Preparation of Gaussian wave packets for the Schwinger model in 1 + 1 dimensions

For the Schwinger model with the θ-term in 1 + 1 dimensional U(1) gauge theory, the Lagrangian density is

$$\mathcal{L}_0 = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\substack{\text{Kinetic term} \\ \text{for the electromagnetic} \\ \text{gauge field}}} + \underbrace{\frac{g\theta}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu}}_{\text{Topological term}} + \underbrace{i\overline{\psi}\gamma^\mu(\partial_\mu + igA^\mu)\psi}_{\substack{\text{Fermion kinetic} \\ \text{and gauge interaction term}}} \underbrace{-m\overline{\psi}\psi}_{\text{Fermion mass term}} \tag{1}$$

where $\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We can then transform the Lagrangian density by chiral rotation to remove θ from the Gauge Sector:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi \underbrace{-m\overline{\psi}e^{i\theta\gamma^5}\psi}_{\substack{\text{Modified effective} \\ \text{mass term}}} \tag{2}$$

where $\gamma^5 = \gamma^0\gamma^1$.

Using an adiabatic method [1] or a VQE (Appendix A.2) we can prepare a $|\text{vac}\rangle$ state for the Schwinger model in 1 + 1 dimensions.

Once we have this state, we need to define some creation operators $c_n^\dagger$ and c_n which must obey [2]:

$$\{c_n, c_m^\dagger\} = \delta_{nm} \tag{3}$$

[3]

A Additional Information

A.1 Derivation of the Hamiltonian from the Lagrangian density

From the Lagrangian density (Equation 1) we can derive a Hamiltonian given in [1]. In order to do so we will use the Legendre transformation $H = \int d^D x \mathcal{H}$ with the Hamiltonian density in the temporal gauge ($A_0 = 0$):

$$\mathcal{H} = \Pi^A \dot{A} + \Pi^\psi \dot{\psi} - \mathcal{L}_0 \tag{4}$$

where $\Pi^A = \frac{\partial \mathcal{L}_0}{\partial \dot{A}_1} = F^{01}$ (in 1 + 1D $F^{01} = E$), and $\Pi^\psi = \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} = i\overline{\psi}\gamma^0$

We can now descretize from continuum space to a set of discrete sites n separated by a lattice spacing a: $x \to na$.

- In temporal gauge, only remaining derivatives are $\partial_1\psi(x) \to \frac{\psi_{n+1} - \psi_n}{a}$
- Integrals become sums over lattice sites: $\int dx \mathcal{H}(x) \to a \sum_n \mathcal{H}_n$
- Gauge field A_1 becomes a link variable between sites n and $n + 1$: $A_1 \to U_n = e^{iagA_1(x_n)}$
- The electric field E becomes an operator L_n between sites n and $n + 1$: $E \to L_n = -i\frac{\partial}{\partial\varphi_n}$ so $U_n = e^{-i\varphi_n}$

  as such, the continuous energy $\int \frac{1}{2}E^2 dx \to a \sum_n \frac{1}{2}E_n^2 \to a \sum_n \frac{1}{2}(-gL_n)^2 = \sum_n \frac{g^2 a}{2} L_n^2$
- Using the staggered fermion formulation, the two-component Dirac spinor $\psi(x) \to \chi_n$

$$
\begin{aligned}
H = -i \sum_{n}^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) \left[\chi_n^\dagger e^{i\varphi_n} \chi_{n+1} - \text{h.c.} \right] \\
+ m \cos(\theta) \sum_{n}^{N} (-1)^n \chi_n^\dagger \chi_n + \frac{g^2 a}{2} \sum_{n}^{N-1} L_n^2
\end{aligned}
\tag{5}
$$

A.2 VQE for the Schwinger model

Using the Jordan-Wigner transformations, we can rewrite the fermions in terms of spin variables like:

$$
\chi_n = \left(\prod_{l<n} -iZ_l \right) \frac{X_n - iY_n}{2} \tag{6}
$$

and as in [1]:

$$
L_n = L_0 + \frac{1}{2} \sum_{l}^{N} \left(Z_l + (-1)^l \right) \tag{7}
$$

where we can take $L_0 = 0$ since the model with $(\theta, L_0) \equiv (\theta + 2\pi L_0, 0)$ and redefine χ_n to eliminate φ_n: $\chi_n \to \prod_{l<n} [e^{i\varphi_l}] \chi_n$.

Thus, the Hamiltonian becomes:

$$
\begin{aligned}
H = \frac{g^2 a}{4} \sum_{n=2}^{N-1} \sum_{1 \le k < l \le n} Z_k Z_l \\
+ \frac{1}{2} \sum_{n=1}^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) \left[X_n X_{n+1} + Y_n Y_{n+1} \right] \\
+ \frac{m \cos(\theta)}{2} \sum_{n=1}^{N} (-1)^n Z_n - \frac{g^2 a}{4} \sum_{n=1}^{N-1} (n \bmod 2) \sum_{l=1}^{n} Z_l
\end{aligned}
\tag{8}
$$

Using this Hamiltonian, we can use the VQE to find it's ground state.

1. State preparation
   - Use some ansatz ($U(\theta)$) and apply it to some initial state: $U(\theta)|0\rangle^{\otimes N} = |\psi(\theta)\rangle$, where θ is a vector of parameters.
2. Expectation value measurement
   - Calculate the expectation value. $E(\theta) = \langle \psi(\theta)|H|\psi(\theta)\rangle$
3. Classical optimization
   - The expectation value is then minimized using a classical optimizer. This is done since it is known that the measured energy must be greater than or equal to the true ground state energy.
4. Iterate
   - Repeat steps 2 and 3 until convergence at a minimum.

Our ground state is then defined as $|\psi(\theta_{\text{min}})\rangle$.

Bibliography

[1] B. Chakraborty, M. Honda, T. Izubuchi, Y. Kikuchi, and A. Tomiya, *Classically Emulated Digital Quantum Simulation of the Schwinger Model with Topological Term via Adiabatic State Preparation*, https://arxiv.org/abs/2001.00485.

[2] Y. Chai, A. Crippa, K. Jansen, S. Kühn, V. R. Pascuzzi, F. Tacchino, and I. Tavernelli, Fermionic wave packet scattering: a quantum computing approach, Quantum **9**, 1638 (2025).

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