Taylor's theorem for function of two independent variable :

If $f(x,y)$ admits of continuous partial derivatives upto nth order in some neighbourhood of point (a,b) and $(a+h, b+k)$ be any point of this neighbourhood, then there exists $\theta \in (0,1)$ such that

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$

Note → (I) As $d^r f = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f \ \forall r$,

Taylor's theorem can be written more compactly as

$$f(x+h, y+k) - f(x,y) = df(x,y) + \frac{1}{2!} d^2 f(x,y) + \cdots + \frac{1}{(n-1)!} d^{n-1} f(x,y) + R_n$$

where $R_n = \frac{1}{n!} d^n f(x+\theta h, y+\theta k)$ for some $\theta \in (0,1)$

(II) The importance of Taylor's theorem lies in the fact that the increament $f(x+h, y+k) - f(x,y)$ of f is split up into increaments of df, d^2f, ... of different orders.

(ii) Taylor's theorem can be mentioned in the following form also

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$
$$+ \cdots + \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n$$

where $R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n$

$f(a+\theta(x-a),\ b+\theta(y-b))$

for some $\theta \in (0,1)$.

Ex 1. Find the polynomial of second degree that best approximates $\sin x \sin y$ in neighbourhood of origin.

$\rightarrow$ $f(x,y) = \sin x \sin y$

$f_x = \cos x \sin y \quad f_y = \sin x \cos y$

$f_{xx} = -\sin x \sin y \quad f_{yy} = -\sin x \sin y$

$f_{yx} = \cos x \cos y \qquad f_{xy} = \cos x \cos y$

All are continuous in the neighbourhood of $(0,0)$.

$$df\Big|_{(0,0)} = \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)\Big|_{(0,0)}$$
$$= 0$$

$$d^2f\Big|_{(0,0)} = \left(x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial x \partial y} + yx\frac{\partial^2 f}{\partial y \partial x} + y^2\frac{\partial^2 f}{\partial y^2}\right)\Big|_{(0,0)}$$

$$= x^2 \cdot 0 + xy \cdot 1 + yx \cdot 1 + y^2 \cdot 0$$
$$= 2xy$$

So the required polynomial is

$$f(x,y) = f(0,0) + \frac{1}{1!} df\Big|_{(0,0)} + \frac{1}{2!} d^2f\Big|_{(0,0)}$$

$$= 0 + 0 + \frac{1}{2}\cdot 2xy = xy.$$

Ex. Compute Taylor's expansion of the function f defined by $f(x,y) = \sin x \sin y \sin(x+y)$ about the point $(0,0)$ including all terms of degree ≤ 2.

Example: Show that the expansion of $\sin xy$ in powers of $(x-1)$ and $(y-\pi/2)$ upto and including second degree terms is

$$1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{1}{2}\pi(x-1)(y-\tfrac{\pi}{2}) - \frac{1}{2}(y-\tfrac{\pi}{2})^2$$

→ $f(x,y) = \sin xy$. All partial derivatives are continuous in neighbourhood of $(1, \frac{\pi}{2})$.

$f_x = y\cos xy \qquad f_y = x\cos xy,$

$f_{xx} = -y^2 \sin xy \qquad f_{yy} = -x^2 \sin xy$

$f_{yx} = \cos xy - xy\sin xy = f_{xy}$

$$\left[(x-1)\frac{\partial}{\partial x} + (y-\tfrac{\pi}{2})\frac{\partial}{\partial y}\right] f\Big|_{(1,\pi/2)}$$

$$= (x-1)\frac{\pi}{2}\cos\frac{\pi}{2} + (y-\tfrac{\pi}{2})\cos\frac{\pi}{2} = 0$$

$$(x-1)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{(1,\frac{\pi}{2})} + 2(x-1)(y-\tfrac{\pi}{2})\frac{\partial^2 f}{\partial x\partial y}\Big|_{(1,\frac{\pi}{2})} + (y-\tfrac{\pi}{2})^2\frac{\partial^2 f}{\partial y^2}\Big|_{(1,\pi/2)}$$

$$= (x-1)^2 (-y^2\sin xy)\Big|_{(1,\frac{\pi}{2})} + (y-\tfrac{\pi}{2})^2(-x^2\sin xy)\Big|_{(1,\frac{\pi}{2})}$$

$$+ 2(x-1)(y-\tfrac{\pi}{2})(-xy\sin xy + \cos xy)\Big|_{(1,\pi/2)}$$

$$= (x-1)^2\left(-\frac{\pi^2}{4}\right) + (y-\tfrac{\pi}{2})^2(-1) +$$

$$2(x-1)(y-\tfrac{\pi}{2})\cdot\left(-\frac{\pi}{2}\right)$$

$$f(x,y) = f\left(1,\tfrac{\pi}{2}\right) + \frac{1}{1!}\left[(x-1)\frac{\partial}{\partial x} + \left(y-\tfrac{\pi}{2}\right)\frac{\partial}{\partial y}\right]f\Big|_{(1,\frac{\pi}{2})}$$
$$+ \frac{1}{2!}\left[(x-1)\frac{\partial}{\partial x} + \left(y-\tfrac{\pi}{2}\right)\frac{\partial}{\partial y}\right]^2 f\Big|_{(1,\frac{\pi}{2})}$$
$$= 1 - \frac{(x-1)^2\pi^2}{8} - \frac{\left(y-\frac{\pi}{2}\right)^2}{2} - \frac{(x-1)\left(y-\frac{\pi}{2}\right)\pi}{2}$$

[Ans]

Ex. Let $F(x,y) = \log(x+e^y)$. Expand according to Taylor's theorem in power of $(x-1)$ and y going far enough to include all terms of degree 2.

(Ans: $\log 2 + [(x-1)+y]\frac{1}{2} - \frac{1}{8}[(x-1)^2 + 2(x-1)y - y^2] + \cdots$)

Ex. Show that the expansion of $\cos xy$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$ upto and including second degree terms is

$$-\frac{\pi}{2}(x-1) - \left(y-\frac{\pi}{2}\right) - (x-1)\left(y-\frac{\pi}{2}\right) + \cdots$$

Ex. Let $F(x,y) = \sin(e^y + x^2 - 2)$. Expand according to Taylor's series in powers of $x-1$ and y upto second degree term.

(Ans. $[2(x-1)+y] + \frac{1}{2}[2(x-1)^2 + y^2] + \cdots$)

Extreme values of functions of two variable:

Definition: Let $S \subset R^2$ and $f: S \to R$ be a function of two independent variables x and y. Let (a, b) be an interior point of S. f is said to have a extreme value (a local extreme) value at (a, b) or equivalently $f(a, b)$ is said to be an extreme value of f, if there exists a suitable neighbourhood $N(a, b)$ of (a, b) such that for all $(x, y) \in N(a, b)$, $f(x, y) - f(a, b)$ does not change sign.

f is said to have a local maximum or local minimum value at (a, b) according as $f(x, y) - f(a, b) \leq$ or ≥ 0 for all $(x, y) \in N(a, b)$

f is said to have a global extremum at $(a, b) \in S$ if $f(x, y) - f(a, b)$ does not change sign in S; So f has a global maximum or global minimum at (a, b) according as $f(x, y) - f(a, b) \leq 0$ or ≥ 0 for all $(x, y) \in S$.

Necessary condition for existence of extreme value:

Let $S \subset R^2$, $f: S \to R$ and (a, b) be an interior point of S. If the partial derivatives f_x, f_y exist at (a, b) and f has an extreme value at (a, b), then $f_x(a, b) = 0$, $f_y(a, b) = 0$.

Remark: The vanishing of partial derivatives $f_x(a,b)$, $f_y(a,b)$ is only necessary condition for the existence of extreme values of a function provided f_x, f_y exist at (a,b). If atleast one f_x, f_y does not exist at (a,b), then also f may possess extreme value at (a,b).

Example 1. Show that the function $f(x,y) = |x| + |y|$, $(x,y) \in R^2$ posses an extreme value (minimum value) at $(0,0)$ although $f_x(0,0)$, $f_y(0,0)$ ~~may~~ do not exist.

$$\rightarrow \quad f_x(0,0) = \lim_{h \to 0} \frac{f(0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = 1 \text{ when } h \to 0^+$$
$$= -1, \text{ when } h \to 0^-$$

So, $f_x(0,0)$ does not exist.

Similarly $f_y(0,0)$ does not exist.

But $f(x,y) = |x| + |y| \geq f(0,0)$ for every point $(x,y) \in N(0,0)$ confirming that f has a minimum value at $(0,0)$.

Definition: Let $S \subset R^2$ and $f: S \to R$. An interior point (a,b) of S is said to be a stationary point of f in S if both $f_x(a,b)$ and $f_y(a,b)$ exist and $f_x(a,b) = 0 = f_y(a,b)$.

Example 2. Find all the stationary points of $f(x,y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$.

$\rightarrow$ Since f is a polynomial in x and y, f possesses partial derivatives f_x and f_y at every point (x,y) of its domain $\mathbb{R}^2$

$$f_x = 8x - y + 3x^2y + y^3$$

$$f_y = -x + 8y + x^3 + 3xy^2$$

$$f_x = 0 = f_y \Rightarrow (x-y)^3 - 9(x-y) = 0.$$

$$\Rightarrow (x-y)\{(x-y)^2 - 9\} = 0.$$

Either $x - y = 0 \Rightarrow x = y$.

or $(x-y+3)(x-y-3) = 0$

$$\Rightarrow x - y = -3 \text{ or } x - y = 3.$$

For $x = y$, $f_x = 7x + 3x^3 + x^3 = 0$

$$\Rightarrow x(4x^2 + 7) = 0$$

$$\Rightarrow x = 0.$$

$f_y = 7xy + y^3 + 3y^3 \Rightarrow y(4y^2+7) = 0$

$$\Rightarrow y = 0.$$

For $x = y - 3$,

$8(y-3) - y + 3(y-3)^2y + y^3 = 0$

$\Rightarrow 7y - 24 + 3y(y^2 - 6y + 9) + y^3 = 0$

$\Rightarrow 4y^3 - 18y^2 + 34y - 24 = 0$

$y = ~~.09269~~ \cdot 3/2$ $= ~~2.90731~~ - 3/2$

$x = 3/2 ~~.09269~~ - 3$

$x = y+3$, $8(y+3) - y + 3(y+3)^2y + y^3$

$\Rightarrow y^3 + 3(y^2 + 6y + 9)y + 7y + 24 = 0$

$\Rightarrow 4y^3 + 18y^2 + 34y + 24 = 0$

$\Rightarrow y = -3/2$ $\therefore x = \frac{3}{-2} + 3 = 3/2$

So $(0,0)$, $\left(\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, -\frac{3}{2}\right)$ are the stationary points for function.

Definition: An interior point (a,b) of the domain of a function f of two independent variables x and y is said to be a saddle point of f if it is a stationary point of f i.e. $f_x(a,b)=0$, $f_y(a,b)=0$ but f has neither maximum nor minimum value.

Ex. Show that $(0,0)$ is a saddle point of $f(x,y) = x^6 + (x-y)^3$.

$f_x(x,y) = 6x^5 + 3(x-y)^2$

$f_y(x,y) = -3(x-y)^2$

At $(0,0)$, both f_x and f_y will vanish, So $(0,0)$ is a stationary point.

Now in the neighbourhood of $(0,0)$, there are points where $x > y$ and $x < y$. Thus $(x-y) > 0$ for some points in the neighbourhood of $(0,0)$ and $(x-y) < 0$ for some points in the neighbourhood of $(0,0)$. Since $x^6 + (x-y)^3 > 0$ when $x > y$ and $x^6 + (x-y)^3 < 0$ when $y > 0$ and $x = 0$, which shows that $f(x,y) - f(0,0)$ does change sign in any neighbourhood of $(0,0)$. Thus f has no extrema at $(0,0)$. $(0,0)$ is saddle point

Example 2. Show that the function $f(x,y) = (x-y)^3 + (2-x)^2$ has a saddle point at $(2,2)$.

$\rightarrow$ $f_x(x,y) = 3(x-y)^2 - 2(2-x)$

$f_y(x,y) = -3(x-y)^2$

At $(2,2)$, $f_x(2,2) = 0$, $f_y(2,2) = 0$.

$(2,2)$ is a ~~saddle~~ stationary point of $f(x,y)$.

For any point $(2+h, 2+k)$ in any neighbourhood of $(2,2)$,

$f(2+h, 2+k) - f(2,2) = (h-k)^3 + h^2$.

For $h=0$, $k>0$, the difference is negative

for $h>0$, $k=0$, the difference is positive.

$\therefore f(2+h, 2+k) - f(2,2)$ changes its sign in the neighbourhood of $(2,2)$ which implies f has no extreme value at $(2,2)$. This proves that $(2,2)$ is a saddle point of f.

Sufficient Condition for existence of extreme value:

Let $S \subset \mathbb{R}^2$ and $f: S \to \mathbb{R}$ be a function of two independent variables x and y. Let (a,b) be an interior point of S such that $f_x(a,b) = 0$, $f_y(a,b) = 0$. Let f prossess continuous second order partial derivatives in a certain neighbourhood $N(a,b)$ of (a,b) such that $f_{xx}(a,b)$, $f_{xy}(a,b)$, $f_{yy}(a,b)$ are not all zero. Then

Then

(i) f has no extreme value at (a,b) if

$H = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 < 0$

(ii) f has extreme value at (a,b) if $H > 0$ and in particular f has maximum value at (a,b) if $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$ and f has minimum value at (a,b) if $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$.

(iii) If $H = 0$, then f may or may not have extreme value at (a,b).

Working procedure to examine the existence of extreme values of function:

Step 1 → Find (a,b) where $f_x(a,b) = 0$, $f_y(a,b) = 0$

Step 2 → Find the principal minors of the ~~te~~ matrix

$$[H] = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$$ of

the quadratic form $d^2f = Ah^2 + 2Bhk + Ck^2$

where $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$,

$C = f_{yy}(a,b)$ and

$$\det H = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

Step 3 → If $A > 0$, $H > 0$, then

$d^2 f\big|_{(a,b)} = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$

is positive definite and hence f has minimum value at (a,b). If $H > 0$ and $A < 0$, then $d^2 f$ is negative definite, and f has maximum.

If $H < 0$, then $d^2 f\big|_{(a,b)}$ is neither positive definite nor negative definite, So f has no extreme value at (a,b).

If $H = 0$ and $A > 0$, then $d^2 f$ is positive semidefinite and if $H = 0$, $A < 0$, $d^2 f$ is ~~posi~~ negative semidefinite. In these cases, no definite conclusion can be drawn.

Example: Let $f(x,y) = x^4 + y^4 - 2x^2$. Show that f has a local minimum at $(-1,0)$ and $(1,0)$. $(0,0)$ is a saddle point of f.

→ $f_x = 4x^3 - 4x = 4x(x^2-1)$

$f_y = 4y^3$

$f_x = 0 = f_y \Rightarrow 4x(x^2-1) = 0 \Rightarrow x = 0$ or $x = \pm 1$

$y^3 = 0 \Rightarrow y = 0$.

So, $(0,0)$, $(1,0)$, $(-1,0)$ are stationary points

$f_{xx} = 12x^2 - 4$ $\quad f_{xy} = 0$, $\quad f_{yy} = 12y^2$

For ~~f_{xx}~~ $(1,0)$

$A = f_{xx}(1,0) = 12 - 4 = 8 > 0$

$B = f_{xy}(1,0) = 0$; $\quad C = f_{yy}(1,0) = 0$.

$H = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = 8 \cdot 0 - 0 = 0$.

Step 3 → If $A > 0$, $H > 0$, then $d^2 f\Big|_{(a,b)} = h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)$ is positive definite and hence f has minimum value at (a,b). If $H > 0$ and $A < 0$, then $d^2 f$ is negative definite, and f has maximum. If $H < 0$, then $d^2 f\Big|_{(a,b)}$ is neither positive definite nor negative definite, So f has no extreme value at (a,b).

If $H = 0$ and $A > 0$, then $d^2 f$ is positive semidefinite and if $H = 0$, $A < 0$, $d^2 f$ is ~~posi~~ negative semidefinite. In these cases, no definite conclusion can be drawn.

Example: Let $f(x,y) = x^4 + y^4 - 2x^2$. Show that f has a local minimum at $(-1,0)$ and $(1,0)$. $(0,0)$ is a saddle point of f.

→ $f_x = 4x^3 - 4x = 4x(x^2-1)$

$f_y = 4y^3$

$f_x = 0 = f_y \Rightarrow 4x(x^2-1) = 0 \Rightarrow x = 0$ or $x = \pm 1$

$y^3 = 0 \Rightarrow y = 0$.

So, $(0,0)$, $(1,0)$, $(-1,0)$ are stationary points

$f_{xx} = 12x^2 - 4$ $\quad f_{xy} = 0$, $\quad f_{yy} = \cancel{8}12y^2$

For ~~f_{xx}~~ $(1,0)$

$A = f_{xx}(1,0) = 12 - 4 = 8 > 0$

$B = f_{xy}(1,0) = 0$; $\quad C = f_{yy}(1,0) = 0$.

$H = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = 8 \cdot 0 - 0 = 0$.

For $(-1,0)$

$A = f_{xx}(-1,0) = 12-4 = 8 > 0$

$B = f_{xy}(-1,0) = 0$

$C = f_{yy}(-1,0) = 0.$

$H = AC - B^2 = 8.0 - 0 = 0.$

At this stange no definite conclusion can be drawn for the points $(1,0)$ and $(-1,0)$.

$f(1+h, 0+k) - f(1,0) = (1+h)^4 + k^2 - 2(1+h)^2 + 1$

$= [(1+h)^2 - 1]^2 + k^2 > 0$ in any neighbourhood of $(1,0)$ since h, k are not simultaneously zero.

Hence f has local minimum at $(1,0)$

$f(-1+h, 0+k) - f(-1,0)$

$= (-1+h)^4 + k^2 - 2(-1+h)^2 + 1$

$= [(h-1)^2 - 1]^2 + k^2 > 0$ in any neighbourhood of $(-1,0)$ since h, k are not simultaneously zero.

Thus f has local minimum at $(-1,0)$.

For the point $(0,0)$

$f_{xx}(0,0) = -4$, $f_{xy}(0,0) = 0$, $f_{yy}(0,0)$

~~$f(0+h, 0+k$~~ $H = AC - B^2 = -4.0 - 0$

$= 0$

No conclusion can be drawn.

$f(0+h, 0+k) - f(0,0) = h^4 + k^4 - 2h^2$
$= h^2(h^2-2) + k^4.$

For (h,k) in the neighbourhood of $(0,0)$, when $0 < |h| < \sqrt{2}$, $k=0$, $f(h,k) - f(0,0) < 0$.

When $h=0$, $k \neq 0$, $f(h,k) - f(0,0) > 0$.

Thus $f(h,k) - f(0,0)$ does change sign in the neighbourhood of $(0,0)$. So f has no extreme value at $(0,0)$. $(0,0)$ is a saddle point.

Ex. Show that if $f(x,y) = 2x^4 - 3x^2y + y^2$, then $f_{xx}f_{yy} - (f_{xy})^2 = 0$ at $(0,0)$ but f has no extrema at $(0,0)$.

Ex. Show that for the function f, where $f(x,y) = y^2 + 2x^2y + 2x^4$, $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$ and f has minimum at $(0,0)$.

Method of Lagrange's Multiplier

Find the maxima/minima of the function $u = f(x,y)$ with the constraint $\phi(x,y) = 0$.

Using the chain rule,

$$\frac{du}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

At the point of extrema, $\frac{du}{dx} = 0$

$$\therefore \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{---(1)}$$

The equation $\phi(x,y) = 0$ is satisfied at any point and so at the point of extrema

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \quad \text{--- (2)}$$

~~Elimi~~ For elemination of $\frac{dy}{dx}$ from (1) and (2), we get

$$\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right)\cdot\left(-\underbrace{\frac{\frac{\partial f}{\partial y}}{\frac{\partial \phi}{\partial y}}}_{=\lambda}\right) + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \text{--- (3)}$$

Similarly taking ~~p~~ the derivative w.r.t of u and ϕ, we obtain

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \text{--- (4)}$$

So the extreme point satisfy the following equations

$$\left.\begin{aligned} &\phi(x,y) = 0 \\ &\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \\ &\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \end{aligned}\right\} \quad (A)$$

Working rule for method of Lagrang's multiplier:

- max/min $u = f(x,y)$ with the constraint $\phi(x,y) = 0$
- Define an auxiliary fn

$$F(x,y,\lambda) = f(x,y) + \lambda\phi(x,y)$$

- Necessary conditions for extrema F,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \phi(x,y) = 0$$

Remark: Using the method of Lagrange's multiplier, we obtain stationary point. Usually we donot determine the nature of the stationary points. In ~~par~~ practical life, we are interested in finding max/min value of function under given constraint. Usually there are few stationary points, we can evaluate f at all of them and choose the largest and smallest values.

Example-1: Find maximum / minimum of the function x^2-y^2-2x in the region $x^2+y^2 \le 1$.

Problem is max/min x^2-y^2-2x
Subject to $x^2+y^2=1$

Auxiliary function for the Lagrange multiplier

$$F(x,y,\lambda) = (x^2-y^2-2x) + \lambda(x^2+y^2-1)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + 2\lambda x = 2 \Rightarrow x(1+\lambda) = 1$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow -2y + 2\lambda y = 0 \Rightarrow y(\lambda-1) = 0$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2+y^2-1=0$$

$y(\lambda-1) = 0 \Rightarrow y=0$ or $\lambda = 1$.

If $y=0$, $x = \pm 1$.
If $x=1$, $\lambda = 0$
If $x=-1$, $\lambda = -2$

$\Rightarrow$ Stationary point $(\pm 1, 0)$ for $\lambda = 0, -2$

If $\lambda = 1$, $x = \frac{1}{2}$, $y = \pm\frac{\sqrt{3}}{2}$ $\Rightarrow$ stationary point $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, $\lambda = 1$.

Functional values (x^2+y^2-2x)	Stationary point
-1	$(1, 0)$
3	$(-1, 0)$
$-\frac{3}{2}$	$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Maximum value of the function 3 for the point $(-1, 0)$

Minimum value of the function $-\frac{3}{2}$ for the point $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

2. Use Lagrange's multiplier method to find the shortest distance between $(-1, 4)$ and the straight line $12x - 5y + 71 = 0$.

→ Let (x, y) be any point on the given Straight line. Then square of its distance from the point $(-1, 4)$ is

$f(x, y) = (x+1)^2 + (y-4)^2$ whose minimum value is to be determined subject to the condition $12x - 5y + 71 = 0$.

We construct the lagrangian function as

$$L(x, y) = (x+1)^2 + (y-4)^2 + \lambda(12x - 5y + 71) \quad \text{---(1)}$$

where λ is the Lagrange's multiplier.

$L_x = 2(x+1) + 12\lambda = 0$, $L_\lambda = 12x - 5y + 71 = 0$

$L_y = 2(y-4) - 5\lambda = 0$,

$$\Rightarrow \frac{\lambda}{2} = \frac{x+1}{\cancel{4}-12} = \frac{y-4}{5}$$

Since (x,y) is a point on the straight line we have

$$12(-1-6\lambda) - 5\left(4+\frac{5}{2}\lambda\right) + 71 = 0$$

$$\Rightarrow \frac{169}{2}\lambda = 39 \Rightarrow \lambda = \frac{6}{13}.$$

$x = -\frac{49}{13}$, $y = \frac{67}{13}$. Hence $\left(-\frac{49}{13}, \frac{67}{13}\right)$ is the stationary point.

$$d^2L = L_{xx}(dx)^2 + 2L_{xy}\,dx\,dy + L_{yy}(dy)^2$$

$$= 2[(dx)^2 + (dy)^2] > 0.$$

$\therefore$ d^2L is positive definite. $\therefore$ f has minimum value at $\left(-\frac{49}{13}, \frac{67}{13}\right)$.

$f_{min} = 9$ $\therefore$ Shortest distance will be $\sqrt{f_{min}} = 3$ unit.

Ex. Use Lagrange's method to find the shortest distance from the point $(0,b)$ to the parabola $x^2 = 4y$.

Ex. Find the maximum and minimum values of the function $f(x,y) = x^2+y^2$ in the region $(x-2)^2 + (y-1)^2 \le 20$.