CS/ECE 374 Spring 2017 Birgit Alitz (alitz2)

Homework 0 Problem 1

Let $G = (V, E)$ be an undirected graph. Unless we say otherwise, a graph has no loops or parallel edges.

- Prove that if $|V| \geq 2$ there are two distinct nodes u and v such that degree of u is equal to degree of v. Recall that the degree of a node x is the number of edges incident to x.
- Prove that if G has at least one edge then there is a path between two distinct nodes u and v such that degree of u is equal to degree of v.

Solution:

(a) In a graph with $|V| \geq 2$, the number of degrees of any vertex has to be between 0 and $|V|-1$ because the definition of an edge requires it to exist between two vertices. This means that there are $|V|$ different options for numbers of degrees that each vertex can have. However, in the case of vertices, none of the vertices can have degree $|V|-1$ because that would force there to be no vertex with degree 0. This leaves $|V|-1$ different options for number of degrees for $|V|$ different vertices. In order for no two vertices to have the same number of degrees, each of the $|V| \geq 2$ vertices has to have a different number of degrees, but since there are only $|V|-1$ options, there have to be at least two with the same number by the pigeon-hole principle.

(b) A graph with $n = 1$ edges, there are at least two vertices, as per the definition of an edge. For every consequential edge added, in a graph with no parallel edges and no loops, at least one vertex has to be added. Thus, in a graph with n edges, there are at least $n + 1$ vertices. In order for all of the vertices connected to edges to have a different number of degrees, they have to have between 1 and n degrees (as a vertex with degree zero would be unconnected to the path). However, for the n edges, there are n+1 connected vertices, so again, by the pidgeon-hole principle, two of the vertices must have the same degree.

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The ***plus one***, w^+, of a string $w \in \{\texttt{0}, \texttt{1}, \texttt{2}\}^*$ is obtained from w by replacing each symbol a in w by the symbol corresponding $a + 1 \mod 3$. For example, $\texttt{0102101}^+ = \texttt{1210212}$. The plus one function is formally defined as follows:

$$w^+ := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \texttt{1} \cdot x^+ & \text{if } w = \texttt{0}x \\ \texttt{2} \cdot x^+ & \text{if } w = \texttt{1}x \\ \texttt{0} \cdot x^+ & \text{if } w = \texttt{2}x \end{cases}$$

1. **Not to submit:** Prove by induction that $|w| = |w^+|$ for every string w.
2. Prove by induction that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{\texttt{0}, \texttt{1}, \texttt{2}\}^*$.

Your proofs must be formal and self-contained, and they must invoke the *formal* definitions of length $|w|$, concatenation $x \bullet y$, and plus one w^+. Do not appeal to intuition!

Solution:

Inductive Base:

$$x = \epsilon$$

$(x \bullet y)^+ = (y)^+$ by definition of concatenation with $x = \epsilon$

$\epsilon^+ = \epsilon$ from the definition of ***plus one***

$(x \bullet y)^+ = \epsilon^+ \bullet y^+ = x^+ \bullet y^+$

Inductive Hypothesis:

$\forall$ strings w with $|w| < n$, and all strings $y \in \Sigma^*$, $(w \bullet y)^+ = w^+ \bullet y^+$

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Homework 0 Problem 3

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be two fixed vectors in the real plane. Recursively define a set $L_n \subseteq \mathbb{R}^2$ as follows.

- $L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$. ($\mathbf{0}$ denotes the zero vector $(0,0)$ in $\mathbb{R}^2$.)
- For integer $n > 0$, $L_n = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in L_{n-1}\}$.

Let $L = \bigcup_{n=0}^{\infty} L_n$. Also, let $D = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{Z}\}$ be the set of vectors obtained as integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

1. Prove that $D \subseteq L$, by giving, for each $a, b \in \mathbb{Z}$, an explicit value of n such that $a\mathbf{u} + b\mathbf{v} \in L_n$. (You don't need to minimize the value of n; but you must argue why $a\mathbf{u} + b\mathbf{v} \in L_n$ for your choice of n.)
2. Use mathematical induction to prove that for all integers $n \geq 0$, $L_n \subseteq D$, and hence $L \subseteq D$.

Solution (induction): Let k be an arbitrary non-negative integer. There are several cases to consider:

- Blah
- Snort
  - Squee
  - Flub
- Kronk

In all cases, we conclude that when k 5-card poker hands are dealt from a standard shuffled deck, the player with the Big Blind gets the cards 7♠, 4♦, 5♥, 3♣, and 2♥ with probability $(\sqrt{5} - 1)/2 = 0.618033989$. ■

Solution (combinatorial): This result follows immediately from Flobbersnort's Fundamental Theorem of negative-dimensional motivic k-schemes, which is in turn an obvious consequence of Flibbertygibbet's Cocohohomomolology Lemma, as described in footnote 17 on the back of page 213 of the 1865 edition of Jeff's induction notes (in the original Flemish). ■