Function of two variables

Explicit and implicit functions:

If we consider a set of n independent variable $x, y, z, \ldots t$ and one independent variable u, the equation

$$u = f(x, y, z, \ldots t) \qquad (1)$$

denotes the functional relation; the function represented by (1) is called the explicit function. Ex. $u = x^2 + y^2 + z^2 + t^2$

When the several variables are concerned, it is rarely possible to obtain an equation expressing one of the variables in terms of others. Thus most of the functions of more than one variable are implicit functions, given by a functional relation:

$$\phi(x, y, z, \ldots t) = 0.$$

Ex: $\underbrace{xyzt^2 + y^2xzt + t^2yx^2z}_{\phi(x,y,z,t)} = 0$

We consider the properties of function of two variables now. i.e. $z = f(x, y)$

The neighbourhood of point:

The neighbourhood of (a,b)

Let $(a,b) \in R^2$. The neighbourhood of (a,b) is a subset S of R^2 s.t. $(a,b) \in S$. If the point (a,b) is not included in the neighbourhood, the neighbourhood is called the deleted neighbourhood of (a,b).

Let $\delta > 0$ be chosen arbitrarily.
Then the subset $\{(x,y) : |x-a| < \delta, |y-b| < \delta\}$ of R^2 is called square δ-nbd of (a,b) and is denoted by $R_\delta(a,b)$.

Again the subset $\{(x,y) : (x-a)^2 + (y-b)^2 < \delta^2\}$ of R^2 is called a circular neighbourhood of (a,b) with radius δ and is denoted by $B_\delta(a,b)$.

If $\alpha, \beta, \gamma, \delta \in R$ such that $\alpha < a < \beta$, $\gamma < b < \delta$, then the subset $R = [\alpha, \beta] \times [\gamma, \delta] = \{(x,y) : \alpha < x < \beta, \gamma < y < \delta\}$ of R^2 is a rectangular neighbourhood of (a,b).

Interior point: Let $S \subset R^2$ and $(a,b) \in S$. Then (a,b) is called an interior point of S if $\exists\, \delta > 0$ s.t. $N_\delta(a,b) \subset S$.

Open Set: If every point of S is an interior point of S, then S is called an open set.

Limit point: A point $(a,b) \in R^2$ is said to be a limit point of a set $S \subset R^2$ if $\forall\, \delta > 0$, $N_\delta'(a,b) \cap S \neq \phi$, where $N_\delta'(a,b) = N_\delta(a,b) \setminus \{(a,b)\}$.

Note → If (a,b) be a limit point of S, then the neighbourhood of (a,b) will contain an infinite no of points in S.

Limit and Continuity in Higher Dimension

Limit of function of two variable:

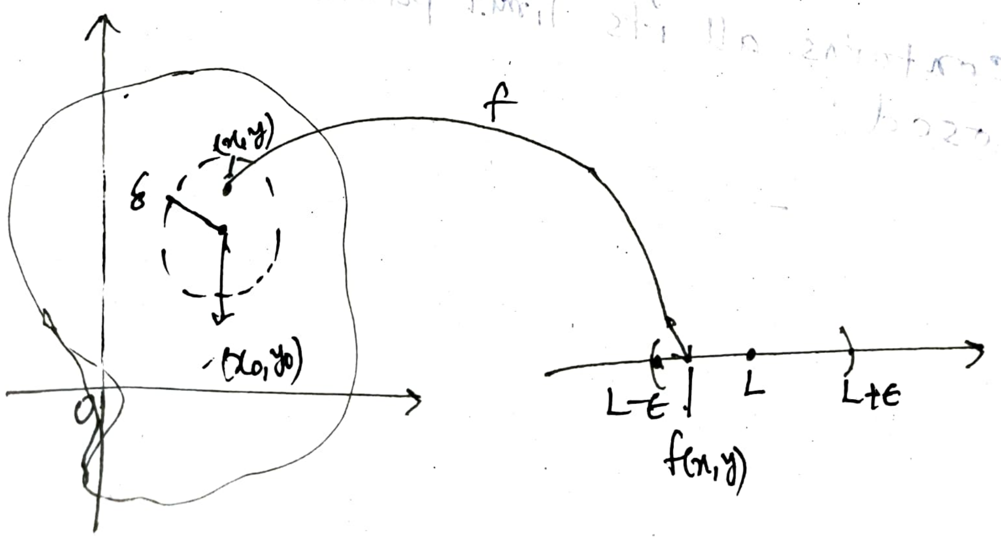
If the values of $f(x,y)$ lie arbitrarily close to fixed real number L for all points (x,y), sufficiently close to a point (x_0,y_0). When (x_0,y_0) lies in the interior of the domain of f, (x,y) can approach (x_0,y_0) in any direction, not just from left or right. For the existence of limit, same limiting value must be obtained whatever direction of approach is taken.

Definition: We say that a function $f(x,y)$ approaches the limit L as (x,y) approaches (x_0,y_0), i.e. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if corresponding to arbitrarily chosen $\epsilon>0$, there exists a ~~corresponding~~ $\delta>0$ such that for all $(x,y) \in D$, the domain of f

$|f(x,y) - L| < \epsilon$ whenever

$$0 < \sqrt{(x-x_0)^2 + (y+y_0)^2} < \delta.$$



In the limit definition, δ is the radius of disk centred at (x_0, y_0). For all points (x, y) in the disk, the functional value $f(x, y)$ lie inside $(L-\epsilon, L+\epsilon)$.

This limit is called the double limit or simultaneous limit.

Non existence of limit: In order to show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist, it is sufficient to show that the above definition does not hold or it is possible to find two different paths in the domain of f for which the function as $(x,y) \to (a,b)$ for which the function $f(x,y)$ has two different limits along those paths.

Suppose if we take $y = f_1(x)$, $y = f_2(x)$, then $f(x, f_1(x))$, $f(x, f_2(x))$ tend separately to different limit as $x \to a$.

$$\therefore \lim_{x\to a} f(x, f_1(x)) \neq \lim_{x\to a} f(x, f_2(x)).$$

Iterated or Repeated limit:

Let $f(x, y)$ be defined in a certain neighbourhood of (a, b). For a fixed value of y, $\lim_{x\to a} f(x,y)$, if exists, will involve y.

$\therefore \lim_{x\to a} f(x,y)$ will be different for different values of y. Let $\lim_{x\to a} f(x,y) = \phi(y)$.

If then, $\lim_{y\to b} \phi(y)$ exists and is equal to A, then $\lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\} = A$ ——— (1)

Now, We change the order of obtaining limits

Keeping x fixed, if $\lim_{y \to b} f(x,y)$ exists it will be a function of x say $\varphi(x)$.
If $\lim_{x \to a} \varphi(x)$ exists and equals to B,

$$\lim_{x \to a} \left\{ \lim_{y \to b} f(x,y) \right\} = B \quad \text{---} (2)$$

The limits given in (1) & (2) are called iterated or repeated limits.

Note → Existence of double limit will imply equality of repeated limits if they exist but not the converse are
If however repeated limits are not equal, double limit cannot exist.

Example 1: Find $\lim_{(x,y) \to (0,0)} f(x,y)$ where

$$f(x,y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

→ Let us put $x = r\cos\theta$, $y = r\sin\theta$
$\therefore x^2 + y^2 = r^2$. This implies $r \to 0$ as $(x,y) \to (0,0)$

$$\therefore \quad f(x,y) = r^2 \sin\theta \cos\theta \cos 2\theta .$$

Now $\left| xy \frac{x^2 - y^2}{x^2 + y^2} \right| = \left| \frac{r^2}{4} \sin 4\theta \right| \leq \frac{r^2}{4} < \epsilon$.

if $r^2 < 4\epsilon$ or $0 < x^2 + y^2 < \delta^2 (= 4\epsilon)$

$\therefore$ By definition

$$\lim_{(x,y) \to (0,0)} f(x,y) = 0 .$$

Alternatively
We know $|x| \leq \sqrt{x^2 + y^2}$
$|y| \leq \sqrt{x^2 + y^2}$

Also $\left|\frac{x^2-y^2}{x^2+y^2}\right| \le 1$.

Therefore $|f(x,y)-0| = \left|xy\,\frac{x^2-y^2}{x^2+y^2}\right|$

$$= |x|\,|y|\left|\frac{x^2-y^2}{x^2+y^2}\right| \le x^2+y^2 < \epsilon.$$

if $0 < x^2+y^2 < \delta^2$ where $\delta = \sqrt{\epsilon}$

By definition, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Example 2. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. But the repeated limits donot exist where

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & \text{when } xy \ne 0 \\ 0, & \text{when } xy = 0. \end{cases}$$

Here $|f(x,y)-0| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$$\le |x|\left|\sin\tfrac{1}{y}\right| + |y|\left|\sin\tfrac{1}{x}\right|$$

$$\le |x| + |y| \quad \left(\because \left|\sin\tfrac{1}{x}\right| \le 1\right)$$

$$< \epsilon \text{ if } |x-0| < \delta,\ |y-0| < \delta$$

where $\delta = \epsilon/2$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Now $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist

$\lim\limits_{x\to 0}\left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right)$ does not exist when y is fixed.

Also $\lim\limits_{y\to 0}\left(x\sin\frac{1}{y} + y\sin\frac{1}{x}\right)$ does not exist when x is fixed.

Therefore the repeated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ does not exist.

Example 3: Let $f(x,y) = \begin{cases} x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2}, & y\neq 0 \\ 0, & y=0 \end{cases}$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\rightarrow$ Let $(x,y)\to(0,0)$ along $y=mx$.

$$\therefore \frac{x^2-y^2}{x^2+y^2} = \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}$$

$$\underset{(x,y)\to(0,0)}{Lt} \frac{x^2-y^2}{x^2+y^2} = \underset{x\to 0}{Lt} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}$$

which is different for different values of m.

$\therefore \underset{(x,y)\to(0,0)}{Lt} \frac{x^2-y^2}{x^2+y^2}$ does not exist, since it is path dependent.

$f(x,y) = f_1(x,y) + f_2(x,y)$

where $f_1(x,y) = x\sin\frac{1}{y}$, $f_2(x,y) = \frac{x^2-y^2}{x^2+y^2}$

$\lim\limits_{(x,y)\to(0,0)} f_2(x,y)$ does not exist, so,

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Problem Set

Use definition to establish the following limits.

(i) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$

(iii) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0.$

(iv) $\lim\limits_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = 1.$

Solution of Problem Set

(i) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$.

To prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

$$\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} < \epsilon \text{ if}$$

$$x^2+y^2 < \delta^2 (= \epsilon^2).$$

(ii) $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$

To prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$

$$\left| \frac{x^3-y^3}{x^2+y^2} \right| \leq \frac{|x|^3+|y|^3}{x^2+y^2} \leq \frac{(x^2+y^2)^{3/2}+(x^2+y^2)^{3/2}}{x^2+y^2}$$

$$= 2(x^2+y^2)^{1/2} < \epsilon$$

$$\text{if } x^2+y^2 < \delta^2 \left(= \frac{\epsilon^2}{4}\right).$$

(iii) $f(x,y) = \dfrac{x^4+y^4}{x^2+y^2}$

$$\left|\frac{x^4+y^4}{x^2+y^2} - 0\right| = \left|\frac{x^4+y^4}{x^2+y^2}\right|$$

$$\leq \frac{|x|^4+|y|^4}{x^2+y^2}$$

$$\leq 2(x^2+y^2) < \epsilon$$

[Since $|x| \leq \sqrt{x^2+y^2}$, $|y| \leq \sqrt{x^2+y^2}$

$|x|^4 + |y|^4 \leq (x^2+y^2)^2$]

if $x^2+y^2 < \delta^2 \left(= \frac{\epsilon}{2}\right)$.

(iv) $f(x,y) = e^{-(x^2+y^2)}$

$$\left|e^{-(x^2+y^2)} - 1\right| \simeq x^2+y^2 < \epsilon$$

if $|x| < \frac{\sqrt{\epsilon}}{2}$, $|y| < \frac{\sqrt{\epsilon}}{2}$.

We choose $\delta = \frac{\sqrt{\epsilon}}{2}$

$\therefore \left|e^{-(x^2+y^2)} - 1\right| < \epsilon$ whenever

$|x| < \delta$, $|y| < \delta$.

Example: Show that the following limits do not exist.

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$

(2) $\lim\limits_{(x,y)\to(0,0)} (x+y)\dfrac{y+(x+y)^2}{y-(x+y)^2}$

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

(4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x-y}$.

Soln

(1) $f(x,y) = \dfrac{xy^3}{x^2+y^6}$

Let $(x,y)\to(0,0)$ along ~~$y=0$ and~~ $x=my^3$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{my^6}{(m^2+1)y^6}$$

[Substituting $x=my^3$. $y\to 0$ as $x\to 0$]

$$= \lim_{y\to 0} \frac{m}{m^2+1} = \frac{m}{m^2+1},$$ which

is a function of m, ~~an~~ therefore this limit has different values for different m.

So, this limit does not exist.

(2)

$$f(x,y) = (x+y)\,\frac{y+(x+y)^2}{y-(x+y)^2}$$

Let $(x,y) \to (0,0)$ along $y=0$ and $y=x^2$ separately.

Along $y=0$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} x \cdot \frac{x^2}{-x^2}$$

$$= 0.$$

Along $y=x^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x^2)$$

$$= \lim_{x\to 0} (x+x^2)\cdot\frac{x^2+(x+x^2)^2}{x^2-(x+x^2)^2}$$

$$= \lim_{x\to 0} (x+x^2)\cdot\frac{x^2+x^2(1+x)^2}{x^2-x^2(1+x)^2}$$

$$= \lim_{x\to 0} x(1+x)\cdot\frac{x^2\left[1+(1+x)^2\right]}{x^2\cdot(2+x)\cdot(-x)}$$

$$= \lim_{x\to 0} \frac{(1+x)\left[1+(1+x)^2\right]}{-(2+x)}$$

$$= \frac{2}{-2} = -1.$$

Therefore the limiting value is different along the different path $y=x$ and $y=x^2$ respectively.

$\therefore \lim_{(x,y)\to(0,0)} (x+y)\,\frac{y+(x+y)^2}{y-(x+y)^2}$ doesnot exist.

(3) $f(x,y) = \dfrac{x^2 y^2}{x^2 y^2 + (x^2 - y^2)^2}$ (Exercise)

[Hint: Take two different path along $y = x$ and $y = 0$ separately]

(4) $f(x,y) = \dfrac{x^3 + y^3}{x - y}$

Let $(x,y) \to (0,0)$ along $y = x - mx^3$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y} = \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{x - x + mx^3}$$

[As $y \to 0$, $x \to 0$]

$$= \lim_{x\to 0} \frac{x^3[1 + (1 - mx^2)^3]}{mx^3}$$

$= \dfrac{2}{m}$ which has different values for different m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Ex: $f(x,y) = \dfrac{\sin x + \sin 2y}{\tan 2x + \tan y}$ (i) Show that $\lim_{y\to 0} \lim_{x\to 0} f(x,y) \neq \lim_{x\to 0} \lim_{y\to 0} f(x,y)$.

(ii) Also prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

[Hint: Take two different path along $y = x$ and $y = 2x$]

Continuity of function of two variable:

Definition: Let $f: S \to R$ and $S \subseteq R^2$. Let $(a,b) \in S$. Then f is called continuous at (a,b) if

(i) (a,b) is an accumulation point of S and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

(ii) (a,b) is an isolated point of S.

If given $\epsilon > 0$, there exists $\delta > 0$ such that

$|f(x,y) - f(a,b)| < \epsilon$ whenever $(x,y) \in N((a,b),\delta) \cap S$.

We say that f is continuous at (a,b).

This $N((a,b),\delta)$ in general can be taken as $(x-a)^2 + (y-b)^2 < \delta^2$

or $|x-a| < \delta$, $|y-b| < \delta$

Note → From above definition, it follows that

$g(x) = f(x,b)$ is continuous at 'a'

$h(y) = f(a,y)$ is continuous at 'b'.

But the converse part is not true.

The following example illustrates that the converse part is not true.

Consider $f(x,y) = \begin{cases} 0, & \text{if } xy \neq 0 \\ 1, & \text{if } xy = 0 \end{cases}$.

Here $f(x,0) = 1$ for all $x \in R$.

$f(0,y) = 1$ for all $y \in R$.

So, $g(x) = f(x,0)$ is continuous at $x=0$ and $h(y) = f(0,y)$ is continuous at $y=0$.

But f is not continuous at $(0,0)$.

If possible let f be continuous at $(0,0)$.

Then for $0 < \epsilon < \frac{1}{2}$, there exists $\delta > 0$ such that $|f(x,y) - 1| < \epsilon$ whenever $(x^2+y^2) < \delta^2$.

If we take $x = \frac{\delta}{2}$, $y = \frac{\delta}{2}$ (in particular)

$\left| f\left(\frac{\delta}{2}, \frac{\delta}{2}\right) - 1 \right| = |0-1| = 1 \nless \frac{1}{2}$, which is absurd.

Since $0 < \epsilon < \frac{1}{2}$

Therefore our assumption is wrong.

So, f is not continuous at $(0,0)$.

Example 1. Check the continuity of the function $f(x,y) = \begin{cases} (ax+by)\sin\dfrac{x}{y}, & y \neq 0 \\ 0, & y = 0, \ \forall x \end{cases}$

$a, b \in \mathbb{R}$

$\rightarrow$ Let $\epsilon > 0$ be chosen arbitrarily.

$$|f(x,y) - f(0,0)| = \left|(ax+by)\sin\frac{x}{y} - 0\right|$$

$$\leq |ax+by|$$

$$\leq |a||x| + |b||y|$$

$$< \epsilon$$

if $|x| < \dfrac{\epsilon}{2(|a|+1)}$, $|y| < \dfrac{\epsilon}{2(|b|+1)}$

whenever $|x-0| < \delta_1 = \dfrac{\epsilon}{2(|a|+1)}$

$|y-0| < \delta_2 = \dfrac{\epsilon}{2(|b|+1)}$

If $\delta = \min\{\delta_1, \delta_2\}$, we have

$|f(x,y) - f(0,0)| < \epsilon$ whenever $|x-0| < \delta$, $|y-0| < \delta$.

So, f is continuous at $(0,0)$.

Ex 2. Show that ~~lim~~ the function g is not continuous at $(0,0)$ where

(i) $g(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{when } x \neq y. \\ 0, & \text{when } x = y. \end{cases}$

Hint: [Show that $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist along the path $y = x - mx^3$.]

(ii) $g(x,y) = \begin{cases} \dfrac{x^4+y^4}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$

[Hint: consider $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$ along $y = mx^4$]

(iii) $g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0. \\ 0, & x^2+y^2 = 0. \end{cases}$

Ex 3. Prove that the function

(i) $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(ii) $h_1(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$

(iii) $h_2(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

are continuous at $(0,0)$.

Partial derivatives

Definition: Let $f: S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^2$. Let (a,b) be an interior point of S.

(i) If $\lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}$ exists finitely, then it is called that first order partial derivative of f with respect to x at (a,b) exists. This is denoted by

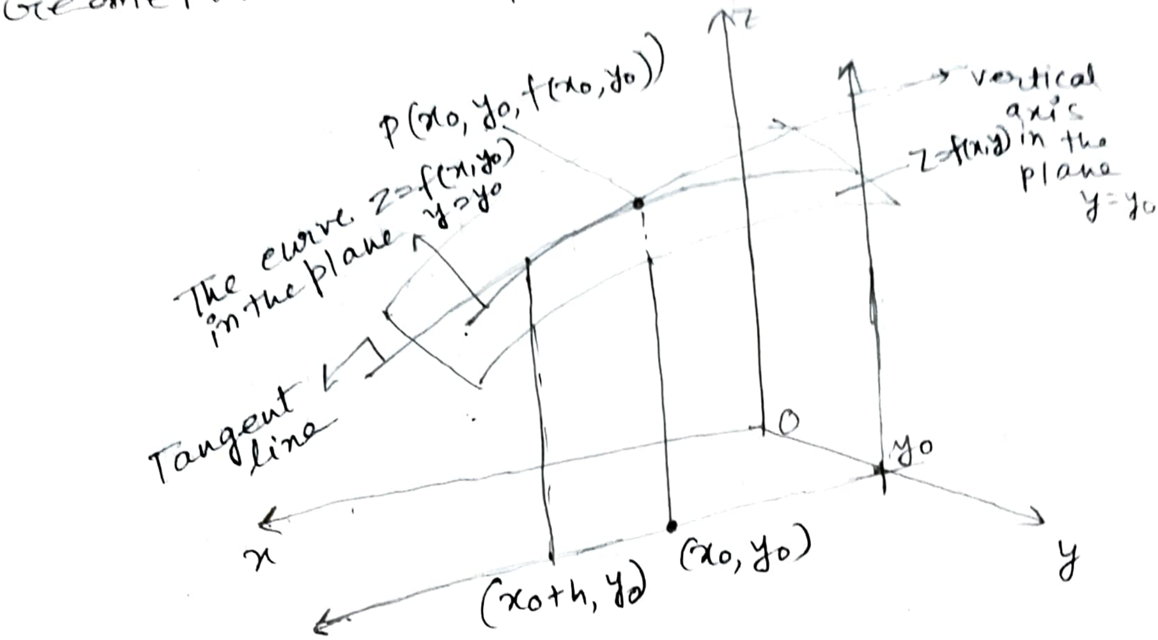
$f_x(a,b)$ or $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$

(ii) If $\lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}$ exists finitely then it is called that first order partial derivative of f with respect to y at (a,b) exists. This is denoted by

$f_y(a,b)$ or $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$

Note → From the definition, it is obivious that the rule for computing partial derivative coincide with the rules given for the derivative of function of one variable. For finding f_x at any point, we have to regard y as constant. For finding f_y at any point, we have to treat x as constant.

Geometrical interpretation:



If (x_0, y_0) is a point in the domain of $f(x,y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$. This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x, the vertical coordinate is z. The y-value is held constant at y_0, so y is not a variable..

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\frac{\partial f}{\partial x}(x_0, y_0)$ gives the rate of change of f w.r.t x when y is held fixed at y_0.

Similarly the plane $x = x_0$ cuts the surface $z = f(x, y)$ ~~ad~~ in the curve $z = f(x_0, y)$ at $x = x_0$. The partial derivative of f w.r.t y, $\frac{\partial f}{\partial y}$ at (x_0, y_0) is the slope of the tangent line to the curve $z = f(x_0, y)$ at $P(x_0, y_0, f(x_0, y_0))$. So $\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$ gives the rate of change of f w.r.t y ~~at~~ when x is held fixed at x_0.

Example 1: If $u = (1 - 2xy + y^2)^{-1/2}$, show that

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right) = 0.$$

$\rightarrow$ $u = (1 - 2xy + y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = \left(-\frac{1}{2}\right)\frac{(-2y)}{(1-2xy+y^2)^{3/2}}$$

$$= \frac{y}{(1-2xy+y^2)^{3/2}} = yu^3.$$

$$\frac{\partial u}{\partial y} = \left(-\frac{1}{2}\right)\frac{(2y-2x)}{(1-2xy+y^2)^{3/2}}$$

$$= (x-y)\,u^3$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = y \cdot 3u^2 \cdot \frac{\partial u}{\partial x} = 3y^2 u^5$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = (x-y)\cdot 3u^2 \frac{\partial u}{\partial y} - u^3$$

$$= 3(x-y)^2 u^5 - u^3$$

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\left(y^2\frac{\partial u}{\partial y}\right)\right\}$$

$$= (1-x^2)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - 2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} + y^2\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= 3y^2(1-x^2)u^5 - 2xyu^3 + 2y(x-y)u^3 + 3y^2(x-y)^2u^5 - y^2u^3$$

$$= 3y^2u^5 - \cancel{3x^2y^2u^5} - \cancel{2xyu^3} + \cancel{2xyu^3} - 2y^2u^3 + \cancel{3x^2y^2u^5} - 6xy^3u^5 + 3y^4u^5 - y^2u^3$$

$$= 3y^2u^5 - 3y^2u^3 - 6xy^3u^5 + 3y^4u^5$$

$$= -3y^2u^3 + 3y^2u^5(1-2xy+y^2)$$

$$= -3y^2u^3 + 3y^2u^5\cdot\frac{1}{u^2}$$

$$= 0$$

Example 2: If $f(x,y) = (|x+y| + x + y)^k$, $(x,y)\in\mathbb{R}^2$. Show that if $0<k<1$, f_x and f_y donot exist at $(0,0)$. When do they exist at $(0,0)$?

→ The function can be defined only when $k>0$ for all $(x,y)\in\mathbb{R}^2$.

$$\lim_{h\to 0+}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0+} 2^k h^{k-1}$$

exists only when $k\geq 1$.

If $0<k<1$, the above limit does not exist.

If $k>1$, the above limiting value is 0 and if $k=1$, it will be 2.

$$\lim_{h\to 0^-} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h}$$

$$= 0.$$

So, for the existence of limit i.e f_x at $(0,0)$, k must be greater than 1. Similarly for the existence of f_y at $(0,0)$, k must be greater than 1.

Example 3: Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist at $(0,0)$. but f is not continuous

→ $$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h^2} = 0.$$

$$\lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k^2} = 0.$$

So, $f_x(0,0)$ and $f_y(0,0)$ exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$ Consider $y = mx$

$$\lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2)}$$ depends on m.

So f is not continuous at $(0,0)$.

Note : Unlike the situation for functions of one variable, the existence of the first order partial derivatives at a point does not imply the continuity of the function at a point.

The following theorem gives the sufficient cond^n of continuity of a function $f(x,y)$ in a region $S \subset R^2$

→ Sufficient condition of continuity:

If the partial derivatives f_x and f_y exist and are bounded in a region $S \subset R^2$, then f is continuous in S.

→ Mean value theorem for function of two variable:

Let $D \subseteq R^2$. $f: D \to R$ be such that f_x exists throughout a neighbourhood $N \subset D$ of a point $(a,b) \in D$ and $f_y(a,b)$ exist, then for any point $(a+h, b+k)$ of N, there exists θ, $(0<\theta<1)$ such that

$$f(a+h, b+k) - f(a,b) = k f_y(a+h, b+\theta k) + h\{f_x(a,b) + \eta(h)\}$$

where $\eta(h) \to 0$ as $h \to 0$.

If further f_y be bounded in N, then f is continuous at (a,b).

Example: Show that for the function $f(x,y) = |x| + |y|$, partial derivatives f_x, f_y do not exist at $(0,0)$ but $f(x,y)$ is continuous at $(0,0)$.

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{|x|}{x} = \begin{cases} 1, & \text{as } x\to 0^+ \\ -1, & \text{as } x \to 0^- \end{cases}$$

$f_x(0,0)$ does not exist.

Similarly $f_y(0,0)$ does not exist.
Here

$$|f(x,y)-f(0,0)| = ||x|+|y||$$
$$= |x|+|y| < \epsilon$$
$$\text{if } |x| < \frac{\epsilon}{2}, \quad |y| < \frac{\epsilon}{2}$$

If we choose $\delta = \epsilon/2$,
then $|f(x,y) - f(0,0)| < \epsilon$ whenever
$$|x| < \delta, \ |y| < \delta.$$

Therefore $f(x,y)$ is continuous at $(0,0)$.

Ex: If $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0. \end{cases}$

Show that $f_x(0,0) = 1$ and $f_y(0,0) = -1$.

Ex: Let $f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & x \neq 0, y \neq 0 \\ x^2 \sin\frac{1}{x}, & y = 0, x \neq 0 \\ y^2 \sin\frac{1}{y}, & x = 0, y \neq 0 \\ 0, & x = 0, y = 0. \end{cases}$

Find $f_x(0,y)$ and $f_y(x,0)$.

[Hint: Find $\lim\limits_{x\to 0} \dfrac{f(x,y) - f(0,y)}{x}$, y fixed

$\lim\limits_{y\to 0} \dfrac{f(x,y) - f(x,0)}{y}$, x fixed]

Differentiability of a function $f(x,y)$:

Let $D \subseteq R^2$ be the domain of $f(x,y)$.
Let (a,b) be an interior point of D and N be the neighbourhood (a,b) lying in D.
Let $(a+h, b+k) \in N$.

Then $\Delta f = f(a+h, b+k) - f(a,b)$ is called the increament of $f(x,y)$ at (a,b).

We say that f is differentiable at (a,b) if

$$\Delta f = f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k) \quad \text{(1)}$$

where $(a+h, b+k) \in D$. A, B are independent of h, k and $\phi(h,k)$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$

Necessary condition for differentiability:

If f be differentiable at an interior point (a,b) of its domain, then (i) f_x and f_y exist at point (a,b) (ii) f is continuous at ~~at~~ that point.

Proof: Since f is differentiable at (a,b), by hypothesis (1),

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

where $(a+h, b+k) \in \text{dom} f$, A, B are independent of h, k and $\phi \to 0$, $\psi \to 0$ as $(h,k) \to (0,0)$.

We take $k=0$, $h \neq 0$.

$$\frac{f(a+h, b) - f(a,b)}{h} = A + \phi(h)$$

As $h \to 0$, $\phi(h) \to 0$.

$\therefore \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$ exists and $= f_x(a,b)$

Next we take $h = 0, k \neq 0$

$$\frac{f(a, b+k) - f(a,b)}{k} = B + \psi(k)$$

As $k \to 0$, $\psi(k) \to 0$

So $\lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$ exists

and equals to $f_y(a,b)$

$\therefore$ $f_y(a,b) = B$.

Hence f_x and f_y at (a,b) exists.

(ii) $\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$ [from (i)]

$\therefore$ f is continuous at (a,b).

Note → (1) We can write,

$$\Delta f = f(a+h, b+k) - f(a,b)$$

$$= h f_x(a,b) + k f_y(a,b) + h\phi(h,k) + k\psi(h,k)$$

where $\phi(h,k), \psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

(2) The following example shows that the mere existence of partial derivative at a point does not ensure the differentiability of a function at a point.

Example: $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

$$\lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$\therefore$ $f_x(0,0) = 0$.

$$\lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\Rightarrow f_y(0,0) = 0$$

In order to be differentiable at $(0,0)$, one must have

$$f(0+h, 0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + h\phi(h,k) + k\psi(h,k)$$

where $\phi(h,k)$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$

In particular

$$\frac{hk}{\sqrt{h^2+k^2}} = h \cdot 0 + k \cdot 0 + h\,\phi(h,k) + k\psi(h,k)$$

If $h = k \neq 0$,

$$\frac{h^2}{\sqrt{2}\,h} = h\left(\phi(h,h) + \psi(h,h)\right)$$

As $h \to 0$, $\phi(h,h) + \psi(h,h) \to 0$.

$$\lim_{h \to 0} \frac{h}{\sqrt{2}\,h} = \frac{1}{\sqrt{2}}$$

So, left hand side $\to \frac{1}{\sqrt{2}}$ as $h \to 0$

But right hand side $\to 0$ as $h \to 0$.

Thus we arrive at a contradiction.

Therefore f is not differentiable at $(0,0)$ though $f_x(0,0)$, $f_y(0,0)$ exist.

Ex: Show that f is not differentiable at $(0,0)$.

$$f(x,y) = \begin{cases} x, & |y| < |x| \\ -y, & |y| \geq |x|. \end{cases}$$

$$\lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$$

$$\lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k}$$

$$= -1.$$

In order to differentiable at $(0,0)$, we must have

$$f(0+h, 0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + h\phi(h,k) + k\psi(h,k)$$

So,

$$f(h,k) - f(0,0) = h - k + h\phi(h,k) + k\psi(h,k)$$

In particular if $k = -h$, we get

$$-k = -2k + k(\psi(-k,k) - \phi(-k,k))$$

$$1 = \psi - \phi.$$

But $\psi(-k,k) - \phi(-k,k) \to 0$ as $k \to 0$.

As $k \to 0$, R.H.S $\to 0$,

But L.H.S $\not\to 0$, which is absurd.

$\therefore$ f is not differentiable at $(0,0)$.

Example: If $f(x,y) = \begin{cases} \dfrac{x^6 - 2y^4}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that $f(x,y)$ is differentiable at $(0,0)$.

$\rightarrow \lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{h^6/h^2}{h} = 0$

$\therefore f_x(0,0) = 0$.

$\lim_{k \to 0} \dfrac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \dfrac{-2k^4}{k^3}$

$= 0$.

$f_y(0,0) = 0$.

In order to show f is differentiable at $(0,0)$

~~$f(0+h, 0) - f$~~

$f(0+h, 0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0)$
$+ h\phi(h,k) + k\psi(h,k)$

where $\phi(h,k)$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\dfrac{h^6 - 2k^4}{h^2+k^2} = h \cdot 0 + k \cdot 0 + h\,\phi(h,k)$
$+ k\,\psi(h,k)$

Let $\phi(h,k) = \begin{cases} \dfrac{h^5}{h^2+k^2}, & h^2+k^2 \neq 0 \\ 0, & h^2+k^2 = 0. \end{cases}$

$\psi(h,k) = \begin{cases} \dfrac{-2k^3}{h^2+k^2}, & h^2+k^2 \neq 0 \\ 0, & h^2+k^2 = 0. \end{cases}$

To show $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

put $h = r\cos\theta$, $k = r\sin\theta$

Let $\epsilon > 0$ be chosen arbitrarily

$\left|\phi(h,k) - 0\right| = \left|\dfrac{r^5\cos^5\theta}{r^2} - 0\right|$

$= \left|r^3\cos^5\theta\right| \le r^3 < \epsilon$

whenever $0 < \sqrt{h^2+k^2} < \epsilon^{1/3}$

If we choose $\delta = \epsilon^{2/3}$

Then $|\phi(h,k) - 0| < \epsilon$ whenever $0 < h^2+k^2 < \delta \; (=\epsilon^{2/3})$

So $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$$|\psi(h,k) - 0| = \left| -\frac{2k^3}{h^2+k^2} \right|$$

Put $h = r\cos\theta$, $k = r\sin\theta$

$$= \left| -2 \cdot \frac{r^3 \sin^3\theta}{r^2} \right|$$

$$= 2|r \sin\theta| \leq 2r < \epsilon$$

whenever $0 < \sqrt{x^2+y^2} < \delta \left(= \frac{\epsilon}{2}\right)$.

$\therefore \psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\therefore f$ is differentiable at $(0,0)$.

Ex. Let $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

Show that f is not differentiable at $(0,0)$ though f is continuous at $(0,0)$.

Example: If $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Show that f is differentiable at $(0,0)$

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \cancel{\lim_{h\to 0} \frac{0 \cdot \frac{h^2-k^2}{h^2+k^2}}{h}}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0 \Rightarrow f_x(0,0) = 0$$

$$\lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k}$$

$$= 0.$$

In order to show, f is differentiable at $(0,0)$,

$f(0+h, 0+k) - f(0,0) = f(h,k) - f(0,0)$

$= \cancel{h f_x(0,0) + k f_y(0,0)}$

$= hk \cdot \dfrac{h^2-k^2}{h^2+k^2}$

$hk \cdot \dfrac{h^2-k^2}{h^2+k^2} = h f_x(0,0) + k f_y(0,0) + h\,\phi(h,k) + k\,\psi(h,k)$

where $\phi, \psi \to 0$ as $(h,k) \to (0,0)$

$= h \cdot 0 + k \cdot 0 + h \cdot \dfrac{k}{2} \dfrac{h^2-k^2}{h^2+k^2} + k \cdot \dfrac{h}{2} \dfrac{h^2-k^2}{h^2+k^2}$

$$\phi(h,k) = \begin{cases} \dfrac{k}{2} \cdot \dfrac{h^2-k^2}{h^2+k^2}, & h^2+k^2 \neq 0 \\ 0, & h^2+k^2 = 0 \end{cases}$$

$$\psi(h,k) = \begin{cases} \dfrac{h}{2} \dfrac{h^2-k^2}{h^2+k^2}, & h^2+k^2 \neq 0 \\ 0, & h^2+k^2 = 0 \end{cases}$$

Let $\epsilon > 0$ be chosen arbitrarily.

Put $h = r\cos\theta$, $k = r\sin\theta$.

$\left|\phi(h,k) - 0\right| = \left|\frac{1}{2} r\sin\theta \cdot \cos 2\theta\right|$

$= \dfrac{r}{2} \left|\sin\theta \cos 2\theta\right|$

$\leq \dfrac{r}{2}$ $\quad [\because |\sin\theta| \leq 1,\ |\cos 2\theta| \leq 1]$

$< \epsilon$ whenever $0 < \sqrt{h^2+k^2} < \delta\,(=2\epsilon)$

$\therefore \phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Similarly

$\left|\psi(h,k) - 0\right| = \left|\frac{1}{2} r\cos\theta \cdot \cos 2\theta\right|$

$= \dfrac{r}{2} \left|\cos\theta \cdot \cos 2\theta\right|$

$\leq \dfrac{r}{2} < \epsilon$

whenever $0 < \sqrt{h^2+k^2} < \delta\,(=2\epsilon)$.

$\therefore \psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Sufficient condition for differentiability at a point :

Theorem: Let $f: S \to R$ where $S \subset R^2$. If (a,b) be a point of the domain of function f such that

(i) f_y is continuous at (a,b)

(ii) f_x exists at (a,b), then f is differentiable at (a,b).

Note → (1) If f is not differentiable at (a,b), the partial derivatives cannot be continuous at (a,b).

(2) The condition of continuity of one of the partial derivatives at a point is not necessary for differentiability of a function at a point.

Example: $$f(x,y) = \begin{cases} x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y}, & xy \neq 0 \\ x^2\sin\frac{1}{x}, & x\neq 0, y=0 \\ y^2\sin\frac{1}{y}, & x=0, y\neq 0 \\ 0, & x=0=y \end{cases}$$

Show that f is differentiable at $(0,0)$ but ~~f be~~ the partial derivatives are not continuous at $(0,0)$.

→ Verify that $f_x(0,0) = 0 = f_y(0,0)$.

as $\lim\limits_{t\to 0} t\sin\frac{1}{t} = 0$.

$\forall\ hk \neq 0$

$f(h,k) - f(0,0) = h\phi(h,k) + k\psi(h,k)$

$h^2\sin\frac{1}{h} + k^2\sin\frac{1}{k} = h\cdot\phi(h,k) + k\psi(h,k)$

$$\phi(h,k) = \begin{cases} h\sin\frac{1}{h}, & h\neq 0 \\ 0, & h=0 \end{cases}$$

$\therefore\ \phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$$\psi(h,k) = \begin{cases} k \sin\frac{1}{k}, & k \neq 0 \\ 0, & k = 0 \end{cases}$$

$\therefore$ $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\therefore$ f is differentiable at $(0,0)$.

$$f_x(x,y) = \begin{cases} 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right), \\ = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \; x \neq 0 \\ 0, \; x = 0. \end{cases}$$

$$f_y(x,y) = \begin{cases} 2y \sin\frac{1}{y} - \cos\frac{1}{y}, & y \neq 0 \\ 0, & y = 0. \end{cases}$$

For $x \neq 0$

Now $\lim\limits_{x \to 0} \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist as $\lim\limits_{x \to 0} \cos\frac{1}{x}$ does not exist.

So $f_x(x,y)$ is not continuous at $(0,0)$

Similarly $f_y(x,y)$ is not continuous at $(0,0)$.

Ex: $f(x,y) = \begin{cases} (x^2+y^2) \sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$

Show that f is differentiable at $(0,0)$ but partial derivatives of f are not continuous at $(0,0)$.

Total differential:

Definition: If $f(x,y)$ be a differentiable function, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is called the total differential of f.

[$\Delta f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y)$ is the total increament]

Sometimes we call the total differential of as df.

$$df = f_x dx + f_y dy \rightarrow (1).$$

Note: (1) A function has a total differential if its partial derivatives are continuous.

(2) The differentials of independent variables coincide with their increament i.e. $dx = \Delta x$, $dy = \Delta y$.

Example 1. Find $df(1,2)$:

$$f(x,y) = x^2 + xy + y^2 - 4\ln x - 10\ln y$$

$$\frac{\partial f(x,y)}{\partial x} = 2x + y - \frac{4}{x}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 2 + 2 - 4 = 0.$$

$$\frac{\partial f(x,y)}{\partial y} = x + 2y - \frac{10}{y}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 1 + 4 - \frac{10}{2} = 0.$$

$$df(1,2) = \left.\frac{\partial f}{\partial x}\right|_{(1,2)} dx + \left.\frac{\partial f}{\partial y}\right|_{(1,2)} dy$$

$$= 0.$$

Example: Show that the expression $(3x+y)\,dx + (x+3y)\,dy$ is a total differential of some function

Let $du(x,y) = (3x+y)\,dx + (x+3y)\,dy$

$$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

As x, y are independent variables

$$\frac{\partial u}{\partial x} = 3x+y, \quad (1) \qquad \frac{\partial u}{\partial y} = x+3y \quad (2)$$

Integrating (1) w.r.t x

$$u(x,y) = \frac{3x^2}{2} + xy + \phi(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = x + \phi'(y) \longrightarrow (3)$$

comparing (2) and (3)

$$\phi'(y) = 3y$$

$$\Rightarrow \phi(y) = \frac{3y^2}{2} + c,$$

c is real constant

Hence $u(x,y) = \dfrac{3x^2}{2} + xy + \dfrac{3y^2}{2} + c$.

Chain rule:

Theorem: Let (i) $x = \phi(u,v)$, $y = \psi(u,v)$ be two functions of u, v defined in a domain $S\ (\subset \mathbb{R}^2)$ and differentiable at a point $(u,v) \in S$. (ii) $z = f(x,y)$ be defined on $S_1\ (\subset \mathbb{R}^2)$ and differentiable at (x,y) of S_1. (iii) S_1 be the image set of S, then z is defined as a function of u, v, is differentiable at the corresponding point (u,v).

So, $z = F(u,v)$ is differentiable function of $\{u,v\}$ and we can write

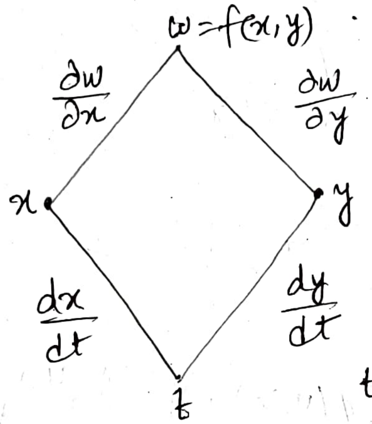
$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\}\ (1)$$

The results (1) is called as chain rule.

Corollary: If in this theorem, x and y be differentiable functions of single variable t, so that the composite function $z = f(\phi(t), \psi(t)) = F(t)$ can be defined, then $z = F(t)$ is differentiable function.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Diagram of chain rule:



$w \to$ Dependent variable

$x, y \to$ Intermediate variable

$t \to$ Independent variable

$$w = f(x(t), y(t))$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

Example 1. Use the chain rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What's the derivative value at $t = \pi/2$.

$\rightarrow$ We apply the chain rule to find $\frac{dw}{dt}$ as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(xy)\cdot\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy)\cdot\frac{d}{dt}(\sin t)$$

$$= (+y)(-\sin t) + x\cdot\cos t$$

$$= \sin t\cdot(-\sin t) + (\cos t)\cdot(\cos t)$$

$$= \cos^2 t - \sin^2 t = \cancel{\sin 2t}\ \cos 2t.$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\cdot\frac{\pi}{2}\right) = \cos\pi = -1.$$

Example 2. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$w = x + 2y,\quad x = \frac{r}{s},\quad y = r^2 + \ln s.$

$$\rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

$$\frac{\partial w}{\partial r} = 1\cdot\frac{1}{s} + 2\cdot 2r = \frac{1}{s} + 4r$$

$$\frac{\partial w}{\partial s} = 1\cdot\left(-\frac{r}{s^2}\right) + 2\cdot\frac{1}{s} = \frac{2}{s} - \frac{r}{s^2}.$$

Ex. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2,\ x = r - s,\ y = r + s.$

Implicit differentiation:

Suppose that (i) The function $F(x, y)$ is differentiable (ii) The equation $F(x,y)=0$ defines y implicitly as a differentiable function of x.

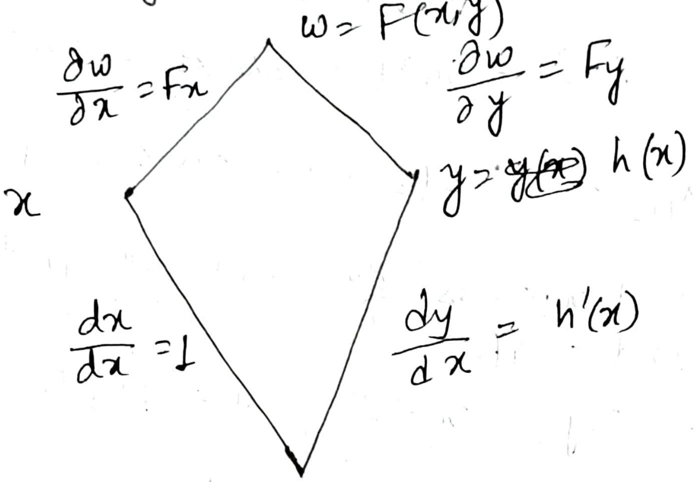
Since $w = F(x, y) = 0 \Rightarrow 0 = \frac{dw}{dx} = F_x\frac{dx}{dx} + F_y\frac{dy}{dx}$

$$\Rightarrow F_x\cdot 1 + F_y\frac{dy}{dx} = 0$$

If $F_y \neq 0$, Then $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

This is called the implicit differentiation for a function $F(x, y)$.

Dependency diagram



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \frac{dy}{dx}$$

Example 3: Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

$\rightarrow$

Let $F(x,y) = y^2 - x^2 - \sin xy$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{-x\cos y + 2y}$$

Example: The relationships $u = f(x, y)$, $v = F(x, y)$, where f and F are differentiable function of x and y. Specify x and y as differentiable function of u, v. Prove that

$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) = 1$$

$\rightarrow u = f(x,y) \Rightarrow du = u_x\, dx + u_y\, dy$ — (1)

$v = F(x,y) \Rightarrow dv = v_x\, dx + v_y\, dy$ — (2)

From (1) and (2),

$$dx = \frac{v_y\, du - u_y\, dv}{u_x v_y - u_y v_x} \quad \text{— (3)}, \quad u_x v_y - u_y v_x \neq 0$$

Since x is differentiable function of u and v, $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$ (4)

From (3) and (4),

$$\frac{\partial x}{\partial u} = \frac{v_y}{u_x v_y - u_y v_x}, \quad \frac{\partial x}{\partial v} = -\frac{u_y}{u_x v_y - u_y v_x}$$

In a similar way, from (1) and (2)

$$dy = \frac{-v_x\, du + u_x\, dv}{u_x v_y - u_y v_x} \quad \text{— (5)}$$

Since y is differentiable function of u, and v, $dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$ — (6)

So from (5) and (6)

$$\frac{\partial y}{\partial u} = -\frac{v_x}{u_x v_y - u_y v_x}, \quad \frac{\partial y}{\partial v} = \frac{u_x}{u_x v_y - u_y v_x}$$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = \frac{u_x v_{xy}}{(u_x v_y - u_y v_x)^2} - \frac{u_y v_x}{(u_x v_{xy} - u_y v_x)^2}$$

$$= \frac{1}{u_x v_y - u_y v_x}$$

$$\Rightarrow (x_u y_v - x_v y_u)(u_x v_{xy} - u_y v_x) = 1$$

[proved]

Ex: If $x = cuv$, $y = c\{(1+u^2)(1-v^2)\}^{1/2}$; c is a nonzero constant, show that

$$\frac{1}{y}\left\{y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y}\right\} = \frac{\left(v\frac{\partial V}{\partial u} + u\frac{\partial V}{\partial v}\right)}{c(u^2+v^2)}$$

where V is any differentiable function of x and y.

→ By Chain rule:

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= V_x \cdot cv + V_y \cdot \frac{c(1-v^2)^{1/2}\cdot(2u)}{2(1+u^2)^{1/2}} \longrightarrow (1)$$

$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= V_x \cdot cu + V_y \cdot \frac{c(1+u^2)^{1/2}\cdot(-2v)}{2(1-v^2)^{1/2}} \longrightarrow (2)$$

$$\cancel{v\frac{\partial V}{\partial u} + u}\frac{\partial V}{\partial v} =$$

From (1)

$$v\cdot\frac{\partial V}{\partial u} = cv^2 V_x + \frac{cuv(1-v^2)^{1/2}}{(1+u^2)^{1/2}} V_y$$

$$= cv^2 V_x + \frac{x}{y}\cdot c(1-v^2)\, V_y$$

$$u\frac{\partial V}{\partial v} = cu^2 V_x - \frac{x}{y}\cdot c(1+u^2)\, V_y$$

$$v\frac{\partial V}{\partial u} + u\frac{\partial V}{\partial v} = c(v^2+u^2)V_x + \frac{cx}{y}V_y(1-v^2-1-u^2)$$

$$= c(u^2+v^2)\left[V_x - \frac{x}{y}V_y\right]$$

$$\frac{v\frac{\partial V}{\partial u}+u\frac{\partial V}{\partial v}}{c(u^2+v^2)}=\frac{1}{\gamma}\left\{y\frac{\partial V}{\partial x}-x\frac{\partial V}{\partial y}\right\}$$

[proved]

§ Directional derivative:

If $f(x,y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x=g(t)$, $y=h(t)$ is

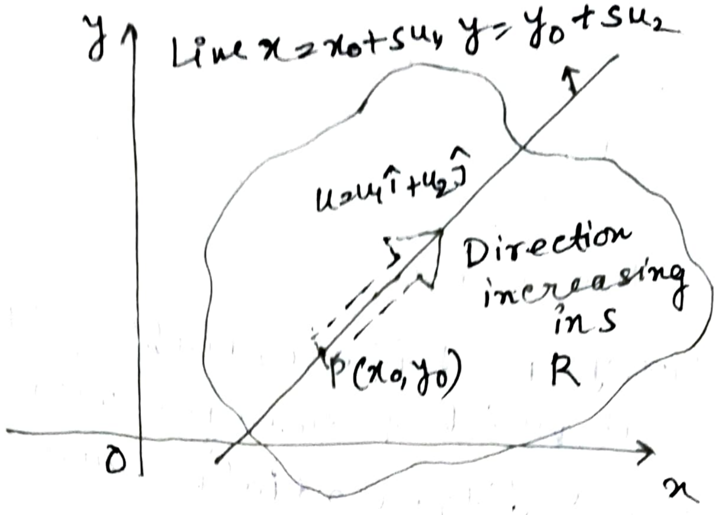
$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

At any point $P_0(x_0,y_0)=P_0(g(t_0),h(t_0))$, this equation gives rate of change of f with respect to increasing and therefore depends on the direction of motion along the curve.

Suppose the function $f(x,y)$ is defined throughout a region R in the xy plane that $P(x_0,y_0)$ in R and $u=u_1\hat{i}+u_2\hat{j}$ is a unit vector, then the equation

$$x=x_0+su_1,\quad y=y_0+su_2$$

parametrize the line through P_0 parallel to u. If the parameter s measures the are length from P_0 in the direction u, we find the rate of change of f at P_0 in the direction of u by calculating $\frac{df}{ds}$ at P_0.



Definition: The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $u = u_1\hat{i} + u_2\hat{j}$ is the number

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \rightarrow (1)$$

provided the limit exists.

The directional derivative defined by equation (1) is also denoted by $D_u f(P_0)$ or $D_u f|_{P_0}$: The derivative of f in the direction of u at P_0

Example: Using the definition, find the derivative of $f(x,y) = x^2 + xy$ ~~in the~~ at $P_0(1,2)$ in the direction of unit vector $u = (1/\sqrt{2})\hat{i} + (1/\sqrt{2})\hat{j}$

$\rightarrow$ Applying definition in eqn (1)

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}},\ 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s}$$

$$= \lim_{s \to 0} \frac{\left(1 + 2\frac{s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \frac{5}{\sqrt{2}}$$

$\therefore$ The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction u is $5/\sqrt{2}$

Another form of directional derivative :

Let f be a function of two independent variables x and y defined over a domain $D \subset R^2$ and $P(a, b) \in D$. Let N be a nbd of (a, b) s.t. $N \subset D$ and $Q(a+h, b+k) \in N$. If α be the angle that the line joining $\overrightarrow{PQ}$ makes with the positive direction of x axis, the direction cosines of $\overrightarrow{PQ}$ are $l = \cos\alpha$, $m = \sin\alpha$

If $\rho = \sqrt{h^2 + k^2}$

Then $h = \rho\cos\alpha$, $k = \rho\sin\alpha$ and $\rho \to 0$ as $Q \to P$

$$\lim_{\rho \to 0} \frac{f(a + \rho\cos\alpha,\ b + \rho\sin\alpha) - f(a, b)}{\rho}$$ if it exists

is called the derivative of $f(x, y)$ at (a, b) in the direction α and denoted by $D_\alpha f(a, b)$. If $\alpha = 0$, the derivative is denoted by $\frac{\partial f(a,b)}{\partial x}$ and if $\alpha = \frac{\pi}{2}$, the derivative is $\frac{\partial f(a,b)}{\partial y}$ and these derivatives are called partial derivatives of $f(x,y)$ at (a,b) w.r.t x and y resp.

Example: Show that $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

has partial derivatives at $(0,0)$ but not the directional derivative in any arbitrary direction. The function is also not continuous at $(0,0)$.

→ Now $\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \frac{0-0}{h} = 0$

$\therefore f_x(0,0) = 0$.

Similarly $\lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$

$= \lim_{k \to 0} \frac{0-0}{k} = 0$

$f_y(0,0) = 0$.

Let us take any arbitrary direction making an angle α with x-axis, then

$D_\alpha f(0,0) = \lim_{\rho \to 0} \frac{f(\rho\cos\alpha, \rho\sin\alpha) - f(0,0)}{\rho}$

$= \lim_{\rho \to 0} \frac{\frac{\rho^2 \cos\alpha \sin\alpha}{\rho^2(\sin^2\alpha + \cos^2\alpha)} - 0}{\rho}$

$= \lim_{\rho \to 0} \frac{\cos\alpha \sin\alpha}{\rho}$, this limit does not exist.

$\therefore D_\alpha f(0,0)$ does not exist.

Also $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist (check) So f is not continuous.

Gradient vector:

Parametric equation of a line:

$$x = x_0 + su_1, \quad y = y_0 + su_2 \quad \text{--- (1)}$$

through $P_0(x_0, y_0)$, which is parametrised with the arc length parameter s increasing in direction of the unit vector $u = u_1\hat{i} + u_2\hat{j}$

$$\left(\frac{df}{ds}\right)_{u,P_0} = \frac{df}{dx}\Big|_{P_0} \frac{dx}{ds} + \frac{df}{dy}\Big|_{P_0} \frac{dy}{ds}$$

[Using chain rule for differentiable fn f]

$$= \frac{df}{dx}\Big|_{P_0} u_1 + \frac{df}{dy}\Big|_{P_0} u_2$$

$$= \left[\frac{df}{dx}\Big|_{P_0} \hat{i} + \frac{df}{dy}\Big|_{P_0} \hat{j}\right] \cdot \left[u_1\hat{i} + u_2\hat{j}\right] \quad \text{--- (2)}$$

From (2), we can say that the derivative of a differentiable function f in the direction u at P_0 is the dot product of u with a special vector, which now we define:

Definition: The gradient vector (or gradient) of $f(x,y)$ is the vector

$$\nabla f = \frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j}$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f\Big|_{P_0} \quad \text{or} \quad \nabla f(x_0, y_0).$$

Theorem: If $f(x,y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{u,P_0} = \vec{\nabla} f\Big|_{P_0} \cdot \vec{u}$$

the dot product of the gradient ∇f at P_0 with the vector u. In brief, $D_u f = \vec{\nabla f} \cdot \vec{u}$

Properties of directional derivative $D_u f = \nabla f \cdot u = |\nabla f| \cos\theta$:

(1) The function f increases most rapidly ~~while~~ when $\cos\theta = 1$, which means that $\theta = 0$ and u is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_u f = |\nabla f| \cos\theta = |\nabla f| \cos 0 = |\nabla f|.$$

(2) f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_u f = |\nabla f| \cos(\pi)$

$$= -|\nabla f|$$

(3) Any direction u orthogonal to a gradient $\nabla f \neq 0$ is direction of zero change in f because θ then equals $\pi/2$ and

$$D_u f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example: Find the derivative of $f(x,y)$ where $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction $v = 3\hat{i} - 4\hat{j}$.

→ Direction of $\vec{v}$ vector is the unit vector along $\vec{v}$

$$\vec{u} = \frac{\vec{v}}{|v|} = \frac{3}{\sqrt{3^2+4^2}}\hat{i} - \frac{4}{\sqrt{3^2+4^2}}\hat{j}$$

$$= \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

The partial derivatives of f ~~are~~ at (2,0) are

$$f_x(x,y)\Big|_{(2,0)} = e^y - y\sin(xy)\Big|_{(2,0)}$$

$$= 1 - 0 = 1$$

$$f_y(x,y)\Big|_{(2,0)} = xe^y - x\sin(xy)\Big|_{(2,0)}$$

$$= 2\cdot 1 - 2\cdot 0 = 2.$$

The gradient of f at (2,0) is

$$\vec{\nabla} f\Big|_{(2,0)} = f_x(2,0)\,\hat{i} + f_y(2,0)\,\hat{j}$$

$$= 1.\hat{i} + 2\hat{j} = \hat{i} + 2\hat{j}.$$

The derivative of at (2,0) in the direction $\vec{u}$

$$= \vec{\nabla} f\Big|_{(2,0)} \cdot \vec{u} = (\hat{i} + 2\hat{j})\cdot\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= \frac{3}{5} - \frac{8}{5} = -1$$

Example: Find the directions in which $f(x,y) = (x^2/2) + (y^2/2)$

(i) increases most rapidly at (1,1)

(ii) decreases most rapidly at (1,1).

(iii) What are the directions of zero change in f at (1,1)?

$\rightarrow \quad f(x,y) = x^2/2 + y^2/2$

$$\nabla f\Big|_{(1,1)} = \cancel{2x} \; \frac{\partial f}{\partial x}\Big|_{(1,1)} \hat{i} + \frac{\partial f}{\partial y}\Big|_{(1,1)} \hat{j}$$

$$= \hat{i} + \hat{j}.$$

Its direction is

$$\vec{u} = \cancel{\tfrac{1}{\sqrt{2}}}\; \frac{1}{\sqrt{1^2+1^2}} \hat{i} + \frac{1}{\sqrt{1^2+1^2}} \hat{j}$$

$$= \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j}.$$

The function increases most rapidly in the direction of $\vec{\nabla f}$ at $(1,1)$

(ii)

The function decreases most rapidly in the direction of $-\vec{\nabla f}$ at $(1,1)$.

$$-\vec{u} = -\frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}.$$

(iii) The direction of zero change at $(1,1)$ are the directions orthogonal to $\vec{\nabla f}$.

Let $\vec{v}$ be direction of unit vector orthogonal to $\vec{u}$

$\vec{v} = v_1 \hat{i} + v_2 \hat{j}$

$$\vec{u} \cdot \vec{v} = 0$$

$$\frac{v_1}{\sqrt{2}} + \frac{v_2}{\sqrt{2}} = 0.$$

$$v_2 = -v_1$$

$$\vec{v} = -\frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} \quad \text{or} \quad \vec{v} = \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}.$$

Ex 1. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of vector $(2,5)$.

Ex 2: Let $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that all directional derivatives exist at (0,0) but f is not differentiable at (0,0)

→ Directional derivative of f at (0,0) along the direction of unit vector is

$$\lim_{h\to 0} \frac{f(ha, hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3 a^3}{h^2(a^2+b^2)}}{h}$$

$$= \frac{a^3}{a^2+b^2}$$

$f_x(0,0) = 1$ [Take $a = \cos\alpha$, $b = \sin\alpha$, $\alpha = 0$]

$f_y(0,0) = 0$ [$\alpha = \frac{\pi}{2}$]

In order to f to be differentiable at (0,0),

$f(0+h, 0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + h\phi(h,k) + k\psi(h,k)$

where $\phi(h,k), \psi(h,k) \to 0$ as $(h,k)\to(0,0)$

So $\frac{h^3}{h^2+k^2} = h\cdot 1 + k\cdot 0 + h\phi(h,k) + k\psi(h,k)$

In particular $h = 2k$.

$\frac{8k^3}{5k^2} = 2k + 2k\phi(h,k) + k\psi(h,k)$

$\Rightarrow \frac{8}{5} = 2 + 2\phi(h,k) + \psi(h,k)$

$\Rightarrow 2\phi(h,k) + \psi(h,k) = 8/5 - 2 = -\frac{2}{5}$

L.H.S $\to 0$ as $(h,k)\to(0,0)$ but R.H.S $= -2/5$ which is a contradiction.

So f is not differentiable at (0,0).

Ex Examine the existence of directional derivative of f at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x}{|y|}\sqrt{x^2+y^2}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Also show that f is not differentiable at $(0,0)$.

Higher Order Partial Derivatives:

The partial derivatives of a function $f(x,y)$; i.e. $f_x(x,y)$ and $f_y(x,y)$ are themselves functions of x and y and in some cases partial derivatives of f_x and f_y w.r.t both x and y may exist at that point. We may define the second order partial derivatives as follows:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

provided the limit exists.

$$\frac{\partial^2 f}{\partial y^2} = f_{yx}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$$

provided the limit exists.

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

provided limit exists.

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{h}$$

provided limit exists.

Ex 1. Let $f(x,y) = \begin{cases} xy, & \text{if } |x| \geq |y| \\ -xy, & \text{if } |x| < |y| \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$

$$\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h \cdot 0 - 0}{h} = 0$$

$$\Rightarrow f_x(0,0) = 0$$

$$\lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{-0 \cdot k - 0}{k}$$

$$= 0$$

$$\Rightarrow f_y(0,0) = 0$$

$$\lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{-hk - 0}{h} \quad [\because h \to 0, k \not\to 0 \;\; |h| < |k|]$$

$$= -k$$

$$f_x(0,k) = -k$$

$$\lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k - 0}{k} = -1$$

$$\Rightarrow f_{yx}(0,0) = -1$$

$$\lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{hk - 0}{k} \quad [\because h \not\to 0 \;\; k \to 0 \;\; |h| > |k|]$$

$$= h$$

$$\Rightarrow f_y(h,0) = h$$

$$\lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h}$$

$$= 1$$

$$\Rightarrow f_{xy}(0,0) = 1$$

So $f_{yx}(0,0) \neq f_{xy}(0,0)$.