

Preparation of Gaussian wave packets for the Schwinger model in 1 + 1 dimensions

For the Schwinger model with the θ -term in 1 + 1 dimensional U(1) gauge theory, the Lagrangian density is

$$\mathcal{L}_0 = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\substack{\text{Kinetic term} \\ \text{for the electromagnetic} \\ \text{gauge field}}} + \underbrace{\frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}}_{\text{Topological term}} + \underbrace{i\bar{\psi}\gamma^\mu(\partial_\mu + igA^\mu)\psi}_{\substack{\text{Fermion kinetic} \\ \text{and gauge interaction term}}} - \underbrace{m\bar{\psi}\psi}_{\text{Fermion mass term}} \quad (1)$$

where $\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We can then transform the Lagrangian density by chiral rotation to remove θ from the Gauge Sector:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi - \underbrace{m\bar{\psi}e^{i\theta\gamma^5}\psi}_{\text{Modified effective mass term}} \quad (2)$$

where $\gamma^5 = \gamma^0\gamma^1$.

Using an adiabatic method [1] or a VQE (Appendix A.2) we can prepare a $|\text{vac}\rangle$ state for the Schwinger model in 1 + 1 dimensions.

Once we have this state, we need to define some creation operators χ_n^\dagger and χ_n which must obey [2]:

$$\{\chi_n, \chi_m^\dagger\} = \delta_{nm} \quad (3)$$

From the Jordan-Wigner transformation, these creation operators could be represented as:

$$\begin{aligned} \chi_n &= \left(\prod_{l < n} -iZ_l \right) (X_n - iY_n) \\ \chi_n^\dagger &= \left(\prod_{l < n} iZ_l \right) (X_n + iY_n) \end{aligned} \quad (4)$$

Which obeys Equation 3:

1. For $n = m$ the Z strings cancel and the resulting expression is:

$$\{\chi_n, \chi_n^\dagger\} = \frac{1}{2}\{(X_n - iY_n), (X_n + iY_n)\} \quad (5)$$

2. We can then solve using Pauli matrices:

$$\begin{aligned} \frac{X - iY}{2} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \frac{X + iY}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \frac{1}{2}((X - iY)(X + iY) + (X + iY)(X - iY)) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (6)$$

therefore $\chi_n\chi_n^\dagger + \chi_n^\dagger\chi_n = 1$
and $\chi_n\chi_m^\dagger + \chi_m^\dagger\chi_n = 0$

Now that we have the creation and annihilation operators, we can create a pair of operators that both act on $|\text{vac}\rangle$ to create a Gaussian wave packet of either fermions or anti-fermions.

$$C^\dagger = \sum_{n=1}^N f_n^c \chi_n^\dagger, \quad D^\dagger = \sum_{n=1}^N f_n^d \chi_n \quad (7)$$

where $f_n^{c(d)}$ is the Gaussian coefficient at site n .

$$\begin{aligned} f_n^c(n_0, \sigma, k_0) &= A^c \cdot \exp \left[-\frac{(n - n_0)^2}{4\sigma^2} \right] \cdot \exp[ik_0 a(n - n_0)] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot (\Pi_{n0} + v_{k_0} \Pi_{n1}) \\ f_n^d(n_0, \sigma, k_0) &= A^d \cdot \exp \left[-\frac{(n - n_0)^2}{4\sigma^2} \right] \cdot \exp[-ik_0 a(n - n_0)] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot (\Pi_{n1} + v_{k_0} \Pi_{n0}) \end{aligned} \quad (8)$$

Where A is the normalization constant for the condition $\sum_{n=1}^N |f_n|^2 = 1$, n_0 is the center position of the wave packet, σ is the standard deviation, k_0 is the average momentum of the wave packet, a is the lattice spacing, $m_{\text{eff}} = m \cos(\theta)$ is the effective mass, $w_{k_0} = \sqrt{(m \cos \theta)^2 + \sin^2 k_0}$ and $v_{k_0} = \frac{\sin k_0}{m \cos \theta + w_{k_0}}$ are the group and phase velocity and Π_{nl} are the projection operators defined as in [2]:

$$\Pi_{nl} = \frac{1 + (-1)^{n+l}}{2}, \quad l \in \{0, 1\} \quad (9)$$

Deriving A^c :

$$\begin{aligned} \sum_{n=1}^N \left| A^c \cdot \exp \left[-\frac{(n - n_0)^2}{4\sigma^2} \right] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot (\Pi_{n0} + v_{k_0} \Pi_{n1}) \cdot \underbrace{\exp[ik_0 a(n - n_0)]}_{\text{magnitude of 1}} \right|^2 &= 1 \\ (A^c)^2 \cdot \sum_{n=1}^N \exp \left[-\frac{(n - n_0)^2}{2\sigma^2} \right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot (\Pi_{n0} + v_{k_0} \Pi_{n1})^2 &= 1 \\ A^c(N, \sigma, n_0) &= \left(\sum_{n=1}^N \exp \left[-\frac{(n - n_0)^2}{2\sigma^2} \right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot (\Pi_{n0} + v_{k_0} \Pi_{n1})^2 \right)^{-\frac{1}{2}} \end{aligned} \quad (10)$$

Which then follows that A^d is:

$$A^d(N, \sigma, n_0) = \left(\sum_{n=1}^N \exp \left[-\frac{(n - n_0)^2}{2\sigma^2} \right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot (\Pi_{n1} + v_{k_0} \Pi_{n0})^2 \right)^{-\frac{1}{2}} \quad (11)$$

We can now use these operators to create a fermion, anti-fermion wave packet on top of the vacuum state:

$$|\psi(0)\rangle = D^\dagger C^\dagger |\text{vac}\rangle \quad (12)$$

A Additional Information

A.1 Derivation of the Hamiltonian from the Lagrangian density

From the Lagrangian density (Equation 1) we can derive a Hamiltonian given in [1]. In order to do so we will use the Legendre transformation $H = \int d^D x \mathcal{H}$ with the Hamiltonian density in the temporal gauge ($A_0 = 0$):

$$\mathcal{H} = \Pi^A \dot{A} + \Pi^\psi \dot{\psi} - \mathcal{L}_0 \quad (13)$$

where $\Pi^A = \frac{\partial \mathcal{L}_0}{\partial A_1} = F^{01}$ (in 1 + 1D $F^{01} = E$), and $\Pi^\psi = \frac{\partial \mathcal{L}_0}{\partial \psi} = i\bar{\psi}\gamma^0$

We can now discretize from continuum space to a set of discrete sites n separated by a lattice spacing a : $x \rightarrow na$.

- In temporal gauge, only remaining derivatives are $\partial_1 \psi(x) \rightarrow \frac{\psi_{n+1} - \psi_n}{a}$
- Integrals become sums over lattice sites: $\int dx \mathcal{H}(x) \rightarrow a \sum_n \mathcal{H}_n$
- Gauge field A_1 becomes a link variable between sites n and $n + 1$: $A_1 \rightarrow U_n = e^{iagA_1(x_n)}$
- The electric field E becomes an operator L_n between sites n and $n + 1$: $E \rightarrow L_n = -i \frac{\partial}{\partial \varphi_n}$ so $U_n = e^{-i\varphi_n}$
as such, the continuous energy $\int \frac{1}{2} E^2 dx \rightarrow a \sum_n \frac{1}{2} E_n^2 \rightarrow a \sum_n \frac{1}{2} (-gL_n)^2 = \sum_n \frac{g^2 a}{2} L_n^2$
- Using the staggered fermion formulation, the two-component Dirac spinor $\psi(x) \rightarrow \chi_n$

$$H = -i \sum_n^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) [\chi_n^\dagger e^{i\varphi_n} \chi_{n+1} - \text{h.c.}] \\ + m \cos(\theta) \sum_n^N (-1)^n \chi_n^\dagger \chi_n + \frac{g^2 a}{2} \sum_n^{N-1} L_n^2 \quad (14)$$

A.2 VQE for the Schwinger model

Using the Jordan-Wigner transformations, we can rewrite the fermions in terms of spin variables like:

$$\chi_n = \left(\prod_{l < n} -iZ_l \right) \frac{X_n - iY_n}{2} \quad (15)$$

and as in [1]:

$$L_n = L_0 + \frac{1}{2} \sum_l^N (Z_l + (-1)^l) \quad (16)$$

where we can take $L_0 = 0$ since the model with $(\theta, L_0) \equiv (\theta + 2\pi L_0, 0)$ and redefine χ_n to eliminate φ_n : $\chi_n \rightarrow \prod_{l < n} [e^{i\varphi_l}] \chi_n$.

Thus, the Hamiltonian becomes:

$$H = \frac{g^2 a}{4} \sum_{n=2}^{N-1} \sum_{1 \leq k < l \leq n} Z_k Z_l \\ + \frac{1}{2} \sum_{n=1}^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) [X_n X_{n+1} + Y_n Y_{n+1}] \\ + \frac{m \cos(\theta)}{2} \sum_{n=1}^N (-1)^n Z_n - \frac{g^2 a}{4} \sum_{n=1}^{N-1} (n \bmod 2) \sum_{l=1}^n Z_l \quad (17)$$

Using this Hamiltonian, we can use the VQE to find its ground state.

1. State preparation
 - Use some ansatz ($U(\theta)$) and apply it to some initial state: $U(\theta)|0\rangle^{\otimes N} = |\psi(\theta)\rangle$, where θ is a vector of parameters.
2. Expectation value measurement
 - Calculate the expectation value. $E(\theta) = \langle \psi(\theta) | H | \psi(\theta) \rangle$
3. Classical optimization

- The expectation value is then minimized using a classical optimizer. This is done since it is known that the measured energy must be greater than or equal to the true ground state energy.
4. Iterate
- Repeat steps 2 and 3 until convergence at a minimum.

Our ground state is then defined as $|\psi(\theta_{\min})\rangle$.

Bibliography

- [1] B. Chakraborty, M. Honda, T. Izubuchi, Y. Kikuchi, and A. Tomiya, *Classically Emulated Digital Quantum Simulation of the Schwinger Model with Topological Term via Adiabatic State Preparation*, <https://arxiv.org/abs/2001.00485>.
- [2] Y. Chai, A. Crippa, K. Jansen, S. Kühn, V. R. Pascuzzi, F. Tacchino, and I. Tavernelli, Fermionic wave packet scattering: a quantum computing approach, *Quantum* **9**, 1638 (2025).