Grover algo

Basic Idea

Zeno effect

Hare $t_1 \xrightarrow{d} t_2 \xrightarrow{d_2} t_4$

$t_1 >> t_0$

Tortoise $t_0 \xrightarrow{d} t_1 \xrightarrow{d_2} t_3 \xrightarrow{d_3}$

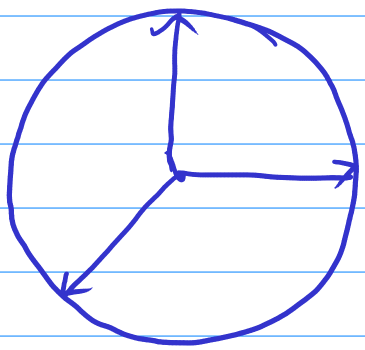
$t_0, t_1, t_2, \ldots t_n$

$t_1, t_2, \ldots$

Zeno paradox

$\longrightarrow$ distance

$|0\rangle$



$|1\rangle$

$$|\psi\rangle = |0\rangle \xrightarrow{T} |\psi\rangle = |1\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$R(\theta)|\psi\rangle = |\psi'\rangle$$

But we don't want just 1 transformation

$$|\psi_n\rangle = R(\theta_n)R(\theta_{n-1})R(\theta_{n-2})R(\theta_{n-3})\cdots R(\theta_3)R(\theta_2)R(\theta_1)|\psi\rangle$$

Grover's Algo

n items represented by n qubits

Each qubit in a 2D hilbert space

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Total dim of space is $2^n = N$

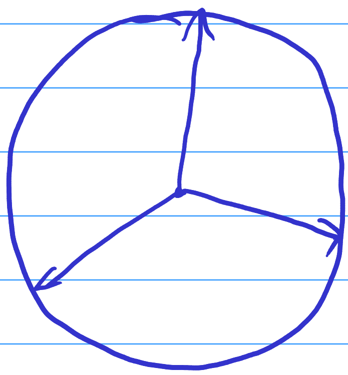
$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H \otimes H |00\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|0\rangle^{\otimes n} = |0\rangle \otimes |0\rangle \otimes |0\rangle \cdots |0\rangle = |0000\cdots 0\rangle$$

$$|\psi\rangle = H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{n=0}^{n-1} |x\rangle$$



$|\psi\rangle = |0\rangle$

$|\psi'\rangle = |1\rangle$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$R(\theta)|\psi\rangle = |\psi'\rangle$

$|\psi_1\rangle = R(\theta_1)|\psi\rangle \quad |\psi_2\rangle = R(\theta_2)|\psi_1\rangle$

$\vdots$

$|\psi_n\rangle \; R(\theta_n)|\psi_{n-1}\rangle$

$\Rightarrow |\psi_n\rangle = R(\theta_n) R(\theta_{n-1}) R(\theta_{n-2}) \cdots R(\theta_1)|\psi\rangle$

We are making small rotations

2 Questions

(1) $|\psi\rangle \xrightarrow{R(\theta_1)\cdots R(\theta_n)} |\psi_n\rangle$

$|0\rangle \qquad\qquad |1\rangle$

(a) measurement is made @ end collapses @ $|1\rangle$ (more probable)

(b) measurement is made @ each $R(\theta_i)$

collapses @ $|0\rangle$ initially (more probable)

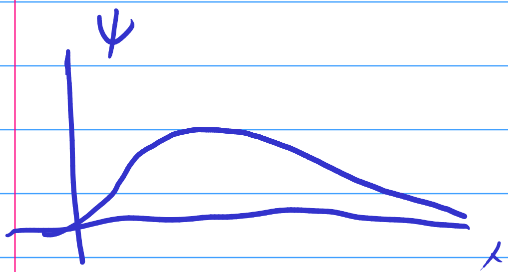
then collapses everytime

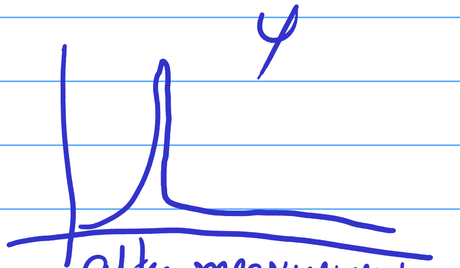
$|\psi\rangle = 0$

$R_{\theta_1}|\psi\rangle = |\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$

Prob $\alpha \approx 1$ (small rotation)

Prob $\beta \approx 0$





after measurement

After some time

A classical way of checking won't work

$\sqrt{n}$ searches

n terms are made by 'n' qubits
each qubit are _ on a two dim
hilbert space

$$|\psi\rangle = \frac{1}{2}|0\rangle + |1\rangle$$

Total dimention of span is $2^n \leq N$

(Look at class)

Classically 'n' dim. N

$$|0\rangle^{\otimes n} = |0\rangle \otimes |0\rangle \cdots |0\rangle$$
$$= |000 \ldots 0 \ldots 0\rangle$$

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

$$H \otimes H |00\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle).$$

$$= \frac{1}{2}[|00\rangle + |10\rangle + |01\rangle + |11\rangle]$$

$$|\psi\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{n-1} |x\rangle.$$

$$|\psi\rangle = H^{\otimes n} |0\rangle^{\otimes n}.$$

$$= [H \otimes H \otimes H \otimes \cdots \otimes H][|0\rangle \otimes |0\rangle \otimes \cdots |0\rangle].$$

$$= [H|0\rangle \otimes H|0\rangle \otimes H|0\rangle \cdots H|0\rangle]$$

$$= \frac{1}{\sqrt{2^n}} [(|0\rangle+|1\rangle)(|0\rangle+|1\rangle) \cdots (|0\rangle+|1\rangle).]$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{n-1} |x\rangle.$$

dim 1	dim 2	dim 3
$\|0\rangle, \|1\rangle$	$\|00\rangle\ \|10\rangle$ $\|01\rangle\ \|11\rangle$	$\|000\rangle$

$\frac{\pi}{4}\sqrt{2^n}$ (proof later) $\Rightarrow \sqrt{N}$ iterations

Quantum oracle

$O \rightarrow \quad |x\rangle \xrightarrow{O} (-1)^{f(x)} |x\rangle$

$|\psi\rangle = |x_i\rangle \quad f(x) = 1$ x is correct state

$f(x) = 0$ x is incorrect

$$|0\rangle^{\otimes 3} = |000\rangle$$

$$|\psi\rangle = H^{\otimes} |0\rangle^{\otimes 3} = H|0\rangle \otimes H|0\rangle \otimes H|0\rangle$$

$$= \frac{1}{2\sqrt{2}} [|0\rangle + |1\rangle] \otimes [|0\rangle + |1\rangle] \otimes [|0\rangle + |1\rangle]$$

$$= \frac{1}{2\sqrt{2}} [|000\rangle + |001\rangle + |010\rangle + |100\rangle + |011\rangle + |101\rangle + |110\rangle + |111\rangle]$$

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left[|000\rangle + |001\rangle + |010\rangle + |100\rangle + |101\rangle + |110\rangle + |011\rangle + |111\rangle\right]$$

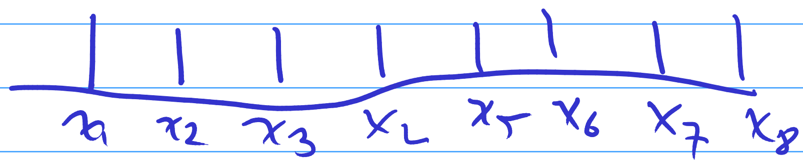
Search state is one of the state
$|101\rangle$

$$|\psi\rangle \longrightarrow \boxed{O} \longrightarrow |\psi\rangle$$

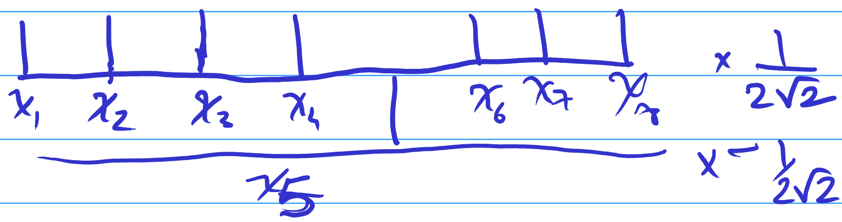
$$(-1)^{f(x)}|x\rangle$$

Output $|\psi\rangle$: $\frac{1}{2\sqrt{2}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle \underbrace{-|101\rangle}_{\text{sign change}} + |110\rangle + |111\rangle\right]$

All states



After O



Apply Grover diffusion operator

$$\left[2|0\rangle\langle 0| - I\right]|0\rangle = 2|0\rangle\overbrace{\langle 0|0\rangle}^{1} - I|0\rangle$$

$$= 2|0\rangle - |0\rangle = |0\rangle$$

$$[2|0\rangle\langle 0| - 1]\,|x\rangle = 2|0\rangle\langle \overset{0}{\widetilde{0|x}}\rangle - |x\rangle$$

$$= 0 - |x\rangle$$

$$\left(H^{\otimes n}[2|0\rangle\langle 0| - I]H^{\otimes n}\right) = [2|\psi\rangle\langle\psi| - I]$$

$[\because |0\rangle^{\otimes n} = |0\rangle]$

Step $[2|\psi\rangle\langle\psi| - I]\,O|\psi\rangle$

$$[2|\psi\rangle\langle\psi| - I]\left[|\psi\rangle - \frac{2}{2\sqrt{2}}|101\rangle\right]$$

$$= 2|\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1} - I|\psi\rangle - \frac{2}{\sqrt{2}}|\psi\rangle\langle\psi|101\rangle + \frac{1}{\sqrt{2}}|101\rangle$$

$$= 2|\psi\rangle - |\psi\rangle - \frac{1}{2}|\psi\rangle + \frac{1}{\sqrt{2}}|101\rangle$$

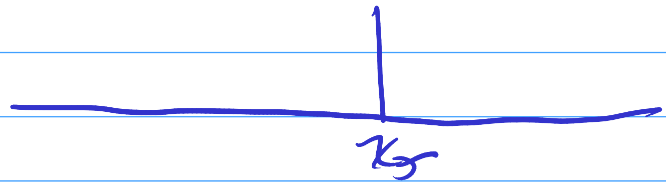
$$= \frac{1}{2}|\psi\rangle + \frac{1}{\sqrt{2}}|101\rangle$$

$$|\psi\rangle\underbrace{\langle\psi|101\rangle}$$

$$\langle\psi|101\rangle = \langle 01|\psi\rangle = \frac{\langle 101|101\rangle}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$\frac{2}{\sqrt{2}}|\psi\rangle\langle\psi|101\rangle = \frac{2}{\sqrt{2}}\,\frac{|\psi\rangle}{2\sqrt{2}} = \frac{1}{2}|\psi\rangle$$

$$\frac{1}{4\sqrt{2}}|000\rangle + \frac{1}{4\sqrt{2}}|001\rangle + \dots + \underbrace{\frac{5}{4\sqrt{2}}}_{\frac{5}{4\sqrt{2}} \text{ alone}}|101\rangle + \dots + \frac{1}{4\sqrt{2}}|111\rangle$$



This is for 1 unitary operation

apply twice $\Rightarrow \frac{11}{8\sqrt{2}} \Rightarrow$ probability $\frac{121}{128}$

pattern searching works differently though, they're using DL to solve this problem

$$|0\rangle \equiv |0000\dots0\rangle \equiv |0\rangle^{\otimes n}.$$

$$H|0\rangle = [H \otimes H \otimes H \otimes \dots \otimes H]|000\dots0\rangle$$

$$= H|0\rangle \otimes H|0\rangle \otimes H|0\rangle \dots \otimes H|0\rangle$$

$$= \underline{\underline{H^{\otimes n}|0\rangle^{\otimes n}}}.$$

$$|ab\rangle = |a\rangle \otimes |b\rangle.$$

$$[A \otimes B](|a\rangle \otimes |b\rangle) = A|A\rangle \otimes B|B\rangle$$

construct the state from $|0\rangle$

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{n-1}|x\rangle$$

Step 2 What we are searching

We associate that search state any one of the Eigen states of $\sum_{x=0}^{n-1}|x\rangle$ $\boxed{|x_i\rangle}$

Third step is to apply O(oracle)

$$\sum_{x=0}^{n-1}|x\rangle \rightarrow \boxed{O_{|x_i\rangle}} \longrightarrow (-1)^{f(x)}|x\rangle$$

$$f(x) @ |x_i\rangle = 1$$

Then apply Diffusion operation

Repeat etc.

Eg: $n=3$

$$(H\otimes H\otimes H)|000\rangle$$

$$=\frac{1}{2\sqrt{2}}[|0\rangle+|1\rangle][|0\rangle+|1\rangle][|0\rangle+|1\rangle]$$

$$=\frac{1}{2\sqrt{2}}[\ldots\ldots]$$
$$|000\rangle \ldots\ldots |111\rangle$$

Let $|x_{(w)}\rangle = |101\rangle$

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\sum_{n=0}^{7}|x_i\rangle$$