Tuesday, July 19, 2022 9:43 AM

Transient and Recurrent state

Recurrent $\downarrow$ Persistent

A state j is said to be persistent if $f_{jj} = 1$

A state j is said to be transient if $f_{jj} < 1$.

State j

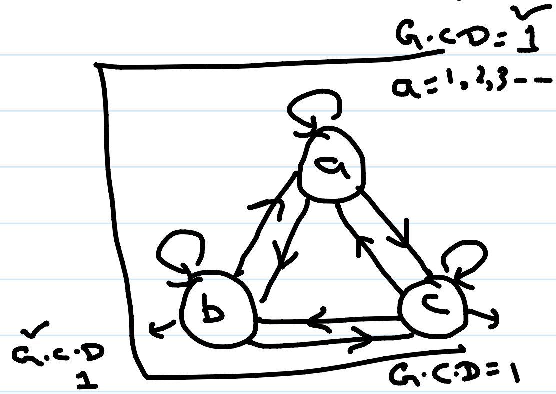
- Persistent ($f_{jj} = 1$)
  - Null Persistent ($\mu_{jj} = \infty$)
  - Non-Null Persistent (Positive Persistent) ($\mu_{jj} < \infty$)
- transient ($f_{jj} < 1$).

[μ = Mean recurrence time].

Ergodic state :-

If state j is (+ve) persistent and aperiodic, then state j is called an -

aperiodic, then state j is called an ergodic state.



Mean Recurrence time

Expected time of return to state i starting from state i.

$$\mu_i = \begin{cases} \sum_{n=1}^{\infty} n \cdot f_{ii}^{(n)} & \text{if } i \text{ is recurrent} \\ \infty & \text{if } i \text{ is transient} \end{cases}$$

Example

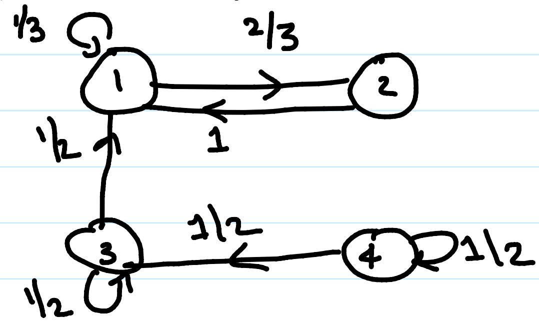
Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

Find transient, recurrent and ergodic states.

1/3 2/2

Find transient, recurrent and ergodic states.



$\boxed{1 \to 2 \to 1} \to 1$

one column

Recurrent (Persistent) State = 1 and 2.

Transient State = 3 and 4.

<u>Theorem</u> $i \leftrightarrow j$

i is recurrent $\leftrightarrow$ j is also recurrent.

<u>State 1</u> :-

$f_{11}^{(1)} = \frac{1}{3}$.

$f_{11}^{(2)} = \frac{2}{3} \times 1 = \frac{2}{3}$.

$f_{11}^{(3)} = 0$.

$f_{11}^{(4)} = 0$

$\vdots$

$$F_{11} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + \cdots$$

$$= \frac{1}{3} + \frac{2}{3} + 0 + 0 + \cdots$$

$$= \frac{3}{3} = 1$$

State 1 is recurrent (Persistent)

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State 2: -

$f_{22}^{(1)} = 0$

$f_{22}^{(2)} = 1 \times \frac{2}{3} = \frac{2}{3}.$

$f_{22}^{(3)} = 1 \times \frac{1}{3} \times \frac{2}{3}$

$f_{22}^{(4)} = 1 \times \left(\frac{1}{3}\right)^2 \times \frac{2}{3}.$

$f_{22}^{(5)} = 1 \times \left(\frac{1}{3}\right)^3 \times \frac{2}{3}.$

$f_{22}^{(6)} = 1 \times \left(\frac{1}{3}\right)^4 \times \frac{2}{3}$

$\vdots$

$F_{22} = f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + \cdots$

$= 0 + \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \left(\frac{1}{3}\right)^2 \times \frac{2}{3} + \left(\frac{1}{3}\right)^3 \times \frac{2}{3} + \cdots$

$= \frac{2}{3}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right]$ → G.P.

$= \frac{2}{3}\left[\frac{1}{1-\frac{1}{3}}\right]$

$\frac{a}{1-r}$ (a → First term, r → common ratio.)

$= \frac{2}{3} \times \frac{1}{\frac{2}{3}} = \cancel{\frac{2}{3}} \times \cancel{\frac{3}{2}} = 1.$

State 2 is recurrent (Persistent)

State 3 $\rightarrow$

$f_{33}^{(1)} = \frac{1}{2} < 1. \implies F_{33} = \frac{1}{2} + 0 + 0 = \frac{1}{2}.$

State 3 is transient state.

$f_{44}^{(1)} = \frac{1}{2}$

$f_{44}^{(2)} = 0$

$\vdots$

$F_{44} = \frac{1}{2} < 1$

State 4 is Transient

<u>Ergodic state</u>

Positive Persistent + aperiodic

Persistent = 1 and 2.

Positive Persistent = $\mu_{jj} < \infty.$

$$\mu_{jj} = \sum_{n=1}^{\infty} n . f_{jj}^{(n)}.$$

$j = 1.$
$[n = 1, 2, 3, \ldots .$

$\mu_{11} = 1 . f_{11}^{(1)}$

$= 1 \cdot \ldots$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

$$\mu_{11} = 2 \cdot f_{11}^{(2)}$$

$$= 2 \times \frac{2}{3} = \frac{4}{3}$$

$$\mu_{11} = 3 \cdot f_{11}^{(3)}$$

$$= 0$$

$$\vdots$$

$$\mu_{11} = \frac{1}{3} + \frac{4}{3} + 0 + 0$$

$$= \frac{5}{3} < \infty. \quad [\text{+ve persistent}].$$

Aperiodic $\longrightarrow$ 1 is aperiodic.

Hence state is ergodic state.

Similarly

State 2 $\longrightarrow$ (+ve) persistent and aperiodic

State 2 is also ergodic state.

<u>Example 2</u>

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2/3 & 0 & 1/3 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{array}$$

1) Transient
2) Recurrent
3) Ergodic state.

$$\begin{array}{c} 2 \\ 3 \end{array}\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \quad \text{State.}$$

for $\quad f_{00}^{(1)} = \frac{2}{3}.$

$$f_{00}^{(2)} = \frac{1}{3} \times \frac{1}{2}$$

$$f_{00}^{(3)} = \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2}.$$

$$f_{00}^{(4)} = \frac{1}{3} \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2}$$

$$\vdots$$

$$F_{00} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + \cdots$$

$$= \frac{2}{3} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} + \cdots$$

$$\frac{2}{3} + \frac{1}{3} \times \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right].$$

↳ G.P.

$$\frac{2}{3} + \frac{1}{6}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{3} + \frac{1}{6}\left[\frac{1}{\frac{1}{2}}\right]$$

$$\frac{2}{3} + \frac{\cancel{2}\,1}{\cancel{6}\,3} = \frac{3}{3} = 1 \quad \text{(recurrent)}$$

$\therefore$ State 2 is also recurrent

$i \leftrightarrow j$ &