ON PRICING THE DATA PRIVACY: ENDOGENOUS EVOLUTION, OPTIMAL STOPPING, AND INCENTIVE COMPATIBILITY *

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Abstract.
Privacy is an essential issue in data trading markets. This work uses mechanism design approach to study the optimal market model to economize the value of privacy of personal data, using differential privacy. Motivated by the discovery of an individual's dual motives for privacy protection, we consider that each data owner privately possesses an intrinsic motive and an instrumental motive. We study optimal market design in a dynamic environment by determining the privacy assignment rule that specifies the privacy protection at each usage of data and the payment rules to compensate the privacy loss of the data owners, when the data owners' instrumental motive experiences endogenous dynamics due to the data buyer's dynamic activities. Due to the fundamental tradeoff between privacy and data utility of differential privacy, there is inevitable privacy loss when data is traded with privacy protection. To mitigate the risk of uncertainties, we allow the data owners to leave the market by solving an optimal stopping problem if the accumulated privacy loss is beyond their privacy budgets that depend on their intrinsic motives. In order to influence the data owners' stopping decisions, the data buyer uses a stopping payment rule that is independent of the data owners' preferences and specifies a monetary transfer to a data owner only at the period when he decides to stop at the end of that period.

The research desideratum of this work is to characterize the theoretical design regime of optimal dynamic market models when each data owner makes coupled decisions of stopping times and the reporting of his true instrumental motives and study the influence of data owners' endogenously dynamic private information on the design of the dynamic market model. We introduce the notion of dynamic incentive compatibility to capture the joint deviations from optimal stopping and truthful reporting. Under a monotonicity assumption about the data owners' evolution of instrumental motives, the optimal stopping rule of the data owners can be formulated as a threshold-based rule. A theoretical design principle is provided by a sufficient condition of dynamic incentive compatibility. We relax the data buyer's optimal market design problem by characterizing the monetary transfer rules in terms of privacy assignment rule and the threshold functions. To address the unavailable analytical intractability, we provide a sufficient condition for an approximated dynamic incentive-compatible market model.

Key words. Data Pricing, Dynamic Mechanism Design, Optimal Stopping

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction.

Big data is proving itself as the biggest promising opportunity for businesses, research communities, and governments since the Internet went mainstream about two decades ago. Gigabytes, terabytes, and petabytes of industrial, commercial, and personal data rush into a great wave of opportunities. Business leaders are seeking actionable methods to exploit the enormous value of data to promote financial gains by improving customer management, enhancing risk analysis, placing accurate marketing strategies, and so on. Meanwhile, data marketization is attracting increasing attention in response to the valuable benefits and the keen demand of data. Designing effective data market models is critical to efficiently utilize the data by enabling data trading between data owners and data buyers. Commoditization of data in a digital market can incentivize data owners' participation through monetary transfers and thus enables the data buyers to access data of higher quality

*Submitted to the editors DATE.

Funding: This work was funded by the Fog Research Institute under contract no. FRI-454.

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and larger quantity. Furthermore, data marketization also provides opportunities to adapt governing regime and market standardization into the digital domain. Efforts in researches of data market modelings including analysis and pricing data trading have been invested for, for example, financial data (e.g., [3, 4, 11]), IoT data (e.g., [39, 40, 60]), and medical data ([34, 53]).

However, privacy issues naturally follow. It is critical to provide privacy protection for any forms of data releasing in the data market. Hence, privacy-preserving schemes should be an indispensable component of data market model. Due to the natural tradeoff between the privacy and the utility (accuracy) of data usages, however, private data releasing without any privacy loss in general unavoidably eliminates useful value of data. As a result, privacy-preserving data market model has to take into account the privacy-utility tradeoff and provide incentives for both data owners and data buyers to participate. Yet, the tradeoff is in general difficult to model explicitly and uncertain to both data owners and buyers. Owners of database with sensitive information often inevitably release more information than intended even under carefully crafted privacy protection ([27]). For example, ineffectiveness of anonymization has been shown in literature that a small amount of auxiliary information is sufficient for an adversary to de-anonymize an individual in database consisting of anonymized data about that individual's personal information (see, for example, [36, 8, 28]). Another main challenges for privacy-preserving data releasing is due to the limited information about the adversary's knowledge and ability. Hence, rigorous quantification of privacy loss and its influence on data utility and the robustness to adversarial privacy pry with heterogeneous knowledge and prior information become important in accurate design of monetization and incentivization.

To this end, differential privacy ([19]) (also refers to ϵ-differential privacy, with $\epsilon \in \mathbb{R}_+$) is widely used as the privacy notion for the privacy-preserving schemes of data processing. Differential privacy provides strong privacy guarantees such that whether an individual data point is in the database or not is near-indistinguishable based on the output information released by randomized processing of the database regardless of what auxiliary knowledge or side information available to the adversary. Differential privacy has been studied in a significant amount of work in noise-perturbed data-releasing mechanism and data-learning algorithms such as empirical risk minimization (e.g., [15]), statistical learning (e.g., [21]), and deep learning (e.g., [1]). The rigorous mathematical formulation of differential privacy provides an elegant framework to quantify the individual privacy loss. In particular, the parameter ϵ quantifies the degree of privacy and accuracy by characterizing the upper bound of privacy loss that any individual data point can suffer by participating in any ϵ-differential private data processing. The parameter ϵ also gracefully parameterizes the tradeoff between privacy and accuracy. Basically, the privacy of data usages increases when ϵ decreases at the expense of decreasing the accuracy or the accuracy of data usages increases when ϵ increases at the expense of decreasing the privacy. Hence, the design of privacy-preserving schemes in the data market can be characterized by the craft of ϵ.

As remarked in [18], the choice of ϵ is essentially a social question. Choosing an acceptable ϵ may depend on the risk of privacy leakage. Suppose that an ϵ' has been chosen for a data usage process. In situations with very low risks, it may be tolerable to a higher value $\epsilon'' = k\epsilon'$, for some $k = 2$ or 3, while in cases when the risk is very high, increasing ϵ' by a very small factor, e.g., 1.01 may be intolerable. The value of ϵ may also depend on data owners' preferences over privacy protection, or equivalently, data owners' valuation of privacy. However, the preference of privacy is subjective ([56, 2]). Hence, the choice of ϵ for a database consisting data points from multiple data owners needs to leverage the data owners' preference deviations over the consequences of privacy leakage.

In reality, data buyers necessarily request multiple accesses to the data. This induces other challenges of privacy preservation design in the data market. First, an individual owner's valuation of privacy can change over time due to learning-by-doing, context-dependence of privacy, or influence from external factors. For example, monetary compensation at one data access may influence a data owner's valuation of privacy at the next round of data access. The same data owner may in some cases severely concern about, but under some other circumstances be indifferent to, privacy leakages ([2]). Also, an individual's preference over privacy loss tends to be influenced by external aspects that aim to activate or suppress privacy concern, for example, fake news is spread to create illusions of safe (resp. risky) cyber environment to encourage (resp. discourage) data sharing. As a result, the change of context or time-evolution of external influences may lead to dynamics of personal valuation of privacy. Therefore, one-shot or myopic privacy protection (e.g., fixed or independent policy over time) no longer provide adequate protection according to the perspective of dynamic data usages. Second, the total privacy loss might be amplified and accumulative when the number of data accesses increases due to the inevitable privacy loss at each single access. Technically, this suggests that a budget or tolerance of the total privacy loss of data owners has to be considered in the design of the market model.

In this paper, we study the economy of data privacy and consider a privacy trading market as a part of data market activities. The market consists of a group of data owners (owners), who are the prospective participants to contribute private data by surrendering limited amount of privacy, and a single data buyer (buyer), who requests multiple accesses to the owners' private data by providing differential privacy guarantees and monetary compensation for the corresponding privacy loss. We capture the fundamental tradeoff between the differential privacy and the accuracy of data analysis, which characterizes the conflict between owners' preferences and the buyer's desire: owners prefers a smaller ϵ to obtain a certain degree of privacy while the buyers prefers a larger ϵ to extract more value from the data (through more accurate data analysis).

Motives for an owner to protect his data privacy can arise because the privacy itself is valued as an *intrinsic* right [54, 33]. They can also emerge as an *instrumental* value, the owners' expected economic losses from releasing their privacy through data usage to the buyer [51, 46, 33]. Hence, in this work, we characterize that each owner has a dual preference of data privacy: an *intrinsic preference* and an *instrumental preference*. Each owner's intrinsic preference is utility primitive and is independent of how the data is used by the buyer. His instrumental preference, on the other hand, arises endogenously from the buyer's usage of his data. Therefore, the multiple usages of data in a dynamic environment inevitably causes the instrumental preference evolves dynamically over time.

By conceptualizing the relationship between the owners and the buyer by a principal-multiagent model in a finite horizon, this paper proposes a theoretical framework for pricing differential privacy of data in a dynamic environment, where owners' instrumental preference over privacy is time-evolving due to endogenous progresses as well as exogenous impacts. The model applies mechanism design approaches to engineer the *rules of encounter* for the privacy trading activities of the owners and the buyer. The buyer takes the role of the mechanism designer, whose goal is to minimize the cost by specifying allocation rules that allocate a privacy level at ϵ at each period of time (each data access request) and payment rules that determine a payment to each participating owner at each period to compensate the owner's privacy loss based on the valuation of privacy reported by each owner.

The proposed model highlights the owners' personal choices and autonomy in the privacy trading activity by making a *take-it-or-leave-it* offer to the potential participating owners with a soft commitment and entitling each owner to adopt a stopping rule that decides whether or not to leave the market at each round of data access after a time-varying deadline determined by the historical allocations of ϵ and the individual tolerance of the maximum privacy loss. We restrict attention to direct mechanisms, in which each owner truthfully reveals the instrumental preference over privacy protection at each period to the buyer by characterizing the *incentive compatibility* constraints to the buyer's optimization problem such that truthful reporting is in each owner's best interest. The autonomy raised by allowing stopping time rules fundamentally complicates the guarantee of incentive compatibility in dynamic settings.

This work studies the design of a dynamic market for trading data privacy and focuses on the theoretical analysis of how to optimally design the mechanism rules provided by the buyer and how the mechanism influences the owners' coupled decision makings of reporting and stopping. The contributions of this paper are summarized as follows.

1. We propose a dynamic market model with a soft-commitment for trading data privacy based on differential privacy, when each owner has a dual preference over privacy protection, i.e., the static intrinsic preference and the time-evolving instrumental preference. Our model captures the fundamental tradeoff between privacy and utility of data in differential privacy and allows each owner to leave the market before the finite-horizon trading relationship naturally ends.
2. We model the owners' decision makings as a dynamic Bayesian game when each owner makes coupled decision makings of reporting and stopping. We define truthful perfect Bayesian equilibrium as the solution concept of the buyer's optimal market design. We define an optimal stopping problem for each owner when he dynamically chooses how to report his instrumental preference to the buyer at each period.
3. We define a new notion of dynamic incentive compatibility that captures coupled deviations from truthful reporting decisions and optimal stopping behaviors by establishing a Bellman-equation based relationship.
4. We characterize the DIC and transform the owners' optimal stopping problem into a threshold-based rule under a monotonicity assumption about owners' change of instrumental preferences. A theoretical design regime is established by formulating the preference-dependent monetary transfer rules in terms of the privacy assignment rule and the preference-independent payment rules in terms of the privacy assignment rule and the threshold functions.
5. Based on the design regime, we relax the buyer's optimal market design problem by determining four decision rules and two constraints to a problem of finding the privacy assignment rules and the threshold functions with a single constraint for each owner. A notion of approximated dynamic incentive compatibility is proposed to address the inevitable violations of incentive compatibility due to the analytical intractability of mechanism design problems in general.

1.1. Related Work. There is literature on the interactions of differential privacy and mechanism design. [25] have initiated the study of private data markets. They have treated differential privacy of data as a commodity and applied traditional static mechanism design approaches to model one-query private data trading as a variant of a multi-unit procurement auction. They have considered the cost of privacy loss as each owner's private information and studied truthful mechanism in which each owner is incentivized to truthfully release his private information. Works following [25] include [22, 17, 32, 48, 6], which have studied how to determine ϵ through auctions. Other literature of studying how rational agents evaluate differential privacy loss and choose ϵ includes, e.g., [38, 27, 16, 55]. Authors of [27] have proposed a framework to choose differential privacy parameters through a simple static economic model with complete information of data owners and buyers based on quantities that can be estimated in practice. [30]

There is also related work in dynamic settings. Authors of [31] have studied an orthogonal problem to [25]: owners' valuations are public knowledge and there are multiple queries of data usages. Besides the accuracy of query outputs, [31] have also considered unbiasedness. Their model allows the data buyers to get an arbitrary number of queries and provides arbitrage-free pricing scheme for the buyers that is balanced by taking into account the compensation for privacy loss and the profits from data usages. Other line of work in dynamic setting concerns optimal pricing in a time-evolving environment. There is literature considering posted price models that do not require truthful revealing of private information (e.g., [50]) and models that require incentive compatibility (e.g., [5]). Authors of [57] have proposed a dynamic privacy pricing framework in a market where a data buyer repeatedly buys data from a group of data owners, whose valuations of privacy are randomly drawn from an unknown distribution. They have treated each candidate price as one arm and modeled a multi-armed bandit problem to dynamically adjust the prices to compensate the data owners.

In contrast, we consider a dynamic market framework, in which each owner can learn and update new his valuation of privacy (his private information). We use mechanism design approaches to dynamically set the value of ϵ and the price of privacy as a compensation for each owner's privacy loss at each period through a dynamic optimization problem that minimizes the buyer's cost by taking into account the incentive compatibility, individual rationality, and the buyer's accuracy requirement. Our model offers a flexible commitment and allows each owner to leave the market by adopting a stopping rule once his pre-determined privacy budget is exceeded.

There is a significant amount of work on dynamic mechanism design problems. The literature on dynamic mechanism designs can be divided into two classes. Those are (1) mechanisms with dynamic population and static private information and (2) mechanisms with dynamic private information and static population. Authors of [43] have studied a sequential allocation problems when the participating population is dynamic. In particular, their model has considered the environment when each self-interested agent arrives and departs dynamically overtime. The information possessed by each agent is static and includes the arrival and the departure time as well as her valuation about allocation outcomes. Other works consider this class of dynamic settings include, e.g., [41, 23, 24, 49, 42, 14]. Orthogonal to the dynamic population mechanisms, there are other works considering mechanisms, in which the underlying model is dynamic due to the time-evolution of agents' private information. There is a large number of works lying in this category that studies for example, the dynamic pivot mechanisms (e.g., [10, 29]), dynamic team mechanisms (e.g., [9, 7, 37]), and more generally (e.g., [44, 59]). [7] have considered a dynamic team problem and proposed a balanced team mechanism to implement dynamic efficiency with a balanced budget. Each agent observes private signals over time and decisions are made periodically. Their mechanism provides each agent an incentive payment in each period, which equals to the expected present value of the other agents' payoffs induced by this agent's current period report, to establish an equilibrium in truthful strategies.

The theoretical framework of our mechanism model lies in the interaction of mechanism design with dynamic population and with time-evolving private information. In particular, each data owner's private information (i.e., valuation of privacy) changes over time and the population is dynamic due to the stopping time rule adopted by each data owner. Unlike the aforementioned works with dynamic population, we do not consider the arrival of new data owners and the departure time is determined by the stopping rule (depends on the owner's valuation and the privacy guarantees) and is not treated as private information.

1.2. Preliminaries. In this section, we review basic concepts in differential privacy and preliminary model of the data market to properly situate the contributions of this paper.

1.3. Differential Privacy. Let $\mathcal{D} \equiv \{D_1, D_2, \dots, D_n\}$, where each $D_k \in \mathbb{D}$ is a single data point, denote the database consisting of n data points. Let $\mathcal{A} : \mathbb{D}^n \mapsto \mathcal{S}$ denote a randomized algorithm such that $\mathcal{A}(\mathcal{D}) \in \mathcal{S}$ is the output of the algorithm with $\mathcal{D}$ as the input data. The following definition defines indistinguishability of any algorithm.

Definition 1.1. *Let $\mathcal{D} \in \mathbb{D}^n$ and $\mathcal{D}' \in \mathbb{D}^n$ be any two databases. We say the randomized algorithm $\mathcal{A}$ ϵ-indistinguishable (or indistinguishable) for these two databases if*

$$P_r(\mathcal{A}(\mathcal{D}) \in \mathcal{S}) \leq \exp(\epsilon) P_r(\mathcal{A}(\mathcal{D}') \in \mathcal{S}). \tag{1.1}$$

Basically, a higher degree of indistinguishability (i.e., smaller ϵ) implies a higher degree of privacy. Let $\mathcal{D}' \equiv \{D'_1, \dots, D'_n\}$ be another database that differs from $\mathcal{D}$ in one data point, i.e., $D_k \neq D'_k$ and $D_j = D'_j$, for all $j \neq k$. In other words, the Hamming Distance, which is defined as $\mathrm{HD}(\mathcal{D}, \mathcal{D}') = \sum_{i=1}^{n} \mathbf{1}\{D_i \neq D'_i\}$, is 1. The notion of differential privacy is developed in [19]. Specifically, the algorithm $\mathcal{A}$ is differentially private if the probability likelihood of $\mathcal{A}(\mathcal{D}) \in \mathcal{S}$ is close to the the probability likelihood of $\mathcal{A}(\mathcal{D}') \in \mathcal{S}$. Basically, differential privacy captures the indistinguishability of the algorithm in the *worst-case scenario*, in which the adversary knows every data points other than *any* single sensitive D_k, and guarantees that any single data point does not influence the distribution of algorithm outcome by much and the adversary cannot obtain much information about the sensitive data point by observing the output of the algorithm. Definition 1.2 formally describes the concept of differential privacy.

Definition 1.2. *(ϵ-Differential Privacy.) A randomized algorithm $\mathcal{A} : \mathbb{D}^n \mapsto \mathcal{S}$ is ϵ-differentially private if for any pair of database $\mathcal{D}$ and $\mathcal{D}'$ with $HD(\mathcal{D}, \mathcal{D}') = 1$,*

$$P_r(\mathcal{A}(\mathcal{D}) \in \mathcal{S}) \leq \exp(\epsilon) \mathcal{A}(\mathcal{D}' \in \mathcal{S}), \tag{1.2}$$

where $\epsilon \in \mathbb{R}_+$.

Differential privacy is a strong privacy notion that protects any single sensitive data point in the worst-case scenario. In particular, any ϵ-differentially private algorithm $\mathcal{A}$ that is robust to the adversary who targets on knowing the k-th data point

D_k of the input database $\mathcal{D}$ is also robust to any other adversaries who have different target data points $D_j \in \mathcal{D}$, for any $j \neq k$. It is difficult to know what information the adversary could have about the target database. By considering the worst-case scenario, differential privacy makes no assumptions about the knowledge set of the adversary. The standard randomization approach for promoting differential privacy is perturbation with Laplacian noise (see, e.g., [20, 15, 58]).

Next, we consider a different scenario, i.e., there are $m > 1$ sensitive data points and the adversary knows all other $n - m$ data points except these m points. Let $\mathcal{D}^m$ be any database such that $\mathrm{HD}(\mathcal{D}^m, \mathcal{D}) = m$. The following corollary directly follows Definition 1.2 (see, e.g., [20, 25]).

Corollary 1.3. *Let $\mathcal{A}$ be any ϵ-differentially private algorithm defined in Definition 1.2. Let $\mathcal{D}$ and $\mathcal{D}^m$ be any two database with $HD(\mathcal{D}^m, \mathcal{D}) = m$. Then the following holds:*

$$P_r(\mathcal{A}(\mathcal{D}) \in \mathcal{S}) \leq \exp(m \times \epsilon) P_r(\mathcal{A}(\mathcal{D}^m) \in \mathcal{S}). \tag{1.3}$$

Let any randomized algorithm $\mathcal{A}$ satisfying (1.3) be named as $m\epsilon$-indistinguishable.

Proof. Let $\mathcal{D}^{-1,0} = \mathcal{D}$, $\mathcal{D}^{0,1}$, $\mathcal{D}^{1,2}$, ..., $\mathcal{D}^{m-1,m} = \mathcal{D}^m$ be any sequence of data base such that each pair $\mathcal{D}^{k-1,k}$ and $\mathcal{D}^{k,k+1}$ have $\mathrm{HD}(\mathcal{D}^{k-1,k}, \mathcal{D}^{k,k+1}) = 1$, for all $0 \leq k \leq m-1$ Then, we have

$$\frac{P_r(\mathcal{A}(\mathcal{D}) \in \mathcal{S})}{P_r(\mathcal{A}(\mathcal{D}^m) \in \mathcal{S})} = \prod_{k=0}^{m-1} \frac{P_r(\mathcal{A}(\mathcal{D}^{k-1,k}) \in \mathcal{S})}{P_r(\mathcal{A}(\mathcal{D}^{k,k+1}) \in \mathcal{S})} \leq \prod_{k=0}^{m-1} \exp(\epsilon) = \exp(m \times \epsilon).$$

In the *non-worst-case scenario* when there are m sensitive data points, i.e., when the adversary does not know m data points in the private database, Corollary 1.3 states that any algorithm $\mathcal{A}$ that is ϵ-indistinguishable can provide $\frac{\epsilon}{m}$-differential privacy (i.e., in the worst-case scenario). In other words, raising indistinguishability for multiple sensitive data points guarantees a higher degree of differential privacy (i.e., indistinguishability in the worst-case scenario).

Another important feature of differential privacy is its *compositionality*. In particular, running the ϵ-differentially private algorithm $\mathcal{A}$ with input $\mathcal{D}$ k times induces a $k\epsilon$-differential process. More generally, let the algorithm $\mathcal{A}$ be ϵ_t-differentially private at the t-th round of running, for some $\epsilon_t \in \mathbb{R}_+$. Let $\mathcal{A}^k$ denote the sequence of k rounds of running the algorithm $\mathcal{A}$ with $\epsilon_1, \ldots, \epsilon_k$. Then, the following holds, for any pair $\mathcal{D}, \mathcal{D}^1 \in \mathbb{D}^n$ with $\mathrm{HD}(\mathcal{D}, \mathcal{D}^1) = 1$,

$$P_r(\mathcal{A}^k(\mathcal{D}) \in \mathcal{S}^k) \leq \exp(\sum_{i=1}^{k} \epsilon_i) P_r(\mathcal{A}^k(\mathcal{D}^1) \in \mathcal{S}^k). \tag{1.4}$$

2. Model. In this section, we describe the basic model of the private data market. With reference to Figure 1, the market consists of two parties: those are (1) n data owners (*owner*, he), denoted as $\mathbb{I} \equiv [n]$ and (2) a data buyer (*buyer*, she); a typical owner is indexed by $i \in \mathbb{I}$. The buyer provides *take-it-or-leave-it* (i.e., the mechanism) offer to the owners. The buyer requests multiple data accesses and each single data access requirement initiates one period of time. We consider a finite-horizon data trading market. Let $\mathbb{T} \equiv \{0, 1, \ldots, T\}$, with $0 \leq T < \infty$, denote the *life cycle* of data trading. Since each owner is allowed to leave the market, it is straightforward to see that the dynamics of valuations may lead to a dynamic population. To make owners'

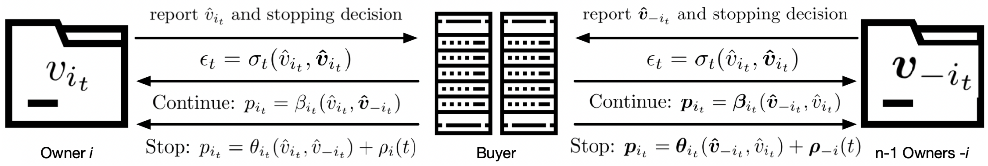

FIG. 1. *Market model for data privacy: At each period t, each owner i reports his instrumental preference v_{i_t} as $\hat{v}_{i_t} = \chi_{i_t}(v_{i_t}; v_i^{t-1}, \hat{v}_i^{t-1}, \epsilon^t)$ and his stopping decision to the buyer. Given the reports $\hat{\boldsymbol{v}}_{i_t} = \{\hat{v}_{i_t}, \hat{\boldsymbol{v}}_{-i_t}\}$, the buyer assigns privacy protection as ϵ_t-differential privacy, where $\epsilon_t = \sigma(\hat{v}_{i_t}, \hat{\boldsymbol{v}}_{-i_t})$ and specifies compensation $p_{i_t} = \beta_{i_t}(\hat{v}_{i_t}, \hat{\boldsymbol{v}}_{-i_t})$ to each owner i for his privacy loss at this period if owner i continues; the buyer specifies a compensation for privacy and stopping payment $p_{i_t} = \theta_{i_t}(\hat{v}_{i_t}, \hat{\boldsymbol{v}}_{-i_t}) + \rho_i(t)$ if owner i stops at t.*

indices (initially given by $\mathbb{I}$) consistent with population dynamics, we re-index each owner participating at period t. Let $N_t \equiv [n_t]$ with $N_0 = \mathbb{I}$ denote the set of participating at period t, where $n_t \geq 0$ is the number of of owners at period t. Clearly, n_t is weakly decreasing in t. For simplicity, let $\mathcal{I}_t : \mathbb{I} \mapsto N_t$ be the *identity transition mapping*, with $\mathcal{I}_t^{-1} : N_t \mapsto \mathbb{I}$, such that the index of owner i at period t is $i_t = \mathcal{I}_t(i) \in N_t$ and $\mathcal{I}_t^{-1}(i_t) = i \in \mathbb{I}$. For a typical owner $i \in \mathbb{I}$, we use $\{i_0 = i, i_1, \ldots, i_{s-1}, i_s, \ldots i_T\}$ to denote owner i in different period, i.e., $i_t = \mathcal{I}_t(i)$, for all $t \in \mathbb{T}$.

Privacy loss and payment . After the agents accept the offer at the beginning, the buyer pays each owner i a *payment for the purchase of data*, denoted as $b_i \in \mathcal{J} \equiv [0, \bar{b}] \subset \mathbb{R}_+$, where $\bar{b} < \infty$. In this work, we consider that the data market operates without taking into account the quality of the data. Hence, the payment b_i depends only on the quantity of the data provided by owner i. Additionally, since the worst-case scenario of privacy loss is independent of the data quantity, we assume, without loss of generality, that each owner provides the same quantity of data. Therefore, the payment for the purchase of data is the same for every owner, i.e., $b_i = b_j = b \in \mathcal{J}$, for all $i \neq j \in \mathbb{I}$. At each period t, the buyer uses differential privacy to protect the data privacy and quantify a *degree of privacy protection* by the (bounded) privacy loss of differential privacy that is parameterized by a scalar $\epsilon_t \in \mathcal{E} \equiv (0, \bar{\epsilon}] \subset \mathbb{R}_+$, $t \in \mathbb{T}$, where $\bar{\epsilon} < \infty$[1]. Since the buyer uses the data from all owners together, she takes advantage of the robustness of differential privacy and specifies ϵ_t that is the same for every owner. The privacy loss is inevitable. As a result, the buyer specifies a *payment to compensate the privacy loss* of each owner i at each period t, denoted as $p_{i_t} \in \mathcal{P} \equiv [0, \bar{p}] \subset \mathbb{R}_+$, where $\bar{p} < \infty$.

Dual preference of privacy . The motive of the owners to protect their privacy in the data trading is based on their preference of privacy. In this work, the buyer specifies the privacy parameter ϵ_t and the compensation $p_t \equiv \{p_{i_t}\}_{i_t \in N_t}$ according to owners' preference of privacy. We consider that each owner has a dual preference of privacy in the following definition.

DEFINITION 2.1 (**Dual Preference of Privacy**). *Owner i's dual preference of privacy consists of (i) an intrinsic preference $c_i \in \mathcal{C}$, and (ii) a time-varying instrumental preference $v_{i_t} \in V_i \subset \mathbb{R}$, which is distributed according to a transition kernel K_{i_t}, for all $i \in \mathbb{I}$, $i_t = \mathcal{I}_t(i)$, $t \in \mathbb{T}$.*

[1]We eliminate $\epsilon_t = 0$ for all $t \in \mathbb{T}$ because when $\epsilon_t = 0$, the buyer cannot extract any utility from the data.

Here, owner i's intrinsic preference captures his intrinsic motives to protect his data privacy, regardless of how his data is traded and used in the data market. Owner i's instrumental preference, on the other hand, represents his motives of privacy protection due to participating in the data market. In other words, owner i's instrumental preference v_{i_t} is endogenously affected by the buyer's dynamic mechanism. In this data market, we assume that the time-evolution of v_{i_t} is due to the accumulated privacy losses, which is characterized by *endogenous dynamics* defined as follows.

DEFINITION 2.2 (**Endogenous Dynamics**). *The time evolution of $v_{i,t}$ is endogenous dynamics if the transition kernel $K_{i_t} : V_i^{t-1} \times \mathcal{E}^{t-1} \mapsto \Delta(V_i)$, such that $K_{i_t}(v_i^{t-1}, \epsilon^{t-1})$ specifies the probability distribution of v_{i_t}, for $t \in \mathbb{T}$, $i \in \mathbb{I}$, when owner i's history of instrumental preferences is v_i^{t-1} and the accumulated privacy losses is (parameterized by) ϵ^{t-1}.*

The endogenous dynamics shown in Definition 2.2 captures the endogenous effect of the accumulated privacy losses ϵ^{t-1} on v_{i_t}. The dependence of v_{i_t} on the history v_{i_t} can be relaxed by Markovian endogenous dynamics:

DEFINITION 2.3 (**Markovian Endogenous Dynamics**). *The endogenous dynamics are Markovian if the transition kernel $k_{i_t} : V_i \times \mathcal{E}^{t-1} \mapsto \Delta(V_i)$ depends on the history v_i^{t-2} only through $v_{i_{t-1}}$, i.e., $k_{i_t}(v_{i_{t-1}}, \epsilon^{t-1})$, specifies the probability distribution of v_{i_t} when owner i's current instrumental preference is $v_{i_{t-1}}$ and the accumulated privacy losses is ϵ^{t-1}.*

Both the intrinsic and the instrumental preferences are the private information of each owner. We assume that each owner is aware of his intrinsic and the instrumental preferences through, e.g., data privacy audition, and all owners' preferences are evaluated according to the same standard. Additionally, we assume that each owner's intrinsic preference is independent of other owners' and his period-t instrumental preference is independent of other owners' period-t instrumental preferences.

Decision Makings. Since the dual preference of privacy is the private information of each owner, the data market requires each owner to report his intrinsic and instrumental preferences. Since the intrinsic preference is independent of the data market, owner i is required to report his c_i at the *prior ex-ante period*, when the take-it-or-leave-it offer is not observed by the owners. The instrumental preference v_{i_t}, however, needs to be reported at the beginning of each period t. We consider a *direct mechanism*, in which each owner chooses report from the spaces of preference. Specifically, each owner i chooses how to report his preference by using *reporting strategies* $\mathring{\chi}_i : \mathcal{C} \mapsto \mathcal{C}$ and $\chi_{i_t} : V_i^t \times V_i^{t-1} \times \epsilon^{t-1} \mapsto V_i$, such that $\hat{c}_i = \mathring{\chi}_i(c_i)$ is the reported intrinsic preference, when his true preference is c_i, and $\hat{v}_{i_t} = \chi_{i_t}(v_{i_t}; v_i^{t-1}, \hat{v}_i^{t-1}, \epsilon^t)$ is his period-t reported instrumental preference, when his history of true instrumental preference is v_i^t, his history of reports is $\hat{v}_i^{t-1}$, and the history of privacy losses is ϵ^{t-1}. Here, owner i chooses his period-t report $\hat{v}_{i_t}$ of v_{i_t} by taking into account the histories, where ϵ^{t-1} is the only source of information he can obtain that contains the realizations of other owners' private intrinsic preferences. To simplify the notation, we omit the dependence of the reporting strategies on the history, i.e., $\chi(v_{i_t}) = \chi_{i_t}(v_{i_t}; v_i^{t-1}\hat{v}_i^{t-1}, \epsilon^t)$, unless otherwise stated. Owner i is allowed to leave the data market by choosing an optimal stopping time once his privacy loss is beyond his privacy tolerance characterize by his intrinsic preference c_i. Since the intrinsic preference is independent of the market model, we assume that the owners truthfully report their intrinsic preference.

To support the owners' stopping decisions, we introduce the notions of *tolerance* *of privacy loss* and *commitment period.*

Definition 2.4 (**Tolerance of Privacy Loss**). *Owner i's tolerance of privacy loss (tolerance) $B_i \in \mathcal{B} \subset \mathbb{R}_+$ is a scalar such that he can bear privacy loss up to B_i, i.e., $\sum_{s=0}^{t} \epsilon_s \leq B_i$.*

Definition 2.5 (**Commitment Period**). *The commitment period $\mathcal{T}_i^c \in \mathbb{T}$ of owner i is the smallest t such that the privacy loss exceeds his tolerance, i.e., $\mathcal{T}_i^c = \min\{t \in \mathbb{T} : \sum_{s=0}^{t} \epsilon_s \geq B_i\}$ denote the commitment period.*

The proposed data market requires a *soft commitment* from owners. Specifically, each owner i is allowed to leave the market after his commitment period (included) is reached. The commitment is soft in the sense that $\mathcal{T}_i^c$ is stochastic due to the endogenous dynamics of v_{i_t}. Let $\mathcal{T}_{i_t}^c$ represent owner i's period-t *expected* commitment period (expected commitment). It is straightforward to see that in general $\mathcal{T}_{i_t}^c \neq \mathcal{T}_{i_{t'}}^c$, for any $t \neq t' \in \mathbb{T}$, $i_t \in N_t$, $i_{t'} \in N_{t'}$, $i_t = \mathcal{I}_t(\mathcal{I}_{t'}^{-1}(i_{t'}))$. The commitment period $\mathcal{T}_i^c$ is realized at t if and only if $\mathcal{T}_{i_t}^c = t$. The following assumption describes the relationship between each owner's intrinsic preference and his tolerance.

Assumption 1. *There exists a mapping, $\lambda : \mathcal{C} \mapsto \mathcal{B}$, such that $B_i = \lambda(c_i)$ and owner i's payoff depends on c_i only through B_i, for all $i \in \mathbb{I}$.*

Assumption 1 implies that owner i's commitment period is partially intrinsic and partially instrumental. That is, owner i's expected commitment $\mathcal{T}_{i_t}^c$ is jointly determined by his intrinsic preference c_i through B_i, current-period privacy loss ϵ_t, and history of privacy losses ϵ^{t-1}.

We focus on when the buyer's specifications of ϵ_t and $\{p_{i_t}\}_{i_t \in N_t}$ based only on owners' reports of their instrumental preferences $\{\hat{v}_{i_t}\}_{i_t \in N_t}$. This setting captures the utility primitiveness of the intrinsic motives of privacy protection and the endogenous nature of the instrumental ones. The buyer specifies the payment b by the *payment rule* $\mathring{\beta} : \mathcal{C}^n \mapsto \mathcal{J}$ such that $b = \mathring{\beta}(\hat{\boldsymbol{c}})$ when the agents report $\hat{\boldsymbol{c}} \equiv \{\hat{c}_i\}_{i \in \mathbb{I}}$. Due to the endogenous dynamics, the compensation p_{i_t} is dynamically assigned at each period t. After receiving all the reports $\hat{\boldsymbol{v}}_t \equiv \{\hat{v}_{i_t}\}_{i_t \in N_t}$, the buyer uses a *privacy assignment rule* (assignment rule) $\sigma_t : \times_{i_t \in N_t} V_{i_t} \mapsto \mathcal{E}$ and a *compensation rule* $\beta_{i_t} : \times_{i_t \in N_t} V_{i_t} \mapsto \mathcal{P}$, respectively, to specify $\epsilon_t = \sigma_t(\hat{\boldsymbol{v}}_t)$ and $p_{i_t} = \beta_{i_t}(\hat{\boldsymbol{v}}_t)$.

To influence owners' stopping decision, the buyer uses a preference-independent *stopping payment rule* $\rho_{i_t} : \mathbb{T} \mapsto \mathbb{R}$, such that $\rho_{i_t}(t)$ specifies an additional payment to owner i if he decides to stop at period t. To distinguish the stopping period and the continuing period, let $\theta_{i_t} : \times_{i_t \in n_t} V_{i_t} \mapsto \mathbb{R}$ denote the preference-dependent compensation rule that specifies a compensation for owner i's privacy loss at the period when he decides to stop at t. Hence, $p_{i_t} = \theta_{i_t}(\hat{\boldsymbol{v}}_t) + \rho_{i_t}(t)$ is the payment owner i receives when he stops at t and the joint reports of the owners are $\hat{\boldsymbol{v}}_t$. Let $\sigma \equiv \{\sigma_t\}_{t \in \mathbb{T}}$, $\boldsymbol{\beta} \equiv \{\beta_{i,t}\}_{i \in \mathbb{I}, t \in \mathbb{T}}$, $\boldsymbol{\theta} \equiv \{\theta_{i,t}\}_{i \in \mathbb{I}, t \in \mathbb{T}}$, and $\boldsymbol{\rho} \equiv \{\rho_{i,t}\}_{i \in \mathbb{I}, t \in \mathbb{T}}$ [2]. The buyer's rule profile $< \sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho} >$ constitute the dynamic mechanism of the data market.

2.1. Dynamic Bayesian Game. We restrict attention to the Markovian endogenous dynamic environment as described in Definition 2.3. Given the mechanism $< \sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho} >$, each owner's independent private information and his reporting strategy determine the distribution of the payoff he obtains from the data trading. The strategic interactions among the owners along with their beliefs about others' private

[2]The rules provided by the buyer as a take-it-or-leave-it offer at the ex-ante stage are specified for all the initial participants at each period t, for all $t \in \mathbb{T}$, no matter whether any individual leaves at any period.

information influences their decision makings (i.e., reporting and stopping decisions) in the data market. Since each owner's private information is dynamic, we naturally obtain a dynamic Bayesian game among the owners in our dynamic mechanism. A standard game-theoretic tool used to analyze such dynamic game is the solution concept known as *perfect Bayesian equilibrium* (PBE). In this work, we study the implementability of the dynamic mechanism in PBE, in which each owner reports truthfully, i.e., $\chi_{i_t}(v_{i_t}) = v_{i_t}$, $i_t \in N_t$, $t \in \mathbb{T}$, when all other owners report truthfully.

Let $\mu_i(\cdot|\boldsymbol{K}) : V^t_{i_t} \times \mathcal{E}^{t-1} \mapsto \Delta(\boldsymbol{V}^t_{-i_t})$ denote owner i's belief about other owners' realized instrumental preferences, such that $\mu_i(\hat{v}^t_i, \epsilon^{t-1})$ is his belief of $\hat{\boldsymbol{v}}^t_{-i}$, when he has reported $\hat{v}^{t-1}_i$, he reports $\hat{v}_{i_t}$, and the history of privacy losses ϵ^{t-1}, given the joint transition kernel $\boldsymbol{K}$ (when all other owners report truthfully.) Based on his estimation of $\boldsymbol{v}^t_{-i}$ using the belief μ_i, owner i estimates the population dynamics due to the optimal stopping rule. Let N^i_t denote the set of participating owners at period t estimated by owner i and let $h_{i_t} \equiv \{N^i_s\}_{s\in\mathbb{T}_{t+1}}$ denote the sequence of populations of participating owners from period t to period T that is estimated by owner i at period t. With a slight abuse of notation, we refer to $\boldsymbol{v}_{-i_t}$ as the joint instrumental preference of owners other than i in N^i_t.

Given the privacy loss ϵ_t and his instrumental preference v_{i_t}, owner i obtains a monetary *flow loss*, given by a *loss function* $\ell_{i_t} : V_i \times \mathcal{E} \mapsto \mathbb{R}$, defined as follows [25]:

$$\ell(v_{i_t}, \epsilon_t) \equiv v_{i_t}\big[\exp(\epsilon_t) - 1\big]. \tag{2.1}$$

The loss function ℓ is increasing in ϵ_t, i.e., the larger (resp. smaller) ϵ_t is, the less (resp. more) private the data becomes; when $\epsilon_t \to 0$, there is no private loss, i.e., $\lim_{\epsilon_t\to 0} \ell(v_{i_t}, \epsilon_t) = 0$. Since v_{i_t} and ϵ_t are finite, $\ell(v_{i_t}, \epsilon_t)$ is bounded, i.e., $|\ell(v_{i_t}, \epsilon_t)| < \infty$, for all $v_{i_t} \in V_i$ and $\epsilon_t \in \mathcal{E}$. This captures the tradeoff between owners' privacy and the buyer's utility extracted from the data because larger (resp. smaller) ϵ_t gives the buyer more (resp. less) utility from the data. Given the compensation p_{i_t}, owner i's period-t *utility* is defined as follows:

$$z_{i_t}(v_{i_t}, \epsilon_t, p_{i_t}) \equiv -\ell(v_{i_t}, \epsilon_t) + p_{i_t}. \tag{2.2}$$

Since ℓ is bounded and p_{i_t} is finite , the utility $z_{i_t}(v_{i_t}, \epsilon_t, p_{i_t})$ is bounded, i.e., $|z_{i_t}(v_{i_t}, \epsilon_t, p_{i_t})| < \infty$, for all $v_{i_t} \in V_i$, $\epsilon_t \in \mathcal{E}$, and $p_{i_t} \in \mathcal{P}$.

According to Ionescu Tulcea theorem (see, e.g., [26]), the kernels $\boldsymbol{K}$, the assignment rules σ, and the reporting strategy χ_i define a unique probability measure $P[\sigma, \chi_i]$ on V^T_i. Similarly, at any period t, current-period v_{i_t}, $\boldsymbol{K}^T_t$, σ, and χ_i define a unique probability measure $P[\sigma, \chi_i]|v_{i_t}$ on $V^T_{i,t+1}$. Given owner i's belief μ_i, the expectations with respect to $P[\sigma, \chi_i]$ and $P[\sigma, \chi_i]|v_{i_t}$, respectively, are denoted as $\mathbb{E}^{\sigma,\chi_i;\mu_i}$ and $\mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t}}_t$. Given $<\sigma, \beta_i, \theta_i, \rho_i>$ and the reporting strategy χ_i, owner i's *ex-ante expected payoff function* and *interim expected payoff function*, respectively, for any fixed time horizon are defined as follows: for any $\tau' \in \mathbb{T}$,

$$\begin{aligned} J^{\sigma,\beta_i,\theta_i,\rho_i;\chi_i,\mu_i}_i(\tau') \equiv& \mathbb{E}^{\sigma,\chi_i;\mu_i}\Big[\sum_{t=0}^{\tau'-1} z_{i_t}\Big(\tilde{v}_{i_t}, \sigma_t\big(\chi_{i_t}(\tilde{v}_{i_t}), \tilde{\boldsymbol{v}}_{-i_t}\big), \beta_{i_t}\big(\chi_{i_t}(\tilde{v}_{i_t}), \tilde{\boldsymbol{v}}_{-i_t}\big)\Big) \\ &+ z_{i_{\tau'}}\Big(\tilde{v}_{i_\tau}, \sigma_{\tau'}\big(\chi_{i_s}(\tilde{v}_{i_{\tau'}}), \tilde{\boldsymbol{v}}_{-i_{\tau'}}\big), \theta_{\tau'}\big(\chi_{i_s}(\tilde{v}_{i_{\tau'}}), \tilde{\boldsymbol{v}}_{-i_{\tau'}}\big) + \rho_{i_t}(\tau')\Big)\Big], \end{aligned} \tag{2.3}$$

and, for any $\tau \in \mathbb{T}_{\mathcal{T}^c_{i_t}}$,

$$
\begin{aligned}
&J^{\sigma,\beta_i,\theta_i,\rho_i;\chi_i,\mu_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)\\
\equiv&\mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t}}_t\Big[\sum_{s=0}^{\tau-1} z_{i_s}\Big(\tilde{v}_{i_t},\sigma_s\big(\chi_{i_s}(\tilde{v}_{i_s}),\tilde{\boldsymbol{v}}_{-i_s}\big),\beta_{i_s}\big(\chi_{i_s}(\tilde{v}_{i_s}),\tilde{\boldsymbol{v}}_{-i_s}\big)\Big)\\
&+z_{i_\tau}\Big(\tilde{v}_{i_\tau},\sigma_\tau\big(\chi_{i_s}(\tilde{v}_{i_\tau}),\tilde{\boldsymbol{v}}_{-i_\tau}\big),\theta_\tau\big(\chi_{i_s}(\tilde{v}_{i_\tau}),\tilde{\boldsymbol{v}}_{-i_\tau}\big)+\rho_{i_t}(\tau)\Big)\Big].
\end{aligned}
\tag{2.4}
$$

Owner i's expected payoff functions in (2.3) and (2.4) are defined when all other owners reporting truthfully. For simplicity, we remove the mechanism rules $<\sigma,\beta_i,\theta_i,\rho_i>$ and owner i's belief μ_i in the superscript of the expected payoff functions, e.g., $J^{\chi_i}_i(\tau')=J^{\sigma,\beta_i,\theta_i,\rho_i;\chi_i,\mu_i}_i(\tau')$ with $J_i(\tau')=J^{\chi^*_i}_i(\tau')$ when χ^*_i is truthful, unless otherwise stated. Also, we remove the belief μ_i in the expectation, i.e., $\mathbb{E}^{\sigma,\chi_i|v_{i_t}}_t=\mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t}}_t$.

Definition 2.6 (**Truthful Perfect Bayesian Equilibrium**). *A Truthful perfect Bayesian equilibrium (TPBE) of the mechanism $<\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$ is a profile $<\boldsymbol{\mu},\boldsymbol{\chi}>$, where each belief μ_i updates $\Delta(\boldsymbol{V}_{-i_t})$ according to Bayes' rule and each reporting strategy χ_{i_t} is truthful and*

$$
\chi_i=\arg_{\chi'_i}\max J^{\sigma,\beta_i,\theta_i,\rho_i;\chi'_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i),
$$

for all $v_{i_t}\in V_i$, $\epsilon^{t-1}\in\mathcal{E}^{t-1}$, $i\in\mathbb{I}$, $t\in\mathbb{T}$, when all other owners' reporting strategies are truthful at all periods. The mechanism $<\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$ is implementable in TPBE (TPBE-implementable, TPBE-implementability) if it induces a game, in which the equilibrium is a TPBE.

2.2. The Buyer's Mechanism Design Problem. The privacy-utility tradeoff implies that the buyer suffers losses of utility she can extract from the data by providing differential privacy. In this work, we ignore the utility loss from the reduce of data due to owners' stopping decisions and restrict attention to the utility loss caused only by differential privacy protection. For any $\epsilon_t\in\mathcal{E}$, $t\in\mathbb{T}$, define the buyer's *utility loss* as:

$$
a_t(\epsilon_t)\equiv L\exp(-\epsilon_t), \tag{2.5}
$$

where $L\in\mathbb{R}_{++}$ represents the maximum utility loss when $\epsilon_t\to 0$.

As a mechanism designer, the buyer aims to determine the rule profile $<\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$ and provide these rules as a take-it-or-leave-it offer to the owners at the ex-ante stage. The buyer's decision makings take into account the expected population dynamics due to owners' stopping rule. Let $\bar{\boldsymbol{\tau}}\equiv\{\bar{\tau}_i\}_{i\in\mathbb{I}}\in\mathbb{T}^n$, where each $\bar{\tau}_i$ denotes the *expected stopping time* of owner i evaluated at the ex-ante stage. Additionally, let $\boldsymbol{N}\equiv\{N_t\}_{t\in\mathbb{T}}$ denote the sequence of population sets evaluated by the buyer. The buyer mechanism design problem is to minimize her *ex-ante expected cost*,

$$
\begin{aligned}
C^{\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}}(\bar{\boldsymbol{\tau}};\boldsymbol{N})\equiv&\mathbb{E}^{\sigma,\chi_i;\mu_i}\Big[\sum_{t\in\mathbb{T}\backslash\bar{\boldsymbol{\tau}}}\Big(L\exp(-\sigma_t(\tilde{\boldsymbol{v}}_t))+\sum_{i_t\in N_t}\beta_{i_t}(\tilde{\boldsymbol{v}}_t)\Big)\\
&+\sum_{\tau\in\bar{\boldsymbol{\tau}}}\Big(L\exp(-\sigma_\tau(\tilde{\boldsymbol{v}}_\tau))+\sum_{i_\tau\in N_\tau}\big(\beta_{i_\tau}(\tilde{\boldsymbol{v}}_{i_\tau})+\rho_{i_\tau}(\tau)\big)\Big)\Big],
\end{aligned}
\tag{2.6}
$$

by determining the rule profile $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$[3] that is TPBE-implementable, i.e., satisfying the *dynamic incentive compatibility* (DIC) for each owner i (DIC$_i$), and is *individual rational* for each each owner i (IR$_i$). That is, the buyer aims to solve a constrained optimization problem:

$$\min_{\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}} C^{\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}}(\bar{\boldsymbol{\tau}}; \boldsymbol{N}), \text{ s.t. } \mathrm{IR}_i,\ \mathrm{DIC}_{i,t}, \forall i \in \mathbb{I}, t \in \mathbb{T}. \tag{2.7}$$

The IR constraint is imposed to guarantee that each owner i has incentive to participate in the data market. This is captured by a non-negative ex-ante expected payoff, i.e., $J_i(\bar{\tau}_i) + b \geq 0$. The DIC constraint ensures the TPBE-implementability of the mechanism, such that each owner i has incentive to report his instrumental preference truthfully at every period. In the rest of the paper, we focus of the theoretical characterizations of DIC$_{i,t}$.

As described in Section 2.1, the TPBE solution concept of the dynamic Bayesian game induced by the mechanism $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ requires a strong rationality of owners: each owner i is rational in the sense that *(i)* he maintains beliefs about all that is unknown (but payoff-relevant) to him and *(*ii) he can accurately forecast and estimate how other owners will respond to any decisions he makes at each period. Likewise, as shown in Section 2.2, the buyer's mechanism design problem also requires her strong rationality to adopt accurate beliefs in regard to the dynamics of the environment (owners' instrumental preferences and the population evolution) and to the decision makings of each owner including his beliefs about others' instrumental preferences and the population dynamics. Our theoretical characterizations in this work are conducted based on these assumptions of strong rationality.

2.3. Optimal Stopping Rule. Given the mechanism $<\sigma, \beta_i, \theta_i, \rho_i>$, owner i makes coupled decisions of reporting and stopping at each period. For any reporting strategy χ_i, owner i's stopping rule is optimal if there exists $\bar{\tau}_i \in \mathbb{T}$ such that

$$\sup_{\tau\in\mathbb{T}} J_i^{\chi_i}(\tau) = J_i^{\chi_i}(\bar{\tau}_i). \tag{2.8}$$

Basically, a stopping rule is optimal if there exists a time horizon $\bar{\tau}_i$ such that owner i's ex-ante expected payoff is maximized, given $<\sigma, \beta_i, \theta_i, \rho_i>$ and fixed χ_i.

To solve the optimal stopping problem (2.8), we introduce the *value function* $U_{i_t}^{\chi_i} : V_i \mapsto \mathbb{R}$ as follows: for any $v_{i_t} \in V_i$, $v_i^{t-1} \in V_i^{t-1}$, $t \in \mathbb{T}$, $i \in \mathbb{I}$,

$$U_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1}) \equiv \sup_{\tau\in\mathbb{T}_t} J_{i_t}^{\chi_i}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1}), \tag{2.9}$$

with $U_{i_t}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1}) = U_{i_t}^{\chi_i^*}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})$, when owner i uses truthful reporting strategy χ_i^*. Here, the value $U_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})$ is owner i's maximum period-t interim expected payoff given any $<\sigma, \beta_i, \theta_i, \rho_i>$ and χ_i. Since the utility z_{i_t} is bounded and the time horizon is finite, owner i's value $U_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})$ is also bounded, i.e., $|U_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})| < \infty$, for all $v_i^t \in V_i^t$, $t \in \mathbb{T}$, $i \in \mathbb{I}$.

LEMMA 2.7 ([45]). *Fix any $<\sigma, \beta_i, \theta_i, \rho_i>$ and χ_i. The followings are true.*
(i) the value function can be represented as, for for any $v_{i_t} \in V_i$, $v_i^{t-1} \in V_i^{t-1}$,

[3]Recall that in footnote 2, each rule in $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ well define the assignment and payments at every period for every owner though each owner may leave the market at any period.

$t \in \mathbb{T}$, $i \in \mathbb{I}$,

$$U^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) = \max\Big\{J^{\chi_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i), \mathbb{E}^{\sigma,\chi_i|v_{i_t}}_t\Big[U^{\chi_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big]\Big\}. \tag{2.10}$$

(ii) The optimal stopping rule $\phi^{\chi_i}_i$ *is given as follows:*

$$\phi^{\chi_i}_i : \exists\, \tau_i \in \mathbb{T},\ s.t., \quad \tau_i = \inf\big\{t \in \mathbb{T} : U^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) = J^{\chi_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i)\big\}, \tag{2.11}$$

with $\phi^*_i = \phi^{\chi^*_i}_i$, *where* χ^*_i *is owner* i*'s truthful reporting strategy.*

Lemma 2.7 shows the optimal stopping rule ϕ_i in terms of owner i's value function and his interim expected payoff. Here (2.10) reformulates the value function $v^{\chi_i}_{i_t}$ as a Bellman equation. The optimal stopping rule ϕ_i in (2.11) is established based on the Bellman equation (2.10) and suggests a stopping decision at period t if the owner i's period-t value equals his period-t interim expected payoff if he stops immediately at t. Define, for any $v_{i_t} \in V_i$, $v^{t-1}_i \in V^{t-1}_i$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$

$$G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) \equiv \mathbb{E}^{\sigma,\chi^{[t]}_i|v_{i_t}}_t\Big[\sup_{\tau'\in\mathbb{T}_{t+1}} J^{\chi^{[t]}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}},\tau';\epsilon^t,v^t_i)\Big] - J^{\chi^{[t]}_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i) + \rho_i(t), \tag{2.12}$$

with $J^{\chi^{[T]}_i}_{i_{T+1}} = 0$, such that $G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) - \rho_i(t)$ captures owner i's incentive of stopping or continuing, i.e., the optimal stopping rule $\phi^{\chi_i}_i$ in (2.11) can be rewritten as follows:

$$\phi^{\chi_i}_i : \exists\, \tau_i \in \mathbb{T},\ \text{s.t., } \tau_i = \inf\big\{t \in \mathbb{T} : G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) \le \rho_i(t)\big\}, \tag{2.13}$$

with $\phi^*_i = \phi^{\chi^*_i}_i$ when owner i uses truthful reporting strategy χ^*_i. Since $G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i)$ is independent of $\rho_i(t)$, the buyer can influence each owner i's optimal stopping decision by adjusting $\rho_i(t)$. When $G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) = \rho_i(t)$, owner i with χ_i is indifferent between stopping and continuing.

2.4. Dynamic Incentive Compatibility. In TPBE (Definition 2.6), owner i truthfully reveals his true instrumental preference v_{i_t} when he has reported truthfully at all past periods and plans to report truthfully at all future periods, when all other owners report truthfully at all periods. Each owner i's incentive of choosing a reporting strategy χ_{i_t} at each period t depends on how much interim expected payoff he can obtain by using χ_{i_t} to report his period-t true instrumental preference v_{i_t}. In a Markovian endogenous dynamic environment (Definition 2.3), the probability distribution of owner i's future instrumental preference depends on v_{i_t}, $\{\hat{v}^{t-1}_i, \hat{v}_{i_t} = \chi_{i_t}(v_{i_t})\}$, and ϵ^{t-1}; its dependence on past true instrumental preference v^{t-1}_i only through current v_{i_t}. As a result, if owner i is incentivized to report truthfully when he has reported his instrumental preferences truthfully in all the past periods, then he is also incentivized to report truthfully when has lied in the past. Each owner i's incentive to report truthfully at each period t is guaranteed by *dynamic incentive compatibility* (DIC) defined as follows.

DEFINITION 2.8 (**Dynamic Incentive Compatibility**). *Let $\chi_i = \{\chi^*_{i;0:t-1}, \chi_{i;t:T}\}$, where $\chi^*_{i;0:t-1} = \{\chi^*_{i_s}\}_{s\in\mathbb{T}_{0,t-1}}$ is the sequence of truthful reporting strategies up to period $t-1$ and $\chi_{i;t:T} = \{\chi_{i_s}\}_{s\in\mathbb{T}_{t,T}}$ is any reporting strategy sequence from t to T. The mechanism $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ with exist option is dynamic incentive-compatible in TPBE (DIC, equivalently, implementable in TPBE or TPBE-implementable) with belief $\boldsymbol{\mu} = \{\mu_i\}_{i\in\mathbb{I}}$ defined in Definition 2.6 if, for any $\epsilon^{t-1} \in \mathcal{E}^{t-1}$ when all the owners have reported truthfully in all the past $t-1$ periods for any true $v_i^{t-1} \in V_i^{t-1}$, $v_{i_t} \in V_i$, $t \in \mathbb{T}$, $i \in \mathbb{I}$,*

$$
\begin{aligned}
&\max\Big\{J_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}), \mathbb{E}_t^{\sigma|v_{i_t}}\Big[U_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^t, \tilde{v}_i^t)\Big]\Big\} \\
&\quad\geq \max\Big\{J_{i_t}^{\chi_i}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}), \mathbb{E}_t^{\sigma,\chi_i|v_{i_t}}\Big[U_{i_{t+1}}^{\chi_i}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^t, \tilde{v}_i^t)\Big]\Big\},
\end{aligned}
\tag{2.14}
$$

with $J_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}) \geq J_{i_t}^{\chi_i}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1})$.

The DIC condition shown in (2.14) defines an equilibrium, in which each owner i with belief μ_i *(i)* has no incentive to deviate from truthful reporting at period t and planned truthful reporting in future when he has reported truthfully in all past periods and all other owners report truthfully at all periods and *(ii)* chooses optimal stopping time according to ϕ_i^* given in (2.11).

Define owner i's period-t *one-shot deviation strategy* $\chi_i^{[t]} \equiv \{\{\chi^*_{i_s}\}_{s\in\mathbb{T}_{0,t-1}}, \chi_{i_t}^{[t]}, \{\chi^*_{i_s}\}_{s\in\mathbb{T}_{t+1}}\}$, such that $\chi_i^{[t]}$ reports owner i's instrumental preferences at all periods except period t. Let $\hat{v}_{i_t} = \chi_{i_t}^{[t]}(v_{i_t})$ denote owner i's period-t report of his true v_{i_t} using $\chi_i^{[t]}$.

PROPOSITION 2.9. *The mechanism $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ with exist option is DIC with belief $\boldsymbol{\mu}$ if and only if, for any $v_i^t \in V_i^t$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $t \in \mathbb{T}$, $i \in \mathbb{I}$,*

$$
\begin{aligned}
&\max\Big\{J_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}), \mathbb{E}_t^{\sigma|v_{i_t}}\Big[U_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^t, \tilde{v}_i^t)\Big]\Big\} \\
&\quad\geq \max\Big\{J_{i_t}^{\chi_i^{[t]}}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}), \mathbb{E}_t^{\sigma,\chi_i^{[t]}|v_{i_t}}\Big[U_{i_{t+1}}^{\chi_i}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^t, \tilde{v}_i^t)\Big]\Big\},
\end{aligned}
\tag{2.15}
$$

with $J_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}) \geq J_{i_t}^{\chi_i}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1})$.

Proof. See Appendix A.

Proposition 2.9 establishes a *one-shot deviation principle* (see, e.g., [13]) for our dynamic model that enables us to restrict attention on the characterizations of DIC when each owner i deviates from truthfulness by using any one-shot deviation strategy $\chi_i^{[t]}$ for every $t \in \mathbb{T}$, while the optimality of stopping decision is maintained. The stopping rule ϕ_i defined in (2.11) is optimal for any reporting strategy χ_i. However, the realization of owner i's stopping time at period t according to ϕ_i depends on his current reporting strategy χ_{i_t} and planned $\{\chi_{i_s}\}_{s\in\mathbb{T}_i}$, true instrumental preference v_{i_t}, histories ϵ^{t-1} and v_i^{t-1}, i.e., owner i with χ_i is optimal to stop at t if $\tau_i = t$, where τ_i is given in the definition of ϕ_i in (2.11). Let τ_i^* denote τ_i when owner uses χ_i^*. Proposition 2.9 implies that when $\tau_i^* = t$, owner i has no incentives to use $\chi_i^{[t]}$ and stop at t, use $\chi_i^{[t]}$ and continue, or use χ_i^* and continue; when $\tau_i^* > t$, owner i has no incentives to use $\chi_i^{[t]}$ and stop at t or use $\chi_i^{[t]}$ and continue.

3. Characterization of DIC. In this section, we characterize the DIC of our dynamic data market by providing formulations of the monetary transfer rules (i.e., the compensation rules and the stopping payment rule) in terms of the assignment rule and the sufficient and the necessary conditions for DIC. The following assumption holds for this section.

Assumption 2. *The probability density* $f_{i_t}(v_{i_t}|v_{i_{t-1}}, \epsilon^{t-1}) > 0$ *for all* $v_{i_{t-1}} \in V_i$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $t \in \mathbb{T}\backslash\{0\}$.

Assumption 2 considers a full support environment, in which each of owner i's instrumental preferences has a strictly positive probability to occur at every period.

Given any DIC market model $< \sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho} >$, truthful reporting strategy $\chi^*_{i_t}$ is optimal for each owner i, for all $t \in \mathbb{T}$. For simplicity, let $J_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1})$ denote owner i's period-t interim expected payoff when he uses any one-shot deviation strategy $\chi_i^{[t]}$ with $\hat{v}_{i_t} = \chi_{i_t}^{[t]}(v_{i_t})$; When owner i reports truthfully, $J_{i_t}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1}) = J_{i_t}(v_{i_t}, v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1})$ Then, we have the following lemma based on the envelope theorem (see, e.g., [35, 44]).

Lemma 3.1. *Suppose Assumption 2 holds. Then, in any DIC market model* $< \sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho} >$, *we have, for all* $\tau \in \mathbb{T}_t$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $v_i^{t-1} \in V_i^{t-1}$,

$$\frac{\partial J_{i_t}(x, \tau; \epsilon^{t-1}, v_i^{t-1})}{\partial x}\Big|_{x=v_{i_t}} = \mathbb{E}^{\sigma; v_{i_t}}\Big[\sum_{s=t}^{\tau}\big(1-\exp(\sigma_s(\tilde{\boldsymbol{v}}_s))\big)\mathcal{G}_t^s(\tilde{\boldsymbol{v}}_t^s|\sigma)\Big],\ \textit{where} \tag{3.1}$$

$$\mathcal{G}_t^s(\tilde{\boldsymbol{v}}_t^s|\sigma) \equiv \prod_{k=t}^{s}\frac{-\partial F_{i_t}(x|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})}{f_{i_t}(\tilde{v}_{i_k}|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})\partial x}\Big|_{x=\tilde{v}_{i_k}}. \tag{3.2}$$

Proof. See Appendix B. ∎

Lemma 3.1 provides a first-order necessary condition for the optimality of each owner's truthful reporting strategy. Since $\epsilon_t > 0$, the term $1 - \exp(\epsilon_t) < 0$, for all ϵ_t, $t \in \mathbb{T}$. Then, the monotonicity of J_{i_t} with respect to owner i's instrumental preference is determined by the sign of $\mathcal{G}_t^s$. With a slight abuse of notation, let $J_{i_t}(v_{i_t}, \hat{v}_{i_t}; \epsilon^{t-1}, v_i^{t-1})$

Assumption 3. *For any* $v'_{i_t} \geq v_{i_t} \in V_i$, $v_{i_{t+1}} \in V_i$, $\epsilon^t \in \mathcal{E}^t$, $t \in \mathbb{T}\backslash T$, $i \in \mathcal{I}$,

$$F_{i_{t+1}}(v_{i_{t+1}}|v'_{i_t}, \epsilon^t) \leq F_{i_{t+1}}(v_{i_{t+1}}|v_{i_t}, \epsilon^t). \tag{3.3}$$

Assumption 3 imposes a monotonicity condition to the probability distribution function of each owner in the sense of first-order stochastic dominance, i.e., higher instrumental preference at current period t leads to a higher instrumental preference at the next period $t+1$ probabilistically, while keeping the same ϵ^t. In other words, Assumption 3 assumes that owners who value their privacy more at the current period will most probability continue to value their privacy at the next period more than other owners with relatively lower valuation of privacy at the current period. The following lemma gives the monotonicity of J_{i_t}.

Lemma 3.2. *Suppose Assumption 3 holds. Then, in any DIC market model* $< \sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho} >$, $J_{i_t}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1})$ *is weakly decreasing in* v_{i_t} *for any* $\tau \in \mathbb{T}_t$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $v_i^{t-1} \in V_i^{t-1}$, $t \in \mathbb{T}$, $i \in \mathbb{I}$.

Proof. See Appendix C. ∎

Based on Assumption 3.3, Lemma 3.2 shows that increasing an owner's instrumental preference at any period decreases his interim expected payoff. In other words, owners with relatively higher instrumental preference over privacy protection incline to stop than owners with a relatively lower instrumental preference.

With a slight abuse of notation, define

$$G^{\chi_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)\equiv \mathbb{E}^{\sigma,\chi^{[t]}_i|v_{i_t}}_t\Big[J^{\chi^{[t]}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}},\tau;\epsilon^t,v^t_i)\Big]-J^{\chi^{[t]}_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i)+\rho_i(t), \tag{3.4}$$

where $\tau=\inf\arg_{\tau'\in\mathbb{T}_T}\sup J^{\chi^{[t]}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}},\tau';\epsilon^t,v^t_i)\Big]$. From Lemmas 3.1 and 3.2, we have that in any DIC market model $G^{\chi_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)$ is non-increasing, i.e.,

$$\frac{\partial G^{\chi_i}_{i_t}(x,\tau;\epsilon^{t-1},v^{t-1}_i)}{\partial x}\Big|_{x=v_{i_t}}\leq 0.$$

Hence, it is straightforward to see that the term $G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i)$ in (2.12) is also non-increasing. From the optimal stopping rule (2.13), we define the *stopping region* as follows:

$$\mathcal{R}^{\chi_i}_{i_t}\equiv\{v_{i_t}\in V_i: G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i)\leq\rho_i(t)\}. \tag{3.5}$$

Define the *indifference region* of the stopping region $\mathcal{R}^{\chi_i}_{i_t}$ is given as

$$\mathcal{S}^{\chi_i}_{i_t}\equiv\{v_{i_t}\in V_i: G^{\chi_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i)=\rho_i(t)\}.$$

At any $v_{i_t}\in\mathcal{S}^{\chi_i}_{i_t}$, each owner i is indifferent between stopping and continuing.

Proposition 3.3. *Suppose Assumptions 2 and 3 hold. In any DIC market model* $<\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$, *the optimal stopping rule* ϕ^{χ_i} *(2.13) is a threshold rule, i.e., the indifference region* $\mathcal{S}^{\chi_i}_{i_t}$ *is a unique interval* $[\kappa^l_i(t),\kappa^r_i(t)]=\mathcal{S}^{\chi_i}_{i_t}$ *with* $\kappa^l_i(t)\leq\kappa^r_i(t)$ *and* $\kappa^l_i(T)=\kappa^r_i(T)=\underline{v}_i$, *such that the stopping region* $\mathcal{R}^{\chi_i}_{i_t}$ *in (3.5) is equivalent to*

$$\mathcal{R}^{\chi_i}_{i_t}\equiv\{v_{i_t}\in V_i: v_{i_t}\geq\kappa^l_i(t)\}, \tag{3.6}$$

for any reporting strategy χ_i, $t\in\mathbb{T}$. *Each agent* i *is indifferent between stopping and continuing when his instrumental preference* $v_{i_t}\in[\kappa^l_i(t),\kappa^r_i(t)]$. *We refer to* $\kappa^l_i:\mathbb{T}\mapsto V_i$ *as the threshold function of owner* i *and the optimal stopping rule is called threshold rule.*

Proof. See Appendix D. ∎

With threshold rule, it is optimal for each owner i using any reporting strategy χ_i to stop at t when his instrumental preference $v_{i_t}\geq\kappa^l_i(t)$, for all $t\in\mathbb{T}$. Let $\bar{J}^{\chi_i}_{i_t}(\cdot,\tau;\cdot,\cdot)$, $\tau>t$, denote owner i's period-t interim expected payoff $J^{\chi_i}_{i_t}(\cdot,\tau;\cdot,\cdot)$ without current-period compensation rule β_{i_t}, i.e.,

$$\begin{aligned}\bar{J}^{\chi_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)\equiv&-\ell(v_{i_t},\sigma_t(\chi_{i_t}(v_{i_t}),\boldsymbol{v}_{-i_t}))\\&+\mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t}}_t\Big[\sum_{s=t+1}^{\tau-1}z_{i_s}\Big(\tilde{v}_{i_t},\sigma_s\big(\chi_{i_s}(\tilde{v}_{i_s}),\tilde{\boldsymbol{v}}_{-i_s}\big),\beta_{i_s}\big(\chi_{i_s}(\tilde{v}_{i_s}),\tilde{\boldsymbol{v}}_{-i_s}\big)\Big)\\&+z_{i_\tau}\Big(\tilde{v}_{i_\tau},\sigma_\tau\big(\chi_{i_s}(\tilde{v}_{i_\tau}),\tilde{\boldsymbol{v}}_{-i_\tau}\big),\theta_\tau\big(\chi_{i_s}(\tilde{v}_{i_\tau}),\tilde{\boldsymbol{v}}_{-i_\tau}\big)+\rho_{i_t}(\tau)\Big)\Big].\end{aligned} \tag{3.7}$$

Let $\bar{J}_{i_t}(v_{i_t},\hat{v}_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)$ denote $\bar{J}^{\chi^{[t]}_i}_{i_t}(v_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i)$ when owner i's reporting strategy is a one-shot deviation strategy $\chi^{[t]}_i$ with $\hat{v}_{i_t}=\chi^{[t]}_{i_t}(v_{i_t})$. Suppose owner i adopts any one-shot deviation strategy. Define the *distances*, for any $t\in\mathbb{T}$, $i\in\mathbb{I}$,

$$d^S_{i_t}(\hat{v}_{i_t},v_{i_t})\equiv-\ell\big(\hat{v}_{i_t},\sigma_t(\hat{v}_{i_t},\boldsymbol{v}_{-i_t})\big)+\ell\big(v_{i_t},\sigma_t(\hat{v}_{i_t},\boldsymbol{v}_{-i_t})\big), \tag{3.8}$$

and, for any $\tau \in \mathbb{T}_t$,

$$d^{-S}_{i_t}(\hat{v}_{i_t}, v_{i_t}; \tau) \equiv \bar{J}^{\chi_i}_{i_t}(\hat{v}_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) - \bar{J}^{\chi_i}_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i). \tag{3.9}$$

We have the following theorem.

THEOREM 3.4. *Suppose Assumptions 2 and 3 hold. Define* $\Lambda^{\sigma}_i : V_i \times V_i \mapsto \mathbb{R}$ *as:*

$$\Lambda^{\sigma}_i(v_{i_t}, v'_{i_t}; \tau) \equiv \int^{v_{i_t}}_{v'_{i_t}} \mathbb{E}^{\sigma; v_{i_t}, x}\Big[\sum_{s=t}^{\tau} \big(1 - \exp(\sigma_s(\tilde{\boldsymbol{v}}_s))\big)\mathcal{G}^s_t(\tilde{\boldsymbol{v}}^s_t|\sigma)\Big]dx, \tag{3.10}$$

where $\mathcal{G}^s_t$ *is defined in (3.2). In any DIC market model, the followings hold:*

(i) The compensation rule $\boldsymbol{\beta}$ *can be represented in terms of the assignment rule* σ_t*, i.e.,*

$$\begin{aligned}\beta_{i_t}(\boldsymbol{v}_t) = \sup_{\tau \in \mathbb{T}_t} \Lambda^{\sigma}_i(v_{i_t}, \bar{v}_i; \tau) - \mathbb{E}^{\sigma; \mu_i | v_{i_t}}\Big[\sup_{\tau \in \mathbb{T}_{t+1}} \Lambda^{\sigma}_i(\tilde{v}_{i_{t+1}}, \bar{v}_i; \tau)\Big] \\ + \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})).\end{aligned} \tag{3.11}$$

(ii) When owner i *decides to stop at* t*, the compensation rule and the the stopping payment rule, respectively, are given in terms of* σ *as follows:*

$$\theta_{i_t}(\boldsymbol{v}_t) = \Lambda^{\sigma}_i(v_{i_t}, \bar{v}_i; t) + \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})), \tag{3.12}$$

$$\begin{aligned}&\rho_i(t) \\ =&\mathbb{E}^{\sigma; \mu_i | \kappa^l_i(t)}\Big[\sum_{s=t}^{T-1} \Big(\Lambda^{\sigma}_i(\tilde{v}_{i_{s+1}} \wedge \kappa^l_i(s+1), \bar{v}_i; s+1) - \Lambda^{\sigma}_i(\tilde{v}_{i_s} \wedge \kappa^l_i(s), \bar{v}_i; s)\Big) \\ &-\Big(\sup_{\tau' \in \mathbb{T}_{s+1}} \Lambda^{\sigma}_i(\tilde{v}_{i_{s+1}} \wedge \kappa^l_i(s+1), \bar{v}_i; \tau') - \sup_{\tau' \in \mathbb{T}_s} \Lambda^{\sigma}_i(\tilde{v}_{i_s} \wedge \kappa^l_i(s), \bar{v}_i; \tau')\Big)\Big].\end{aligned} \tag{3.13}$$

(iii) the assignment rule σ *satisfies the following conditions:*

$$\Lambda^{\sigma}_i(\hat{v}_{i_t}, v_{i_t}; t) \leq d^S_{i_t}(\hat{v}_{i_t}, v_{i_t}), \tag{3.14}$$

$$\sup_{\tau \in \mathbb{T}_t} \Lambda^{\sigma}_i(\hat{v}_{i_t}, \bar{v}_i; \tau) - \sup_{\tau \in \mathbb{T}_t} \Lambda^{\sigma}_i(v_{i_t}, \bar{v}_i; \tau) \leq \inf_{\tau \in \mathbb{T}_t} \Big\{d^{-S}_{i_t}(\hat{v}_{i_t}, v_{i_t}; \tau)\Big\} - \sup_{\tau \in \mathbb{T}_t} \rho_i(\tau). \tag{3.15}$$

We write $\mathtt{DIC}[\boldsymbol{\kappa}]$ *as a set of all assignment rules that satisfy (3.14) and (3.14), when* $\boldsymbol{\rho}$ *is constructed in (3.13) given threshold functions* $\boldsymbol{\kappa}$*.*

Proof. See Appendix E. ∎

Theorem 3.4 establishes a design regime for DIC market model. Specifically, (3.11) and (3.12) give the designs of preference-related compensation rules $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ in terms of the assignment rule σ, respectively, while (3.13) constructs the preference-independent stopping payment rule $\boldsymbol{\rho}$ in terms of σ. Given the constructions (3.11)-(3.13), the conditions (3.14) and (3.15) compose a sufficient condition for DIC. Here, $\Lambda^{\sigma}_i(v_{i_t}, \bar{v}_i; \tau)$ is the *information rent* for any $\tau \in \mathbb{T}_t$ of owner i with v_{i_t} that captures the maximum privacy protection (equivalently, least privacy loss) he can obtain by pretending an owner with the highest instrumental preference $\bar{v}_i \geq v_{i_t}$ due to the buyer's not knowing his true private information v_{i_t} (while keeping other owners' behaviors fixed). From Lemma 3.2, we have that owner i's information rent $\Lambda^{\sigma}_i(v_{i_t}, \bar{v}_i; \tau)$ is non-negative, for all $v_{i_t} \in V_i$, $i \in \mathcal{N}$, $\tau \in \mathbb{T}_t$, $t \in \mathbb{T}$. Hence, each owner i with $v_{i_t} = \bar{v}_i$ has no information rent for privacy protection, which coincides with the setting that owners have tendency for more privacy protection. Given the information rents, we can interpret each compensation rules as follows. The compensation rule β_{i_t} in (3.11) is constructed by the maximum information rent given the optimal stopping rule, the expected future information rent, and the current-period immediate privacy loss. The compensation rule θ_{i_t} is (3.12) is constructed by the current-period one-period information rent and the immediate privacy loss. The stopping payment rule $\boldsymbol{\rho}$ is independent of any realizations of owners' instrumental preferences. For a typical owner i, $\rho_i(t)$ in (3.13) is formulated as an expected combination of information rent, in which the period-t interim expectation is taken by letting current-period instrumental preference be the threshold $\kappa^l_i(t)$ and the stochastic process from $t+1$ onward is constrained, i.e., forcing the realization of $\tilde{v}_{i_s}$ to be the threshold $\kappa^l_i(s)$ if it is above $\kappa^l_i(s)$. The formulation (3.13) obtains a relationship between the stopping payment rule ρ_i and the threshold function κ^l_i, given the assignment rule σ, such that the design of ρ_i can be equivalent to the design of κ^l_i, for $i \in \mathcal{N}$. The following corollary directly follows Theorem 3.4.

Corollary 3.5. *Suppose Assumptions 2 and 3 hold. In any DIC market model without the stopping payment rule $\boldsymbol{\rho}$ (i.e., $\rho_i(t) = 0$, for all $t \in \mathbb{T}$, $i \in \mathcal{N}$), the stopping rule is optimal if and only if there exists a threshold function κ_i that solves the following equation:*

(3.16)
$$\begin{aligned}\mathbb{E}^{\sigma;\mu_i|\kappa^l_i(t)}\Big[\sum_{s=t}^{T-1}\Big(\Lambda^{\sigma}_i(\tilde{v}_{i_{s+1}}\wedge\kappa^l_i(s+1),\bar{v}_i;s+1)-\Lambda^{\sigma}_i(\tilde{v}_{i_s}\wedge\kappa^l_i(s),\bar{v}_i;s)\Big)\\ -\Big(\sup_{\tau'\in\mathbb{T}_{s+1}}\Lambda^{\sigma}_i(\tilde{v}_{i_{s+1}}\wedge\kappa^l_i(s+1),\bar{v}_i;\tau')-\sup_{\tau'\in\mathbb{T}_s}\Lambda^{\sigma}_i(\tilde{v}_{i_s}\wedge\kappa^l_i(s),\bar{v}_i;\tau')\Big)\Big]=0.\end{aligned}$$

Let $e_i = \{e_{i_0}, e_{i_1}, \dots, e_{i_T}\}$ denote a sequence of threshold value generated by a threshold function κ^l_i, where each $e_{i_t} = \kappa^l_i(t)$. From the stopping region $\mathcal{R}^{\chi_i}_{i_t}$ given in (3.6), if the buyer can freely choose $\kappa^l_i(t) \in \{\underline{v}_i, \bar{v}_i\}$, for all $t \in \mathbb{T}$, then we say the buyer can control owner i's stopping decision, i.e., she can make owner i stop or continue at any period t.

Corollary 3.6. *Suppose Assumptions 2 and 3 hold. Suppose, additionally, when each owner is indifferent between stopping or continuing, he chooses to stay in the market. The buyer is able to prevent owner i to leave the market before $t = T$ if and only if there exists $\sigma \in \mathtt{DIC}[\{\bar{v}^T_i, \boldsymbol{\kappa}^l_{-i}\}]$. The buyer is able to make owner i to leave the market at any specific period $t \in \mathbb{T}$ (not before t) if and only if there exists $\sigma \in \mathtt{DIC}[\{\{\bar{v}^{t-1}_i, \underline{v}_i, v^T_{i_{t+1}}\}, \boldsymbol{\kappa}^l_{-i}\}]$, where $v^T_{i_{t+1}} = \{v_{i_s}\}^T_{s=t+1}$.*

Corollary 3.6 shows restrictions of the buyer's ability to control any owner's stopping decision. These restrictions are specified by $\mathtt{DIC}[\{\bar{v}^T_i, \boldsymbol{\kappa}^l_{-i}\}]$ and $\mathtt{DIC}[\{\{\bar{v}^{t-1}_i, \underline{v}_i, v^T_{i_{t+1}}\}, \boldsymbol{\kappa}^l_{-i}\}]$, which require the design of σ and the choice of thresholds $\{\bar{v}^T_i, \boldsymbol{\kappa}^l_{-i}\}$ or $\{\{\bar{v}^{t-1}_i, \underline{v}_i, v^T_{i_{t+1}}\}, \boldsymbol{\kappa}^l_{-i}\}$ to satisfy the conditions constructed in Theorem 3.4.

We establish a necessary condition of DIC based on the result obtained in Lemma 3.1.

Proposition 3.7. *Suppose Assumptions 2 and 3 hold. Let the compensation rules $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ be formulated in (3.11) and (3.12), respectively, and let the stopping* *rule $\boldsymbol{\rho}$ be formulated in (3.13). In any DIC market model, the assignment rule σ satisfies the followings:*

$$\Lambda_i^{\sigma}(\hat{v}_{i_t}, \bar{v}_i; t) - \Lambda_i^{\sigma}(v_{i_t}, \bar{v}_i; t) \leq d_{i_t}^{S}(\hat{v}_{i_t}, v_{i_t}), \tag{3.17}$$

$$\sup_{\tau\in\mathbb{T}_t} \Lambda_i^{\sigma}(\hat{v}_{i_t}, \bar{v}_i; \tau) - \sup_{\tau\in\mathbb{T}_t} \Lambda_i^{\sigma}(v_{i_t}, \bar{v}_i; \tau) \leq \sup_{\tau\in\mathbb{T}_t} d_{i_t}^{-S}(\hat{v}_{i_t}, v_{i_t}; \tau). \tag{3.18}$$

Proof. See Appendix F. ∎

4. Optimal Market Design Problem and Its Relaxation. From the formulations of $<\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ in Theorem 3.4, $\bar{\boldsymbol{\tau}} = \{\bar{\tau}_i\}_{i\in\mathcal{N}}$ can be characterized by the assignment rule σ, the threshold rule $\boldsymbol{\kappa}^l$, and the endogenous dynamics; i.e., we can write (with a slight abuse of notation) $\bar{\tau}_i = \bar{\tau}_i(\sigma, \kappa_i^l) = \sum_{t=0}^{T} t \cdot P\big(v_{i_t} \geq \kappa_i^l(t)\big) = \mathbb{E}^{\sigma;\mu_i}\Big[\sum_{t=0}^{T} t\cdot F_{i_t}(\kappa_i^l(t)|v_{i_{t-1}}, \sigma^{t-1}(\boldsymbol{v}^{t-1}))\Big]$. Since our market model is finite-horizon, $\bar{\tau}_i$ exists. Based on Theorem 3.4, we apply a first-order approach [47, 52] to rewrite the buyer's objective function (2.6) as follows (by integration by parts):

$$\begin{aligned} C^{\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}}(\bar{\boldsymbol{\tau}};\boldsymbol{N}) = \mathbb{E}^{\sigma;\mu_i}\Big[&\sum_{t=0}^{\bar{\boldsymbol{\tau}}} L\exp(-\sigma_t(\tilde{\boldsymbol{v}}_t)) + \sum_{i\in N, \tau_i'\in\bar{\boldsymbol{\tau}}}\sum_{t=0}^{\tau_i'} \tilde{v}_{i_t}[\exp(\sigma_t(\tilde{\boldsymbol{v}}_t)) - 1] \\ &+ \sum_{i\in N,\tau_i'\in\bar{\boldsymbol{\tau}}}\sum_{t=0}^{\tau_i'} \frac{[\exp(\sigma_t(\tilde{\boldsymbol{v}}_t))) - 1](1-F_{i_0}(\tilde{v}_{i_0}))}{f_i(\tilde{v}_{i_0})/\mathcal{G}_0^t(\tilde{\boldsymbol{v}}_0^t|\sigma)}\Big] + J_{i_0}(\underline{v}_i, \bar{\tau}_i). \end{aligned} \tag{4.1}$$

Here, (4.1) is a relaxed objective function by making owners to make decisions at stationary points and substituting $<\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ given in (3.11)-(3.13), respectively. From Lemma 3.2, we have hi J_{i_t} is weakly decreasing. Hence, if we make $J_{i_0}(\bar{v}_i, \tau_i') \geq 0$, for all $i\in N$, the IR constraint is satisfied, i.e., $J_i(\tau_i') + b \geq 0$ for all $\tau_i' \in \mathbb{T}$. Let $\bar{C}^{\sigma,\boldsymbol{\kappa}}(\bar{\tau};\boldsymbol{N}) = C^{\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}}(\bar{\boldsymbol{\tau}};\boldsymbol{N}) - J_{i_0}(\underline{v}_i, \bar{\tau}_i)$. Hence, based on (4.1) we can relax the buyer's mechanism design problem (2.7) as follows:

$$\min_{\sigma,\boldsymbol{\kappa}} \ \bar{C}^{\sigma,\boldsymbol{\kappa}}(\bar{\boldsymbol{\tau}};\boldsymbol{N}), \text{ s.t., } J_{i_0}(\bar{v}_i, \tau_i') \geq 0. \tag{4.2}$$

Therefore, the buyer's mechanism design problem of finding $<\sigma, \boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\rho}>$ by satisfying the DIC and IR constraints is relaxed into solving (4.2), which requires $J_{i_0}(\bar{v}_i, \tau_i') \geq 0$, for all $i\in N$.

Due to the dynamics of owners' instrumental preferences, however, the constrained optimization problem (4.2) is in general intractable. Computationally solving (4.2) in general involves approximations, which may inevitably violate the strong DIC conditions we obtained in Theorem 3.4 and Proposition 3.7. It is beyond the scope of this paper to conduct algorithmic analysis of such computational approximations and to design efficient algorithms to solve (4.2) numerically and we put them to our future work. To address the analytical intractability in practial mechanism design, one way is to use a weaker versions of incentive compatibility, referred to as δ-incentive compatibility or δ approximate incentive compatibility. In our dynamic environment, we define a notion of $<\delta_{i_t}^{S}, \delta_{i_t}^{-S}>$-DIC: the market model is $\delta_{i_t}^{S}$-DIC when it is optimal for owner i to stop; the market model is $\delta_{i_t}^{-S}$-DIC when it is optimal for owner i to continue. Let

$$h_{i_t}^{S} \equiv \sup_{v_{i_t},\hat{v}_{i_t}} \Big\{\ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big) - \ell\big(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big) + \Lambda_i^{\sigma}(\hat{v}_{i_t}, v_{i_t}; t)\Big\}, \tag{4.3}$$

and

$$
\begin{aligned}
h^{-S}_{i_t} \equiv \sup_{v_{i_t},\hat{v}_{i_t}} \Big\{ & \sup_{\tau\in\mathbb{T}} \bar{J}^{\chi_i}_{i_t}(v_{i_t},\hat{v}_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i) - \sup_{\tau\in\mathbb{T}} \bar{J}^{\chi_i}_{i_t}(\hat{v}_{i_t},\hat{v}_{i_t},\tau;\epsilon^{t-1},v^{t-1}_i) \\
& + \sup_{\tau\in\mathbb{T}_t} \Lambda^{\sigma}_i(\hat{v}_{i_t},\bar{v}_i;\tau) - \sup_{\tau\in\mathbb{T}_t} \Lambda^{\sigma}_i(v_{i_t},\bar{v}_i;\tau) \Big\},
\end{aligned} \tag{4.4}
$$

where $\bar{J}^{\chi_i}_{i_t}$ is defined in (3.7). We have the following proposition.

PROPOSITION 4.1. *Suppose Assumptions 2 and 3 hold. Let* $<\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$ *be constructed in (3.11)-(3.13), respectively. Then, the market model is* $<\delta^S_{i_t},\delta^{-S}_{i_t}>$*-DIC, with* $\delta^S_{i_t}=h^S_{i_t}$ *and* $\delta^{-S}_{i_t}=h^{-S}_{i_t}+\sup_{\tau\in\mathbb{T}_t}\rho_i(\tau)$*, when* $\delta^S_{i_t}>0$ *and* $h^{-S}_{i_t}+\sup_{\tau\in\mathbb{T}_t}\rho_i(\tau)>0$*; the market model is DIC, when* $\delta^S_{i_t}\leq 0$ *and* $h^{-S}_{i_t}+\sup_{\tau\in\mathbb{T}_t}\rho_i(\tau)\leq 0$.

Proof. See Appendix G. ∎

Proposition 4.1 establishes a sufficient condition for a relaxed DIC criterion for our dynamic market model. The result can used as a worst-case analysis of the dynamic market model when there are opportunities for the owners to misreport their instrumental preferences.

5. Conclusion. This work has proposed a dynamic market model for trading data privacy when the data owners have the intrinsic preferences and the instrumental preferences over privacy protections offered by the data buyer, by designing one privacy assignment rule, two preference-dependent monetary transfer rules, and one preference-independent stopping payment rule. The preference-dependent monetary transfer rules are used to compensate for the privacy loss of the data owners, while the stopping payment rule is used by the data buyer to influence the data owners' stopping decisions. We have studied mechanism designs in a dynamic environment when data owners' instrumental preferences experience an endogenous evolution due to the data buyer's dynamic usage of their data. The data owners is allowed to leave the market after the accumulated privacy loss is beyond the privacy budget depending on their intrinsic preferences. An optimal stopping problem has been modeled for data owners' optimal leave of the market. Under a monotonicity assumption about the time evolution of the data owners' instrumental preferences, the optimal stopping rule has been transformed into a threshold-based stopping rule with a set of threshold functions. By taking into consideration the owners' coupled deviations from optimal stopping and truthful reporting, a notion of dynamic incentive compatibility based on the Bellman equation has been defined as an essential design restriction of the buyer's optimal market design.

We have provided a solid theoretic design regime for dynamic incentive-compatible market models by characterizing the preference-dependent monetary transfer rules in terms of the privacy assignment rule. The stopping payment rule has been characterized in terms of the privacy assignment rule and the threshold functions to maintain the optimality of the data owners' stopping decision and to support the guarantee of dynamic incentive compatibility. A restriction of the data buyer's ability to control the data owners' stopping decisions has been characterized in terms of the relationships between the privacy assignment rule and the threshold functions. The data buyers' optimal market design problem by determining the four decision rules with the individual rational and the dynamic incentive compatibility constraints has been relaxed to an optimization problem of determining the privacy assignment rule and the threshold functions with one modified individual rational constraint. An approximating dynamic incentive compatible market design principle has been provided to address the inevitable violation of incentive compatibility due to analytical intractability of mechanism design problems. Designing efficient algorithms to computationally solve the data buyer's optimal dynamic market model with an analysis of the violation of the theoretical implementability is our natural future work.

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Appendix A. Proof of Proposition 2.9.

The *only if* part is straightforward due to the optimality of truthful reporting. Hence, we omit it here and focus on the *if* part. The proof is constructed by establishing contradictions. Fix a profile $<\sigma,\boldsymbol{\beta},\boldsymbol{\theta},\boldsymbol{\rho}>$. Suppose that the truthful reporting strategy χ^*_i satisfies (2.15) for any period-t one-shot deviation strategy $\chi^{[t]}_i$ for any $t\in\mathbb{T}$, but it violates the DIC defined in (2.14). In other words, there exists another reporting strategy $\chi^{(1)}_i\equiv\{\chi^{(1)}_{i_t}\}$ and some instrumental preference v_{i_t} such that $U^{\chi^{(1)}_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i) > U^{\chi^*_i}_{i_t}(v_{i_t};\epsilon^{t-1},v^{t-1}_i)$. Let $\phi^{\chi^\star_i}_i$ and $\phi^{\chi^{(1)}_i}_i$ denote the optimal stopping rules given $\chi^\star_i$ and $\chi^{(1)}_i$, respectively. Suppose that at period t, $\phi^{\chi^\star_i}_i$ calls for stopping but $\phi^{\chi^{(1)}_i}_i$ calls for continuing, i.e.,

$$J^{\chi^*_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i) < \mathbb{E}^{\sigma,\chi^{(1)}_i|v_{i_t}}_t\Big[U^{\chi^{(1)}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big].$$

Equivalently, there exists some constant $\eta>0$ such that

$$J^{\chi^*_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i) + 2\eta \leq \mathbb{E}^{\sigma,\chi^{(1)}_i|v_{i_t}}_t\Big[U^{\chi^{(1)}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big]. \tag{A.1}$$

Consider another reporting strategy $\chi^{(2)}_i\equiv\{\chi^{(2)}_{i_t}\}_{t\in\mathbb{T}}$, such that $\chi^{(2)}_{i_s}=\chi^{(1)}_{i_s}$, for all $s\in\mathbb{T}_{t,t+k}$, for some $k>0$, and

$$\mathbb{E}^{\sigma,\chi^{(1)}_i|v_{i_t}}_t\Big[U^{\chi^{(1)}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big] - \eta \leq \mathbb{E}^{\sigma,\chi^{(2)}_i|v_{i_t}}_t\Big[U^{\chi^{(2)}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big]. \tag{A.2}$$

Hence, (A.1) and (A.2) yield:

$$J^{\chi^*_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i) + \eta \leq \mathbb{E}^{\sigma,\chi^{(2)}_i|v_{i_t}}_t\Big[U^{\chi^{(2)}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big]. \tag{A.3}$$

Let $\chi'_i\equiv\{\chi'_{i_t}\}_{t\in\mathbb{T}}$ denote any reporting strategy, such that $\chi'_{i_s}=\chi^{(2)}_{i_s}$, for all $s\in\mathbb{T}_{t,t+k}$ and $\chi'_{i_{s'}}$ is truthful for all $s'\in\mathbb{T}\backslash\mathbb{T}_{t,t+k}$, for some $k>0$. Hence, (A.3) tells us that a deviation using any such χ'_i is enough to obtain a non-negative profit. Next, consider a one-shot deviation reporting strategy $\chi^{[s]}_i$ for some $s\in\mathbb{T}_{t,t+k}$, for $k>0$, such that $\chi^{[s]}_{i_s}=\chi^{(2)}_{i_s}$. Then, (A.3) gives

$$J^{\chi^*_i}_{i_t}(v_{i_t},t;\epsilon^{t-1},v^{t-1}_i) < \mathbb{E}^{\sigma,\chi^{[t+k-1]}_i|v_{i_t}}_t\Big[U^{\chi^{[t+k-1]}_i}_{i_{t+1}}(\tilde{v}_{i_{t+1}};\tilde{\epsilon}^t,\tilde{v}^t_i)\Big]. \tag{A.4}$$

Let $t' = t + k$. From (2.10) in Lemma 2.7, we have, for all $v_{i_{t'}} \in V_i$,

(A.5)
$$\begin{aligned} &U^{\chi_i^{[t'-1]}}_{i_{t'-1}}(v_{i_{t'-1}}; \epsilon^{t'-2}, v_i^{t'-2}) \\ &= \max\Big\{J^{\chi_i^{[t'-1]}}_{i_{t'-1}}(v_{i_{t'-1}}, t'-1; \epsilon^{t'-2}, v_i^{t'-2}), \mathbb{E}^{\sigma,\chi_i^{[t'-1]}|v_{i_{t'-1}}}_{t'-1}\Big[U^{\chi_i^{[t'-1]}}_{i_{t'}}(\tilde{v}_{i_{t'}}; \tilde{\epsilon}^{t'-1}, \tilde{v}_i^{t'-1})\Big]\Big\}. \end{aligned}$$

With a slight abuse of notation, let $U^{\chi_i}_{i_t}()$ Since the truthful reporting strategy χ^* satisfies (2.15),

$$U^{\chi_i^{[t'-2]}}_{i_{t'-1}}(v_{i_{t'-1}}; \epsilon^{t'-2}, v_i^{t'-2}) \geq U^{\chi_i^{[t'-1]}}_{i_{t'-1}}(v_{i_{t'-1}}; \epsilon^{t'-2}, v_i^{t'-2}). \tag{A.6}$$

Hence, we have,

$$\mathbb{E}^{\sigma,\chi_i^{[t'-2]}|v_{i_t}}_{t}\Big[U^{\chi_i^{[t'-2]}}_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^{t}, \tilde{v}_i^{t})\Big] \geq \mathbb{E}^{\sigma,\chi_i^{[t'-1]}|v_{i_t}}_{t}\Big[U^{\chi_i^{[t'-1]}}_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^{t}, \tilde{v}_i^{t})\Big]. \tag{A.7}$$

From (A.4), we have

$$\mathbb{E}^{\sigma,\chi_i^{[t'-2]}|v_{i_t}}_{t}\Big[U^{\chi_i^{[t'-2]}}_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^{t}, \tilde{v}_i^{t})\Big] > J^{\chi_i^*}_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}). \tag{A.8}$$

Backward induction yields:

$$\mathbb{E}^{\sigma,\chi_i^{[t]}|v_{i_t}}_{t}\Big[U^{\chi_i^{[t]}}_{i_{t+1}}(\tilde{v}_{i_{t+1}}; \tilde{\epsilon}^{t}, \tilde{v}_i^{t})\Big] > J^{\chi_i^*}_{i_t}(v_{i_t}, t; \epsilon^{t-1}, v_i^{t-1}), \tag{A.9}$$

which contradicts the setting that χ^* satisfies (2.15). Similar procedures can be used for the other cases: *(i)* $\phi_i^{\chi_i^*}$ calls for stopping and $\phi_i^{\chi_i^{(1)}}$ calls for stopping, *(ii)* $\phi_i^{\chi_i^*}$ calls for continuing but $\phi_i^{\chi_i^{(1)}}$ calls for stopping, *(iii)* $\phi_i^{\chi_i^*}$ calls for continuing and $\phi_i^{\chi_i^{(1)}}$ calls for continuing.

□

Appendix B. Proof of Lemma 3.1.

In DIC, truthful reporting is optimal for all owners. Hence, from the envelope theorem, we have, for all $\tau \in \mathbb{T}_t$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $v_i^{t-1} \in V_i^{t-1}$,

$$\frac{\partial J_{i_t}(x, \tau; \epsilon^{t-1}, v_i^{t-1})}{\partial x}\Big|_{x=v_{i_t}} = \mathbb{E}^{\sigma; v_{i_t}}\Big[\sum_{s=t}^{\tau}\big(1 - \exp(\sigma_s(\tilde{\boldsymbol{v}}_s))\big)\frac{\partial \tilde{v}_{i_s}}{\partial x}\Big|_{x=v_{i_t}}\Big]. \tag{B.1}$$

From Kolmogorov's Existence Theorem [12], we have, for $v_{i_t} \in V_i$, any $t \in \mathbb{T}\backslash\{T\}$,

$$v_{i_{t+1}} = \inf\{v'_{i_{t+1}} \in V_i : F_{i_{t+1}}(v'_{i_{t+1}}|v_{i_t}, \epsilon^{t-1}) \geq c_{i_{t+1}}\},$$

where $c_{i_{t+1}}$ is uniformly drawn from $(0, 1)$. From Assumption 2, we have

$$\begin{aligned} \frac{\partial \tilde{v}_{i_s}}{\partial x}\Big|_{x=v_{i_t}} = \frac{\partial \tilde{\boldsymbol{v}}_s}{\partial x}\Big|_{x=v_{i_t}} &= \prod_{k=t}^{s}\frac{-\partial F_{i_t}(x|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})}{f_{i_t}(\tilde{v}_{i_k}|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})\partial x}\Big|_{x=\tilde{v}_{i_k}} \\ &= \mathcal{G}^s_t(\tilde{\boldsymbol{v}}^s_t|\sigma). \end{aligned}$$

□

Appendix C. Proof of Lemma 3.2 .

From Lemma 3.1, we have

$$\frac{\partial J_{i_t}(x, \tau; \epsilon^{t-1}, v_i^{t-1})}{\partial x}\Big|_{x=v_{i_t}} = \mathbb{E}^{\sigma; v_{i_t}}\Big[\sum_{s=t}^{\tau}\big(1 - \exp(\sigma_s(\tilde{\boldsymbol{v}}_s))\big)\mathcal{G}^s_t(\tilde{\boldsymbol{v}}^s_t|\sigma)\Big],$$

where

$$\mathcal{G}_t^s(\tilde{\boldsymbol{v}}_t^s|\sigma) \equiv \prod_{k=t}^{s} \frac{-\partial F_{i_t}(x|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})}{f_{i_t}(\tilde{v}_{i_k}|\tilde{v}_{i_{k-1}}, \tilde{\epsilon}^{k-1})\partial x}\Big|_{x=\tilde{v}_{i_k}}.$$

From Assumption 3, we have $\mathcal{G}_t^s(\boldsymbol{v}_t^s|\sigma) > 0$, for all $\boldsymbol{v}_t^s \in \boldsymbol{V}_t^s$, $s \geq t \in \mathbb{T}$. Since $\exp(x) \geq 1$, for all $x \geq 0$, then $\frac{\partial J_{i_t}(x,\tau;\epsilon^{t-1},v_i^{t-1})}{\partial x}\Big|_{x=v_{i_t}} \leq 0$. Hence, $J_{i_t}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1})$ is weakly decreasing in v_{i_t} for any $\tau \in \mathbb{T}_t$, $\epsilon^{t-1} \in \mathcal{E}^{t-1}$, $v_i^{t-1} \in V_i^{t-1}$, $t \in \mathbb{T}$, $i \in \mathbb{I}$.

□

Appendix D. Proof of Proposition 3.3.

We first prove that $G_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})$ in (2.12) is weakly decreasing. Let χ_i^* and $\chi_i^{[t]}$, respectively, denote owner i's truthful reporting strategy and period-t one-shot deviation strategy. Recall the term $G_{i_t}^{\chi_i}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1})$ given in (3.4). With a slight abuse of notation, let $\tau_i[\chi_i, v_{i_t}]$ denote the minimum time horizon, given the reporting strategy χ_i and owner i's current instrumental preference v_{i_t}, such that

$$G_{i_t}^{\chi_i}(v_{i_t}, \tau_i[\chi_i, v_{i_t}]; \epsilon^{t-1}, v_i^{t-1}) = G_{i_t}^{\chi_i}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1}).$$

From Lemma 3.2, it is straightforward to see that $G_{i_t}^{\chi_i^*}(v_{i_t}; \epsilon^{t-1}, v_i^{t-1})$ is weakly decreasing in any DIC market model.

Suppose that the indifference region contains two different intervals, $[\kappa_i^l(t), \kappa_i^r(t)]$ and $[\bar{\kappa}_i^l(t), \bar{\kappa}_i^r(t)]$ with no intersections ($\kappa_i^l(t) \neq \bar{\kappa}_i^l(t)$). With a slight abuse of notation, let $\phi_i^{\chi_i^*}[\kappa_i^l]$ denote the (optimal) threshold rule in DIC market model and let $\tau_i[\chi_i, v_{i_t}; \kappa_i^l]$ denote the term $\tau_i[\chi_i, v_{i_t}]$ define above, when the threshold function is κ_i^l. Suppose $\tau_i[\chi_i^*, v_{i_t}; \kappa_i^l] = \tau_i[\chi_i^*, v_{i_t}; \bar{\kappa}_i^l]$, for some $v_{i_t} \in V_i$, $t \in \mathbb{T}$. Assume without loss of generality $\kappa_i^l(t) > \bar{\kappa}_i^l(t)$, for some $t \in \mathbb{T}$. Then,

$$\begin{aligned}
P(\tau_i[\chi_i^*, v_{i_t}; \kappa_i^l] = t) =& P(v_{i_t} \geq \kappa_i^l(t), \tau_i[\chi_i^*, v_{i_{t-1}}; \kappa_i^l] > t-1) \\
=& \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{v_{i_t} \geq \kappa_i^l(t)\}}\Big]\mathbf{1}_{\{v_{i_{t-1}} < \kappa_i^l(t-1)\}}\Big].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&P(\tau_i[\chi_i^*, v_{i_t}; \bar{\kappa}_i^l] = t) - P(\tau_i[\chi_i^*, v_{i_t}; \kappa_i^l] = t) \\
=& \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{v_{i_t} \geq \bar{\kappa}_i^l(t)\}}\Big]\mathbf{1}_{\{v_{i_{t-1}} < \bar{\kappa}_i^l(t-1)\}}\Big] - \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{v_{i_t} \geq \kappa_i^l(t)\}}\Big]\mathbf{1}_{\{v_{i_{t-1}} < \kappa_i^l(t-1)\}}\Big] \\
=& \mathbb{E}\Big[\mathbb{E}\Big[\mathbf{1}_{\{\bar{\kappa}_i^l(t) \leq v_{i_t} \leq \kappa_i^l(t)\}}\Big]\mathbf{1}_{\{v_{i_{t-1}} < \bar{\kappa}_i^l(t-1)\}}\Big].
\end{aligned} \tag{D.1}$$

Due to Assumption 2 and the setting $\kappa_i^l(t) > \bar{\kappa}_i^l(t)$, the right-hand side of (D.1) is strictly positive. However, since $\tau_i[\chi_i^*, v_{i_t}; \kappa_i^l] = \tau_i[\chi_i^*, v_{i_t}; \bar{\kappa}_i^l]$, $P(\tau_i[\chi_i^*, v_{i_t}; \bar{\kappa}_i^l] = t) - P(\tau_i[\chi_i^*, v_{i_t}; \kappa_i^l] = t) = 0$, which gives a contradiction. Therefore, the threshold function is unique.

□

Appendix E. Proof of Theorem 3.4 .

We first prove that, given any Λ_i^σ that satisfies the conditions (3.14) and (3.15), the market model with β_{i_t}, θ_{i_t}, and ρ_i constructed in (3.11)-(3.13), respectively, is DIC. After that, we prove that the formulation of Λ_i^σ in (3.10) is valid.

We fix other owners' period-t instrumental preference as $\boldsymbol{v}_{-i_t}$, for any $t \in \mathbb{T}$. Let $v_{i_t} \in V_i$ and $\hat{v}_{i_t} \in V_i$ be any two instrumental preferences at any period $t \in \mathbb{T}$. The formulation of θ_{i_t} in (3.12) yields

$$\begin{aligned}
&\theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \\
=& \Lambda_i^\sigma(\hat{v}_{i_t}, \bar{v}_i; t) + \ell(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})) - \Lambda_i^\sigma(v_{i_t}, \bar{v}_i; t) - \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) \\
=& \Lambda_i^\sigma(\hat{v}_{i_t}, \bar{v}_i; t) - \Lambda_i^\sigma(v_{i_t}, \bar{v}_i; t) - \big(- \ell(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})) + \ell(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}))\big) \\
&- \big(\ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) - \ell(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}))\big)
\end{aligned} \tag{E.1}$$

From the definition of $d^S_{i_t}$ in (3.8) and condition (3.14), the right-hand side (RHS) of (E.1) becomes:

$$\begin{aligned}\theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) =& \Lambda^\sigma_i(\hat{v}_{i_t}, \bar{v}_i; t) - \Lambda^\sigma_i(v_{i_t}, \bar{v}_i; t) - d^S_{i_t}(\hat{v}_{i_t}, v_{i_t}) \\ & - \big(\ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) - \ell(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}))\big) \\ \leq & - \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) + \ell(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})).\end{aligned} \tag{E.2}$$

Rearranging (E.2) gives

$$- \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) + \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \geq -\ell(v_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})) + \theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}). \tag{E.3}$$

Next, we apply similar procedures to β_{i_t}. From the formulation of β_{i_t} in (3.11), we have

$$\begin{aligned}&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \\ &= \sup_{\tau\in\mathbb{T}_t} \Lambda^\sigma_i(\hat{v}_{i_t}, \bar{v}_i; \tau) - \sup_{\tau\in\mathbb{T}_t} \Lambda^\sigma_i(v_{i_t}, \bar{v}_i; \tau) + \ell(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})) - \ell(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})) \\ &+ \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sup_{\tau\in\mathbb{T}_{t+1}} \Lambda^\sigma_i(\tilde{v}_{i_{t+1}}, \bar{v}_i; \tau)\Big] - \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sup_{\tau\in\mathbb{T}_{t+1}} \Lambda^\sigma_i(\tilde{v}_{i_{t+1}}, \bar{v}_i; \tau)\Big].\end{aligned} \tag{E.4}$$

We apply the formulations of β_{i_t} and θ_{i_t}, respectively, in (3.11) and (3.12) to (E.4) and obtain the following, for any $\tau \in \mathbb{T}_t$:

$$\begin{aligned}&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) = \sup_{\tau\in\mathbb{T}_t} \Lambda^\sigma_i(\hat{v}_{i_t}, \bar{v}_i; \tau) - \sup_{\tau\in\mathbb{T}_t} \Lambda^\sigma_i(v_{i_t}, \bar{v}_i; \tau) \\ &+ \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{T} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{T-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T)\Big] \\ &- \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \Lambda^\sigma_i(\tilde{v}_{i_\tau}, \bar{v}_i; \tau)\Big].\end{aligned} \tag{E.5}$$

We apply the condition (3.15) to (E.5) and obtain:

$$\begin{aligned}&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \leq \inf_{\tau\in\mathbb{T}_t} \Big\{d^{-S}_{i_t}(\hat{v}_{i_t}, v_{i_t}; \tau)\Big\} - \sup_{\tau\in\mathbb{T}_t} \rho_i(\tau) \\ &+ \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{T} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{T-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T)\Big] \\ &- \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \Lambda^\sigma_i(\tilde{v}_{i_\tau}, \bar{v}_i; \tau)\Big].\end{aligned} \tag{E.6}$$

From the definition of $d^{-S}_{i_t}$ and $\bar{J}^{\chi_i}_{i_t}$, respectively, in (3.9) and (3.7), we have

$$\begin{aligned}&\inf_{\tau\in\mathbb{T}_t} \Big\{d^{-S}_{i_t}(\hat{v}_{i_t}, v_{i_t}; \tau)\Big\} - \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_\tau}(\tilde{\boldsymbol{v}}_\tau)\Big] \\ &= \inf_{\tau\in\mathbb{T}_t} \Big\{\mathbb{E}^{\sigma,\chi_i;\mu_i|\hat{v}_{i_t}}_t\Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T)\Big] \\ &\quad - \mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t},\hat{v}_{i_t}}_t\Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T)\Big] \\ &\quad - \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau} -\ell(\tilde{v}_{i_t}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \Lambda^\sigma_i(\tilde{v}_{i_\tau}, \bar{v}_i; \tau)\Big]\Big\} \\ &\leq \inf_{\tau\in\mathbb{T}_t} \Big\{ - \mathbb{E}^{\sigma,\chi_i;\mu_i|v_{i_t},\hat{v}_{i_t}}_t\Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \boldsymbol{\sigma}_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T)\Big]\Big\}.\end{aligned}$$

Hence, (E.6) becomes:

$$\begin{aligned}\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \leq & - \sup_{\tau\in\mathbb{T}_t} \rho_i(\tau) \\ & + \inf_{\tau\in\mathbb{T}_t} \Big\{ - \mathbb{E}_t^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T) \Big] \Big\} \\ & + \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{T} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{T-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T) \Big].\end{aligned} \tag{E.7}$$

From the monotonicity of J_{i_t} in Lemma 3.2 and the formulation of ρ_i in (3.13), we have, for any $\tau' \in \mathbb{T}_t$,

$$\begin{aligned}\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \leq & - \sup_{\tau\in\mathbb{T}_t} \rho_i(\tau) \\ & + \inf_{\tau\in\mathbb{T}_t} \Big\{ - \mathbb{E}_t^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_T}(\tilde{\boldsymbol{v}}_T) \Big] \Big\} \\ & + \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'}) + \rho_i(\tau') \Big] \\ \leq & \sup_{\tau'\in\mathbb{T}_t} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'}) + \rho_i(\tau') \Big] \Big\} \\ & + \inf_{\tau\in\mathbb{T}_t} \Big\{ - \mathbb{E}_t^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_\tau}(\tilde{\boldsymbol{v}}_\tau) + \rho_i(\tau) \Big] \Big\} \\ = & \sup_{\tau'\in\mathbb{T}_t} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'}) + \rho_i(\tau') \Big] \Big\} \\ & - \sup_{\tau\in\mathbb{T}_t} \Big\{ \mathbb{E}_t^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_{i_s}(\tilde{v}_{i_s}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_\tau}(\tilde{\boldsymbol{v}}_\tau) + \rho_i(\tau) \Big] \Big\}.\end{aligned} \tag{E.8}$$

Hence, (E.3) and (E.8) show that the market model with β_{i_t}, θ_{i_t}, and ρ_i constructed in (3.11)-(3.13), respectively, is DIC.

Next, we prove that the formulation of Λ_i^σ in (3.10) is valid. Substituting β_{i_t}, θ_{i_t}, and ρ_i constructed in (3.11)-(3.13), respectively, with Λ_i^σ given in (3.10), yields:

$$\begin{aligned}J_{i_t}(v_{i_t}, \tau; \epsilon^{t-1}, v_i^{t-1}) = & \Lambda_i^\sigma(v_{i_t}, v'_{i_t}; \tau) \\ = & \int_{\bar{v}_i}^{v_{i_t}} \mathbb{E}^{\sigma;\mu_i|x} \Big[\sum_{s=t}^{\tau} \big(1 - \exp(\sigma_s(\tilde{\boldsymbol{v}}_s))\big) \mathcal{G}_t^s(\tilde{\boldsymbol{v}}_t^s|\sigma) \Big] dx.\end{aligned} \tag{E.9}$$

From (3.1) of Lemma (3.1), we can see that (E.9) satisfy the envelope condition.

□

Appendix F. Proof of Proposition 3.7 .

We divide the proof into to parts: *(i)* $v_{i_t} \geq \kappa_i^l(t)$ and *(ii)* $v_{i_t} \leq \kappa_i^l(t)$. Let $\boldsymbol{v}_{-i_t} \in \boldsymbol{V}_{-i}$ denote the period-t instrumental preference of owners other than owner i, for any $t \in \mathbb{T}$.

(i) $v_{i_t} \geq \kappa^l_i(t)$**.** Let $\hat{v}_{i_t} \geq v_{i_t} \geq \kappa^l_i(t)$. For the distance $d^S_{i_t}(v_{i_t}, \hat{v}_{i_t})$ we have

$$\begin{aligned}
d^S_{i_t}(v_{i_t}, \hat{v}_{i_t}) =& -\ell\big(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big) + \ell\big(\hat{v}_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big)\\
=& -\ell\big(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big) + \ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big)\\
&-\ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big) + \ell\big(\hat{v}_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big)\\
=& -\ell\big(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big) + \ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big)\\
&+ -\ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big) + \theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) + \ell\big(\hat{v}_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big) - \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t})\\
&+ \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) - \theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\\
(\text{DIC } \rightarrow) \geq& -\ell\big(v_{i_t}, \sigma_t(v_{i_t}, \boldsymbol{v}_{-i_t})\big) + \ell\big(\hat{v}_{i_t}, \sigma_t(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\big)\\
&+ \theta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) - \theta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t})\\
=& \Lambda^\sigma_i(v_{i_t}, \bar{v}_i; t) - \Lambda^\sigma_i(\hat{v}_{i_t}, \bar{v}_i; t).
\end{aligned}$$

Hence, the condition (3.17) is satisfied.

(ii) $v \leq \kappa^l_i(t)$**.** Define,

$$z^m_i(v_{i_t}) \equiv \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}.$$

From the definition of Λ^σ_i in (3.10), we have

$$\begin{aligned}
&\sup_{\tau \in \mathbb{T}_{t+1}} \Lambda^\sigma_i(\hat{v}_{i_t}, \bar{v}; \tau) - \sup_{\tau \in \mathbb{T}_t} \Lambda^\sigma_i(v_{i_t}, \bar{v}; \tau)\\
=& z^m_i(\hat{v}_{i_t}) - z^m_i(v_{i_t})\\
=& \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
&- \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
=& \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
&- \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
&+ \beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t})
\end{aligned} \tag{F.1}$$

From the optimality of truthful reporting in DIC market model,

$$\begin{aligned}
&\text{RHS of (F.1)}\\
\leq& \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
&- \sup_{\tau \in \mathbb{T}_{t+1}} \Big\{ \mathbb{E}^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}} \Big[\sum_{s=t}^{\tau} -\ell_s(\tilde{v}_{i_s}, \sigma_{i_s}(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s)) + \theta(\tilde{\boldsymbol{v}}_\tau)) + \rho_i(\tau) \Big] \Big\}\\
=& \sup_{\tau \in \mathbb{T}_t} \Big\{ \bar{J}^{\chi_i}_{i_t}(\hat{v}_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) \Big\} - \sup_{\tau \in \mathbb{T}_t} \Big\{ \bar{J}^{\chi_i}_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) \Big\}\\
\leq& \sup_{\tau \in \mathbb{T}_t} \Big\{ d^{-S}_{i_t}(v_{i_t}, \hat{v}_{i_t}) \Big\}.
\end{aligned}$$

Hence, the condition (3.18) is satisfied.

□

Appendix G. Proof of Proposition 4.1 .

Fix $\boldsymbol{v}_{-i_t}$ as the instrumental preferences of owners other than owner i. Let $v_{i_t},\, \hat{v}_{i_t} \in V_i$. From the formulation of β_{i_t} in (3.11), we have, for any two $\tau', \tau'' \in \mathbb{T}_t$,

$$\begin{aligned}
&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) = \sup_{\tau\in\mathbb{T}_t} \Lambda^{\sigma}_{i}(\hat{v}_{i_t}, \bar{v}_i; \tau) - \sup_{\tau\in\mathbb{T}_t} \Lambda^{\sigma}_{i}(v_{i_t}, \bar{v}_i; \tau)\\
&+ \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'})\Big]\\
&- \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau''} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau''-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \Lambda^{\sigma}_{i}(\tilde{v}_{i_{\tau''}}, \bar{v}_i; \tau'')\Big].
\end{aligned} \tag{G.1}$$

From the definition of $h^{-S}_{i_t}$ in (4.4), (G.1) becomes

$$\begin{aligned}
&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t})\\
\le& h^{-S}_{i_t} - \sup_{\tau\in\mathbb{T}} \bar{J}^{\chi_i}_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) + \sup_{\tau\in\mathbb{T}} \bar{J}^{\chi_i}_{i_t}(\hat{v}_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i)\\
&+ \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'})\Big]\\
&- \mathbb{E}^{\sigma;\mu_i|\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau''} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau''-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \Lambda^{\sigma}_{i}(\tilde{v}_{i_{\tau''}}, \bar{v}_i; \tau'')\Big].
\end{aligned} \tag{G.2}$$

Since $\Lambda^{\sigma}_{i}(v_{i_t}, \bar{v}_i; \tau) \geq \Lambda^{\sigma}_{i}(v_{i_t}, \bar{v}_i; t)$, for any $v_{i_t} \in V_i$, $t \in \mathbb{T}$, $\tau \in \mathbb{T}_t$, we have $\Lambda^{\sigma}_{i}(v_{i_t}, \bar{v}_i; \tau)\, geq - \ell_t(v_{i_t}, \sigma_t(\boldsymbol{v}_t)) + \theta_{i_t}(\boldsymbol{v}_t))$. Then, (G.2) becomes

$$\begin{aligned}
&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t}) \le - \sup_{\tau\in\mathbb{T}} \bar{J}^{\chi_i}_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) + \sup_{\tau\in\mathbb{T}_t}\Big\{\rho_t(\tau)\Big\}\\
&+ h^{-S}_{i_t} + \mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'})\Big].
\end{aligned} \tag{G.3}$$

From the formulation of ρ_i in (3.13), we can find the upper bound of (G.3) as follows:

$$\begin{aligned}
&\beta_{i_t}(\hat{v}_{i_t}, \boldsymbol{v}_{-i_t}) - \beta_{i_t}(v_{i_t}, \boldsymbol{v}_{-i_t})\\
\le& h^{-S}_{i_t} + \sup_{\tau\in\mathbb{T}_t}\Big\{\rho_t(\tau)\Big\}\\
&+ \sup_{\tau\in\mathbb{T}_t}\Big\{\mathbb{E}^{\sigma;\mu_i|v_{i_t}}\Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'}) + \rho_i(\tau)\Big]\Big\}\\
&\sup_{\tau\in\mathbb{T}_t}\Big\{\mathbb{E}^{\sigma;\mu_i|v_{i_t},\hat{v}_{i_t}}\Big[\sum_{s=t}^{\tau'} -\ell(\tilde{v}_{i_t}, \sigma_s(\tilde{\boldsymbol{v}}_s)) + \sum_{s=t+1}^{\tau'-1} \beta_{i_s}(\tilde{\boldsymbol{v}}_s) + \theta_{i_{\tau'}}(\tilde{\boldsymbol{v}}_{\tau'}) + \rho_i(\tau)\Big]\Big\},
\end{aligned} \tag{G.4}$$

which implies that

$$\sup_{\tau\in\mathbb{T}_t} J^{\chi_i}_{i_t}(v_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i) + h^{-S}_{i_t} + \sup_{\tau\in\mathbb{T}_t}\Big\{\rho_t(\tau)\Big\} \geq \sup_{\tau\in\mathbb{T}_t} J^{\chi_i}_{i_t}(v_{i_t}, \hat{v}_{i_t}, \tau; \epsilon^{t-1}, v^{t-1}_i).$$

Then, it is straightforward to see that the market model is $h^{-S}_{i_t} + \sup_{\tau\in\mathbb{T}_t}\Big\{\rho_t(\tau)\Big\}$-DIC. Similar procedures can be applied to prove the case when the optimal stopping calls for stopping.

□