28th July 2024 **Topics :** *Trignometry, Functions.*

"It is not the answer that enlightens, but the question".
–Eugène Ionesco

1. If the real number α satisfies $\cos\alpha = \tan\alpha$, then the value of $\frac{1}{\sin\alpha} + \cos^4\alpha$ is ___________.

2. Given $\cos\left(\frac{\pi}{4} + x\right) = m$, then $\sin 2x =$ ___________.

3. If $\frac{\pi}{4} < \alpha < \frac{3\pi}{4}, 0 < \beta < \frac{\pi}{4}, \cos\left(\frac{\pi}{4} - \alpha\right) = \frac{3}{5}, \sin\left(\frac{3\pi}{4} + \beta\right) = \frac{5}{13}$, then $\sin(\alpha+\beta) =$ ___________.

4. If $f(\tan x) = \sin 2x$, then the value of $f(-1)$ is ___________.

5. Given $\alpha + \frac{1}{x} = 2\cos\frac{\pi}{24}$, then the value of $x^8 + \frac{1}{x^8}$ is ___________.

6. Suppose $\frac{3\pi}{2} < \alpha < \frac{7\pi}{4}$ and $2\cot^2\alpha + 7\cot\alpha + 3 = 0$, then $\cos 2\alpha =$ ___________.

7. Given $\tan\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$ and $0 < \theta < \frac{\pi}{2}$, then $\tan\theta =$ ___________, $\sin 2\theta =$ ___________, $\cos 2\theta =$ ___________.

8. Calculate $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ =$ ___________.

9. Calculate $\tan 5^\circ + \cot 5^\circ = \frac{2}{\cos 80^\circ} =$ ___________.

10. Given $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$, then $\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta} =$ ___________.

11. Given $\sin\alpha + \sin\beta = \frac{1}{2}$, $\cos\alpha + \cos\beta = \frac{2}{3}$, then $\cos(\alpha - \beta) =$ ___________.

12. If the real number x satisfies $\sin(x+20^\circ) = \cos(x+10^\circ)+\cos(x-10^\circ)$, then $\tan x =$ ___________

13. Given $\sin\alpha + \cos\beta = \frac{\sqrt{3}}{2}$, $\cos\alpha + \sin\beta = \sqrt{2}$, then the value of $\tan\alpha\cot\beta$ is ___________.

14. Suppose $f(x) = \sin^4\frac{kx}{10} + \cos^4\frac{kx}{10}$, where k is a positive integer. If for any real number α, the mean of $f(x)$ in the interval $\alpha < x < \alpha + 1$ is equal to the set of $f(x)$ for $x \in \mathbb{R}$, then the minimum value of k is ___________.

15. Find the value of the following expressions :

i. $\cos^2 24° + \sin^2 6° + \cos^2 18°$;

ii. $4\cos^2 36° - \sin 84°(\sqrt{3} - \tan 6°)$.

16. Suppose $\cos\left(\alpha - \frac{\beta}{2}\right) = -\frac{1}{9}$, $\sin\left(\frac{\alpha}{2} - \beta\right) = \frac{2}{3}$, and $\frac{\pi}{2} < \alpha < \pi$, $0 < \beta < \frac{\pi}{2}$, find the values of $\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$ and $\cos(\alpha + \beta)$.

17. Prove the following identities :

i. $\tan\frac{3x}{2} - \tan\frac{x}{2} = \frac{2\sin x}{\cos x + \cos 2x}$;

ii. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan(A + B)$.

18. Given a and b are integers, and $a > b$, $\sin\theta = \frac{2ab}{a^2+b^2}$, where $\theta \in \left(0, \frac{\pi}{2}\right)$, $A_n = (a^2 + b^2)^n \sin n\theta$. Prove that for all positive integers n, A_n is an integer.

19. Find the value of :

$$\sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \sin^3(3^k\alpha) = \frac{3}{4}\sin\alpha - \frac{1}{4}\cdot\frac{1}{3^n}\sin 3^{n+1}\alpha.$$

20. Find the value of :

i.
$$\cos^4\frac{\pi}{16} + \cos^4\frac{3\pi}{16} + \cos^4\frac{5\pi}{16} + \cdots + \cos^4\frac{15\pi}{16}.$$

ii.
$$\cos^5\frac{\pi}{9} + \cos^5\frac{5\pi}{9} + \cos^5\frac{7\pi}{9}.$$

21. Given $\frac{\sin^2\gamma}{\sin^2\alpha} = 1 - \frac{\tan(\alpha-\beta)}{\tan\alpha}$, prove : $\tan^2\gamma = \tan\alpha \cdot \tan\beta$.

22. Given $\cos\alpha = \tan\beta$, $\cos\beta = \tan\gamma$, $\cos\gamma = \tan\alpha$, then :

$$\sin^2\alpha = \sin^2\beta = \sin^2\gamma = \cos^4\alpha = \cos^4\beta = \cos^4\gamma = 4\sin^2 18°.$$

23. Find the value of :

$$\frac{1}{\cos 0° \cos 1°} + \frac{1}{\cos 1° \cos 2°} + \cdots + \frac{1}{\cos 88° \cos 89°} = \frac{\cos 1°}{\sin^2 1°}.$$

24. Suppose $x \in (0, \pi)$, prove that for all positive integers n :

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is an integer.

25. Suppose $k > 10$, prove :

$$\cos x \cdot \cos 2x \cdot \sin 3x \cdot \cos 4x \cdot \cos 5x \cdots \cos 2^k x \leq \frac{3}{2^{k+1}}.$$