CS/ECE 374 Spring 2017 Birgit Alitz (alitz2)

Homework 0 Problem 1

Let $G = (V, E)$ be an undirected graph. Unless we say otherwise, a graph has no loops or parallel edges.

- Prove that if $|V| \geq 2$ there are two distinct nodes u and v such that degree of u is equal to degree of v. Recall that the degree of a node x is the number of edges incident to x.
- Prove that if G has at least one edge then there is a path between two distinct nodes u and v such that degree of u is equal to degree of v.

Solution:

(a) In a graph with $|V| \geq 2$, the number of degrees of any vertex has to be between 0 and $|V|-1$ because the definition of an edge requires it to exist between two vertices. This means that there are $|V|$ different options for numbers of degrees that each vertex can have. However, in the case of vertices, none of the vertices can have degree $|V|-1$ because that would force there to be no vertex with degree 0. This leaves $|V|-1$ different options for number of degrees for $|V|$ different vertices. In order for no two vertices to have the same number of degrees, each of the $|V| \geq 2$ vertices has to have a different number of degrees, but since there are only $|V|-1$ options, there have to be at least two with the same number by the pigeon-hole principle.

(b) A graph with number of edges $n \geq 1$ has, by definition of an edge, at least two vertices, and contains a connected component, which we can consider a subgraph for the remainder of this problem. Since there are $|V| \geq 2|$ vertices in the connected subgraph, the proof from the previous problem applies, confirming that at least two of the vertices in the subgraph have the same degree. Furthermore, from knowing that the subgraph is connected, we know that there exists a path between these two unique vertices with the same degree as per the definition of connected.

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Homework 0 Problem 2

The ***plus one***, w^+, of a string $w \in \{0,1,2\}^*$ is obtained from w by replacing each symbol a in w by the symbol corresponding $a+1 \mod 3$. For example, $0102101^+ = 1210212$. The plus one function is formally defined as follows:

$$w^+ := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ 1 \cdot x^+ & \text{if } w = 0x \\ 2 \cdot x^+ & \text{if } w = 1x \\ 0 \cdot x^+ & \text{if } w = 2x \end{cases}$$

1. **Not to submit:** Prove by induction that $|w| = |w^+|$ for every string w.
2. Prove by induction that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{0,1,2\}^*$.

Your proofs must be formal and self-contained, and they must invoke the *formal* definitions of length $|w|$, concatenation $x \bullet y$, and plus one w^+. Do not appeal to intuition!

Solution:

Inductive Base: $x = \epsilon$

$$(x \bullet y)^+ = (y)^+ \text{ by definition of concatenation with } x = \epsilon$$

$$\epsilon^+ = \epsilon \text{ from the definition of } \textbf{\textit{plus one}}$$

$$(x \bullet y)^+ = \epsilon^+ \bullet y^+ = x^+ \bullet y^+$$

Inductive Hypothesis:

$\forall$ strings w with $|w| = k < n$, and all strings $y \in \Sigma^*$, $(w \bullet y)^+ = w^+ \bullet y^+$

Inductive Step: For string x, with $|x| = n$ greater than arbitrary k, x can be written as aw, where $a \in \{0,1,2\}$ and w is a string in the language $\{0,1,2\}^*$ with $|w| = k$ for which the inductive hypothesis holds.

$$(x \bullet y)^+ = ((a \bullet w) \bullet y)^+ \text{ by definition of string length}$$

$$((a \bullet w) \bullet y)^+ = (a \bullet (w \bullet y))^+ \text{ by concatenation lemma}$$

$$(a \bullet (w \bullet y))^+ = a^+ \bullet (w \bullet y)^+ \text{ definition of } \textbf{\textit{plus one}}$$

$$a^+ \bullet (w \bullet y)^+ = a^+ \bullet (w^+ \bullet y^+) \text{ by inductive hypothesis}$$

$$a^+ \bullet (w^+ \bullet y^+) = (a^+ \bullet w^+) \bullet y^+ \text{ by concatenation lemma}$$

$$(a^+ \bullet w^+) \bullet y^+ = x^+ \bullet y^+ \text{ by declaration of } x \text{ and definition of } \textbf{\textit{plus one}}$$

$$\text{Thus, } (x \bullet y)^+ = x^+ \bullet y^+.$$

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Homework 0 Problem 3

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be two fixed vectors in the real plane. Recursively define a set $L_n \subseteq \mathbb{R}^2$ as follows.

- $L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$. ($\mathbf{0}$ denotes the zero vector $(0,0)$ in $\mathbb{R}^2$.)
- For integer $n > 0$, $L_n = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in L_{n-1}\}$.

Let $L = \bigcup_{n=0}^{\infty} L_n$. Also, let $D = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{Z}\}$ be the set of vectors obtained as integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

1. Prove that $D \subseteq L$, by giving, for each $a, b \in \mathbb{Z}$, an explicit value of n such that $a\mathbf{u} + b\mathbf{v} \in L_n$. (You don't need to minimize the value of n; but you must argue why $a\mathbf{u} + b\mathbf{v} \in L_n$ for your choice of n.)
2. Use mathematical induction to prove that for all integers $n \geq 0$, $L_n \subseteq D$, and hence $L \subseteq D$.

Solution (1):

Inductive Bases (a):

- For $a = 0$: $0\mathbf{u} = 0$; $0 \in L_0$ by definition of L_0, so $\forall D | a = 0$, $a\mathbf{u} \subseteq L$
- For $a = 1$: $1\mathbf{u} = \mathbf{u}$; $u \in L_0$ by definition of L_0, so $\forall D | a = 1$, $a\mathbf{u} \subseteq L$
- For $a = -1$: $-1\mathbf{u} = -\mathbf{u}$; $-\mathbf{u} \in L_{|-1|}$, so $\forall D | a = -1$, $a\mathbf{u} \subseteq L$
  - Since 0, $\mathbf{u}$ are in L_0, $0 - \mathbf{u} = -\mathbf{u} \in L_1$ by definition of L_n

Inductive Hypothesis:

- $\forall k$ with $1 < |k| < |n|$, $\pm k\mathbf{u} \in L_{|k|}$

Inductive Step:

Proving that $D \subseteq L$ for all a:

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