

28th July 2024

Topics : Trigonometry, Functions.

"It is not the answer that enlightens, but the question".
—Eugène Ionesco

1. If the real number α satisfies $\cos \alpha = \tan \alpha$, then the value of $\frac{1}{\sin \alpha} + \cos^4 \alpha$ is _____.
2. Given $\cos\left(\frac{\pi}{4} + x\right) = m$, then $\sin 2x =$ _____.
3. If $\frac{\pi}{4} < \alpha < \frac{3\pi}{4}$, $0 < \beta < \frac{\pi}{4}$, $\cos\left(\frac{\pi}{4} - \alpha\right) = \frac{3}{5}$, $\sin\left(\frac{3\pi}{4} + \beta\right) = \frac{5}{13}$, then $\sin(\alpha + \beta) =$ _____.
4. If $f(\tan x) = \sin 2x$, then the value of $f(-1)$ is _____.
5. Given $\alpha + \frac{1}{x} = 2 \cos \frac{\pi}{24}$, then the value of $x^8 + \frac{1}{x^8}$ is _____.
6. Suppose $\frac{3\pi}{2} < \alpha < \frac{7\pi}{4}$ and $2 \cot^2 \alpha + 7 \cot \alpha + 3 = 0$, then $\cos 2\alpha =$ _____.
7. Given $\tan \frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$ and $0 < \theta < \frac{\pi}{2}$, then $\tan \theta =$ _____, $\sin 2\theta =$ _____, $\cos 2\theta =$ _____.
8. Calculate $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ =$ _____.
9. Calculate $\tan 5^\circ + \cot 5^\circ = \frac{2}{\cos 80^\circ} =$ _____.
10. Given $\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$, then $\sqrt{1 + \sin \theta} - \sqrt{1 - \sin \theta} =$ _____.
11. Given $\sin \alpha + \sin \beta = \frac{1}{2}$, $\cos \alpha + \cos \beta = \frac{2}{3}$, then $\cos(\alpha - \beta) =$ _____.
12. If the real number x satisfies $\sin(x+20^\circ) = \cos(x+10^\circ) + \cos(x-10^\circ)$, then $\tan x =$ _____.
13. Given $\sin \alpha + \cos \beta = \frac{\sqrt{3}}{2}$, $\cos \alpha + \sin \beta = \sqrt{2}$, then the value of $\tan \alpha \cot \beta$ is _____.
14. Suppose $f(x) = \sin^4 \frac{kx}{10} + \cos^4 \frac{kx}{10}$, where k is a positive integer. If for any real number α , the mean of $f(x)$ in the interval $\alpha < x < \alpha + 1$ is equal to the set of $f(x)$ for $x \in \mathbb{R}$, then the minimum value of k is _____.
15. Find the value of the following expressions :

- i. $\cos^2 24^\circ + \sin^2 6^\circ + \cos^2 18^\circ$;
 ii. $4 \cos^2 36^\circ - \sin 84^\circ (\sqrt{3} - \tan 6^\circ)$.

16. Suppose $\cos\left(\alpha - \frac{\beta}{2}\right) = -\frac{1}{9}$, $\sin\left(\frac{\alpha}{2} - \beta\right) = \frac{2}{3}$, and $\frac{\pi}{2} < \alpha < \pi$, $0 < \beta < \frac{\pi}{2}$, find the values of $\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$ and $\cos(\alpha + \beta)$.

17. Prove the following identities :

- i. $\tan \frac{3x}{2} - \tan \frac{x}{2} = \frac{2 \sin x}{\cos x + \cos 2x}$;
 ii. $\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan(A + B)$.

18. Given a and b are integers, and $a > b$, $\sin \theta = \frac{2ab}{a^2 + b^2}$, where $\theta \in (0, \frac{\pi}{2})$, $A_n = (a^2 + b^2)^n \sin n\theta$. Prove that for all positive integers n , A_n is an integer.

19. Find the value of :

$$\sum_{k=0}^n \left(\frac{1}{3}\right)^k \sin^3(3^k \alpha) = \frac{3}{4} \sin \alpha - \frac{1}{4} \cdot \frac{1}{3^n} \sin 3^{n+1} \alpha.$$

20. Find the value of :

- i. $\cos^4 \frac{\pi}{16} + \cos^4 \frac{3\pi}{16} + \cos^4 \frac{5\pi}{16} + \dots + \cos^4 \frac{15\pi}{16}$.
 ii. $\cos^5 \frac{\pi}{9} + \cos^5 \frac{5\pi}{9} + \cos^5 \frac{7\pi}{9}$.

21. Given $\frac{\sin^2 \gamma}{\sin^2 \alpha} = 1 - \frac{\tan(\alpha - \beta)}{\tan \alpha}$, prove : $\tan^2 \gamma = \tan \alpha \cdot \tan \beta$.

22. Given $\cos \alpha = \tan \beta$, $\cos \beta = \tan \gamma$, $\cos \gamma = \tan \alpha$, then :

$$\sin^2 \alpha = \sin^2 \beta = \sin^2 \gamma = \cos^4 \alpha = \cos^4 \beta = \cos^4 \gamma = 4 \sin^2 18^\circ.$$

23. Find the value of :

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \dots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

24. Suppose $x \in (0, \pi)$, prove that for all positive integers n :

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots + \frac{\sin(2n-1)x}{2n-1}$$

is an integer.

25. Suppose $k > 10$, prove :

$$\cos x \cdot \cos 2x \cdot \sin 3x \cdot \cos 4x \cdot \cos 5x \cdots \cos 2^k x \leq \frac{3}{2^{k+1}}.$$