Reading Time: An initial **2 minutes** to view **BOTH** sections

MATHEMATICS METHODS : UNITS 3 & 4, 2023

Thursday 17th August (T3W5)

Test 4 – Logarithmic Functions, Continuous Random Variables and the Normal Distribution (10%)

4.1.4, 4.1.6 – 4.1.14, 4.2.1 – 4.2.7~~, 4.3.1~~

Time Allowed	First Name	Surname	Marks
20 minutes			/ 23 marks

Circle your Teacher's Name:

Mrs Alvaro	Ms Chua	Mrs Fraser-Jones	Mrs Greenaway
Mr Luzuk	Mrs Murray	Ms Narendranathan	Mr Tanday

Assessment Conditions: ***(N.B. Sufficient working out must be shown to gain full marks)***

- Calculators: Not Allowed
- Formula Sheet: Provided
- Notes: Not Allowed

PART A – CALCULATOR FREE

Question 1 (4 marks)

Determine the expected value and standard deviation for the continuous random variable X with the probability density function $f(x) = 6x(1-x)$, for $0 \le x \le 1$.

$$f(x) = 6x - 6x^2$$

$$E(X) = \int_0^1 x \times f(x)\,dx$$
$$= \int_0^1 6x^2 - 6x^3\,dx$$
$$= \left[2x^3 - \frac{3x^4}{2}\right]_0^1$$
$$= 2 - \frac{3}{2}$$
$$= \frac{1}{2}$$

$$var(X) = \int_0^1 x^2 \times f(x)\,dx - [E(X)]^2$$
$$= \int_0^1 6x^3 - 6x^4\,dx - \left(\frac{1}{2}\right)^2$$
$$= \left[\frac{3x^4}{2} - \frac{6x^5}{5}\right]_0^1 - \frac{1}{4}$$
$$= \frac{3}{2} - \frac{6}{5} - \frac{1}{4}$$
$$= \frac{1}{20}$$

$$\rightarrow SD(X) = \sqrt{\frac{1}{20}}$$
$$= \frac{1}{2\sqrt{5}}$$
$$= \frac{\sqrt{5}}{10}$$

✓ correct limits and antiderivative for E(X)
✓ correct E(X)
✓ correct antiderivative and variance formula
✓ correct SD(X)
doesn't need to be rationalised

$$\therefore E(X) = \frac{1}{2} \ \& \ SD(X) = \frac{\sqrt{5}}{10}$$

Question 2 (6 marks = 2, 2, 2)

Determine the following, expressing your answer in simplest form.

a) $\frac{dy}{dx}$ where $y = 2x^3 \ln(5x - 7)$

$$\frac{dy}{dx} = 6x^2 \times \ln(5x-7) + \frac{5}{5x-7} \times 2x^3$$

$$= 6x^2 \ln(5x-7) + \frac{10x^3}{5x-7}$$

✓ attempts to use product rule
✓ correct simplified derivative

b) $\frac{dy}{dx}$ where $y = \ln\left(\frac{3x+1}{(x^2+3)^5}\right)$

$$y = \ln(3x+1) - 5\ln(x^2+3)$$

$$\frac{dy}{dx} = \frac{3}{3x+1} - 5 \times \frac{2x}{x^2+3}$$

$$= \frac{3}{3x+1} - \frac{10x}{x^2+3}$$

✓ applies log laws to simplify y
✓ correct simplified derivative

OR

✓ applies chain rule (to differentiate original y)
✓ correct simplified derivative

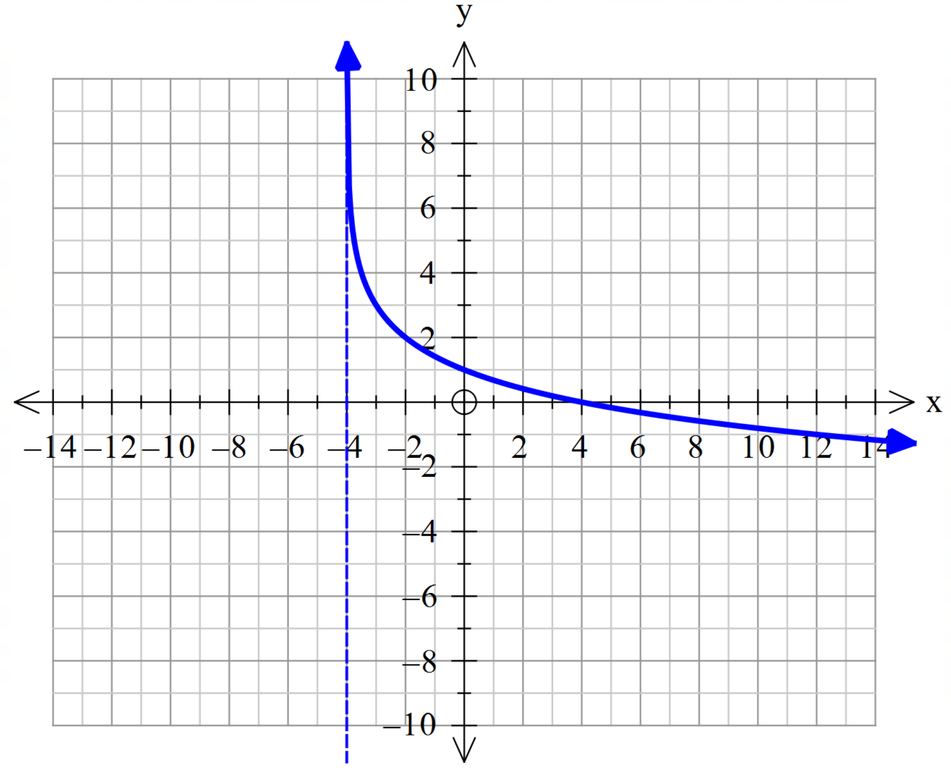
c) $\int \frac{x^2-5x}{2x^3-15x^2-1} dx$

$$= \frac{1}{6} \times \int \frac{6x^2 - 30x}{2x^3 - 15x^2 - 1} dx$$

$$= \frac{1}{6} \ln(2x^3 - 15x^2 - 1) + C$$

✓ indicates appropriate scalar factor
✓ correct antiderivative with +c

Question 3 (3 marks)
On the axes below, sketch the graph of $y = 3 - \log_2(4 + x)$, showing all key features.



✓ vertical asymptote plotted
✓ correct x and y intercepts
✓ smooth curve with correct shape and general location/orientation

Question 4 (4 marks = 2, 2)
A continuous random variable, X, has probability density function $f(x) = 2\cos 2x$, for domain $0 \le x \le \frac{\pi}{4}$.

a) Demonstrate why $f(x)$ is suitable to be used as a probability density function.

$f(x) \ge 0$ over domain $0 \le x \le \frac{\pi}{4}$

$$\int_0^{\pi/4} f(x)\,dx = \int_0^{\pi/4} 2\cos 2x\,dx$$
$$= \left[\sin 2x\right]_0^{\pi/4}$$
$$= \sin\frac{\pi}{2} - \sin 0 = 1$$

$\therefore$ $f(x)$ suitable since $f(x) \ge 0$ & $\int_0^{\pi/4} f(x)\,dx = 1$.

✓ states f(x)>=0 over domain (graph not required)
✓ shows integral over domain is 1

b) Determine $P\left(X \ge \frac{\pi}{6}\right)$.

$$P\left(X \ge \frac{\pi}{6}\right) = \int_{\pi/6}^{\pi/4} 2\cos 2x\,dx$$
$$= \left[\sin 2x\right]_{\pi/6}^{\pi/4}$$
$$= \sin\frac{\pi}{2} - \sin\frac{\pi}{3}$$
$$= 1 - \frac{\sqrt{3}}{2}$$

✓ correct antiderivative and limits
✓ correct evaluation with exact values

Question 5 (6 marks = 2, 4)

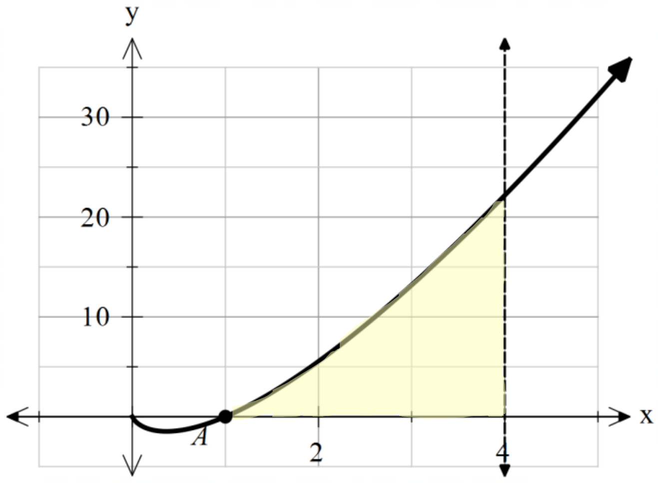
The graph of $y = 4x\ln x$ is shown below and point A is located at $\left(\frac{1}{\sqrt{e}}, 0\right)$.

a) Show that $\frac{d}{dx}(x^2 \ln x) = 2x\ln x + x$

$\frac{d}{dx}(x^2 \ln x) = 2x \times \ln x + \frac{1}{x} \times x^2$

$= 2x\ln x + x$ as req.

✓ clearly indicates use of product rule
✓ simplifies to required form



b) Hence, determine the exact area enclosed by the graph of $y = 4x\ln x$, the x-axis and $x = 4$.

From (a), $\int \frac{d}{dx}(x^2\ln x)\,dx = \int 2x\ln x + x\,dx$

$x^2\ln x + c = \int 2x\ln x\,dx + \frac{x^2}{2}$ by FTC

$\Rightarrow \int 2x\ln x\,dx = x^2\ln x - \frac{x^2}{2} + c$

Area $= \int_{1/\sqrt{e}}^{4} 4x\ln x\,dx$

$= 2\left[x^2\ln x - \frac{x^2}{2}\right]_{1/\sqrt{e}}^{4}$

$= 2\left[(16\ln 4 - 8) - \left(\frac{1}{e}\ln(e^{-1/2}) - \frac{1}{2e}\right)\right]$

$= 32\ln 4 - 16 - 2\left(\frac{1}{e} \times \left(-\frac{1}{2}\right) - \frac{1}{2e}\right)$

$= 32\ln 4 - 16 + \frac{1}{e} + \frac{1}{e}$

$= 32\ln 4 - 16 + \frac{2}{e}$ units^2

✓ indicates use of (a), i.e. applies FTC
✓ uses result to determine correct antiderivative for area integral
✓ correct limits and substitution
✓ correct simplified area
units^2 not needed

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30 minutes			30 marks / ~~28~~ marks

Circle your Teacher's Name:

Mrs Alvaro	Ms Chua	Mrs Fraser-Jones	Mrs Greenaway
Mr Luzuk	Mrs Murray	Ms Narendranathan	Mr Tanday

Assessment Conditions: ***(N.B. Sufficient working out must be shown to gain full marks)***

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PART B – CALCULATOR ASSUMED

Question 6 **(6 marks = 1, 3, 2)**

The continuous random variable X has probability density function $f(x) = \frac{3x^2(4-x)}{64}$, $0 \le x \le 4$.

Determine each of the following.

a) $P(X > 3)$

$$= \int_3^4 \frac{3x^2(4-x)}{64}\,dx$$

$$= \frac{67}{256} = 0.2617 \text{ (4dp)}$$

✓ correct probability (exact or rounded to at least 3 d.p.)

b) The probability of being within one standard deviation of the mean.

$$\text{mean } E(X) = \int_0^4 x \times \frac{3x^2(4-x)}{64}\,dx = 2.4$$

$$SD(X) = \sqrt{\int_0^4 (x-2.4)^2 \times \frac{3x^2(4-x)}{64}\,dx} = 0.8$$

$$\Rightarrow P(\mu-\sigma \le X \le \mu+\sigma) = P(1.6 \le X \le 3.2) = \frac{16}{25} = 0.64$$

✓ determines E(X)
✓ determines SD(X)
✓ determines probability

c) The median of X.

let m be median

$\Rightarrow P(X \le m) = 0.5$ or $P(X \ge m) = 0.5$

$$\int_0^m \frac{3x^2(4-x)}{64}\,dx = 0.5$$

CAS solve: $m = 2.457$, ~~4.9849~~ discard out of domain

$\therefore$ median of X is 2.457 (3dp)

✓ sets up required integral equation for median
✓ solves for the median (at least 2 d.p.)
OK if only the valid solution was stated

Answer only = 1 mark

Question 7 (3 marks)
A logarithmic scale with base 5 is used and the distance measured on the axis between the notches representing 10 and 15 is 2.02 cm. What is the distance, correct to one decimal place, between the notches for 5 and 125?

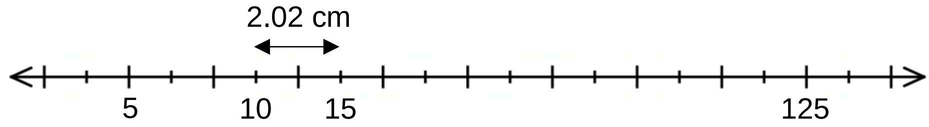


Diagram is not drawn to scale.

2.02 cm relates to $\log_5\left(\frac{15}{10}\right) = 0.2519$

since $\log_5\left(\frac{125}{5}\right) = 2$,

dist. between 5 & 125 $= 2 \times \frac{2.02}{0.2519}$

$= 16.0$ cm (1dp)

✓ determines log_5 (15/10) or equivalent
✓ determines log_5 (125/5) or equivalent
✓ applies ratio/calculation to determine required distance to 1 d.p.
No penalty for missing units

Question 8 (4 marks = 1, 3)
The lifetime, T, of a new battery used for a scientific calculator is a continuous random variable where the probability that it will run flat t weeks after it is first inserted is modelled with the probability density function $f(t) = \frac{e^{-0.25t}}{4}$ for $t \geq 0$.

a) Determine the probability that the battery will last at least 6 weeks.

$P(T \geq 6) = \int_6^\infty \frac{e^{-0.25t}}{4}\, dt$

$= 0.2231$ (4dp)

(exact $e^{-1.5}$)

✓ correct probability
Allow the exact value, but comment

b) A cautious teacher buys 10 spare scientific calculators at the beginning of the term and makes sure they all have a new battery inserted. Determine the probability that more than half of the calculators will last at least 6 weeks.

let X represent how many of the 10 calc. last at least 6 wks

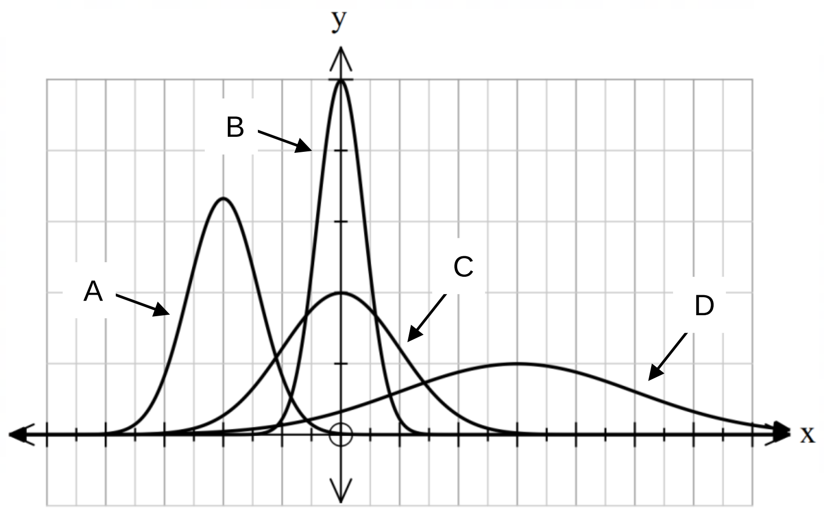
$\Rightarrow X \sim Bin(10, 0.2231)$

$P(X > 5) = P(X \geq 6)$

$= 0.0112$ (4dp)

✓ defines new random variable in words
✓ states new distribution and parameters
✓ calculates probability

Question 9 (4 marks = 2, 2)
The graphs of the probability density functions for four different normal distributions, A, B, C and D, are shown below.



a) Which of the four distributions has the largest mean? Justify your answer.

Distribution D
since it is centred around the highest x-value

✓ identifies distribution D
✓ gives reason
(i.e. refers to centre/peak location)

b) State the four distributions in <u>ascending</u> order of their standard deviations. Justify your answer.

B, A, C, D
The graph for B is narrowest (indicating the smallest SD) & the widths for graphs A, C then D progressively increase.

✓ correct order
✓ gives reason
(i.e. refers to widths)

Question ~~12~~ 10 (3 marks)
Students learning to use a woodworking lathe take an average time of 35 minutes to make a simple figure. 30% of the students could finish their figure within 30 minutes. If the times, T, are normally distributed, determine the probability that a student took more than 38 minutes to make their figure.

$T \sim N(35, \sigma^2)$ where $\sigma = SD(T)$

$P(T < 30) = 0.3$

consider standard normal $Z \sim N(0,1)$

$P(Z < k) = 0.3$

$k = -0.5244$

$\Rightarrow \dfrac{30-35}{\sigma} = -0.5244$

$\sigma = 9.535$

$\therefore T \sim N(35, 9.535^2)$

$P(T > 38) = 0.3765$ (4dp)

✓ equivalent standard normal probability and solves k
✓ uses standard score equation and solves SD
✓ determines probability

11 6 marks

Question ~~10~~ (~~4~~ marks = 2, 1, 3)

The continuous random variable X is normally distributed with mean 18 and variance 5.

a) Determine the following probabilities:

(i) $P(12 < X \leq 17)$

$= 0.3237$ (4dp)

✓ correct prob. (i)
✓ correct prob. (ii)

(ii) $P(X > 20 \mid X < 24)$

$= \frac{P(20<X<24)}{P(X<24)} = \frac{0.1819}{0.9964} = 0.1826$ (4dp)

b) What is the third quartile of X?

$P(X<Q3) = 0.75$

$Q3 = 19.51$ (2dp)

✓ correct Q3

c) A second random variable, Y, is normally distributed with mean -1.6 and variance 7.2. Y can be expressed as a linear transformation of X, i.e. $Y = aX + b$. Determine the values of a and b.

$E(Y) = a \times E(X) + b$

$Var(Y) = a^2 \times Var(X)$

$\Rightarrow \begin{cases} -1.6 = 18a+b \\ 7.2 = 5a^2 \end{cases}$

✓ correct simultaneous equations
✓ solves one pair of values
✓ solves both pairs of values

CAS: $\begin{cases} a = -1.2 \\ b = 20 \end{cases}$ or $\begin{cases} a = 1.2 \\ b = -23.2 \end{cases}$

12

Question ~~11~~ (4 marks = 1, 3)

The weight of a regulation baseball is required to be between 142 grams and 149 grams.

a) The supplier, Rawlings, of baseballs for the national league has found that the weight of their manufactured baseballs, W, is normally distributed with a mean of 145.5 grams and standard deviation 1.8 grams. Determine the probability that one of their baseballs will not meet the regulation requirements.

$W \sim N(145.5, 1.8^2)$

$P(\text{not meet req.}) = 1 - P(142 \leq W \leq 149)$

$= 1 - 0.9482$

✓ correct probability

$= 0.0518$ (4dp)

b) A startup company, Crawlings, is testing their new machinery and find that the weight of their baseballs, C, have a 10.6% chance of being too light, and 4.8% chance of being too heavy. What is the mean and standard deviation for the weight of their baseballs?

let the mean weight be μ grams & std. dev. be σ, i.e. $C \sim N(\mu, \sigma^2)$

$P(C<142) = 0.106$ $\quad P(C>149) = 0.048$

$Z \sim N(0,1)$, $P(Z<k_1) = 0.106$ $\quad P(Z>k_2) = 0.048$

$k_1 = -1.2481$ $\quad k_2 = 1.6646$

$Z = \frac{C-\mu}{\sigma} \Rightarrow \begin{cases} \frac{142-\mu}{\sigma} = -1.2481 \\ \frac{149-\mu}{\sigma} = 1.6646 \end{cases}$

✓ sets up one of the probabilities with the standard normal and solves k
✓ solves second k-value and sets up simultaneous equations
✓ solves mean and SD with units

CAS: $\mu = 144.9995$ $\quad \sigma = 2.403$

$\therefore$ mean weight 145g & SD of 2.40g (3SF)