Birth-Death Process (M/M/1)

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Q1: Customers arrive at a counter managed by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of ~~36~~ 36 per hour. Find the average waiting time of a customer.

Ans: - (M|M|1)

Given that

Mean arrival rate $(\lambda) = 20$ per hour.

Mean service rate $(\mu) = 36$ per hour.

$\therefore$ The average waiting time of a customer in ~~a~~ a queue

$$E[W_q] = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{36(36-20)} \text{ hour.}$$

$$= \frac{20}{36 \times 16} = \frac{20}{576} = 0.0347 \text{ hours.}$$

$$= 0.0347 \times 60 = 2.08 \text{ mins.}$$

$$= 2.08 \times 60 = 125 \text{ seconds.}$$

Q2: Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call assumed to be distributed exponentially, with mean 3 minutes. Find the followings:

A) what is the probability that a person arriving at the booth will have to wait.

B) what is the average length of the queues that form from time to time?

C) the telephone company will install a second booth when convinced that an arrival would expect to have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

D) find the average number of units in the system

F) what is the probability that an arrival will have to wait more than 10 minutes before the phone is free.

E) what is the probability that it will take, an arrival, more than 10 minutes altogether to wait for the phone and complete his call?

Ans 2 :- Mean arrival rate $(\lambda) = \frac{1}{10}$.

Mean service rate $(\mu) = \frac{1}{3}$.

(a) Prob. (arrival will have to wait)

$$\rho = 1 - P_0 = \frac{\lambda}{\mu} = \frac{3}{10} = 0.3.$$

(b) Average length of the queue

$$= \frac{\mu}{\mu - \lambda}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} - \frac{1}{10}} = \frac{10}{7} = 1.43$$

(c) Average waiting time of an (customers) arrival in the queue.

$$E[W_q] = \frac{\lambda}{\mu(\mu - \lambda)} \quad \text{—— (1)}$$

According to the condition

$E[W_q] = 3.$

$\mu = \frac{1}{3}$, $\lambda = ?$ $(\lambda = \lambda' \text{ say})$

From eq. (1)

$$3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3} - \lambda'\right)}$$

$$3 = \frac{\lambda'}{\frac{1}{9} - \frac{\lambda'}{3}} \Rightarrow 3 = \frac{\lambda'}{\frac{1 - 3\lambda'}{9}} \Rightarrow 3 = \frac{9\lambda'}{1 - 3\lambda'}.$$

$$\cancel{3}(1 - 3\lambda') = \overset{3}{\cancel{9}}\lambda'$$

$\cancel{3}(1-3\lambda') = \overset{3}{\cancel{9}}\lambda'$

$\Rightarrow 1-3\lambda' = 3\lambda'$.

$\Rightarrow 1-3\lambda' - 3\lambda' = 0$.

$\Rightarrow 1 = 6\lambda'$

$\boxed{\lambda' = \frac{1}{6}}$

Increase in mean arrival rate $= \lambda' - \lambda$

$= \frac{1}{6} - \frac{1}{10}$.

$= \frac{1}{15} = 0.067$

$= 0.067$ arrival per min.

Increase in the flow of arrival: $\frac{1}{15} \times 60 = 4$ per hour.

$\therefore$ The second booth is justified it the increase in arrival rate is 0.667 customer per min (or 4 customer per hour).

(d) Average no. of units in the system

$$E[L_s] = \frac{\lambda}{\mu-\lambda} = \frac{\frac{1}{10}}{\frac{1}{3}-\frac{1}{10}} = \frac{3}{7} = 0.43 \text{ customers.}$$

(e) Prob. (time that a unit spend in the system ≥ 10)

$\int^{\infty}$

$$= \int_{10}^{\infty} \theta(v)\,dv.$$

$$= \int_{10}^{\infty} (\mu-\lambda)\, e^{-(\mu-\lambda)v}\,dv.$$

$$= \left[-e^{-(\mu-\lambda)v}\right]_{10}^{\infty}$$

$$= \left[e^{-(\mu-\lambda)10}\right] = e^{-\left(\frac{1}{3}-\frac{1}{10}\right)10}$$

$$= e^{-7/3} = e^{-2.33} = \underline{\underline{0.10}}$$

(f) Prob. [Waiting time of an arrival in queue ≥ 10]

$$= \int_{10}^{\infty} \phi(\omega)\,d\omega.$$

$$= \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)\, e^{-(\mu-\lambda)\omega}\,d\omega.$$

$$= \frac{\lambda}{\mu} e^{-10(\mu-\lambda)}$$

$$= \frac{3}{10} e^{-10\left(\frac{1}{3}-\frac{1}{10}\right)}$$

$$= \frac{3}{10} e^{-2.33} = \frac{3}{10} \times 0.10 = 0.03.$$

<u>Q3</u> :— A T.V repairman finds that the time spent on his job has an exponential dist. with

on this job has an exponential dist. with mean 30 minutes. If he repairs sets in the order in which the customers they come in, and if the arrival of the sets is approximately Poisson with an average of 10 per 8 hour day. What is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought it.

Ans:- $\mu = \frac{1}{\cancel{30}} \times \cancel{60}^{2} = 2$ sets per hour.

$\lambda = \frac{10}{8} = \frac{5}{4}$ per hour.

$\rho = \frac{\mu}{\lambda} = \frac{\frac{5}{4}}{\frac{2}{1}} = \frac{5}{4} \times \frac{1}{2} = \frac{5}{8}.$

Now the Prob. that there is no unit in the queue $= P_0 = 1 - \rho = 1 - \frac{5}{8} = \frac{3}{8}.$

Hence the expected idle time for the repairman in 8 hour day.

$= 8 \times \underline{3} = 3$ hrs.

$= 8 \times \frac{3}{8} = 3$ hrs.

Again, expected no. of jobs.

$$E[L_s] = \frac{\lambda}{\mu-\lambda} = \frac{5/4}{2-(5/4)} = 5/3.$$

Q4 There is congestion on the platform of a railway station. The trains arrive at the rate of 30 trains per day. The waiting time for any train to hump is exp. dist. with an average of 36 minutes. Calculate the following

(a) the mean queue size.

(b) the prob. that queue size exceeds 10.

If the input of trains increases to an average 33 per day. What is the change in (a) and (b)

Ans 4 The trains arrive at the rate of 30 trains per day.

λ = 30 trains per day.

1 day = 30 trains

$$\Rightarrow \frac{1}{\lambda} = \frac{60 \times 24}{30} = 48 \text{ mins.}$$

or $\lambda = \frac{1}{48}$ min.

$$\Rightarrow \frac{1}{\mu} = 36 \text{ mins.}$$

or

$$\mu = \frac{1}{36} \text{ mins.}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1/48}{1/36} = 0.75.$$

(a) Mean queue size

$$E[L_s] = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(b) Prob. (queue size ≥ 10)

1 − P(queue size is less than 10).

$$1 - (1-\rho^{10})$$

$$= \rho^{10} = (0.75)^{10} = 0.06 \text{ (approx.)}$$

When the input increases to 33 trains per day, we have.

λ = 33 trains / day

$\lambda = 33$ trains / day

$\lambda = \frac{33}{60 \times 24} = \frac{11}{480}$ trains per minute.

$\mu = \frac{1}{36}$ trains per minute.

$\rho = \frac{\lambda}{\mu} = \frac{11}{480} \times 36 = 0.83$.

Thus, we get.

(a) Average queue size $L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.9$
or 5 trains (approx).

(b) $P(\geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (approx).

<u>Q5</u> Let an average 96 patients per 24 hour day require the service of an emergency clinic. Also an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100 per patient treated to obtain

. . .. costs the clinic Rs. 100 per patient treated to obtain on average servicing time of 10 minutes and that each minute of decrease in this average time would cost Rs. 10 per Patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from one and one-third patient to ~~bar~~ half a patient.

Ans

$$\lambda = \frac{96}{24 \times 60} = \frac{1}{15} \text{ Patients/minute}$$

$$\mu = \frac{1}{10} \text{ Patients/minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3}.$$

Expected no. of patients in the waiting line $= \dfrac{\rho^2}{1-\rho} = \dfrac{\left(\frac{2}{3}\right)^2}{1-\frac{2}{3}} = \dfrac{4}{3}.$

Fraction of time during which there

are no patients $= 1-\rho = 1-\frac{2}{3} = \frac{1}{3}$.

Average size of the queue $E[L_s] = \frac{\rho^2}{1-\rho}$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$E[L_q] = \frac{1}{2}$, $\lambda = \frac{1}{15}$, $\mu = ?$

$$E[L_q] = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\left(\frac{1}{15}\right)^2}{\mu\left(\mu-\frac{1}{15}\right)}$$

$$\frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu\left(\mu-\frac{1}{15}\right)}$$

$\mu = \frac{2}{15}$ patients / minute.

$\therefore$ Average rate of treatment required

$$= \frac{1}{\mu} = \frac{15}{2} = 7\frac{1}{2} \text{ minutes.}$$

So the decrease in the average rate of treatment $= 10 - 7\frac{1}{2} = 2\frac{1}{2}$ minutes.

$\therefore$ Budget per patient $=$ Rs $100 + \frac{5}{2} \times 10$

$=$ Rs 125.

In order to get the required Size of the queue the budget should be increased from Rs. 100 per patient to Rs. 125 per patient.