Space Weather Summer School:

Taylor Series Expansion:

$$f(x+\partial x) = f(x) + \partial x \frac{\partial f}{\partial x} + \frac{1}{2}\partial x^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{6}\partial x^3 \frac{\partial^3 f}{\partial x^3} + \dots$$

$$f(x-\partial x) = f(x) - \partial x \frac{\partial f}{\partial x} + \frac{1}{2}\partial x^2 \frac{\partial^2 f}{\partial x^2} - \frac{1}{6}\partial x^3 \frac{\partial^3 f}{\partial x^3} + \dots$$

$+ \Rightarrow f(x+\partial x) + f(x-\partial x) = 2f(x) + \partial x^2 \frac{\partial^2 f}{\partial x^2} + O(4)$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{f(x+\partial x) + f(x-\partial x) - 2f(x)}{\partial x^2}}$$

$- \Rightarrow f(x+\partial x) - f(x-\partial x) = 2\partial x \frac{\partial f}{\partial x} + O(3)$

$$\boxed{\frac{\partial f}{\partial x} = \frac{f(x+\partial x) - f(x-\partial x)}{2\partial x}}$$

functions to try: $f(x) = 2x+1$

$f(x) = 4x^2 - 3x - 7$

$f(x) = 4\sin(x)$

$f(x) = 2x\cos^2(x) - 3x^2 e^{-\frac{x}{\pi}}$

- Evaluate between $[-2\pi, 2\pi]$
  - find $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ analytically & numerically
  - find errors given $\partial x = \pi/2, \pi/4, \pi/8, \dots \pi/64$
    - plot log(error) vs log(∂x) for each

Let's do some physics with This!!

Diffusion/Conduction: $\frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + S(x,t)$

$S(x,t)$ → Source term → dependent on time + location

$\lambda \frac{\partial^2 T}{\partial x^2}$ → Conduction term

$\frac{\partial T}{\partial t}$ → Time derivative

Simplification #1: Steady State!

$$\Downarrow$$

$$0 = \lambda \frac{\partial^2 T}{\partial x^2} + S(x)$$

Simplification #2: No sources!

$$\lambda \frac{\partial^2 T}{\partial x^2} = 0$$

→ Boundary Value Problem! Solution is only dependent on Boundaries!

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x+\partial x) + T(x-\partial x) - 2T(x)}{\partial x^2} = 0$$

$$\Downarrow$$

$$T_{i+1} + T_{i-1} - 2T_i = 0$$

$$\Downarrow$$

$T_{i-1} - 2T_i + T_{i+1} = 0$

$T_i - 2T_{i+1} + T_{i+2} = 0$

$T_{i+1} - 2T_{i+2} + T_{i+3} = 0$

$T_{i+2} - 2T_{i+3} + T_{i+4} = 0$

$\ddots$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = 0$$

→ Tri-Diagonal Matrix!

T_0 & T_6 are boundaries → not physical cells! ghost cells!

For Now, we Fix T_0 & T_6 → Boundary Value Problem!

Google search Tri-diagonal matrix algorithm!

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & & \ddots & \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \partial_n \end{bmatrix}$$

boundary!

$b_1 \cdot x_1 + c_1 \cdot x_2 = \partial_1$

$a_2 + x_1 + b_2 \cdot x_2 + c_2 \cdot x_3 = \partial_2$

$a_3 \cdot x_2 + b_3 \cdot x_3 + c_3 \cdot x_4 = \partial_3$

→ Want to force $x_1 = V_L$

$\ddots$

$c_{n-2} \cdot x_{n-2}$ boundary!

$a_{n-1} x_{n-2} + b_{n-1} \cdot x_{n-1} + c_{n-1} \cdot x_n = \partial_{n-1}$

$a_n \cdot x_{n-1} + b_n \cdot x_n = \partial_n$ → Same

$$\underbrace{b_1}_{1!} \cdot x_1 + \underbrace{c_1}_{0!} \cdot x_2 = \underbrace{\partial_2}_{V_L}$$

$x_1 = V_L$!

$a_n = 0$

$b_n = 1$

$\partial_n = V_U$

all other $\partial = 0$

Add Sources:

$\lambda \frac{\partial^2 T}{\partial x^2} = S(x)$

$\frac{\partial^2 T}{\partial x^2} = \frac{S(x)}{\lambda(x)}$ → Could be $\lambda(x)$!!

$$\Rightarrow \frac{T(x+\partial x) + T(x-\partial x) - 2T(x)}{\partial x^2} = \frac{S(x)}{\lambda(x)}$$

$T_{i+1} + T_{i-1} - 2T_i = \partial x^2 \cdot \frac{S_i}{\lambda_i}$

$T_{i-1} - 2T_i + T_{i+1} = \partial x^2 \cdot \frac{S_i}{\lambda_i}$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \partial x^2 \frac{S_1}{\lambda_1} \\ \partial x^2 \frac{S_2}{\lambda_2} \\ \partial x^2 \frac{S_3}{\lambda_3} \\ \partial x^2 \frac{S_4}{\lambda_4} \\ \partial x^2 \frac{S_5}{\lambda_5} \\ 0 \end{bmatrix}$$

Once again, Fixed Boundary

What about other boundaries???