CS/ECE 374 Birgit Alitz (alitz2@illinois.edu)
Homework 0 Problem 1

Let $G = (V, E)$ be an undirected graph. Unless we say otherwise, a graph has no loops or parallel edges.

- Prove that if $|V| \geq 2$ there are two distinct nodes u and v such that degree of u is equal to degree of v. Recall that the degree of a node x is the number of edges incident to x.
- Prove that if G has at least one edge then there is a path between two distinct nodes u and v such that degree of u is equal to degree of v.

Solution:

(a)

(b)

■

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Homework 0 Problem 2

The ***plus one***, w^+, of a string $w \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^*$ is obtained from w by replacing each symbol a in w by the symbol corresponding $a+1 \mod 3$. for example, $\mathtt{0102101}^+ = \mathtt{1210212}$. The plus one function is formally defined as follows:

$$w^+ := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \mathtt{1} \cdot x^+ & \text{if } w = \mathtt{0}x \\ \mathtt{2} \cdot x^+ & \text{if } w = \mathtt{1}x \\ \mathtt{0} \cdot x^+ & \text{if } w = \mathtt{2}x \end{cases}$$

1. **Not to submit:** Prove by induction that $|w| = |w^+|$ for every string w.
2. Prove by induction that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^*$.

Your proofs must be formal and self-contained, and they must invoke the *formal* definitions of length $|w|$, concatenation $x \bullet y$, and plus one w^+. Do not appeal to intuition!

Solution:

Inductive Bases:

$$x \text{ or } y = \epsilon$$

By definition of concatenation, if x is ϵ, then $x \bullet y = y$.

$$(x \bullet y)^+ = (y)^+$$

$$\epsilon^+ = \epsilon$$

$$(x \bullet y)^+ = \epsilon^+ \bullet y^+ = x^+ \bullet y^+$$

The same is true if x and y are switched in the above.

$$|x| \text{ or } |y| = 1$$

This case is covered in the definition of w^+ for all possible x ($x = \mathtt{0}, \mathtt{1}, \mathtt{2}$). In the case of y,

$$2 \rightsquigarrow 6 \rightsquigarrow 12 \rightsquigarrow 13$$

I told you that I'm crazy for these cupcakes, cousin!

(a) Yo, where's the movie playin'? Upper West Side, dude.

- Well, let's hit up Yahoo! Maps to find the dopest route.
- I prefer MapQuest. That's a good one, too.
- Google Maps is the best. True dat. DOUBLE TRUE!

(b) Yo, stop at the deli. The theater's over-priced. You've got the backpack? Gonna pack it up nice. Don't want security to get suspicious.

$$\boxed{\text{Mr. Pibb} + \text{Red Vines} = \textit{crazy delicious!}}$$

I'll reach in my pocket, pull out some dough. Girl actin' like she never seen a ten before. **It's all about the Hamiltons, baby.** Throw the snacks in a bag, and I'm ghost like Swayze.

Roll up to the theater, ticket buying, what we're handlin'. You can call us Aaron Burr from the way we're droppin' Hamiltons.

```
MovieTrivia(question[1..n]):
  illest ← True
  for i ← 1 to n
      if question[i] = "Which Friends alum starred in films with Bruce Willis?"
          speed ← ∞
          scary ← True
          Shout "Matthew Perry!"
  if quiet ∉ theater
      tragic ← True
  return DreamWorld(magic)
```

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Homework 0 Problem 3

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be two fixed vectors in the real plane. Recursively define a set $L_n \subseteq \mathbb{R}^2$ as follows.

- $L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$. ($\mathbf{0}$ denotes the zero vector $(0, 0)$ in $\mathbb{R}^2$.)
- For integer $n > 0$, $L_n = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in L_{n-1}\}$.

Let $L = \bigcup_{n=0}^{\infty} L_n$. Also, let $D = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{Z}\}$ be the set of vectors obtained as integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

1. Prove that $D \subseteq L$, by giving, for each $a, b \in \mathbb{Z}$, an explicit value of n such that $a\mathbf{u} + b\mathbf{v} \in L_n$. (You don't need to minimize the value of n; but you must argue why $a\mathbf{u} + b\mathbf{v} \in L_n$ for your choice of n.)
2. Use mathematical induction to prove that for all integers $n \geq 0$, $L_n \subseteq D$, and hence $L \subseteq D$.

Solution (induction): Let k be an arbitrary non-negative integer. There are several cases to consider:

- Blah
- Snort
  - Squee
  - Flub
- Kronk

In all cases, we conclude that when k 5-card poker hands are dealt from a standard shuffled deck, the player with the Big Blind gets the cards 7♠, 4♦, 5♥, 3♣, and 2♥ with probability $(\sqrt{5} - 1)/2 = 0.618033989$. ■

Solution (combinatorial): This result follows immediately from Flobbersnort's Fundamental Theorem of negative-dimensional motivic k-schemes, which is in turn an obvious consequence of Flibbertygibbet's Cocohohomomolology Lemma, as described in footnote 17 on the back of page 213 of the 1865 edition of Jeff's induction notes (in the original Flemish). ■