Partial Differential Equations in Modelica

15th July 2013

Contents

Chapter 1

Modelica extension for PDE

Space & coordinates

What should be specified

- Dimension of the problem (1,2 or 3D)
- ?? Coordinates (cartesian, cylindrical, spherical ...) – where this information will be used (if at all):
  - in differential operators as grad, div, rot etc.
  - in visualization of results
  - ?? in computation – perhaps equations should be transformed and the calculation would be performed in cartesian coordinates
- Names of independent (coordinate) variables (x, y, z, r, φ, θ...)

Perhaps all these should be specified within the domain deffinition.
Dimension can be infered from number of return values of shape-function or different properities of the domain in other cases.
The base coordinates would be cartesian and they would be always implicitly defined in any domain. Besides that other coordinate systems could be defined also.
Names of independent variables in cartesian coordinates should be fixed *x, (x,y), (x,y,z)* in 1D, 2D and 3D domains respectively.

Domain & boundary

What should be specified

- the domain where we perform the computation and where equations hold
- boundary and its subsets where particular boundary conditions hold

- normal vector of the boundary

Possible approaches

Parametrization of the domain with shape function and ranges – from The Book (Principles of ...), section 8.5.2

Example from the book:

```
model HeatCircular2D
        import DifferentialOperators2D.*;
        parameter DomainCircular2DGrid omega;
        FieldCircular2DGrid u(domain=omega, FieldValueType=SI.Temperature);
equation
        der (u) = pder (u,D.x2)+ pder (u,D . y 2 )        in omega.interior;
        nder(u) = 0               in omega.boundary;
end HeatCircular2D;

record DomainCircular2DGrid "Domain being a circular region"
        parameter Real radius = 1;
        parameter Integer nx = 100;
        parameter Integer ny = 100;
        replaceable function shapeFunc = circular2Dfunc "2D circular region";
        DomainGe2D interior(shape=shapeFunc,range={{O,radius},{O,l}},geom= ... )
        DomainGe2D boundary (shape=shapeFunc, range={{radius, radius), { 0,1}} ,
        function shapeFunc = circular2Dfunc "Function spanning circular region";
end DomainCircular2DGrid;

function circular2Dfunc "Spanned circular region for v in range 0..1"
        input Real r,v;
        output Real x,y;
algorithm
        x : = r*cos (2*PI*v);
        y := r*sin(2*PI*v);
end circular2Dfunc;

record FieldCircular2DGrid
        parameter DomainCircular2DGrid domain;
        replaceable type FieldValueType = Real;
        replaceable type FieldType = Real[domain.nx,domain.ny,domain.nz];
        parameter FieldType start = zeros(domain.nx,domain.ny,domain.nz.);
        FieldType Val;
end FieldCircularZDGrid;
```

And modified version, where all numerical stuff (grid, number of points – this should be configured using simulation setup or annotations) omitted, modified `pder` operator, `Field` as Modelica build-in type:

```
model HeatCircular2D
        parameter DomainCircular2D omega(radius=2);
        field Real u(domain=omega, start = 0, FieldValueType=SI.Temperature);
equation
        pder(u,time) = pder(u,x)+ pder(u,y) in omega.interior;
        pder(u,omega.boundary.n) = 0    in omega.boundary;
end HeatCircular2D;

record DomainCircular2D
        parameter Real radius = 1;
    parameter Real cx = 0;
        parameter Real cy = 0;
        function shapeFunc
                input Real r,v;
                output Real x,y;
        algorithm
                x := cx + radius*r * cos(2 * C.pi * v);
                y := cy + radius*r * sin(2 * C.pi * v);
        end shapeFunc;
        Domain2DInterior interior(shape = shapeFunc, range = {{O,1},{O,1}});
Domain2DBoundary boundary(shape = shapeFunc, range = {{1,1},{0,1}});
end DomainCircular2D;
```

Description by the boundary Domain is defined by closed boundary curve, which may by composed of several connected curves. Needs new operator *interior* and type *Domain2d* (and *Domain1D* and *Domain3d*). (similarly used in FlexPDE – http://www.pdesolutions.com/.)

```
package BoundaryRepresentation
  partial function cur
    input Real u;
    output Real x;
    output Real y;
  end cur;
  function arc
    extends cur;
    parameter Real r;
    parameter Real cx;
    parameter Real cy;
  algorithm
    x:=cx + r * cos(u);
```

```
  y:=cy + r * sin(u);
end arc;
function line
  extends cur;
  parameter Real x1;
  parameter Real y1;
  parameter Real x2;
  parameter Real y2;
algorithm
  x:=x1 + (x2 - x1) * u;
  y:=y1 + (y2 - y1) * u;
end line;
function bezier3
  extends cur;
  //start-point
  parameter Real x1;
  parameter Real y1;
  //end-point
  parameter Real x2;
  parameter Real y2;
  //start-control-point
  parameter Real cx1;
  parameter Real cy1;
  //end-control-point
  parameter Real cx2;
  parameter Real cy2;
algorithm
  x:=(1 - u) ^ 3 * x1 + 3 * (1 - u) ^ 2 * u * cx1 + 3 *
      (1 - u) * u ^ 2 * cx2 + u ^ 3 * x2;
  y:=(1 - u) ^ 3 * y1 + 3 * (1 - u) ^ 2 * u * cy1 + 3 *
      (1 - u) * u ^ 2 * cy2 + u ^ 3 * y2;
end bezier3;
record Curve
  function curveFun = line;
  // to be replaced with another fun
  parameter Real uStart;
  parameter Real uEnd;
end Curve;
record Boundary
  constant Integer NCurves;
  Curve curves[NCurves];
  //    for i in 1:(NCurves-1) loop
  //assert(Curve[i].curveFun(Curve[i].uEnd) = Curve[i
     +1].curveFun(Curve[i+1].uStart), String(i)+"th
     curve and "+String(i+1)+"th curve are not
     connected.",level = AssertionLevel.error);
```

```
    //      end for;
    //      assert(curves[NCurves].curveFun(curves[NCurves
        ].uEnd) =
    //
        curves[1].curveFun(curves[1].uStart),
    //
        String(NCurves)+"th curve and first curve are not
        connected.",
    //
        level = AssertionLevel.error);
  end Boundary;
  record DomainHalfCircle
    constant Real pi = Modelica.Constants.pi;
    arc myArcFun(cx = 0, cy = 0, r = 1);
    Curve myArc(curveFun = myArcFun, uStart = pi / 2,
        uEnd = (pi * 3) / 2);
    line myLineFun(x1 = 0, y1 = -1, x2 = 0, y2 = 1);
    Curve myLine(curveFun = myLineFun, uStart = 0, uEnd =
         1);
    line myLine2(curveFun = line(x1 = 0, y1 = -1, x2 = 0,
         y2 = 1), uStart = pi / 2, uEnd = (pi * 3) / 2);
    Boundary b(NCurves = 2, curves = {myArc,myLine});
    //new externaly defined type Domain2D and operator
        interior:
    Domain2D d = interior Boundary;
  end DomainHalfCircle;
end BoundaryRepresentation;
```

Constructive solid geometry used in Matlab PDE toolbox, http://en.wikipedia.org/wiki/Constructive_sol

Domain is build from primitives (cuboids, cylinders, spheres, cones, user defined shapes ...) applying boolean operations *union*, *intersection* and *difference.*

How to describe boundaries?

Listing of points – export from CAD

Inequalities

Boundary representation (BRep) (NETGEN, STEP)

Fields

Partial derivative

$\frac{\partial^2 u}{\partial x \partial y}$ ... `pder(u,x,y)`

directional derivative ... `pder(u,omega.boundary.n)`

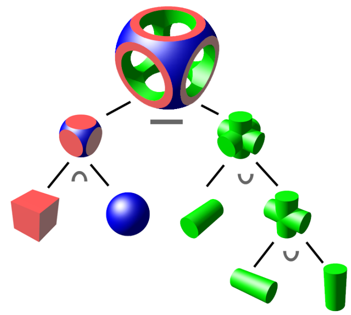

Figure 1.1: constructive solid geometry

Equations, boundary and initial conditions

Use the *in* operator to express where equations hold.

Chapter 2

Numerics

Goals

1. advection equation in 1D and eulerian coordinate, dirichlet BC, explicit solver
2. numann BC
3. automatic dt
4. diffusion or mixed equation
5. implicit solver
6. systems of equations
7. 2D (rectangle), 3D (cube)
8. lagrangian coordinate
9. general domain

difference schemes separated from the rest of solver

Difussion eq:

$$u_t = \alpha u_{xx}$$

or

$$\begin{aligned} u_t &= -w_x \\ w &= -\alpha u_x. \end{aligned}$$

String eq:

$$y_{tt} = k y_{xx}$$

or

$$\begin{aligned} s_x &= kv_t \\ y_t &= v \\ y_x &= s \end{aligned}$$

The description without higher derivative is ugly, we need higher derivatives.

Representation

Explicit

$$u_t = f(u, u_x, t) \tag{2.1}$$

resp.

$$u_t = f(u, u_x, u_{xx}, \dots, t)$$

..

Implicit

$$F(u, u_t, u_x, t) = 0 \tag{2.2}$$

resp.

$$F(u, u_t, u_x, u_{xx}, \dots, t) = 0$$

Solvers

Difference schemes for explicit solver

U denotes discretized u

Time difference from Lax-Friedrichs in explicit form (i.e. with the u_m^{n+1} on LHS):

$$u_m^{n+1} = D_t^{exp}(v, U, n, m) = v\Delta t + \frac{1}{2}(u_{m+1}^n + u_{m-1}^n) \tag{2.3}$$

Space difference from Lax-Friedrichs:

$$D_x(U, n, m) = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{2.4}$$

Explicit solver Lax-Friedrichs

We solve equation (2.1) substituing space difference (2.4) and applying time difference in explicit form (2.3):

$$\begin{aligned} u_m^{n+1} &= D_t^{exp}(f((u_m^n, D_x(U, n, m), t^n))) = \\ &= \Delta t \cdot f(u, \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, t) + \frac{1}{2}(u_{m+1}^n + u_{m-1}^n) \end{aligned}$$

Difference schemes for implicit solver space difference from Crank-Nicolson

$$
\begin{aligned}
D_x(u^n_{m-1}, u^n_m, u^n_{m+1}, u^{n+1}_{m-1}, u^{n+1}_m, u^{n+1}_{m+1}) &= \frac{1}{2}\left(\frac{u^{n+1}_{m+1} - u^{n+1}_{m-1}}{2\Delta x} + \frac{u^n_{m+1} - u^n_{m-1}}{2\Delta x}\right) \qquad (2.5) \\
D_{xx}(u^n_{m-1}, u^n_m, u^n_{m+1}, u^{n+1}_{m-1}, u^{n+1}_m, u^{n+1}_{m+1}) &= \frac{1}{2}\left(\frac{u^{n+1}_{m+1} - 2u^{n+1}_m + u^{n+1}_{m-1}}{2(\Delta x)^2} + \frac{u^n_{m+1} - 2u^n_m + u^n_{m-1}}{2(\Delta x)^2}\right) \qquad (2.6)
\end{aligned}
$$

time difference from Crank-Nicolson

$$D_t(u^n_{m-1}, u^n_m, u^n_{m+1}, u^{n+1}_{m-1}, u^{n+1}_m, u^{n+1}_{m+1}) = \frac{u^{n+1}_m - u^n_m}{\Delta t} \qquad (2.7)$$

Implicit solver Crank-Nicolson
With nonlinear solver:
We solve equation (2.2) substituting space (2.5) and time (2.7) differences

$$F\left(u^n_m, D_t(u^n_{m-1}, ...), D_x(u^n_{m-1}, ...), t^n\right) = 0,\ m \in \hat{M} \qquad (2.8)$$

and than solving the whole system for all unknown $u^{n+1}_{:}$. System has 3-band Jacobian. If F is linear in u_x and u_t, system is also linear with 3-band matrix eventhou is given generaly. Is there any solver eficient in solving linear equations with banded matrix given implicitly? (I hope Newton-Raphson is.) As initial guess for the solution we can use extrapolated values. If solving fails we can try value from the node on left or right (this could help on shocks).
With linear solver:
If F is linear, we expres (2.8) as

$$\boldsymbol{A}\bar{u}^{n+1}_{:} = \bar{b}.$$

$\boldsymbol{A}$ is $M \times M$ 3-diagonal. Functions for evaluation of $\boldsymbol{M}$ and $\bar{b}$ are generated during compilation. In runtime we solve just the linear system. In this aproach difference schema must be chosen before compilation of model.

Implicit solver and systems of PDE If we solve e.g. system with three variables u, v, w, se can sort difference equations in order

$$u_1,\ v_1,\ w_1,\ u_2,\ v_2,\ w_2,\ u_3,\ v_3,\ w_3,\ ...$$

so that the system is stil banded.

Articles and books

I want to read:

A DIFFERENTIATION INDEX FOR PARTIAL DIFFERENTIAL-ALGEBRAIC EQUATIONS [6]

INDEX AND CHARACTERISTIC ANALYSIS OF LINEAR PDAE SYSTEMS [7]

Finite difference methods for ordinary and partial differential equations [5]

Questions:

- Shall we support higher derivatives in solver?
- What about multi step mothods (RK, P-K)?
- How to generate even (or arbitrary) meshes with nonlinea shape functions?
- How to generate mesh points just on the boundary? 1D – simple – just two points. 2D – We can go through the boundary curve and detect crossings of grid lines. 3D – who knows?!
- How to represent on which particular boundary an boundary condition hold in generated code (or even on which interior domain hold which PDE equation system, if we support various interiors)?
  - some domain struct could hold both shapeFunction parameter ranges and pointer (or some index) to function with the corresponding equations.
  - boundary condition function knows on which elements (indexes) of variable arrays should be applied.

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Chapter 3

Example models

3.1 Package PDEDomains

Modelica code of domain definitions:

```
package PDEDomains
  import C = Modelica.Constants;
  record DomainLineSegment1D
    parameter Real l = 1;
    parameter Real a = 0;
    function shapeFunc
      input Real v;
      output Real x = l*v + a;
    end shapeFunc;
    Domain1DInterior interior(shape = shapeFunc, range =
        {0,1});
    Domain1DBoundary left(shape = shapeFunc, range =
        {0,0});
    Domain1DBoundary right(shape = shapeFunc, range =
        {1,1});
  end DomainLineSegment1D;
  record DomainRectangle2D
    parameter Real Lx = 1;
    parameter Real Ly = 1;
    parameter Real cx = 0;
    parameter Real cy = 0;
    function shapeFunc
      input Real v1,v2;
      output Real x = v1 / 2 * Lx + cx,y = v2 / 2 * Ly +
          cy;
    end shapeFunc;
    Domain2DInterior interior(shape = shapeFunc, range =
```

```
      {{-1,1},{-1,1}});
    Domain2DBoundary right(shape = shapeFunc, range =
      {{1,1},{-1,1}});
    Domain2DBoundary bottom(shape = shapeFunc, range =
      {{-1,1},{-1,-1}});
    Domain2DBoundary left(shape = shapeFunc, range =
      {{-1,-1},{-1,1}});
    Domain2DBoundary top(shape = shapeFunc, range =
      {{-1,1},{1,1}});
  end DomainRectangle2D;
  record DomainCircular2D
    parameter Real radius = 1;
    parameter Real cx = 0;
    parameter Real cy = 0;
    function shapeFunc
      input Real r,v;
      output Real x,y;
    algorithm
      x:=cx + radius * r * cos(2 * C.pi * v);
      y:=cy + radius * r * sin(2 * C.pi * v);
    end shapeFunc;
    Domain2DInterior interior(shape = shapeFunc, range =
      {{O,1},{O,1}});
    Domain2DBoundary boundary(shape = shapeFunc, range =
      {{1,1},{0,1}});
  end DomainCircular2D;
  record DomainBlock3D
    parameter Real Lx = 1,Ly = 1,Lz = 1;
    parameter Real cx = 0,cy = 0,cz = 0;
    function shapeFunc
      input Real vx,vy,vz;
      output Real x = vx / 2 * Lx + cx,y = vy / 2 * Ly +
        cy,z = vz / 2 * Lz + cz;
    end shapeFunc;
    Domain3DInterior interior(shape = shapeFunc, range =
      {{-1,1},{-1,1},{-1,1}});
    Domain3DBoundary right(shape = shapeFunc, range =
      {{1,1},{-1,1},{-1,1}});
    Domain3DBoundary bottom(shape = shapeFunc, range =
      {{-1,1},{-1,y},{1,1}});
    Domain3DBoundary left(shape = shapeFunc, range =
      {{-1,-1},{-1,1},{-1,1}});
    Domain3DBoundary top(shape = shapeFunc, range =
      {{-1,1},{-1,1},{1,1}});
    Domain3DBoundary front(shape = shapeFunc, range =
      {{-1,1},{-1,-1},{-1,1}});
```

```
    Domain3DBoundary rear(shape = shapeFunc, range =
        {{-1,1},{1,1},{-1,1}});
  end DomainBlock3D;
  //and others ...
end PDEDomains;
```

Listing 3.1: Standard domains deffinitions: 1D – Line segment, 2D – Rectangle, Circle, 3D – Block

3.2 Simple models

3.2.1 Advection equation (1D)[9]

L .. length
c .. constant, assume $c > 0$
$u \in \langle 0, L\rangle \times \langle 0, T\rangle \to \mathbb{R}$

equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

initial conditions

$$u(x, 0) = 0$$

boundary conditions

$$u(0, t) = \sin(2\pi t)$$

Modelica code

```
model advection "advection equation"
  import C = Modelica.Constants;
  parameter Real L = 1; // length
  parameter Real c = 1;
  parameter DomainLineSegment1D omega(length = L);
  field Real u(domain = omega, start = 1);
equation
  pder(u,time) + c*pder(u,x) = 0  in  omega.
      interior;
  u = cos(2*pi*time) in omega.left;
end advection;
```

Listing 3.2: Advection equation in Modelica

Flat model

```
/*TODO: finish it!!*/

model advection_flat "advection equation"
  import C = Modelica.Constants;
  parameter Real L = 1; // length
  parameter Real c = 1;
//  parameter DomainLineSegment1D omega(length = L
   );
  parameter Real DomainLineSegment1D.l = L;
  parameter Real DomainLineSegment1D.a = 0;
  function DomainLineSegment1D.shapeFunc
    input Real v;
    output Real x = l*v + a;
  end DomainLineSegment1D.shapeFunc;
  Domain1DInterior DomainLineSegment1D.interior(
     shape = shapeFunc, range = {0,1});
  Domain1DBoundary DomainLineSegment1D.left(shape
     = shapeFunc, range = {0,0});
  Domain1DBoundary DomainLineSegment1D.right(shape
     = shapeFunc, range = {1,1});

  field Real u(domain = omega, start = 1);
equation
  pder(u,time) + c*pder(u,x) = 0  in  omega.
     interior;
  u = cos(2*pi*time) in omega.left;
end advection_flat;
```

Listing 3.3: Advection equation – flat model

Generated C code

```
#include "data.h"
#include "PDESolver.h"
#include "model.h"
#include <math.h>
//#define _USE_MATH_DEFINES
//#include <math.h>
double pi = 3.14159265358979323846;
```

```
int setupArrayDimensions(struct DATA* data) {
        data->nStateFields = 1;
        data->nAlgebraicFields= 0;
        data->nParameterFields= 0;
        data->nParameters = 4;
        data->nDomainSegments = 3;
        return 0;
}



int setupModel(struct DATA* data)
{
        int i;
        /*interior:*/
        data->domainRange[0].v0 = 0;
        data->domainRange[0].v1 = 1;
        /*left*/
        data->domainRange[1].v0 = 0;
        data->domainRange[1].v1 = 0;
        /*right*/
        data->domainRange[2].v0 = 1;
        data->domainRange[2].v1 = 1;
        for (i=0; i<data->M; i++){
                data->stateFields[data->M*0 + i] =
                    1;
        }
        /*advection.L*/data->parameters[0] = 1;
        /*advection.c*/data->parameters[1] = 1;
        /*DomainLineSegment1D.l*/data->parameters
           [2] = 1;
        /*DomainLineSegment1D.a*/data->parameters
           [3] = 0;
        data->isBc[data->nStateFields*0 + 0] = 1;
        data->isBc[data->nStateFields*0 + 1] = 0;

        return 0;
}

double shapeFunction(struct DATA *data, double v)
{
        return /*DomainLineSegment1D.l*/data->
           parameters[2]*v + /*DomainLineSegment1D
           .a*/data->parameters[3];
}
```

```
int functionPDE(struct DATA *data)
{
        int i;
        for (i = 0; i<data->M; i++)
                /*u_t*/data->stateFieldsDerTime[
                    data->M*0 + i] = - /*c*/data->
                    parameters[1] * /*u_x*/data->
                    stateFieldsDerSpace[data->M*0 +
                     i];
        return 0;
}

int functionBC(struct DATA *data)
{
        data->stateFields[data->M*0 + 0] = cos(2*
           pi*data->time);
        return 0;
}
```

Listing 3.4: Advection equation – "generated" C code

3.2.2 String equation (1D)[12]

L .. length
$u \in \langle 0, L\rangle \times \langle 0, T\rangle \to \mathbb{R}$ (string position)
c .. constant
equation:

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0$$

initial conditions

$$u(x,0) = \sin\left(\frac{4\pi}{L}x\right)$$

boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0$$

Modelica code

```
model string "model of a vibrating string with fixed ends
   "
  import C = Modelica.Constants;
```

```
  parameter Real L = 1; // length
  parameter Real T = 1; // tension
  parameter Real mu = 1; // linear density
  parameter DomainLineSegment1D omega(length = L);
  function u0
    input Real x;
    output Real u0 := sin(4*C.pi/L*x);
  end u0;
  field Real u(domain = omega, start = u0);
equation
  pder(u,time,time) - T/mu*pder(u,x,x) = 0   in  omega.
      interior;
  u = 0; in omega.left + omega.right;
end string;
```

Listing 3.5: String model in Modelica

Generated C code

```
int const M = 100;
int const nStatesPDE = 1;
int const nAlgebraicsPDE = 0;
int const nParametersPDE = 0;
int const nParameters = 3;

#include "../src/data.h"

int functionInitial(){
  for (i=0; i<M; i++){
    statesPDE[i][1] = ;
  }
  /*L*/parameters[0] = 1;
  /*c*/parameters[1] = 1;
  isBC[0][0] = true;int const M = 100;
int const nStatesPDE = 1;
int const nAlgebraicsPDE = 0;
int const nParametersPDE = 0;
int const nParameters = 2;

#include "data.h"

int functionInitial(){
  for (i=0; i<M; i++){
    statesPDE[i] = 1;
  }
  /*L*/parameters[0] = 1;
  /*c*/parameters[1] = 1;
```

```
  isBC[0][0] = true;
  isBC[1][0] = false;

  return 0;
}

int functionPDE()
{
  /*u_t*/statesDerTime[0] = - /*c*/parameters[1] * /*u_x
      */statesDerSpace[0];
  return 0;
}

int functionBC(){
  statesPDE[0] = cos(2*pi*time);
  return 0;
}

  isBC[1][0] = false;

  return 0;
}

int functionPDE()
{
  /*u_t*/statesDerTime[0] = - /*c*/parameters[1] * /*u_x
      */statesDerSpace[0];
  return 0;
}

int functionBC(){
  statesPDE[0] = cos(2*pi*time);
  return 0;
}
```

Listing 3.6: String equation – "generated" C code

3.2.3 Heat equation in square with sources (2D)

a .. domain square side hlaf length
c .. conductivity quocient
T .. temperature

$$W(x,y) = \begin{cases} 1 & \text{if } |x| < a/10 \text{ and } y < a/10 \\ 0 & \text{else} \end{cases}$$

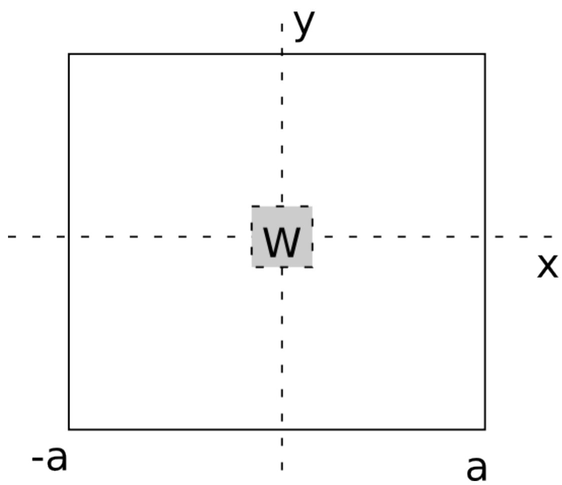

Figure 3.1: Heat eq.

equation

$$\frac{\partial T}{\partial t} + c\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = W$$

initial conditions

$$T(x, y, 0) = 0$$

boundary conditions insulated walls (top, left, bottom)

$$\begin{aligned} \frac{\partial T}{\partial \bar{n}}(x, a, t) &= 0 \\ \frac{\partial T}{\partial \bar{n}}(-a, y, t) &= 0 \\ \frac{\partial T}{\partial \bar{n}}(x, -a, t) &= 0 \end{aligned}$$

fixed temperature (right)

$$T(a, y, t) = 0$$

3.3 More complex realistic models

3.3.1 Henleho klička - protiproudová výměna

$c_{in}(x,t)$ .. koncentrace Na v sestupné části tubulu
$\bar{c}_{in}(x,t)$ .. koncentrace Na ve vzestupné části tubulu

$c_{out}(x,t)$ .. koncentraca Na v dření
$Q(x,t)$ .. tok vody v sestupné části tubulu
$f_{H_2O}(x,t)$ .. tok vody na milimetr délky z sestupné části tubulu do dřeně
f^*_{Na} .. tok sodíku ze vzestupné části tubulu do dřeně na milimetr délky – aktivní transport – parametr
L .. délka tubulu
P_{H_20} .. prostupnost cévy pro vodu (permeabilita)

$$
\begin{aligned}
\frac{\partial Q}{\partial x}(x,t) + f_{H_20}(x,t) = 0 \\
(c_{out}(x,t) - c_{in}(x,t)) \cdot P_{H_2O} &= f_{H_20}(x,t) \\
f_{H_20}(x,t) &= \frac{dV}{dt}(t) \\
Q(L,t) \cdot c_{in}(L,t) &= f^*_{Na} \cdot L + Q(L,t) \cdot c^*(t) \\
\frac{\partial}{\partial x}\left(\bar{c}_{in}(x,t) \cdot Q(x,t)\right) &= f^*_{Na} \\
f^*_{Na} \cdot L &= \frac{dm_{Na}}{dt}
\end{aligned}
$$

3.3.2 Oxygen diffusion in tissue around vessel

polar coordinates (r, φ)

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} + q\left(\frac{1}{r}\frac{\partial \varrho}{\partial r} + \frac{\partial^2 \varrho}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varrho}{\partial \varphi^2}\right) + w &= 0 \\
\varrho(r_0, \varphi) &= \varrho_0 \\
\varrho(r, 0) &= \varrho(r, 2\pi) \\
\varrho_{nnn}(R, \varphi) &= 0 \; (= \varrho_{tn}(R, \varphi))
\end{aligned}
$$

ϱ .. oxygen concentration
ϱ_0 .. concentration in the vessel
q .. diffusion coefficient
w .. local oxygen consumption
R .. Ω diameter
The last equation should simulate infinite continuation of the domain.

3.3.3 Heat diffusion

domain border

$$
\partial\Omega \;=\; (a_b \cos(v),\, b_b \sin(v)),\; v \in \langle 0, 2\pi)
$$

equation [10]

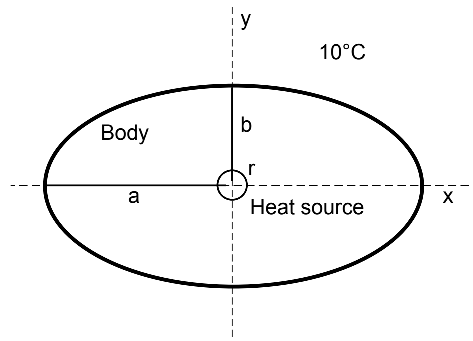

Figure 3.2: Scheme of heat diffusion in body

$$\frac{\partial T}{\partial t} + \frac{\lambda}{c\varrho}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad = \quad W$$

λ .. thermal conductivity
$W(x, y)$ .. heat power density of tissue (input)

$$W(x,y) = \begin{cases} W_0 & \text{if } x^2 + y^2 \leq r^2 \\ 0 & \text{else} \end{cases}$$

boundary condition

$$\lambda \frac{\partial T}{\partial n} = -\alpha(T - T_{out}),\ (x, y) \in \partial\Omega$$

α .. tissue-air thermal transfer coefficient [11]
initial condition

$$T(x, y, 0) = T_0(x, y)$$

3.3.4 Pulse waves in arteries caused by heart beats [1, 8]

$A(x, t)$ .. crossection of vessel
$U(x, t)$ .. average velocity of blood
$Q(x, t)$ .. flux

$$Q = AU$$

$P(x, t)$ .. pressure
P_{ext} .. external pressure
A_0 .. vessel crossection at $(P = P_{ext})$ (24mm)
$\beta = \frac{\sqrt{\pi} h_0 E}{(1-\nu^2) A_0}$
h_0 .. vessel wall thicknes (2mm)
E .. Young's modulus (0.24 - 6.55MPa)[4, 3, 2]

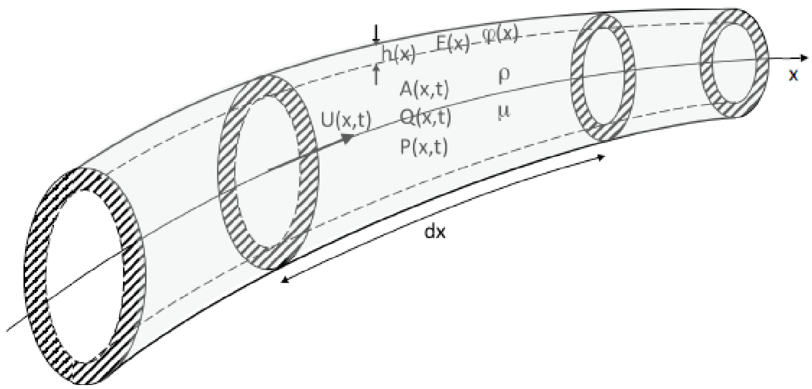

Figure 3.3: Arteria scheme

ν .. Poisson ratin (1/2)
$\varrho = 1050\,\mathrm{kg\,m^{-3}}$ .. blood density
$\mu = 4.0\,\mathrm{mPa\,s}$
α .. other ugly coefficient, let us say its 1
f .. frictional forces per unit length, let us assume inviscide flow $f = 0$

$$
\begin{aligned}
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\alpha \frac{Q^2}{A}\right) + \frac{A}{\varrho}\frac{\partial P}{\partial x} &= \\
= \frac{\partial Q}{\partial t} + \alpha\left(2\frac{Q}{A}\frac{\partial Q}{\partial x} - \frac{Q^2}{A^2}\frac{\partial A}{\partial x}\right) + \frac{A}{\varrho}\frac{\partial P}{\partial x} &= \frac{f}{\varrho} \\
P_{ext} + \beta\left(\sqrt{A} - \sqrt{A_0}\right) &= P
\end{aligned}
$$

Three segment geometry – splitting arteria

We model arteria being splited into two minor arteries. Three same equation systems (super-indexes A, B, C) are solved on three different domains. Systems are connected via BC.

Border conditions

input

$$
\begin{cases}
P^A(0,t) = P_S & t \in \langle 0, \frac{1}{3}T_c) \\
Q^A(0,t) = 0 & t \in \langle \frac{1}{3}T_c, T_c)
\end{cases}
$$

T_c .. cardiac cycle period

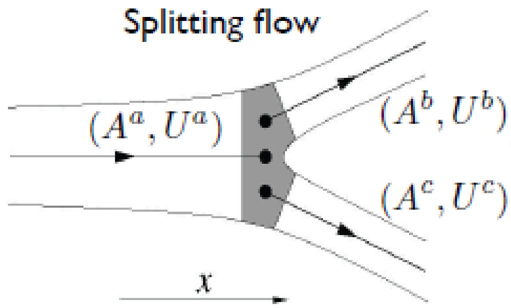

Figure 3.4: Arteria splitting

junction

$$
\begin{aligned}
Q^A(L^A, t) &= Q^B(0, t) + Q^C(0, t) \\
P^A(L^A, t) &= P^B(0, t) \\
P^A(L^A, t) &= P^C(0, t)
\end{aligned}
$$

terminal we simulate the continuation of segments B and C with just a resitance

$$Q(L, t) = \frac{P(L, t)}{R_{out}}, \text{ for } B \text{ and } C$$

For check: the result should be in agreement with Moens–Korteweg equation.

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