MECH 467/541 Project 2 – Prelab 2 (Phase 1)

¡Student Name¿ – ¡Student Number¿

November 13, 2025

System Summary

The open-loop path from the command voltage $v_{in}(t)$ to the measured table position $x_a(t)$ is built from the amplifier, torque constant, rigid rotor, and ballscrew encoder, while Coulomb friction, saturation, and disturbance torque are ignored for this phase. With $K_d = 1\ \mathrm{mm\,V^{-1}}$ to capture the Beckhoff scaling, the numeric constants are listed in Table **??**.

Quantity	Symbol	Value
Amplifier gain	K_a	$0.887\ \mathrm{A\,V^{-1}}$
Torque constant	K_t	$0.72\ \mathrm{N\,m\,A^{-1}}$
Encoder gain	K_e	$\frac{20}{2\pi} = 3.1831\ \mathrm{mm\,rad^{-1}}$
Inertia	J_e	$7 \times 10^{-4}\ \mathrm{kg\,m^2}$
Viscous friction	B_e	$0.00612\ \mathrm{Nm/(rad/s)}$
Sample time	T_s	$2 \times 10^{-4}\ \mathrm{s}$
Ballscrew pitch	h_p	$20\ \mathrm{mm}$

Table 1: Given machine constants for the Beckhoff-driven ballscrew axis.

1 Prelab 1 – Discrete Transfer Function Derivation

1.1 Continuous open-loop model

Applying the cascade in Fig. 1 of the handout gives the continuous plant

$$J_e\dot{\omega}(t) + B_e\omega(t) = K_t K_a\, v_{in}(t), \tag{1}$$

$$\dot{x}_a(t) = K_e\,\omega(t), \tag{2}$$

so the Laplace-domain transfer function is

$$G_{ol}(s) \equiv \frac{X_a(s)}{V_{in}(s)} = \frac{K_d K_a K_t K_e}{s(J_e s + B_e)} = \frac{2.0329}{s(7 \times 10^{-4} s + 0.00612)}. \tag{3}$$

Partial-fraction decomposition of (**??**) (Black's formula) yields

$$G_{ol}(s) = \frac{K_d K_a K_t K_e}{B_e}\left(\frac{1}{s} - \frac{1}{s + \frac{B_e}{J_e}}\right), \tag{4}$$

which explicitly separates the integrator from the viscously damped pole.

1.2 Zero-order-hold derivation

Let the states be $x_1 = \omega$ and $x_2 = x_a$; with $\lambda \equiv B_e/J_e = 8.7429\ \mathrm{s}^{-1}$ the continuous dynamics become

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -\lambda & 0 \\ K_e & 0 \end{bmatrix}}_{A_c} \mathbf{x} + \underbrace{\begin{bmatrix} \dfrac{K_a K_t}{J_e} \\ 0 \end{bmatrix}}_{B_c} v_{in}, \qquad y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C_c} \mathbf{x}. \tag{5}$$

Solving (??) over one sample with a zero-order hold gives the discrete updates

$$\omega[k+1] = \mathrm{e}^{-\lambda T_s}\omega[k] + \frac{K_a K_t}{B_e}\left(1 - \mathrm{e}^{-\lambda T_s}\right) v_{in}[k], \tag{6}$$

$$\begin{aligned} x_a[k+1] = x_a[k] &+ \frac{K_e}{\lambda}\left(1 - \mathrm{e}^{-\lambda T_s}\right)\omega[k] \\ &+ \frac{K_a K_t K_e}{B_e}\left(T_s - \frac{1 - \mathrm{e}^{-\lambda T_s}}{\lambda}\right) v_{in}[k], \end{aligned} \tag{7}$$

which correspond to the closed-form matrices

$$A_d = \begin{bmatrix} \phi & 0 \\ \dfrac{K_e}{\lambda}(1-\phi) & 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} \dfrac{K_a K_t}{B_e}(1-\phi) \\ \dfrac{K_a K_t K_e}{B_e}\left(T_s - \dfrac{1-\phi}{\lambda}\right) \end{bmatrix}, \quad \phi \equiv \mathrm{e}^{-\lambda T_s}. \tag{8}$$

Carrying out the algebra $G_{ol}(z) = C_d(zI - A_d)^{-1}B_d$ leads to

$$G_{ol}(z) = \frac{\beta_1 z + \beta_0}{z^2 - (1+\phi)z + \phi}, \qquad \beta_1 = \frac{K_a K_t K_e}{B_e}\left(T_s - \frac{1-\phi}{\lambda}\right), \ \beta_0 = \frac{K_e}{\lambda}(1-\phi)\frac{K_a K_t}{B_e}(1-\phi) - \phi\beta_1. \tag{9}$$

Substituting the numeric parameters produces

$$G_{ol}(z) = \frac{5.8048 \times 10^{-5}\, z + 5.8014 \times 10^{-5}}{z^2 - 1.998252956\, z + 0.998252956}. \tag{10}$$

1.3 Comparison with MATLAB

The MATLAB script `phase1_prelab.m` implements the same discretization through both the analytical formulas (??)–(??) and the built-in `c2d` command. Table ?? confirms that the coefficients match to numerical precision ($< 1.1 \times 10^{-16}$).

Coefficient	Manual ZOH	MATLAB `c2d`
b_1 (units mm/V)	$5.80477117505705 \times 10^{-5}$	$5.80477117506742 \times 10^{-5}$
b_0 (units mm/V)	$5.80138880862634 \times 10^{-5}$	$5.80138880861596 \times 10^{-5}$
Denominator z^1 term	-1.99825295643180	-1.99825295643180
Denominator z^0 term	0.99825295643180	0.99825295643180

Table 2: Numerical equality between the manual ZOH derivation and `c2d`.

2 Prelab 2 – State-Space Model and Step Response

2.1 Continuous and discrete state-space models

Equations (??) already define the continuous-time state matrices

$$A_c = \begin{bmatrix} -8.7429 & 0 \\ 3.1831 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 912.343 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D_c = 0.$$

Using the closed-form expressions in (??), the discrete model sampled at $T_s = 2 \times 10^{-4}$ s becomes

$$A_d = \begin{bmatrix} 0.9982529564 & 0 \\ 6.3606 \times 10^{-4} & 1 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 1.8230913 \times 10^{-1} \\ 5.8047712 \times 10^{-5} \end{bmatrix}, \qquad C_d = C_c, \quad D_d = 0. \tag{11}$$

The matrix-exponential check inside `phase1_prelab.m` verifies that these hand-derived matrices match the numerical `expm` result to machine precision ($< 10^{-12}$).

2.2 Open-loop step response

Both the discrete transfer function (??) and the discrete state-space model (??) were driven by a 1 V step over 0.02 s. As expected for a type-I plant, the response is a ramp (no finite steady state). The maximum absolute difference between the two simulated trajectories was 7.6×10^{-14} mm, i.e., numerical noise only. The peak displacement at 20 ms is 0.5484 mm for both models; traditional rise/settling metrics are undefined because of the inherent integrator.

3 Prelab 3 – Stability Analysis

3.1 Closed-loop characteristic equations

Closing the loop with a proportional gain K_p (units $\mathrm{V\,mm^{-1}}$) produces the continuous characteristic polynomial

$$J_e s^2 + B_e s + K_p K_d K_a K_t K_e = 0, \tag{12}$$

whose coefficients remain positive for $K_p > 0$; hence both poles stay in the left half-plane and the root locus simply migrates away from the origin (one pole starts at $s = 0$, the other at $s = -B_e/J_e$). In discrete time the closed-loop denominator becomes

$$D(z) + K_p N(z) = z^2 - (1 + \phi)z + \phi + K_p\big(\beta_1 z + \beta_0\big) = 0, \tag{13}$$

with the coefficients defined in (??). Because $N(z)$ adds a finite-delay zero, the discrete root locus wraps around the unit circle; the sample data stored in `results.rlocus` show the discrete poles leaving the unit circle once K_p exceeds roughly 600, while the continuous poles remain damped conjugates for all $K_p > 0$.

3.2 Frequency-response margins

Using the open-loop models,

$$GM_s = \infty, \quad PM_s = 9.28^\circ \text{ at } \omega_{cp} = 53.53 \text{ rad s}^{-1},$$

so the continuous system can tolerate arbitrarily large gain because the integrator ensures 0 dB never occurs at a finite phase crossover. The discrete plant has finite margins due to the sample-and-hold delay:

$$GM_z = 30.11\ (\approx 29.6\ \text{dB}), \quad PM_z = 8.97^\circ \text{ at } \omega_{cp} = 53.54\ \text{rad}\,\text{s}^{-1}.$$

Varying the sample time illustrates how faster sampling improves robustness; Table **??** summarizes the zero-order-hold gain margins from the MATLAB sweep.

T_s [s]	Gain margin	Phase margin [deg]
2.0×10^{-2}	0.310	-20.30
2.0×10^{-3}	3.020	6.21
2.0×10^{-4}	30.11	8.97

Table 3: Discrete gain/phase margins versus sampling period.

The coarse sample ($T_s = 20$ ms) effectively destabilizes the loop (negative phase margin), while the Beckhoff sampling rate used in the lab restores positive phase and a comfortable gain margin. Therefore, stability in the s and z domains is not automatically equivalent; sufficient sampling speed is required to keep discrete poles inside the unit circle even though the underlying continuous dynamics are stable for any positive K_p.

Saved MATLAB artefacts

Running `phase1_prelab.m` generates the file `results/phase1_data.mat` containing:

- Continuous and discrete numerator/denominator coefficients.
- The analytic A_d, B_d matrices and verification data.
- Step-response time histories and comparison metrics.
- Root-locus pole sweeps for both domains.
- Frequency-response margins versus sample time.

These records will be reused in Phase 2 (Simulink controller design) so that the report only needs to include plots/reference tables generated from a single source of truth.