Assignment 1

Vector Spaces

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2018A7PS0172H

Definition of a vector space:

* A vector space V is a set that is closed under finite vector addition and multiplication.
* The scalars are members of a field F for a general vector space, in which case V is called a vector space over F.

We deal with 2 operations as follows:

→ Vector addition: $+ : V \times V \rightarrow V$
Takes two vectors from V and assigns them to a third vector (also in V). Written as $\vec{V_1} + \vec{V_2} = \vec{V_3}$

→ Scalar multiplication $\cdot : F \times V \rightarrow V$
Takes any scalar k and a vector $\vec{V}$ and gives $k\vec{V}$. Not to be confused with dot product of two vectors.

The following conditions must be satisfied for all elements $\vec{x}, \vec{y}, \vec{z} \in V$ and all scalars $r, s \in F$.

1) Associativity of addition: $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

(2) Commutativity of addition: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

(3) Identity element: There exists an element $\vec{0} \in V$ called the zero vector, such that $\vec{x} + \vec{0} = \vec{x}$

(4) Inverse elements of addition: $\forall \vec{v} \in V$, $\exists -\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$

(5) Associativity of scalar multiplication: $(r\,s)\vec{x} = r(s\vec{x})$

(6) Distributivity of scalar sums: $(r+s)\vec{x} = r\vec{x} + s\vec{x}$

(7) Distributivity of vector sums: $r(\vec{x}+\vec{y}) = r\vec{x} + r\vec{y}$

(8) Scalar multiplication identity $1\vec{x} = \vec{x}$

Eg:

$\mathbb{R}^3$ is a vector space over $\mathbb{R}$

Proof:

$+$ is defined for two vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ as $\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$

$\cdot$ is defined for a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and a scalar k as $\begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} = k\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Associativity

$$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) + \begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} + \begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} a+d+g \\ b+e+h \\ c+f+i \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d+g \\ e+h \\ f+i \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \left(\begin{bmatrix} d \\ e \\ f \end{bmatrix} + \begin{bmatrix} g \\ h \\ i \end{bmatrix}\right)$$

Commutativity:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} = \begin{bmatrix} d+a \\ e+b \\ f+c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

(scalar addition is commutative)

Additive identity: Consider a vector $\vec{O} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_1+a \\ k_2+b \\ k_3+c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} k_1+a &= a \\ k_2+b &= b \\ k_3+c &= c \end{aligned} \qquad \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

independent of a, b or c

Additive inverse: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ is $-\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} a+k_1 &= 0 \\ b+k_2 &= 0 \\ c+k_3 &= 0 \end{aligned} \qquad \Rightarrow \quad \begin{aligned} k_1 &= -a \\ k_2 &= -b \\ k_3 &= -c \end{aligned}$$

$\Rightarrow -\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is defined as $\begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$ $\forall\ a, b, c \in V$

Associativity of scalar multiplication: $r(s\vec{X}) = (rs)\vec{X}$

$$kl\begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad k \in \mathbb{R} \quad l \in \mathbb{R}$$

$$= k\begin{bmatrix} la \\ lb \\ lc \end{bmatrix} = \begin{bmatrix} (kl)a \\ (kl)b \\ (kl)c \end{bmatrix} = kl\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

distributivity of scalar sums: $(r+s)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} (r+s)a \\ (r+s)b \\ (r+s)c \end{bmatrix} = \begin{bmatrix} ra+sa \\ rb+sb \\ rc+sc \end{bmatrix}$

$= \begin{bmatrix} ra \\ rb \\ rc \end{bmatrix} + \begin{bmatrix} sa \\ sb \\ sc \end{bmatrix} = r\begin{bmatrix} a \\ b \\ c \end{bmatrix} + s\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

distributivity of vector sums $r\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = r\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} = \begin{bmatrix} ra+rd \\ rb+re \\ rc+rf \end{bmatrix}$

$= r\begin{bmatrix} a \\ b \\ c \end{bmatrix} + r\begin{bmatrix} d \\ e \\ f \end{bmatrix}$

(8) Scalar multiplicative identity $\exists\, I \in \mathbb{R} \ni I\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$I\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} Ia \\ Ib \\ Ic \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{array}{l} a = Ia \\ b = Ib \\ c = Ic \end{array} \Rightarrow I = 1$

Eg 2 Let V be a vector space over $\mathbb{R}$ such that $+$ is defined for 2 vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

as $\begin{bmatrix} a+d+ad \\ b+e+be \\ c+f+cf \end{bmatrix}$

finding an inverse $-\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$

$a + k_1 + ak_1 = 0$

$\begin{bmatrix} -\frac{a}{1+a} \\ -\frac{b}{1+b} \\ -\frac{c}{1+c} \end{bmatrix}$ $\frac{-a}{1+a}$

Then $\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -\frac{a}{1+a} \\ -\frac{b}{1+b} \\ -\frac{c}{1+c} \end{bmatrix}$, which is not defined for $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \in V$

$\Rightarrow V$ is not a vector space