Preparation of Gaussian wave packets for the Schwinger model in 1 + 1 dimensions

For the Schwinger model with the θ-term in 1 + 1 dimensional U(1) gauge theory, the Lagrangian density is

$$\mathcal{L}_0 = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\substack{\text{Kinetic term}\\ \text{for the electromagnetic}\\ \text{gauge field}}} + \underbrace{\frac{g\theta}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu}}_{\text{Topological term}} + \underbrace{i\bar{\psi}\gamma^\mu(\partial_\mu + igA^\mu)\psi}_{\substack{\text{Fermion kinetic}\\ \text{and gauge interaction term}}} \underbrace{-m\bar{\psi}\psi}_{\text{Fermion mass term}} \tag{1}$$

where $\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We can then transform the Lagrangian density by chiral rotation to remove θ from the Gauge Sector:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi \underbrace{-m\bar{\psi}e^{i\theta\gamma^5}\psi}_{\substack{\text{Modified effective}\\ \text{mass term}}} \tag{2}$$

where $\gamma^5 = \gamma^0\gamma^1$.

Using an adiabatic method [1] or a VQE (Appendix A.2) we can prepare a $|\text{vac}\rangle$ state for the Schwinger model in 1 + 1 dimensions.

Once we have this state, we need to define some creation operators $\chi_n^\dagger$ and χ_n which must obey [2]:

$$\{\chi_n, \chi_m^\dagger\} = \delta_{nm} \tag{3}$$

From the Jordan-Wigner transformation, these creation operators could be represented as:

$$\begin{aligned} \chi_n &= \left(\prod_{l<n} -iZ_l\right)\frac{X_n - iY_n}{2} \\ \chi_n^\dagger &= \left(\prod_{l<n} iZ_l\right)\frac{X_n + iY_n}{2} \end{aligned} \tag{4}$$

Which obeys Equation 3 and is shown in Appendix A.3.

Now that we have the creation and annihilation operators, we can create a pair of operators that both act on $|\text{vac}\rangle$ to create a Gaussian wave packet of either fermions or anti-fermions.

$$C^\dagger = \sum_{n=1}^{N} f_n^c \chi_n^\dagger,\ D^\dagger = \sum_{n=1}^{N} f_n^d \chi_n \tag{5}$$

where $f_n^{c(d)}$ is the Gaussian coefficient at site n.

$$\begin{aligned} f_n^c(n_0, \sigma, k_0) &= A^c \cdot \exp\left[-\frac{(n-n_0)^2}{4\sigma^2}\right] \cdot \exp[ik_0 a(n-n_0)] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot \left(\Pi_{n0} + v_{k_0}\Pi_{n1}\right) \\ f_n^d(n_0, \sigma, k_0) &= A^d \cdot \exp\left[-\frac{(n-n_0)^2}{4\sigma^2}\right] \cdot \exp[-ik_0 a(n-n_0)] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot \left(\Pi_{n1} + v_{k_0}\Pi_{n0}\right) \end{aligned} \tag{6}$$

Where A is the normalization constant for the condition $\sum_{n=1}^{N} |f_n|^2 = 1$, n_0 is the center position of the wave packet, σ is the standard deviation, k_0 is the average momentum of the wave packet, a is the lattice spacing, $m_{\text{eff}} = m\cos(\theta)$ is the effective mass, $w_{k_0} = \sqrt{(m\cos\theta)^2 + \sin^2 k_0}$ and $v_{k_0} = \frac{\sin k_0}{m\cos\theta + w_{k_0}}$ are the group and phase velocity and Π_{nl} are the projection operators defined as in [2]:

$$\Pi_{nl} = \frac{1 + (-1)^{n+l}}{2},\ l \in \{0, 1\} \tag{7}$$

Deriving A^c:

$$\sum_{n=1}^{N} \left| A^c \cdot \exp\left[-\frac{(n-n_0)^2}{4\sigma^2}\right] \cdot \sqrt{\frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}}} \cdot \left(\Pi_{n0} + v_{k_0}\Pi_{n1}\right) \cdot \underbrace{\exp[ik_0 a(n-n_0)]}_{\text{magnitude of 1}} \right|^2 = 1$$

$$(A^c)^2 \cdot \sum_{n=1}^{N} \exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot \left(\Pi_{n0} + v_{k_0}\Pi_{n1}\right)^2 = 1 \tag{8}$$

$$A^c(N, \sigma, n_0) = \left(\sum_{n=1}^{N} \exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot \left(\Pi_{n0} + v_{k_0}\Pi_{n1}\right)^2\right)^{-\frac{1}{2}}$$

Which then follows that A^d is:

$$A^d(N, \sigma, n_0) = \left(\sum_{n=1}^{N} \exp\left[-\frac{(n-n_0)^2}{2\sigma^2}\right] \cdot \frac{m_{\text{eff}} + w_{k_0}}{w_{k_0}} \cdot \left(\Pi_{n1} + v_{k_0}\Pi_{n0}\right)^2\right)^{-\frac{1}{2}} \tag{9}$$

We can now use these operators to create a fermion, anti-fermion wave packet on top of the vacuum state:

$$|\psi(0)\rangle = D^\dagger C^\dagger |\text{vac}\rangle \tag{10}$$

A Additional Information

A.1 Derivation of the Hamiltonian from the Lagrangian density

From the Lagrangian density (Equation 1) we can derive a Hamiltonian given in [1]. In order to do so we will use the Legendre transformation $H = \int d^D x \mathcal{H}$ with the Hamiltonian density in the temporal gauge ($A_0 = 0$):

$$\mathcal{H} = \Pi^A \dot{A} + \Pi^\psi \dot{\psi} - \mathcal{L}_0 \tag{11}$$

where $\Pi^A = \frac{\partial \mathcal{L}_0}{\partial \dot{A}_1} = F^{01}$ (in 1 + 1D $F^{01} = E$), and $\Pi^\psi = \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} = i\overline{\psi}\gamma^0$

We can now descretize from continuum space to a set of discrete sites n separated by a lattice spacing a: $x \to na$.

- In temporal gauge, only remaining derivatives are $\partial_1 \psi(x) \to \frac{\psi_{n+1} - \psi_n}{a}$
- Integrals become sums over lattice sites: $\int dx \mathcal{H}(x) \to a \sum_n \mathcal{H}_n$
- Gauge field A_1 becomes a link variable between sites n and $n+1$: $A_1 \to U_n = e^{iagA_1(x_n)}$
- The electric field E becomes an operator L_n between sites n and $n+1$: $E \to L_n = -i\frac{\partial}{\partial \varphi_n}$ so $U_n = e^{-i\varphi_n}$

  as such, the continuous energy $\int \frac{1}{2}E^2 dx \to a \sum_n \frac{1}{2}E_n^2 \to a \sum_n \frac{1}{2}(-gL_n)^2 = \sum_n \frac{g^2 a}{2} L_n^2$
- Using the staggered fermion formulation, the two-component Dirac spinor $\psi(x) \to \chi_n$

$$
\begin{aligned}
H = -i \sum_{n}^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) \left[\chi_n^\dagger e^{i\varphi_n} \chi_{n+1} - \text{h.c.} \right] \\
+ m \cos(\theta) \sum_{n}^{N} (-1)^n \chi_n^\dagger \chi_n + \frac{g^2 a}{2} \sum_{n}^{N-1} L_n^2
\end{aligned}
\tag{12}
$$

A.2 VQE for the Schwinger model

Using the Jordan-Wigner transformations, we can rewrite the fermions in terms of spin variables like:

$$
\chi_n = \left(\prod_{l<n} -i Z_l \right) \frac{X_n - i Y_n}{2}
$$

and as in [1]:

$$
L_n = L_0 + \frac{1}{2} \sum_{l}^{N} \left(Z_l + (-1)^l \right) \tag{13}
$$

where we can take $L_0 = 0$ since the model with $(\theta, L_0) \equiv (\theta + 2\pi L_0, 0)$ and redefine χ_n to eliminate φ_n: $\chi_n \to \prod_{l<n} [e^{i\varphi_l}] \chi_n$.

Thus, the Hamiltonian becomes:

$$
\begin{aligned}
H = \frac{g^2 a}{4} \sum_{n=2}^{N-1} \sum_{1 \le k < l \le n} Z_k Z_l \\
+ \frac{1}{2} \sum_{n=1}^{N-1} \left(\frac{1}{2a} - (-1)^n \frac{m}{2} \sin(\theta) \right) \left[X_n X_{n+1} + Y_n Y_{n+1} \right] \\
+ \frac{m \cos(\theta)}{2} \sum_{n=1}^{N} (-1)^n Z_n - \frac{g^2 a}{4} \sum_{n=1}^{N-1} (n \bmod 2) \sum_{l=1}^{n} Z_l
\end{aligned}
\tag{14}
$$

Using this Hamiltonian, we can use the VQE to find it's ground state.

1. State preparation
   - Use some ansatz ($U(\theta)$) and apply it to some initial state: $U(\theta)|0\rangle^{\otimes N} = |\psi(\theta)\rangle$, where θ is a vector of parameters.
2. Expectation value measurement
   - Calculate the expectation value. $E(\theta) = \langle \psi(\theta) | H | \psi(\theta) \rangle$
3. Classical optimization
   - The expectation value is then minimized using a classical optimizer. This is done since it is known that the measured energy must be greater than or equal to the true ground state energy.
4. Iterate
   - Repeat steps 2 and 3 until convergence at a minimum.

Our ground state is then defined as $|\psi(\theta_{\text{min}})\rangle$.

A.3 Anti-commutation relation for the creation and annihilation operators

Our creation and annihilation operators are:

$$\chi_n = \left(\prod_{l<n} -iZ_l\right)\frac{X_n - iY_n}{2}$$
$$\chi_n^\dagger = \left(\prod_{l<n} iZ_l\right)\frac{X_n + iY_n}{2}$$

And we require that:

$$\{\chi_n, \chi_m^\dagger\} = \delta_{nm}$$

Let's simplify the expression slightly, defining P_n and $P_n^\dagger$:

$$\begin{aligned} P_n &= \prod_{l<n} -iZ_l \\ P_m^\dagger &= \prod_{l<m} iZ_l \end{aligned} \tag{15}$$

Substituting in:

$$\begin{gathered}
\{\chi_n, \chi_m^\dagger\} = \chi_n\chi_m^\dagger + \chi_m^\dagger\chi_n \\
= \left(P_n\frac{X_n - iY_n}{2}\right)\left(P_m^\dagger\frac{X_m + iY_m}{2}\right) + \left(P_m^\dagger\frac{X_m + iY_m}{2}\right)\left(P_n\frac{X_n - iY_n}{2}\right) \\
\text{and since:} \\
\left\{Z_n, \frac{X_n - iY_n}{2}\right\} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 0 \\
\text{we can rewrite as:} \\
\{\chi_n, \chi_m^\dagger\} = \frac{1}{4}\left[\left(P_nP_m^\dagger\right)(X_n - iY_n)(X_m + iY_m) + \left(P_m^\dagger P_n\right)(X_m + iY_m)(X_n - iY_n)\right] \\
\text{and since:} \\
iZ(iZ)^\dagger = (iZ)^\dagger iZ = -(iZ)(iZ) = -i^2Z^2 = I \\
\text{for } n = m \text{ we can rewrite as:} \\
\{\chi_n, \chi_n^\dagger\} = \frac{1}{4}[(X_n - iY_n)(X_n + iY_n) + (X_n + iY_n)(X_n - iY_n)] \\
= \frac{1}{4}[X_n^2 + Y_n^2 + i[X_n, Y_n] + X_n^2 + Y_n^2 - i[X_n, Y_n]] \\
= \frac{1}{4}[2X_n^2 + 2Y_n^2] \\
= I_{nn} = 1
\end{gathered} \tag{16}$$

$$
\begin{gathered}
\text{and for } n \neq m \text{ assuming } n < m \text{ we can rewrite as:} \\
\{\chi_n, \chi_m^\dagger\} = \left(P_n \frac{X_n - iY_n}{2}\right)\left(P_m^\dagger \frac{X_m + iY_m}{2}\right) + \left(P_m^\dagger \frac{X_m + iY_m}{2}\right)\left(P_n \frac{X_n - iY_n}{2}\right) \\
\text{we can the split P\_m into two parts:} \\
P_m^\dagger = P_n^\dagger (iZ_n) P'^\dagger_{n,m} \\
\text{where } P_n^\dagger \text{ acts on sites } l < n \text{ and } P'^\dagger_{n,m} \text{ acts on sites } n < l < m \\
\text{if } F_n = \frac{X_n - iY_n}{2}, F_m^\dagger = \frac{X_m + iY_m}{2} \text{ then:} \\
\left\{\chi_n, \chi_m^\dagger\right\} = \frac{1}{4}\left[\left(P_n P_n^\dagger\right) F_n (iZ_n) P'^\dagger_{n,m} F_m^\dagger + \left(P_n^\dagger P_n\right)(iZ_n) P'^\dagger_{n,m} F_m^\dagger F_n\right] \\
\text{since } P_n P_n^\dagger = P_n^\dagger P_n = I\text{:} \\
\left\{\chi_n, \chi_m^\dagger\right\} = \frac{i}{4}\left[F_n Z_n \left(P'^\dagger_{n,m} F_m^\dagger\right) + Z_n F_n \left(P'^\dagger_{n,m} F_m^\dagger\right)\right] \\
\text{and as in Equation 16, } \{Z_n, F_n\} = 0 \\
\left\{\chi_n, \chi_m^\dagger\right\} = \frac{i}{4}\left[(-Z_n F_n)\left(P'^\dagger_{n,m} F_m^\dagger\right) + Z_n F_n \left(P'^\dagger_{n,m} F_m^\dagger\right)\right] \\
= 0
\end{gathered}
\tag{17}
$$

Therefore, $\left\{\chi_n, \chi_m^\dagger\right\} = \delta_{nm}$.

Bibliography

[1] B. Chakraborty, M. Honda, T. Izubuchi, Y. Kikuchi, and A. Tomiya, *Classically Emulated Digital Quantum Simulation of the Schwinger Model with Topological Term via Adiabatic State Preparation*, https://arxiv.org/abs/2001.00485.

[2] Y. Chai, A. Crippa, K. Jansen, S. Kühn, V. R. Pascuzzi, F. Tacchino, and I. Tavernelli, Fermionic wave packet scattering: a quantum computing approach, Quantum **9**, 1638 (2025).