E11 [Gambler Ruin Problem]

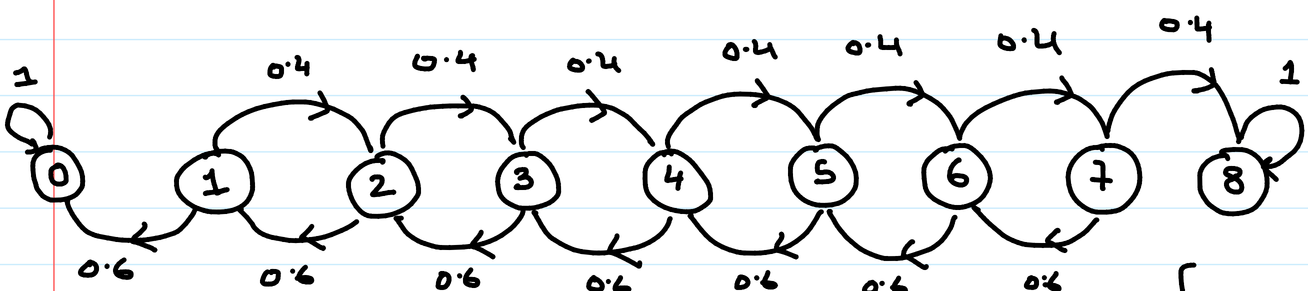
Wednesday, July 27, 2022 2:08 PM

Smith — 3 dollar

get out from jail — 8 dollars

(1) He bets one dollar each time.

[win one dollar = 0.4
lose one dollar = 0.6]



[$i = 1,2,3,4,5,6,7$]

$$\phi_{(i)} = 0.4\phi_{(i+1)} + 0.6\phi_{(i-1)}$$

[$\phi_{(0)} = 0.$ → failure

$\phi_{(8)} = 1$ → success]

$$\phi_{(1)} = 0.4\phi_{(2)} + 0.6\phi_{(0)}$$

$$\phi_{(2)} = 0.4\phi_{(3)} + 0.6\phi_{(1)}$$

$$\phi_{(3)} = 0.4\phi_{(4)} + 0.6\phi_{(2)}$$

$$\phi_{(4)} = 0.4\phi_{(5)} + 0.6\phi_{(3)}$$

$$\phi_{(5)} = 0.4\phi_{(6)} + 0.6\phi_{(4)}$$

$$\phi_{(6)} = 0.4\phi_{(7)} + 0.6\phi_{(5)}$$

$$\phi_{(7)} = 0.4\phi_{(8)} + 0.6\phi_{(6)}$$

Solving the eqs.

$$\phi_{(1)} = 0.4\,\phi_{(2)} + 0.6\,\phi_{(0)} \qquad [\because \phi_{(0)} = 0]$$

$$\boxed{\phi_{(1)} = 0.4\,\phi_{(2)}} \quad \text{—} \quad (1).$$

$$\phi_{(2)} = 0.4\,\phi_{(3)} + 0.6\,\underline{\phi_{(1)}}$$

$$\phi_{(2)} = 0.4\,\phi_{(3)} + 0.6 \times 0.4\,\phi(2).$$

$$\phi_{(2)} - 0.24\,\phi_{(2)} = 0.4\,\phi_{(3)}.$$

$$0.76\,\phi_{(2)} = 0.4\,\phi_{(3)}$$

$$\boxed{\phi_{(2)} = \frac{0.4}{0.76}\,\phi_{(3)}} \quad \text{—} \quad (2)$$

$$\phi(3) = 0.4\,\phi_{(4)} + 0.6\,\phi_{(2)}$$

$$\phi_{(3)} = 0.4\,\phi_{(4)} + 0.6 \times \frac{0.4}{0.76}\,\phi_{(3)}.$$

$$\phi_{(3)} = 0.4\,\phi_{(4)} + 0.316\,\phi_{(3)}$$

$$\phi_{(3)} - 0.316\,\phi_{(3)} = 0.4\,\phi_{(4)}$$

$$0.684\,\phi_{(3)} = 0.4\,\phi_{(4)}.$$

$$\phi_{(3)} = \frac{0.4}{0.684}\,\phi_{(4)}.$$

$$\boxed{\phi_{(3)} = 0.585\,\phi_{(4)}} \quad \text{—} \quad (3)$$

$$\phi(4) = 0.4\,\phi(5) + 0.6\,\phi_{(3)}$$

$$\phi_{(4)} = 0.4\,\phi_{(5)} + 0.6 \times 0.585\,\phi_{(4)}$$

$$\phi_{(4)} - 0.351\,\phi_{(4)} = 0.4\,\phi_{(5)}$$

$$0.649\,\phi_{(4)} = 0.4\,\phi_{(5)}$$

$$\phi_{(4)} = \frac{0.4}{0.649}\,\phi_{(5)}$$

$$\boxed{\phi_{(4)} = 0.616\,\phi_{(5)}} \quad \text{— } 4$$

$$\phi_{(5)} = 0.4\,\phi_{(6)} + 0.6\,\phi_{(4)}$$

$$\phi_{(5)} = 0.4\,\phi_{(6)} + 0.6 \times 0.616\,\phi_{(5)}.$$

$$\phi_{(5)} - 0.3696\,\phi_{(5)} = 0.4\,\phi_{(6)}$$

$$0.6304\,\phi_{(5)} = 0.4\,\phi_{(6)}$$

$$\phi_{(5)} = \frac{0.4}{0.6304}\,\phi_{(6)}$$

$$\boxed{\phi_{(5)} = 0.6345\,\phi_{(6)}} \quad \text{— } 5$$

$$\phi_{(6)} = 0.4\,\phi_{(7)} + 0.6\,\phi_{(5)}$$

$$\phi_{(6)} = 0.4\,\phi_{(7)} + 0.6 \times 0.6345\,\phi_{(6)}$$

$$\phi_{(6)} - 0.3807\,\phi_{(6)} = 0.4\,\phi_{(7)}$$

$$0.6193\,\phi_{(6)} = 0.4\,\phi_{(7)}$$

$$\phi_{(6)} = \frac{0.4}{0.6193} \phi_{(7)}$$

$$\boxed{\phi_{(6)} = 0.646\,\phi_{(7)}} \quad — \text{(6)}$$

$$\phi_{(7)} = 0.4\,\phi_{(8)} + 0.6\,\phi_{(6)}$$

$$\phi_{(7)} = 0.4\,\phi_{(8)} + 0.6 \times 0.646\,\phi_{(7)}$$

$$\phi_{(7)} - 0.3876\,\phi_{(7)} = 0.4. \qquad \left[\because \phi_{(8)} = 1\right]$$

$$0.6124\,\phi_{(7)} = 0.4$$

$$\phi_{(7)} = \frac{0.4}{0.6124} = \underline{\underline{0.653}}$$

from eq. (6)

$$\phi_{(6)} = 0.646 \times 0.653 = 0.422.$$

from eq. (5)

$$\phi_{(5)} = 0.6345 \times 0.422 = 0.268.$$

from eq. (4)

$$\phi_{(4)} = 0.616 \times 0.268 = 0.165.$$

from eq. (3)

$$\phi_{(3)} = 0.585 \times 0.165 = 0.097.$$

from eq. (2)

$$\phi_{(2)} = \frac{0.4}{0.76} \times 0.097 = 0.051.$$

from eq. 1

$$\phi_{(1)} = 0.4 \times 0.051 = 0.0204.$$

Hence.

$\phi_{(0)} = 0$ [First step Analysis Method].

$\phi_{(1)} = 0.0204$

$\phi_{(2)} = 0.051$

$\phi_{(3)} = 0.097$

$\phi_{(4)} = 0.165$

$\phi_{(5)} = 0.268$

$\phi_{(6)} = 0.422$

$\phi_{(7)} = 0.653$

$\phi_{(8)} = 1$

Expected time until the absorbtion occurs.

Example:- A Markov chain with state space $\{1, 2, 3\}$ has transition Prob. matrix

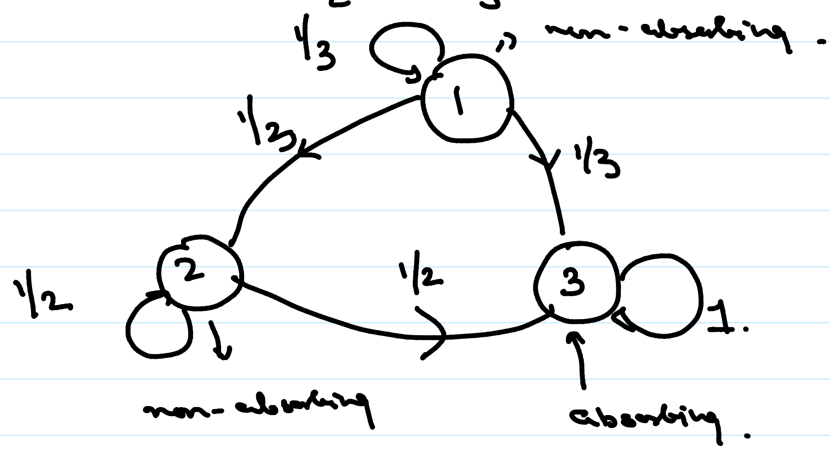
$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that state 3 is absorbing and starting from state 1 find the <u>expected time</u> untill the absorbtion occurs.

<u>Solution</u>

Let $\phi_{(i)}$ be the expected time to reach state 3 starting from state i, where $i \in \{1, 2, 3\}$ we have.



$\phi_{(3)} = 0$ [$\phi_{(3)}$ = Expected time to reach state 3 starting from state 3]

$\phi_{(2)} = 1 + \frac{1}{2}\phi_{(2)} + \frac{1}{2}\phi_{(3)}$ [Law of total Prob. of recursion]

$$\phi_{(1)} = 1 + \frac{1}{3}\phi_{(1)} + \frac{1}{3}\phi_{(2)} + \frac{1}{3}\phi_{(3)}$$

Now,

$$\phi_{(2)} = 1 + \frac{1}{2}\phi_{(2)} + \frac{1}{2}\phi_{(3)} \to 0$$

$$\phi_{(2)} - \frac{1}{2}\phi_{(2)} = 1.$$

$$\frac{1}{2}\phi_{(2)} = 1$$

$$\boxed{\phi_{(2)} = 2} \longrightarrow \text{Expected time}$$

$$\phi_{(1)} = 1 + \frac{1}{3}\phi_{(1)} + \frac{1}{3}\phi_{(2)} + \frac{1}{3}\phi_{(3)} \to 0.$$

$$\phi_{(1)} = 1 + \frac{1}{3}\phi_{(1)} + \frac{1}{3}\times 2$$

$$\phi_{(1)} - \frac{1}{3}\phi_{(1)} = 1 + \frac{2}{3}.$$

$$\frac{2}{3}\phi_{(1)} = \frac{5}{3}.$$

$$\phi_{(1)} = \frac{5}{\cancel{3}} \times \frac{\cancel{3}}{2} = \frac{5}{2}.$$

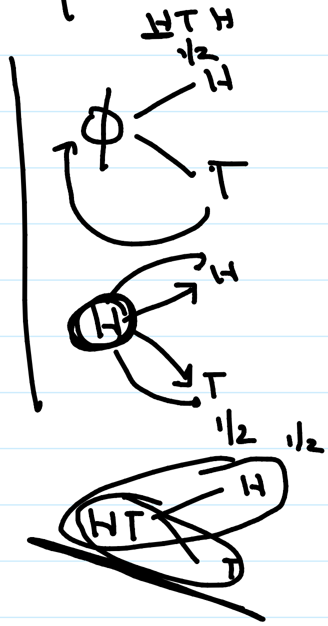
Expected time untill the absorbtion occurs.

$$\phi_{(1)} = \frac{5}{2}, \quad \phi_{(2)} = 2, \quad \phi_{(3)} = 0.$$

<u>Expected no. of steps</u>

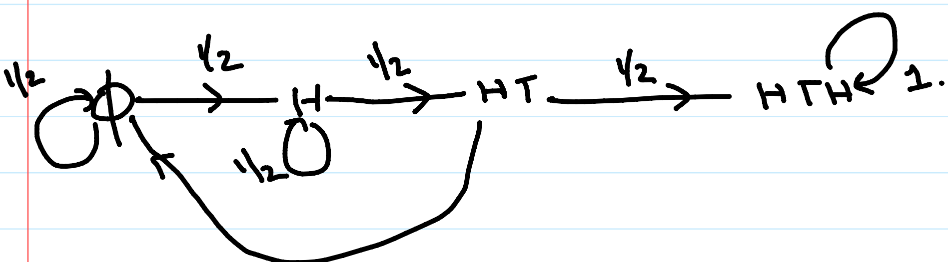
A fair coin is tossed repeatedly and independently. Find the expected no. of tosses till the Pattern HTH appears.

<u>Soln:-</u> The chain starts from ϕ and absorbed at HTH.



$$P(\text{Head}) = \frac{1}{2}$$

$$P(\text{Tail}) = \frac{1}{2}$$



	ϕ	H	HT	HTH
ϕ	1/2	1/2	0	0.
H	0	1/2	1/2	0
HT	1/2	0.	0	1/2
HTH	0	0	0	1

$\phi \longrightarrow 0$

$H \longrightarrow 1$

$HT \longrightarrow 2$

$HTH \longrightarrow 3.$

$$P = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\phi_{(3)}$ = Expected no. of steps to reach states starting from state 3.

$$\phi_{(3)} = 0$$

$$\phi_{(2)} = 1 + \tfrac{1}{2}\phi_{(0)} + \tfrac{1}{2}\phi_{(3)}$$

$$\phi_{(1)} = 1 + \tfrac{1}{2}\phi_{(1)} + \tfrac{1}{2}\phi_{(2)}$$

$$\phi_{(0)} = 1 + \tfrac{1}{2}\phi_{(0)} + \tfrac{1}{2}\phi_{(1)}$$

Now

$$\phi_{(2)} = 1 + \tfrac{1}{2}\phi_{(0)} \quad \text{— } (1)$$

$$\phi_{(1)} = 1 + \tfrac{1}{2}\phi_{(1)} + \tfrac{1}{2}\phi_{(2)}$$

$$\phi_{(1)} - \tfrac{1}{2}\phi_{(1)} = 1 + \tfrac{1}{2}\left[1 + \tfrac{1}{2}\phi_{(0)}\right].$$

$$\tfrac{1}{2}\phi_{(1)} = 1 + \tfrac{1}{2} + \tfrac{1}{4}\phi_{(0)}$$

$$\phi_{(1)} = 2\left[1 + \tfrac{1}{2} + \tfrac{1}{4}\phi_{(0)}\right]$$

$$= 2 + 1 + \tfrac{1}{2}\phi_{(0)}$$

$$\boxed{\phi_{(1)} = 3 + \tfrac{1}{2}\phi_{(0)}} \quad \text{— } (2)$$

$$\phi_{(0)} = 1 + \frac{1}{2}\phi_{(0)} + \frac{1}{2}\phi_{(1)}$$

$$\phi_{(0)} - \frac{1}{2}\phi_{(0)} = 1 + \frac{1}{2}\left[3 + \frac{1}{2}\phi_{(0)}\right]$$

$$\frac{1}{2}\phi_{(0)} = 1 + \frac{3}{2} + \frac{1}{4}\phi_{(0)}$$

$$\frac{1}{2}\phi_{(0)} - \frac{1}{4}\phi_{(0)} = \frac{5}{2}$$

$$\frac{1}{4}\phi_{(0)} = \frac{5}{2}$$

$$\phi_{(0)} = \frac{5}{\cancel{2}_{1}} \times \cancel{4}^{2} = 10.$$

$\phi_{(0)} = 10$ [Expected no. of steps to reach absorbing state starting from state 0]

From eq (2)

$$\phi_{(1)} = 3 + \frac{1}{\cancel{2}} \times \cancel{10}^{5}$$

$$\phi_{(1)} = 8.$$

From eq (1)

$$\phi_{(2)} = 1 + \frac{1}{2}\phi_{(0)}$$

$$\phi_{(2)} = 1 + \frac{1}{\cancel{2}} \times \cancel{10}^{5}$$

$$\phi_{(2)} = 6.$$

Hence, we have

$\phi_{(0)} = 10$, $\phi_{(1)} = 8$, $\phi_{(2)} = 6$.

Checking

$$\phi_{(1)} = 3 + \frac{1}{2}\phi_{(0)}$$

$$8 = 3 + \frac{1}{2} \times \cancel{10}\,5$$

$$\boxed{8 = 8}$$