alpha t}\,dt = \int_0^\infty e^{-(p-\alpha)t}\,dt$$

$$= \lim_{M\to\infty}\left[-\frac{e^{-(p-\alpha)}}{p-\alpha}\right]_0^M$$

$$= \frac{1}{p-\alpha}, \quad p > \alpha.$$

(iv) $$L\{\cosh\alpha t\} = L\left\{\tfrac{1}{2}\left(e^{-\alpha t} + e^{\alpha t}\right)\right\}$$

$$= \frac{1}{2}\left(\frac{1}{p+\alpha} + \frac{1}{p-\alpha}\right)$$

$$= \frac{p}{p^2 - \alpha^2}, \quad p > |\alpha|.$$

(v) $L\{\sinh \alpha t\} = L\{\frac{1}{2}(e^{\alpha t} - e^{-\alpha t})\}$

$= \frac{\alpha}{p^2-\alpha^2}, \quad p > |\alpha|.$

(vi) If $F(t) = \cos \alpha t$, then

$L\{\cos \alpha t\} = \int_0^\infty e^{-pt} \cos \alpha t \, dt$

$= \lim_{M\to\infty} \left[\frac{e^{-pt}}{p^2+\alpha^2} (-p\cos \alpha t + \alpha \sin \alpha t) \right]_0^M$

$= \frac{p}{p^2+\alpha^2}, \quad p > 0.$

Similarly $L\{\sin \alpha t\} = \frac{a}{p^2+\alpha^2}.$

Ex Find the Laplace transforms of $f(t)$ defined as

$$f(t) = \begin{cases} t/k, & \text{when } 0<t<k \\ 1, & \text{when } t>k. \end{cases}$$

$L\{f(t)\} = \int_0^k \frac{t}{k} e^{-pt} dt + \int_k^\infty 1 . e^{-pt} dt$

$= \frac{1}{k} \left[t\frac{e^{-pt}}{-p} \Big|_0^k - \int_0^k \frac{e^{-pt}}{-p} dt \right] + \left[\frac{e^{-pt}}{-p}\right]_k^\infty$

$= \frac{1}{k} \left[\frac{k e^{-pk}}{-p} - \left(\frac{e^{-pt}}{p^2}\right)_0^k \right] + \frac{e^{-kp}}{p}$

$= \frac{e^{-pk}}{-p} - \frac{e^{-pk}}{p^2} + \frac{1}{p^2} + \frac{e^{-kp}}{p}$

$= \frac{1}{p^2}(1-e^{-kp}).$

Inverse of Laplace Transforms

It is defined by $L^{-1}\{f(p)\} = F(t)$, if $f(p)$ is the Laplace transform of $F(t)$

According to the definition of inverse of Laplace transforms, we can state

(a) $L^{-1}\left(\frac{1}{p}\right) = 1$ since $L\{1\} = \frac{1}{p}$.

(b) $L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{n!}$ since $L\{t^n\} = \frac{n!}{p^{n+1}}$

(c) $L^{-1}\left(\frac{1}{p-\alpha}\right) = e^{\alpha t}$, since $L\{e^{\alpha t}\} = \frac{1}{p-\alpha}$.

(d) $L^{-1}\left(\frac{1}{p^2+\alpha^2}\right) = \frac{1}{\alpha}\sin\alpha t$ since $L\{\sin\alpha t\} = \frac{\alpha}{p^2+\alpha^2}$

(e) $L^{-1}\left(\frac{p}{p^2+\alpha^2}\right) = \cos\alpha t$ since $L\{\cos\alpha t\} = \frac{p}{p^2+\alpha^2}$

(f) $L^{-1}\left(\frac{1}{p^2-\alpha^2}\right) = \frac{1}{\alpha}\sinh\alpha t$ since $L\{\sinh\alpha t\} = \frac{\alpha}{p^2-\alpha^2}$

(g) $L^{-1}\left(\frac{p}{p^2-\alpha^2}\right) = \cosh\alpha t$ since $L\{\cosh\alpha t\} = \frac{p}{p^2-\alpha^2}$

Some properties of Laplace Transform:

(a) Linearity property:

We have $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

If a_1, a_2 are constants, then

$L\{a_1F_1(t) + a_2F_2(t)\} = a_1L\{F_1(t)\} + a_2L\{F_2(t)\}$.

(b) First Shifting property

i) If $L\{F(t)\} = f(p)$, then $L\{e^{at}F(t)\} = f(p-a)$.

ii) If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(p-a)\} = e^{at}F(t) = e^{at}L^{-1}\{f(p)\}.$$

(c) Change of scale property

i) If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a}f\left(\frac{p}{a}\right)$.

ii) Similarly if $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(ap)\} = \frac{1}{a}F\left(\frac{t}{a}\right).$$

Laplace transform of derivatives

i) $L\{F'(t)\} = pL\{F(t)\} - F(0)$.

ii) $L\{F''(t)\} = p^2L\{F(t)\} - pF(0) - F'(0)$.

iii) $L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - \dots - pF^{n-2}(0) - F^{n-1}(0)$

In case of inverse Laplace Transforms

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^n(p)\} = L^{-1}\left\{\frac{d^n}{dp^n}(f(p))\right\} = (-1)^n t^n F(t),\ n = 1, 2, \dots$$

Laplace Transform on Integrals

$$L\left\{\int_0^t F(\tau)\,d\tau\right\} = \frac{f(p)}{p} = \frac{1}{p}L\{F(t)\}$$

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\int_0^\infty f(x)\,dx\right\} = \frac{F(t)}{t}.$$

Convolution Theorem →

Let the two functions $F(t)$ and $G(t)$ be defined. Then $F*G$ be defined by

$$F*G = \int_0^t F(x)\,G(t-x)\,dx$$

is called the convolution of the function $F(t)$ and $G(t)$.

Theorem If $f(p)$ and $g(p)$ be the Laplace transforms of $F(t)$ and $G(t)$ resp, then the Laplace transform of the convolution $F*G$ is the product of $f(p)\,g(p)$. $L^{-1}\{f(p)\,g(p)\} = F*G$.

Ex ① Find $L\{\sin^2 at\}$

→ $L\{\sin^2 at\} = L\left\{\frac{1}{2}(1-\cos 2at)\right\}$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2at\}$$

$$= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4a^2}\right), \quad p>0.$$

$$= \frac{2a^2}{p(p^2+4a^2)}$$

② If $L\{F(t)\} = \dfrac{p^2 - p + 1}{(2p+1)(p-1)}$, find $L\{F(2t)\}$ applying the change of scale property.

$\rightarrow$ If $L\{F(t)\} = f(p)$.

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{p}{2}\right)$$

Here $f(p) = \dfrac{p^2 - p + 1}{(2p+1)(p-1)}$

$$f\left(\frac{p}{2}\right) = \frac{\left(\frac{p}{2}\right)^2 - \frac{p}{2} + 1}{\left(2\cdot\frac{p}{2} + 1\right)\left(\frac{p}{2} - 1\right)}$$

$$L\{F(2t)\} = \frac{1}{2} \cdot 2 \frac{p^2/4 - p/2 + 1}{(p+1)(p-2)}$$

$$= \frac{p^2 - 2p + 4}{4(p+1)(p-2)}.$$

③ Find $L\{t\cos at\}$

We know $L\{\cos at\} = \dfrac{p}{p^2+a^2}$, $p > 0$.

$$L\{t\cos at\} = -\frac{d}{dp}\{\cos at\}$$

$$= -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right) = \frac{p^2 - a^2}{(p^2+a^2)^2}$$

④ Evaluate $L^{-1}\left(\dfrac{3p-2}{p^2 - 4p + 20}\right)$.

$$= L^{-1}\left\{\frac{3p-2}{p^2 - 4p + 4 + 16}\right\}$$

$$= L^{-1}\left\{\frac{3(p-2) + 4}{(p-2)^2 + 4^2}\right\}$$

$$= 3L^{-1}\left\{\frac{p-2}{(p-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(p-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{p}{p^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{p^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\sin 4t.$$

(5) Use convolution theorem, find

$$L^{-1}\left\{\frac{1}{(p-1)(p-2)}\right\}$$

$\rightarrow$ Since $L^{-1}\left\{\frac{1}{p-1}\right\} = e^t$, $\quad L^{-1}\left\{\frac{1}{p-2}\right\} = e^{2t}$

Using convolution theorem

$$L^{-1}\left\{\frac{1}{(p-1)(p-2)}\right\} = L^{-1}\left(\frac{1}{(p-1)}\cdot\frac{1}{(p-2)}\right)$$

$$= \int_0^t e^{x} e^{2(t-x)}\,dx$$

$$= \int_0^t e^{2t}e^{-x}\,dx$$

$$= e^{2t}\left[-e^{-x}\right]_0^t$$

$$= e^{2t} - e^{t}$$