Pauli Spin matrices

$\{ \mathbb{1}, \sigma_x, i\sigma_y, \sigma_z \}$

$$i\sigma_y = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha = \frac{1}{\sqrt{2}} = \beta$$

$$|\chi\rangle\langle\psi| = \frac{1}{2}[\,|0\rangle\langle 0| + |1\rangle\langle 1|\,]$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |\psi\rangle\langle\psi| = \frac{1}{2}\big[|0\rangle + |1\rangle\big] \otimes \big[\langle 0| + \langle 1|\big]$$

$$= \frac{1}{2}\big[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big]$$

$$= \frac{1}{2}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(Q) Find the matrix representation of $|\chi\rangle\langle\psi|$ given $\psi = \alpha|0\rangle + \beta|1\rangle$, given $\alpha = \beta = \frac{1}{\sqrt{2}}$ and

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle\langle\psi| = [\alpha|0\rangle + \beta|1\rangle] \otimes [\alpha\langle 0| + \beta\langle 1|]$$

$$= \alpha^2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha\beta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha\beta\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \beta^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \; 0] \quad 0 \; 1 \quad 1 \; 0 \quad 2\times 1$$

$$= \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}$$

Show that (i) $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(ii) $|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(iii) $|0\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $|1\rangle = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$

It is seen that they are linearly independent

We need to prove orthogonality

$\langle 0|0\rangle = \langle 1|1\rangle = 1$ [inner product]

$\langle 0|1\rangle = \langle 1|0\rangle = 0$

$\langle 0|0\rangle = \cos^2\theta + \sin^2\theta = 1$

$\langle 1|1\rangle = (\cos\theta \; -\sin\theta)\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = 1$

$= \cancel{\cos^2}$

$\langle 0|1\rangle = (\cos\theta \;\; \sin\theta)\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \cos 2\theta$

$\neq 0$

Basis only for $\theta = \cancel{\pi/4}$ $(2n+1)\frac{\pi}{4}$

$\frac{1}{\sqrt{2}}$

$\langle 0|1\rangle \; \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1, -1) = 0$

$\langle 0|0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1,1)$

$= 1$

$\langle 1|0\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1,1) = 0$

$\langle 1,1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 \; -1)$

$= \frac{1}{2} \times 2 = 1$