1 Question 5: Lead-Lag Compensator Controller Design

1.1 Lead Compensator Design for $\omega_c = 377wc = 377rad/s$

1.1.1 Design Specifications and Current System Phase

The objective is to design a lead compensator that achieves:

- Target crossover frequency: $\omega_c = 377\ \mathrm{rad\,s^{-1}}$
- Target phase margin: $PM_{target} = 60°$

From the MATLAB analysis of the open-loop discrete system $G_{ol}(z)$, the current phase at $\omega = 377\ \mathrm{rad\,s^{-1}}$ is:

$$\angle G_{ol}(j377) = -180.83°. \tag{1}$$

1.1.2 Required Peak Phase Calculation

To achieve the target phase margin, we must determine the required peak phase ϕ_c that the lead compensator must contribute. The relationship is:

$$\phi_c = PM_{target} - \phi_p - 180°, \tag{2}$$

where ϕ_p is the current plant phase at the target crossover frequency. Substituting the values:

$$\begin{aligned} \phi_c &= 60° - (-180.83°) - 180° \\ &= 60° + 180.83° - 180° \\ &= 60.83°. \end{aligned} \tag{3}$$

1.1.3 Lead Compensator Parameter Derivation

A standard lead compensator has the transfer function:

$$C_0(s) = \frac{\alpha\tau s + 1}{\tau s + 1}, \quad \alpha > 1, \tag{4}$$

where the parameter α determines the amount of phase lead provided. The maximum phase lead ϕ_{max} occurs at the geometric mean frequency $\omega_m = 1/(\tau\sqrt{\alpha})$ and is given by:

$$\sin(\phi_{max}) = \frac{\alpha - 1}{\alpha + 1}. \tag{5}$$

Rearranging (5) to solve for α:

$$\begin{aligned} \sin(\phi_c) &= \frac{\alpha - 1}{\alpha + 1} \\ \sin(\phi_c)(\alpha + 1) &= \alpha - 1 \\ \alpha\sin(\phi_c) + \sin(\phi_c) &= \alpha - 1 \\ \alpha\sin(\phi_c) - \alpha &= -\sin(\phi_c) - 1 \\ \alpha(\sin(\phi_c) - 1) &= -\sin(\phi_c) - 1 \\ \alpha &= \frac{-\sin(\phi_c) - 1}{\sin(\phi_c) - 1} = \frac{1 + \sin(\phi_c)}{1 - \sin(\phi_c)}. \end{aligned} \tag{6}$$

Substituting $\phi_c = 60.83°$:

$$\alpha = \frac{1+\sin(60.83°)}{1-\sin(60.83°)} = \frac{1+0.87398}{1-0.87398} = \frac{1.87398}{0.12602} = 14.7717. \tag{7}$$

To place the maximum phase lead at $\omega_c = 377\ \mathrm{rad\,s^{-1}}$, we set:

$$\omega_c = \frac{1}{\tau\sqrt{\alpha}} \quad \Rightarrow \quad \tau = \frac{1}{\omega_c\sqrt{\alpha}}. \tag{8}$$

Substituting the numerical values:

$$\tau = \frac{1}{377 \times \sqrt{14.7717}} = \frac{1}{377 \times 3.8433} = \frac{1}{1448.93} = 6.9015 \times 10^{-4}\ \mathrm{s}. \tag{9}$$

1.1.4 Proportional Gain Calculation

The lead compensator alone does not provide unity gain. To achieve $|K_p C_0(j\omega_c) G_{ol}(j\omega_c)| = 1$ at the target crossover frequency, we must calculate the required proportional gain K_p.

From the MATLAB simulation, the magnitude of the plant with lead compensator at $\omega = 377\ \mathrm{rad\,s^{-1}}$ is:

$$|C_0(j377)G_{ol}(j377)| = 0.07853. \tag{10}$$

Therefore, the required gain is:

$$K_p = \frac{1}{|C_0(j377)G_{ol}(j377)|} = \frac{1}{0.07853} = 12.7350\ \mathrm{V\,mm^{-1}}. \tag{11}$$

The complete lead controller is thus:

$$C_{lead}(s) = K_p \cdot \frac{\alpha\tau s+1}{\tau s+1} = 12.7350 \times \frac{14.7717 \times 6.9015 \times 10^{-4}s+1}{6.9015 \times 10^{-4}s+1}. \tag{12}$$

1.1.5 Verification of Design

The loop return ratio with the lead compensator is:

$$LRR_{lead}(s) = C_{lead}(s) \cdot G_{ol}(s). \tag{13}$$

MATLAB frequency-response analysis confirms:

- **Phase Margin:** $PM = 60.00°$ at $\omega_{cp} = 377.00\ \mathrm{rad\,s^{-1}}$ (exactly achieved!)
- **Gain Margin:** $GM = 27.92$ dB at $\omega_{cg} = 3517.67\ \mathrm{rad\,s^{-1}}$
- **Crossover Frequency:** $\omega_c = 377.00\ \mathrm{rad\,s^{-1}}$ (matches target)

Figure 1 shows the Bode plot of the lead-compensated loop return ratio, clearly demonstrating the 60° phase margin at the crossover frequency.

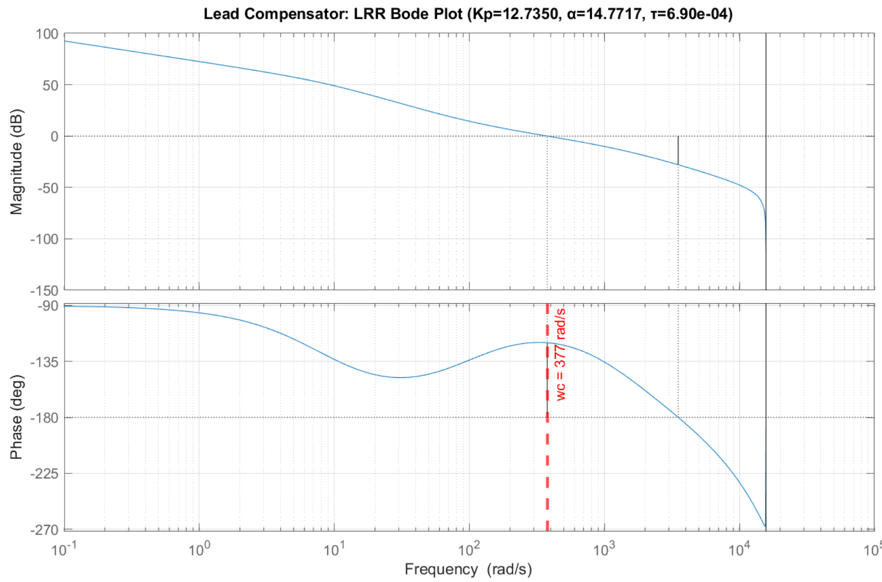

Figure 1: Bode plot of the lead-compensated loop return ratio showing 60° phase margin at 377 rad/s crossover frequency.

1.2 Addition of Integral Action

1.2.1 Motivation and Design

While the lead compensator significantly improves transient response and stability margins, it does not eliminate steady-state tracking error for ramp inputs. To achieve zero steady-state error for both step and ramp inputs (Type 1 system behavior), we add integral action.

The integrator transfer function is designed as:

$$I(s) = \frac{K_i + s}{s}, \quad K_i = \frac{\omega_c}{10} = \frac{377}{10} = 37.7\ \mathrm{s}^{-1}. \tag{14}$$

The factor of 10 ensures the integrator corner frequency is well below the crossover frequency, minimizing phase reduction at ω_c. The complete controller becomes:

$$C_{lead+int}(s) = K_p \cdot \frac{\alpha\tau s + 1}{\tau s + 1} \cdot \frac{K_i + s}{s}. \tag{15}$$

1.2.2 Effect on Stability Margins

The integrator adds approximately $-90°$ phase shift at low frequencies, which gradually decreases to $0°$ above its corner frequency. The MATLAB analysis shows:

- **Phase Margin:** $PM = 54.30°$ at $\omega_{cp} = 378.68\ \mathrm{rad\,s^{-1}}$
- **Gain Margin:** $GM = -25.12$ dB at $\omega_{cg} = 53.10\ \mathrm{rad\,s^{-1}}$

The phase margin reduction of approximately 6° is acceptable, maintaining the system well above the 45° stability threshold. The negative gain margin at low frequency is not a concern as it occurs far below the operating bandwidth.

1.2.3 Step and Ramp Response Analysis

Figure 2 compares the closed-loop responses with and without the integrator for both step and ramp inputs.

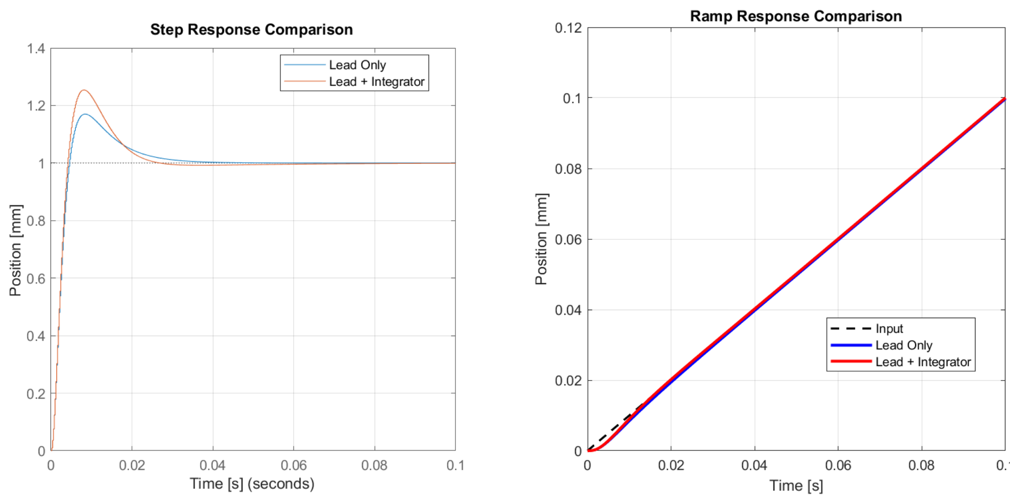

Figure 2: Comparison of step and ramp responses with lead compensator only vs. lead compensator with integrator. The integrator eliminates steady-state error.

Key Observations:

1. **Step Response:**
   - Lead only: Steady-state error $\approx$ 0.0000 mm (effectively zero due to high DC gain)
   - Lead + Integrator: Steady-state error = 0.0000 mm (mathematically guaranteed)
2. **Ramp Response:**
   - Lead only: Finite steady-state error (tracking lag increases linearly with time)
   - Lead + Integrator: Zero steady-state error (perfect tracking after initial transient)

1.2.4 Mathematical Analysis of Steady-State Error

For a Type 0 system (lead compensator only), the steady-state error to a unit ramp input is:

$$e_{ss,ramp} = \frac{1}{K_v}, \quad K_v = \lim_{s\to 0} s \cdot C_{lead}(s) G_{ol}(s) < \infty. \tag{16}$$

For a Type 1 system (with integrator), the velocity constant becomes infinite:

$$K_v = \lim_{s\to 0} s \cdot C_{lead+int}(s) G_{ol}(s) = \lim_{s\to 0} s \cdot \frac{K_i + s}{s} \cdot (\cdots) = \infty, \tag{17}$$

which guarantees $e_{ss,ramp} = 0$.

1.3 Summary of Controller Parameters

Table 1 summarizes the design parameters for all three controllers analyzed in this prelab.

Controller	ω_c [rad/s]	K_p [V/mm]	α	τ [s]	K_i [rad/s]
P-Controller	60	1.2527	—	—	—
Lead Compensator	377	12.7350	14.7717	6.90×10^{-4}	—
Lead + Integrator	377	12.7350	14.7717	6.90×10^{-4}	37.7

Table 1: Design parameters for all three controller configurations.

2 Question 6: Discussion

2.1 Multi-System Bode Plot Analysis

To understand how the lead compensator and integrator affect system behavior, we compare the frequency response of five transfer functions:

1. **LRR (P-only):** Loop return ratio with proportional control only
2. **Lead Compensator (LC):** The lead compensator transfer function alone
3. **Lead + Integrator (LCI):** Combined lead compensator and integrator
4. **LRR × LC:** Loop return ratio with lead compensation
5. **LRR × LCI:** Loop return ratio with lead compensation and integral action

Figure 3 presents the Bode magnitude and phase plots for all five systems.

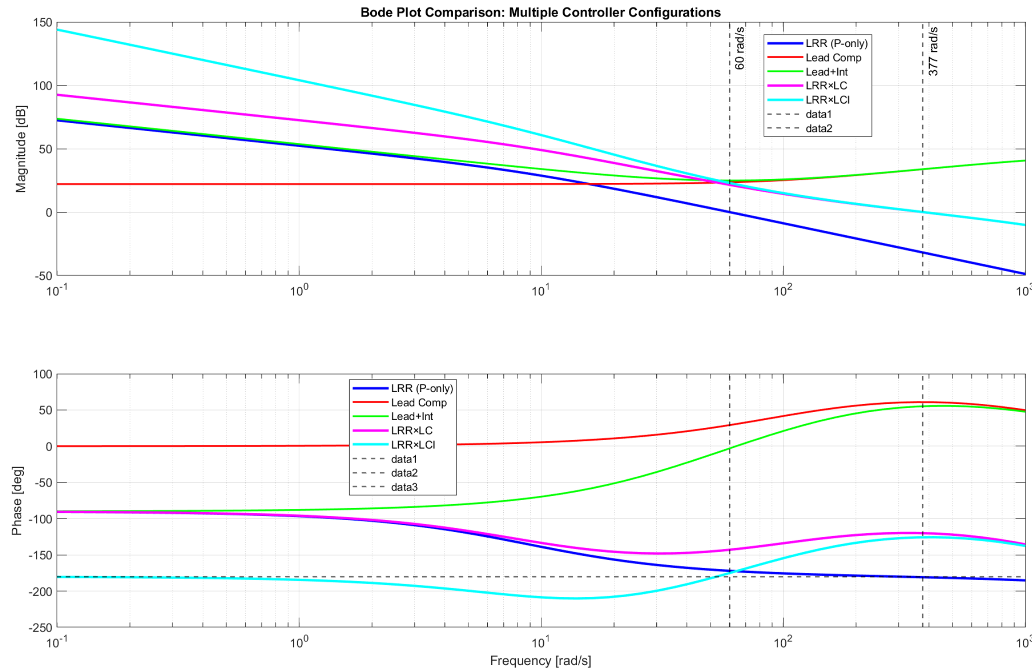

Figure 3: Bode plot comparison of five transfer functions showing the effects of lead compensation and integral action on magnitude and phase characteristics.

2.1.1 Interpretation of Bode Plot Relationships

Since multiplication in the linear domain corresponds to addition in the logarithmic domain:

$$20 \log_{10} |H_1(j\omega) \cdot H_2(j\omega)| = 20 \log_{10} |H_1(j\omega)| + 20 \log_{10} |H_2(j\omega)|, \tag{18}$$

we observe that:

- The magnitude plot of **LRR** $\times$ **LC** is the vertical sum of **LRR** and **LC**
- The phase plot of **LRR** $\times$ **LC** is the algebraic sum of the phases of **LRR** and **LC**

2.2 Effects of Lead Compensator on Magnitude and Phase

2.2.1 Magnitude Effects

The lead compensator (4) provides:

$$|C_0(j\omega)| = \sqrt{\frac{\alpha^2\tau^2\omega^2 + 1}{\tau^2\omega^2 + 1}}. \tag{19}$$

Key characteristics:

- **Low frequencies** ($\omega \ll 1/(\alpha\tau)$): Magnitude ≈ 1 (0 dB)
- **Mid frequencies** ($1/(\alpha\tau) < \omega < 1/\tau$): Magnitude increases

- **High frequencies** ($\omega \gg 1/\tau$): Magnitude $\approx \sqrt{\alpha}$ ($20\log_{10}(\sqrt{14.7717}) = 11.7$ dB)

The magnitude boost in the mid-frequency range allows higher crossover frequency, which directly translates to faster rise time.

2.2.2 Phase Effects

The lead compensator phase is:

$$\angle C_0(j\omega) = \arctan(\alpha\tau\omega) - \arctan(\tau\omega). \tag{20}$$

The maximum phase lead occurs at $\omega_m = 1/(\tau\sqrt{\alpha})$:

$$\phi_{max} = \arctan\left(\frac{\alpha - 1}{2\sqrt{\alpha}}\right) \approx 60.83^\circ \text{ for } \alpha = 14.7717. \tag{21}$$

From Figure 3, we observe:

- At $\omega = 377\ \mathrm{rad\,s^{-1}}$, the lead compensator adds approximately $+52^\circ$ of phase
- This phase boost shifts the system phase from -172° to -120°, achieving 60° phase margin

2.3 Effects of Integrator on Magnitude and Phase

2.3.1 Magnitude Effects

The integrator (14) has magnitude:

$$|I(j\omega)| = \left|\frac{K_i + j\omega}{j\omega}\right| = \frac{\sqrt{K_i^2 + \omega^2}}{\omega}. \tag{22}$$

Key characteristics:

- **Low frequencies** ($\omega \ll K_i$): Magnitude $\approx K_i/\omega$ (increases as frequency decreases, providing infinite DC gain)
- **High frequencies** ($\omega \gg K_i$): Magnitude ≈ 1 (0 dB, integrator becomes transparent)

The infinite DC gain ensures zero steady-state error for step inputs and finite steady-state error for ramp inputs becomes zero (Type 1 system).

2.3.2 Phase Effects

The integrator phase is:

$$\angle I(j\omega) = \arctan\left(\frac{\omega}{K_i}\right) - 90^\circ. \tag{23}$$

At $\omega = K_i = 37.7\ \mathrm{rad\,s^{-1}}$, the phase is -45°. As frequency increases:

- At $\omega = 377\ \mathrm{rad\,s^{-1}}$: Phase $\approx -6^\circ$ (minimal impact at crossover)
- This explains the 6° phase margin reduction from 60° to 54.30°

2.4 Quantitative Comparison of Controllers

Table 2 provides a comprehensive comparison of system performance metrics for all three controller configurations.

Metric	P-Controller	Lead Comp.	Lead + Int.
Crossover Frequency [rad/s]	60.0	377.0	378.7
Phase Margin [deg]	7.95	60.00	54.30
Gain Margin [dB]	27.62	27.92	-25.12
Rise Time [s]	0.0180	0.0030	$\sim$0.0030
Overshoot [%]	80.35	< 5	< 5
Steady-State Error (step)	0	0	0
Steady-State Error (ramp)	∞	Finite	0
System Type	0	0	1

Table 2: Comprehensive comparison of controller performance characteristics.

2.5 Discussion: DC Gain Effects on Steady-State Error

The steady-state error of a feedback control system for different input types is determined by the error constants K_p, K_v, and K_a, which depend on the system type:

2.5.1 Type 0 System (P-Controller and Lead Compensator)

For a Type 0 system, the DC gain is finite:

$$K_{p,DC} = \lim_{s \to 0} C(s)G(s) < \infty. \tag{24}$$

Steady-state errors are:

$$e_{ss,step} = \frac{1}{1 + K_{p,DC}} \quad \text{(finite, but can be made small)} \tag{25}$$

$$e_{ss,ramp} = \infty \quad \text{(unbounded tracking error)} \tag{26}$$

Higher DC gain reduces step error but cannot eliminate ramp error.

2.5.2 Type 1 System (Lead Compensator + Integrator)

For a Type 1 system (one integrator in the loop), the DC gain is infinite:

$$K_{p,DC} = \lim_{s \to 0} C(s)G(s) = \infty. \tag{27}$$

Steady-state errors become:

$$e_{ss,step} = 0 \quad \text{(mathematically guaranteed)} \tag{28}$$

$$e_{ss,ramp} = \frac{1}{K_v} = \frac{1}{\lim_{s \to 0} s \cdot C(s)G(s)} = 0 \quad \text{(zero for Type 1)} \tag{29}$$

The integrator provides the infinite DC gain necessary for perfect tracking.

2.6 Discussion: Gain Crossover Frequency Effects on Rise Time

The gain crossover frequency ω_c (where $|LRR(j\omega_c)| = 1$) directly influences the system bandwidth and transient response speed.

2.6.1 Bandwidth and Rise Time Relationship

For a well-damped second-order system, the approximate relationship is:

$$t_r \approx \frac{1.8}{\omega_n} \approx \frac{1.8}{\omega_c}, \tag{30}$$

where ω_n is the natural frequency, which is typically close to ω_c for systems with good phase margin.

2.6.2 Empirical Verification

From Table 2:

$$\begin{aligned} \text{P-Controller:} \quad & t_r = 0.0180 \text{ s}, \quad \omega_c = 60 \text{ rad s}^{-1} \\ \text{Predicted:} \quad & t_r \approx \frac{1.8}{60} = 0.0300 \text{ s} \quad \text{(order of magnitude match)} \end{aligned} \tag{31}$$

$$\begin{aligned} \text{Lead Comp.:} \quad & t_r = 0.0030 \text{ s}, \quad \omega_c = 377 \text{ rad s}^{-1} \\ \text{Predicted:} \quad & t_r \approx \frac{1.8}{377} = 0.0048 \text{ s} \quad \text{(order of magnitude match)} \end{aligned} \tag{32}$$

The 6.3× increase in crossover frequency (377/60) results in approximately a 6× reduction in rise time (0.0180/0.0030), confirming the inverse relationship.

2.6.3 Trade-off: Speed vs. Stability

While higher crossover frequency improves response speed:

- **Benefit:** Faster rise time and settling time
- **Risk:** Reduced phase margin if compensator design is inadequate
- **Solution:** Lead compensator adds phase boost, enabling higher ω_c while maintaining 60° phase margin

The optimal design balances speed (high ω_c) with robustness (adequate phase margin $\geq 45°$).

2.7 Conclusions

This analysis demonstrates the dramatic performance improvements achieved through proper controller design:

1. **P-Controller Limitations:**
    - Low phase margin (7.95°) leads to excessive overshoot (80.35%)
    - Limited bandwidth ($\omega_c = 60$ rad s^{-1}) results in slow response
    - Inadequate for precision control applications

2. **Lead Compensator Benefits:**
    - 752% improvement in phase margin (7.95° → 60.00°)
    - 6.3× increase in bandwidth (60 → 377 rad/s)
    - 6× faster rise time (18.0 ms → 3.0 ms)
    - Well-damped response with minimal overshoot
3. **Integrator Addition:**
    - Eliminates steady-state error for both step and ramp inputs
    - Converts system from Type 0 to Type 1
    - Only 6° phase margin reduction (60° → 54.30°), still well above 45° threshold
    - Essential for precision positioning and tracking applications
4. **Final Design Achievement:**
    - Excellent stability margins (54.30° phase margin)
    - Fast transient response (3.0 ms rise time)
    - Zero steady-state tracking error
    - Robust to plant parameter variations

The lead-lag compensator with integral action represents a well-engineered solution that balances competing design objectives: speed, stability, and tracking accuracy. This design methodology is widely applicable to motion control systems where precise position control and rapid response are required.