Chapter 1: Single Variable Functions

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1 Single Variable Functions: Continuity

1.1 Basic concepts: Domain, Range, and Graph of a Function

1.1.1 Previous concepts

Real line: types of numbers

- Natural numbers ($\mathbb{N}$): $0, 1, 2, 3, 4, 5, \ldots$
- Integers ($\mathbb{Z}$): $\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$
- Rational numbers ($\mathbb{Q}$): $\frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$
- Irrational numbers ($\mathbb{I}$): $\sqrt{2} \approx 1.41, \pi \approx 3.14, e \approx 2.72 \ldots$
- Real numbers ($\mathbb{R}$): $\mathbb{Q} \cup \mathbb{I}$

Intervals of real numbers:

- Open interval: $(a, b) := \{x \in \mathbb{R} : a < x < b\}$
- Closed interval: $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$
- Half open interval: $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$
- Unbounded interval: $(a, +\infty) := \{x \in \mathbb{R} : a < x < +\infty\}$

Theorem 1. *No number $r \in \mathbb{Q}$ has square equal to 2; i.e. $\nexists r \in \mathbb{Q} : r^2 = 2$, i.e., $\sqrt{2} \notin \mathbb{Q}$.*

Proof. To prove that every r has $r^2 \neq 2$ we can use the actual definition of rational numbers, that is, we know we can express $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Thus, to prove the theorem we could show that $p^2 \neq 2q^2$.

- Assumption: p and q have no common factors, since we could've canceled out beforehand.

Case 1: p is odd. Then, p^2 is also odd, but $2q^2$ is **not** odd. Then, $p^2 \neq 2q^2$.

Case 2: p is even. Since p and q have **no common factors**, q is odd. Then p^2 is divisible by 4 while $2q^2$ is not. Therefore, $p^2 \neq 2q^2$.

Since $p^2 \neq 2q^2 \forall p \in \mathbb{Z} \Rightarrow \nexists r = p/q \ : \ r^2 = 2$ □

Definition 1. Absolute value of a number: three equivalent definitions

1. $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2. $$|x| = +\sqrt{x^2}$$

3. $$|x| = \max\{x, -x\}$$

Using the absolute value, we can define subsets of real numbers:

- Open interval with **center** a and **radius** r: $\{x \in \mathbb{R} : |x-a| < r\} = (a-r, a+r)$
- Closed interval with **center** a and **radius** r: $\{x \in \mathbb{R} : |x-a| \leq r\} = [a-r, a+r]$
- Complementaries:

$$\{x \in \mathbb{R} : |x-a| > r\} = \mathbb{R} \setminus \{x \in \mathbb{R} : |x-a| \leq r\} = (-\infty, a-r) \cup (a+r, +\infty)$$

$$\{x \in \mathbb{R} : |x-a| \geq r\} = \mathbb{R} \setminus \{x \in \mathbb{R} : |x-a| < r\} = (-\infty, a-r] \cup [a+r, +\infty)$$

Example 1. Determine the following sets expressing them in interval or union of intervals form:

1. $\{x \in \mathbb{R} : |x-3| > 2\}$; **Solution:** $(-\infty, 1) \cup (5, +\infty)$
2. $\{x \in \mathbb{R} : |x+1| \geq 5\}$; **Solution:** $(-\infty, -6] \cup [4, +\infty)$
3. $\{x \in \mathbb{R} : |x-2| \leq 8\}$; **Solution:** $[-6, 10]$

Theorem 2. ***Triangle Inequality:*** *For all $x, y \in \mathbb{R}$ we have $|x+y| \leq |x| + |y|$.*

Proof. By transitivity ($x < y < z \Rightarrow x < z$) and translation ($x < y \Rightarrow x + z < y + z$) properties, we have that:

$$x + y \leq |x| + y \leq |x| + |y|$$

$$-x - y \leq |x| - y \leq |x| + |y|$$

Since

$$|x+y| \begin{cases} x+y & \text{if } x+y \geq 0 \\ -x-y & \text{if } x+y \leq 0 \end{cases}$$

And both $x+y$ and $-x-y$ are less than or equal to $|x|+|y|$, we infer that $|x+y| \leq |x|+|y|$. $\square$

Definition 2. The **neighborhood** of a point a is an (open) interval with center a and a radius ε (small close to 0).

$$\mathcal{E}(a,\varepsilon)\{x \in \mathbb{R} : |x-a| < \varepsilon\} = (a-\varepsilon, a+\varepsilon)$$

1.1.2 Cartesian Coordinates: Origin, Axes, Points in the Plane

- Plane $\mathbb{R}$ is the set of ordered pairs of real numbers (x,y) where $x,y \in \mathbb{R}$
- Horizontal axis (abscissa) is formed by the points: $X = \{(x,0) : x \in \mathbb{R}\}$
- Vertical axis (ordinate) is formed by the points: $Y = \{(0,y) : y \in \mathbb{R}\}$

1.1.3 Lines in the plane

Definition 3. A **line** is every set in the plane of the form $\{(x,y) \in \mathbb{R}^2 : Ax + By = c\}$ where A, B, c are known real numbers, with $A, B \neq 0$.

Example 2. Represent the following lines in the plane:

1. $x + y = 1$
2. $y = 2x$
3. $x = 8$
4. $y = -2$

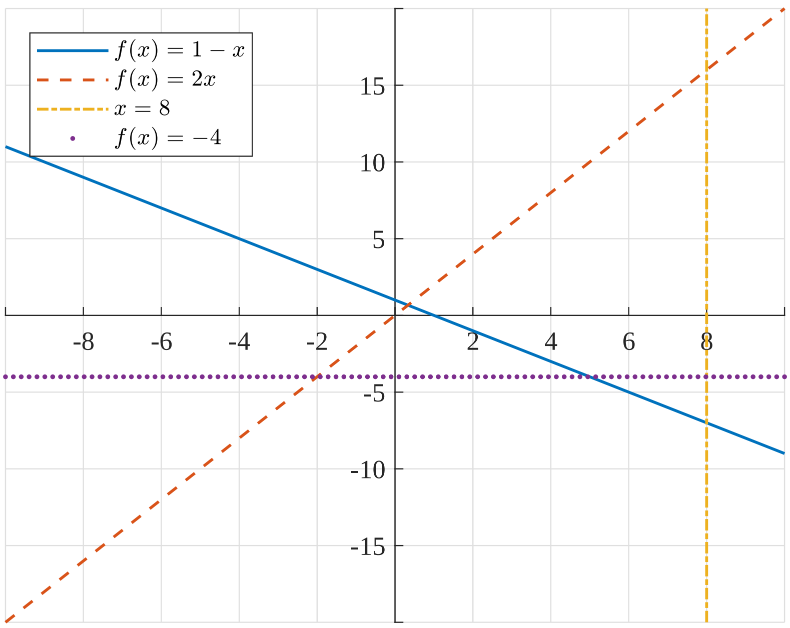


Figure 1: Representing Lines

1.1.4 Slope of a line: Meaning

The main characteristic which distinguishes one line from another is its steepness, the steepness is also called **slope**. If a line is upward sloped, graphically, the line goes up (the function is **increasing**), if the line is downward sloped, it goes down (the function is **decreasing**).

Definition 4. Let the line be $f(x) = a + mx$ where $a, b \in \mathbb{R}$. Let $(x_0, y_0), (x_1, y_1)$ be two arbitrary points on a line ℓ. The *ratio*

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta f(x)}{\Delta x}$$

is called the **slope** of line ℓ. Then, the slope m is defined as the *ratio* of the amount of units that $f(x)$ increases by the amount of units that x increases. For lines, the slope is independent of the two points chosen on ℓ.

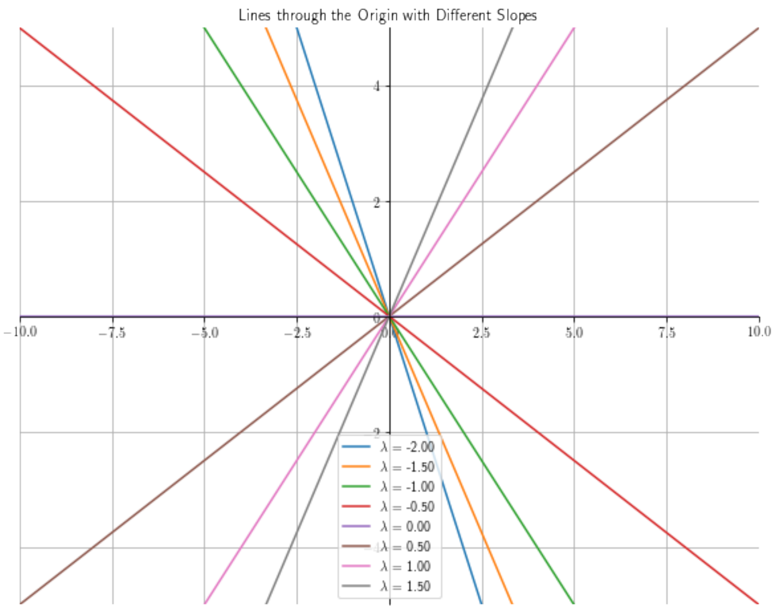


Figure 2: Differences in Slope

Independence of the Points
The slope of a linear function is independent of the points chosen

1.1.5 Equation of a Line

1. Given a point (x_1, y_1) and the slope m, the equation of a line is given by:

$$y - y_1 = m(x - x_1)$$

$$y = \underbrace{(y_1 - mx_1)}_{a} + mx = a + mx$$

2. Given two points $(x_1, y_1), (x_2, y_2)$

   1. First we compute the slope as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

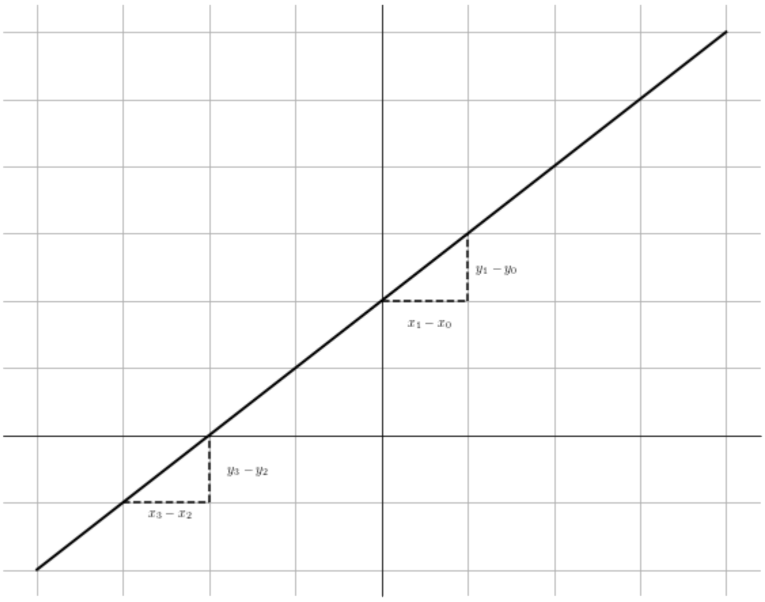


Figure 3: Slope Independence for Linear Functions

2. We choose any of the two known points and apply previous formula.
3. If the line turns to be vertical, the denominator will be 0 but, in this case, we have $x_2 = x_1$ and the equation for a vertical line is just $x = x_1$.

Example 3. Compute the equation of the following lines and draw them.

1. The line that goes through point $(2, -1)$ with slope $m = 5$

$$y - (-1) = 5(x - 2) \Rightarrow y = 5x - 11$$

2. The line that goes through points $(2, -1)$ and $(-1, 2)$

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - (-1)}{-1 - 2} = \frac{3}{-3} = -1$$

Applying $y - y_1 = m(x - x_1) \Rightarrow y - (-1) = (-1)(x - 2) \Rightarrow y = -x + 1$

1.1.6 Euclidean Vector

A Euclidean (geometric or spatial) vector is a geometric object that has **magnitude** and **direction**.

Two points in a plane define a unique line. If we subtract coordinate by coordinate:

$$\vec{\mathbf{u}} = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

This latter vector indicates the *direction* the line follows. Thus, the equation of a line can be *parametrically* written as:

$$\{(x, y) \in \mathbb{R}^2 : (x, y) = (x_1, y_1) + \lambda\vec{\mathbf{u}},\ \lambda \in \mathbb{R}\}$$

Example 4. Euclidean vector and parametric equation of the line that goes through the points $(2,5)$ and $(0,3)$.

1. The Euclidean vector is given by $\vec{\mathbf{u}} = (2,5) - (0,3) = (2,2)$
2. The parametric equation is given by $(x,y) = (2,5) + \lambda(2,2)$

To obtain the standard Cartesian equation of the line from the parametric equation, we just need to solve for λ in one of the coordinates and substitute in the other. For this example, the Cartesian equation is given by $y = x + 3$

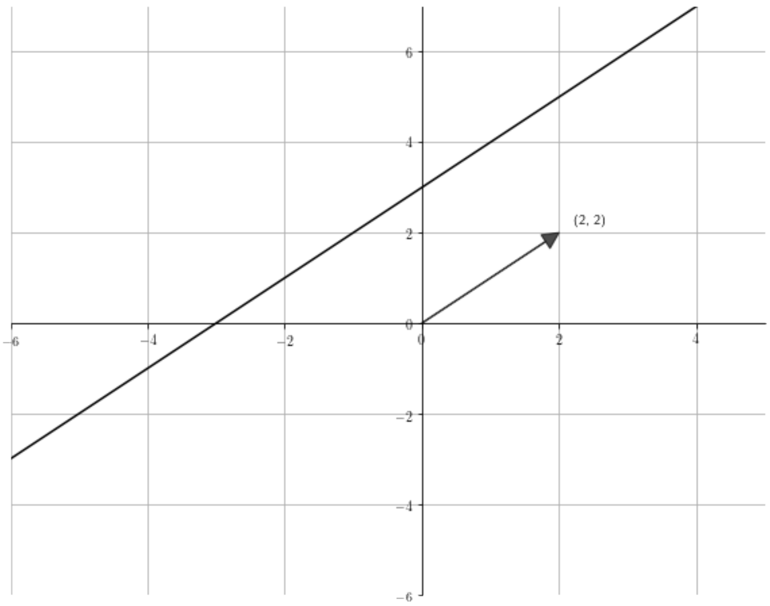


Figure 4: Representation of the Line and its Euclidean Vector

Note that if we have a line given by $y = a + mx$, we know it will go through the points $(0,a)$ and $(1, m+n)$ so we can easily obtain both the Euclidean vector $\vec{\mathbf{u}}$ and the parametric form of this equation.

1.1.7 A Snippet of Trigonometry

Given an angle α, the main trigonometric functions are given by:

$$\sin(\alpha) = \frac{BA}{BO} \; \cos(\alpha) = \frac{AO}{BO} \; \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{BA}{AO}$$

Where BA, BO, AO, BA are the distances represented in Figure 5.

Angles are measured in *radians*. A circle has 380° which corresponds to 2π radians.

As a direct consequence of **Pythagorean Theorem** we get the *fundamental relationship* between sinus and cosinus. Exercise 3 asks you to prove it.

$$\sin^2(\alpha) + cos^2(\alpha) = 1$$

Let's plot the sin and cos functions in Figure 6.

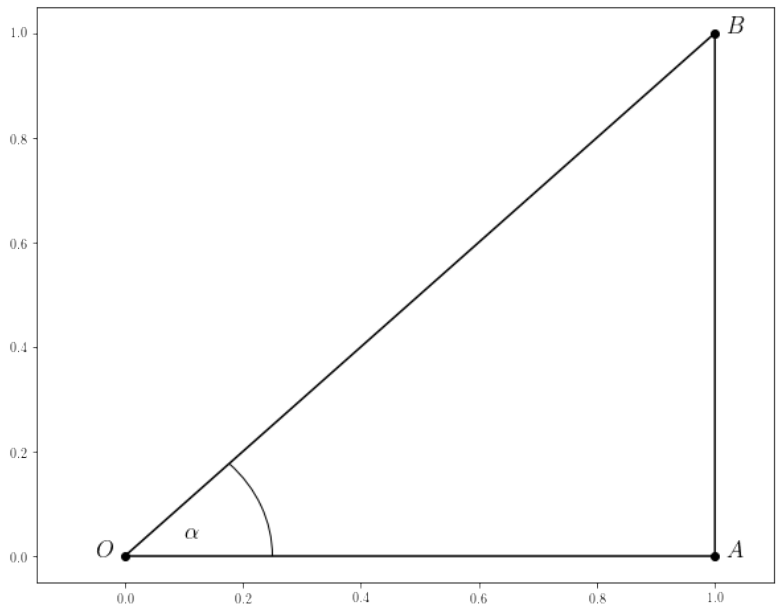


Figure 5: Trigonometric Relationships

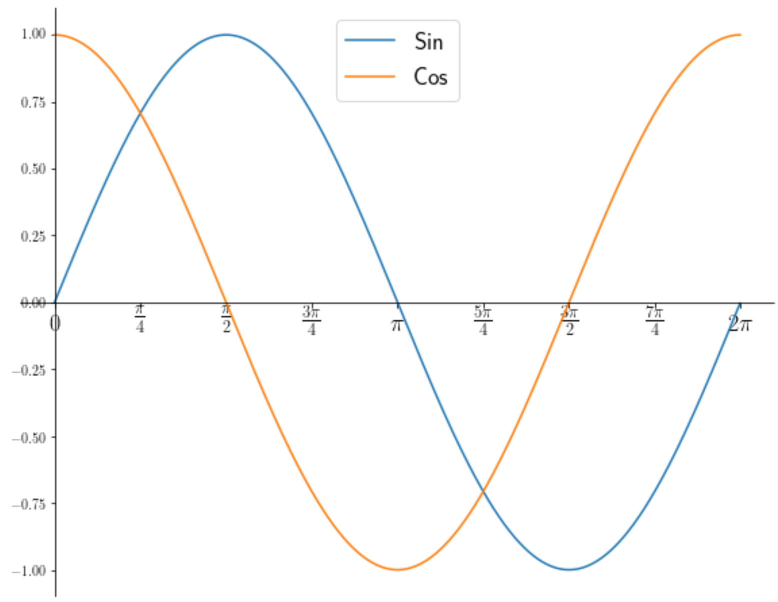


Figure 6: sin and cos functions

1.1.8 Functions of a Real Variable

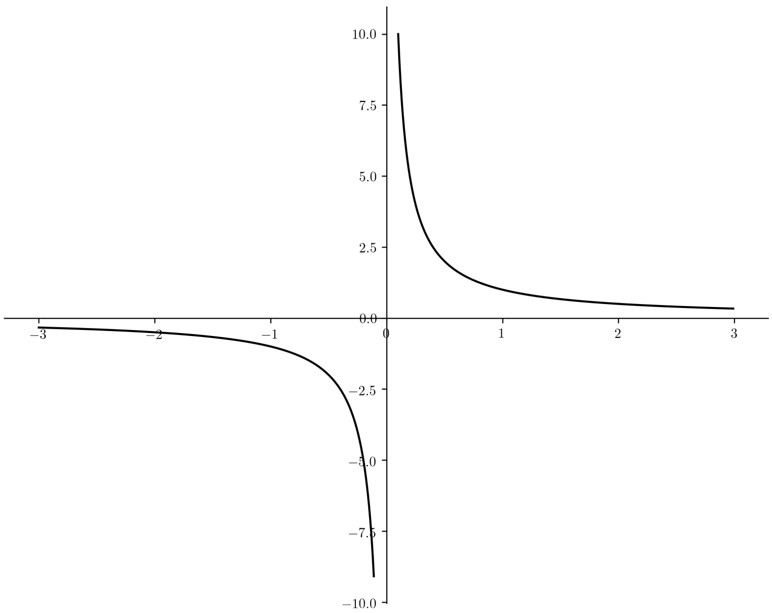
Definition 5. A **function** f is a rule that assigns to each number x in a set $\mathcal{D} \subseteq \mathbb{R}$ a number in $\mathbb{R}$ denoted by $f(x)$. The number $y = f(x)$ is called the **image** of x under f. The pair formed by each real number x and its image $f(x)$ determines a point $(x, f(x))$ in the plane $\mathbb{R}$. The set of all these pairs is called the **graph of a function**.

$$G_f = \{(x, f(x)) \in \mathbb{R}^2 \; ; \; x \in \mathcal{D}\}$$

Where $\mathcal{D}$ is the set where the function is defined, also called **domain**.

Example 5. A basic example can be the **non-vertical** lines. In this case, the equation of the line can be written as $y = f(x) = a + mx$. The domain of this function is $\mathcal{D} = \mathbb{R}$

Example 6. The domain of the function $f(x) = \frac{1}{x}$ is no longer $\mathbb{R}$ since it is not defined for $x = 0$. Therefore, the domain, will be $\mathcal{D} = \{x \in \mathbb{R} \; : \; x \neq 0\} = \mathbb{R} - \{0\}$. And its graph is



Functions can also have restricted domain depending on the application in which the function arises.

Example: Suppose we have a firm with a cost function that is linear, such as $C(q) = a + bq$ where q is the quantity produced. Which is the domain of this function?

The domain of the function will be $\mathcal{D} = \{q \in \mathbb{R}_+\} = \{q \in \mathbb{R} \; : \; q \geq 0\}$

Example: Compute the domain of the following function:

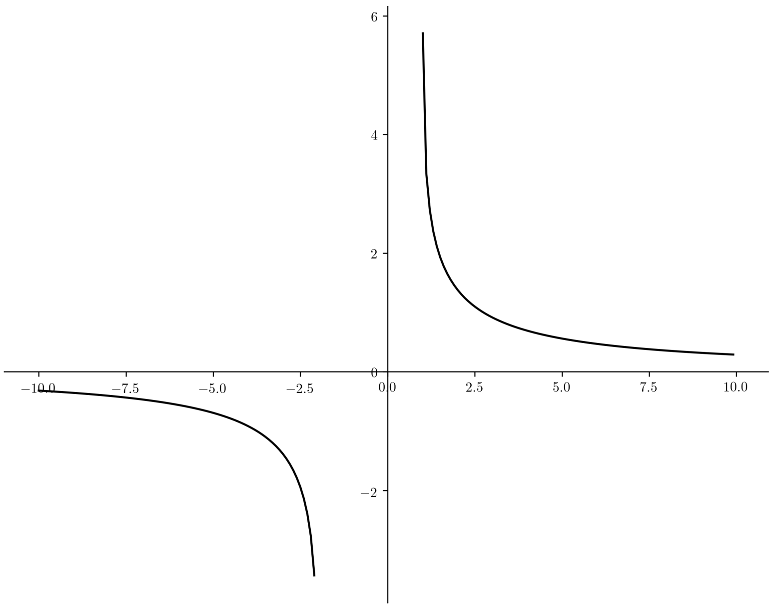
$$f(x) = \log\left(\frac{x+2}{x-1}\right)$$

- The denominator $(x - 1)$ needs to be different from 0, thus, $x \neq 1$.
- Logs can only be computed for positive numbers, thus, both numerator and denominator must have equal signs.

- Both positive $\rightarrow x+2>0\,; x-1>0 \Rightarrow x>1$
- Both negative $\rightarrow x+2<0\,; x-1<0 \Rightarrow x<-2$ Thus, the domain of the function is given by

$$\mathcal{D}=(-\infty,-2)\cup(1,+\infty)$$

Note that neither -2 nor 1 can enter in the domain.
Following figure shows some points of the domain.



1.2 Inverse Function

1.2.1 Composite functions

Definition 6. Given two functions $f,g : \mathbb{R}\rightarrow\mathbb{R}$, the **composite** function of f with g denoted by $(f\circ g)(x)$ is a new function defined by:

$$(f\circ g)(x)=f(g(x))$$

That is, a new number obtained by first applying g to x and f to the result of that.

Example 7. Take the profit function of a firm Π which is generally a function of the output. Suppose that the profit of a firm is given by

$$\Pi(y)=y^2-y+2$$

And the production function of the firm is $y=f(k)=k^{\alpha}$. Where y is output, and k is capital. We can then write the profit function as a function of capital by $\mathcal{P}(k)=(\Pi\circ f)=\Pi(f(k))$. What will be this function?

1.2.2 Inverse Functions

Definition 7. Two functions f, g are said to be **inverse** functions if, for every real number x of their domain:

$$(f \circ g)(x) = (g \circ f)(x) = x$$

The inverse of a function f (g in the definition) is typically represented by $f^{-1}(x)$

Example 8. Compute the inverse function of $f(x) = 3x - 1$.

1. First, write it as $y = f(x) = 3x - 1$.
2. Solve for x:
$$x = \frac{y+1}{3}$$
3. Substitute y by x, giving: $f^{-1}(x) = \frac{x+1}{3}$

We can plot both functions.

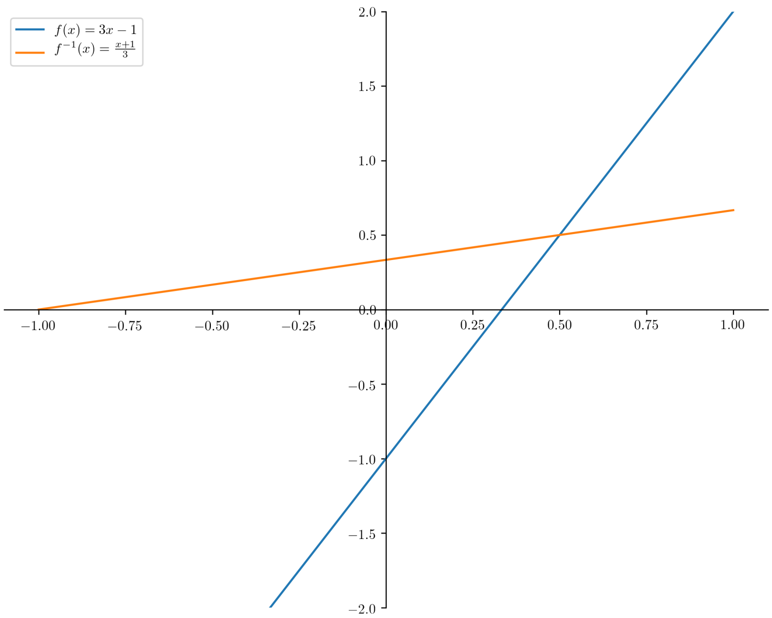


Figure 7: A Function and its Inverse

2 Limits

2.1 Intuition. Sequences of Real Numbers

Take the natural numbers ($x \in \mathbb{N}$). A **sequence** of real numbers is an assignment of a real number to each natural number. We typically write sequences as $\{x_1, x_2, \ldots, x_n, \ldots\}$. Some examples of sequences are:

1. $\{1, 2, 3, 4, \ldots\}$
2. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Note that for each $n \in \mathbb{N}$, there is a well-defined $n-$th number x_n for each sequence. For the first one, the $n-$th element will be simply n, while for the second one, it will be $1/n$.

What is the main difference between the two sequences as we increase the number of elements of each sequence? Note that the first sequence *grows* without a bound, while the second one seems to *approach* 0.

We can *"extend"* this concept to functions. Consider a function $f(x)$ defined in the neighborhood of a point a, that is in the neighborhood $\mathcal{E}(a, \delta) = (a - \delta, a + \delta)$, the idea that the *limit* of $f(x)$ when x gets closer to point a is equal to some number L means that the values of the function are very close to L (as close as we wish). That is represented by:

$$\lim_{x \to a} f(x) = L$$

Definition 8. Given the function $f(x)$ defined in the neighborhood of a point a, we say that

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0 \ : \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Note:

1. Note that the value of the function at the point a itself is not what matters. What matters are the arbitrarily close points *close* to a.
2. The limit of a function and the limit of a sequence are two different concepts but the limit of a sequence in the examples helps clarify the concept of what a limit is in general.

Example 9. Consider the function $f(x) = \frac{1}{x^2}$. It is easy to note that the domain of this function is $\mathcal{D} = (-\infty, 0) \cup (0, +\infty)$. If we would want to compute $\lim_{x \to 0} f(x)$, we cannot compute $f(0)$ since it is not defined. However, we could set values of x closer and closer to 0 without reaching it. What we see is that the function takes larger and larger values, tending to infinity.

2.1.1 Infinite limits

Definition 9. Given a function $f(x)$ defined in the neighborhood of a point a, it is said that:

1. $\lim\limits_{x \to a} f(x) = +\infty \iff \forall K > 0 \ : \ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) > K$
2. $\lim\limits_{x \to a} f(x) = -\infty \iff \forall K > 0 \ : \ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \Rightarrow f(x) < -K$
3. $\lim\limits_{x \to +\infty} f(x) = +\infty \iff \forall K > 0 \ : \ \exists M > 0 \text{ such that} x > M \Rightarrow f(x) > K$

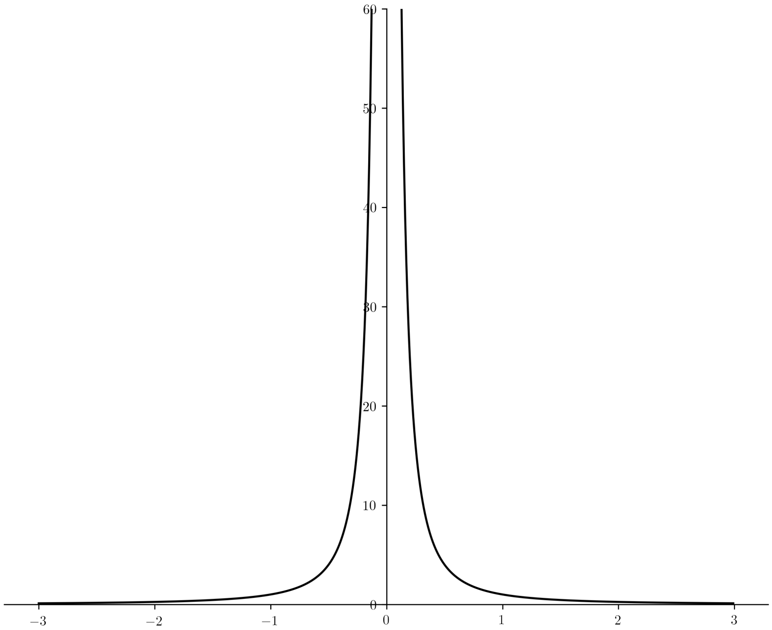


Figure 8: Plot of $f(x) = \frac{1}{x^2}$

2.1.2 Lateral limits

When a function approaches a finite point a, it can approach such a point from the right or from the left (or both simultaneously). If we're only interested in the values at the left side of a $(x < a)$, that limit is being computed *from the left*, that is:

$$\lim_{x \to a^-} f(x) = L^-$$

Similarly, if we're interested only in $x > a$, we have a limit *from the right*, or as before:

$$\lim_{x \to a^+} f(x) = L^+$$

An important property is that if limits from the left and the right do not coincide, then the function **does not have a limit** in that point. That is:

$$L^+ \neq L^- \Rightarrow \nexists \lim_{x \to a} f(x)$$

2.1.3 Asymptotes

Definition 10. The line $x = a$ is a vertical asymptote of function $f(x)$ when at least one of these conditions is satisfied:

1. $$\lim_{x \to a^+} f(x) = +\infty$$

2. $$\lim_{x \to a^-} f(x) = +\infty$$

3.

$$\lim_{x \to a^+} f(x) = -\infty$$

4.

$$\lim_{x \to a^-} f(x) = -\infty$$

Definition 11. The line $y = b$ is a horizontal asymptote of function $f(x)$ when at least one of these conditions is satisfied: 1. $\lim_{x \to +\infty} f(x) = b$ 2. $\lim_{x \to -\infty} f(x) = b$

2.1.4 Properties of limits

Sums of functions. When the limits exist, it is fulfilled that:

1.

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2.

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

3.

$$\lim_{x \to a} (f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$$

4. and **the denominator is different from zero**

$$\lim_{x \to a} (f(x)/g(x)) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$$

5.

$$\lim_{x \to a} (f(x))^k = \left(\lim_{x \to a} f(x)\right)^k$$

6.

$$\lim_{x \to a} (g \circ f)(x) = \lim_{x \to b} g(x) \text{ where } b = \lim_{x \to a} f(x)$$

Examples Compute the following limits:

1.

$$\lim_{x \to -1} \left(x^2 - 3x + 1\right)$$

It is easy to see that $f(-1) = \left((-1)^2 - 3(-1) + 1\right) = 5$

2.

$$\lim_{x \to 0} \left(\frac{|x|}{x}\right)$$

Note that we cannot simply substitute $x = 0$ in this limit, since that would be an indetermination. Instead, we should compute the lateral limits, if they coincide, the limit **exists and takes that value**, otherwise, there is no limit. Thus,

$$\lim_{x \to 0^+} \left(\frac{|x|}{x}\right) = \left(\frac{x}{x}\right) = 1 \; ; \; \lim_{x \to 0^-} \left(\frac{|x|}{x}\right) = \left(\frac{-x}{x}\right) = -1$$

Since the limits do not coincide, there is no limit.

3.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

We cannot substitute here $x = 1$ since we get an indetermination. Thus, we can modify the function a bit as follows

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = 2$$

4.

$$\lim_{x \to +\infty} \frac{x}{e^x}$$

To see this last limit, check out the graph of the numerator and denominator separately. Note that $f(x) = e^x$ grows **much** faster than $g(x) = x$ thus, if the *denominator* is growing much faster than the denominator, when x approaches ∞, the function will tend to 0. This can be seen by using derivation techniques, like *L'Hôpital's* rule (we will see this in the next Chapter).

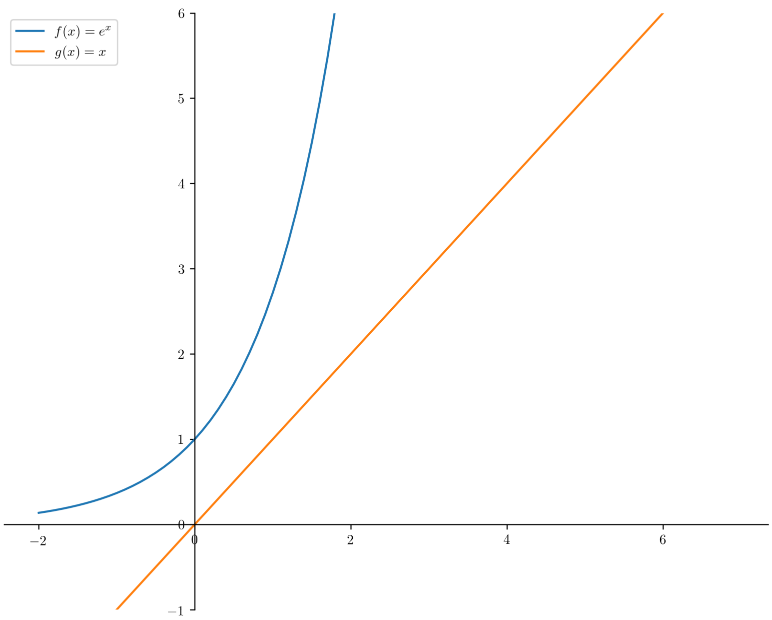


Figure 9: Exponential and Linear Functions

Example: Investigate the following limit.

$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

Theorem 3. ***The Squeeze Theorem:*** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

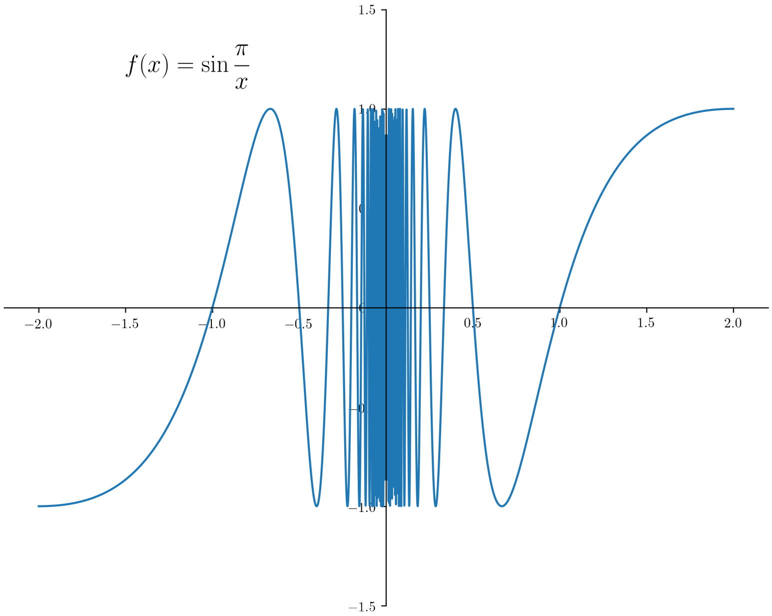


Figure 10: Oscillatory behavior of $\sin(\pi/x)$

Then:

$$\lim_{x\to a} g(x) = L$$

This theorem tells us that if $g(x)$ is *squeezed* between $f(x)$ and $h(x)$ around $x = a$, and if $f(x)$ and $h(x)$ have the same limit L at a, then necessarily, $g(x)$ has the same limit L at a.

Example 10. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

1. We cannot use $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x^2 \times \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$. Why not?
2. Note that $\sin(x) \in [-1, 1]$ for any x, therefore
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$
3. Any inequality will still hold if multiplied by any positive number, and x^2 is positive. Thus
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$
4. If we evaluate the limits for the extremes of the inequality:
$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$
5. Then, applying the Squeeze Theorem (Theorem 3) and taking $f(x) = -x^2, h(x) = x^2$, and $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ we show that
$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

2.2 Continuity

Definition 12. A function $f(x)$ is continuous in the point $x = a$ if three conditions are satisfied:

1. The function exists in $x = a$. That is, $a \in \mathcal{D}_f$ with a finite value $f(a)$.
2. The limit $\lim_{x \to a} f(x) = L \ : \ L < \infty$.
3. $f(a) = \lim_{x \to a} f(x) = L$.

That is all summarized in:

$f(x)$ is continuous in $a \iff \lim_{x \to a} f(x) = f(a) \iff \forall \varepsilon > 0 : \exists \delta > 0 \ : \ 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

If the function $f(x)$ is **not** continuous in $x = a$, we say it is **discontinuous** in that point.

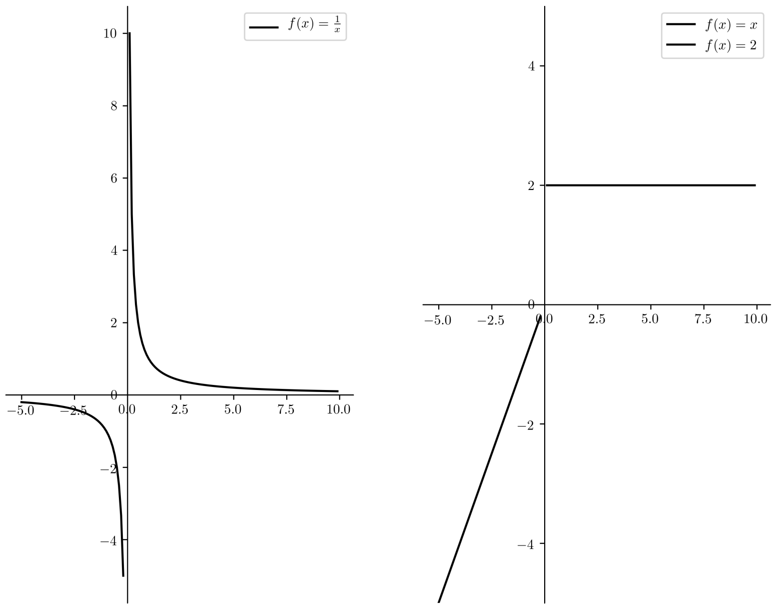


Figure 11: Two Types of Discontinuities

2.2.1 Types of Discontinuity

1. **Finite jump:** lateral limits exist and are finite but different.
2. **Infinite jump:** lateral limits are infinite and with different sign. It can also be that one is finite and another one infinite.
3. **Asymptotic:** Lateral limits are infinite with the same sign.
4. **Avoidable:** function and limit both exist but they take **different values**.

Exercise: Classify the following functions according to the type of discontinuity they show.

1. $f(x) = \frac{1}{x^2}$

2. $f(x) = \frac{1}{x}$

3. $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

4. $f(x) = \begin{cases} -2 & \text{if } x < -3 \\ -1 & \text{if } -3 \leq x < -2 \\ 5 & \text{if } x \geq -2 \end{cases}$

2.2.2 Intermediate Value Theorem

Theorem 4. *If $f(x)$ is a continuous function in a closed interval $[a, b]$ and $k \in [f(a), f(b)]$ then, there exists at least one real number $c \in [a, b]$ such that $f(c) = k$.*

Theorem 5. *(Bolzano's Theorem) If $f(x)$ is a continuous function in a closed interval $[a, b]$, and $f(a)$ has opposite sign than $f(b)$, then there exists at least one real number $c \in (a, b)$ such that $f(c) = 0$.*

Notice that Theorem 5 is a particular case of Theorem 4, given that $f(a)$ and $f(b)$ are of opposite sign, 0 is an intermediate value, thus Theorem 4 applies with $k = 0$.

These two results are of great importance. Take for example $f(x) = x^5 - 20x - 4$. This polynomial has no whole roots (points in which the $f(x) = 0$) thus, it is difficult to compute them. Nevertheless, it is easy to see that $f(-1) = 15$ and $f(0) = -4$, thus, there must exist a real number $c \in (-1, 0)$ such that $f(c) = 0$. And in this case, $c = 0.200016$. We can see this in Figure 12.

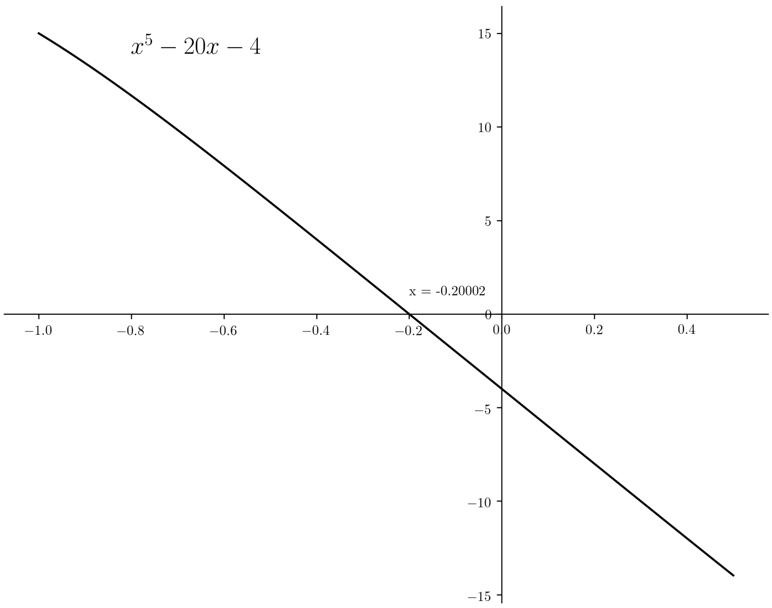


Figure 12: Intermediate Value Visualization

Definition 13. Let $f : \mathcal{D} \to \mathbb{R}$ be a function defined in a domain $\mathcal{D} \subseteq \mathbb{R}$. Then,

1. $f(x)$ achieves a global maximum in $x_M \in \mathcal{D}$ if $f(x_M) \geq f(x)\ \forall x \in \mathcal{D}$

2. $f(x)$ achieves a global minimum in $x_m \in \mathcal{D}$ if $f(x_m) \leq f(x)\ \forall x \in \mathcal{D}$.

Theorem 6. *(Weierstrass' Theorem) If $f(x)$ is a continuous function in a closed interval $[a, b]$, then $f(x)$ achieves a global maximum and minimum in that interval. That is,*

1. $\exists x_M \in [a, b]\ :\ f(x_M) \geq f(x)\ \forall\ x \in [a, b]$
2. $\exists x_m \in [a, b]\ :\ f(x_m) \leq f(x)\ \forall\ x \in [a, b]$

Let's visualize the theorems by plotting $f(x) = \frac{1}{5} x \sin x$

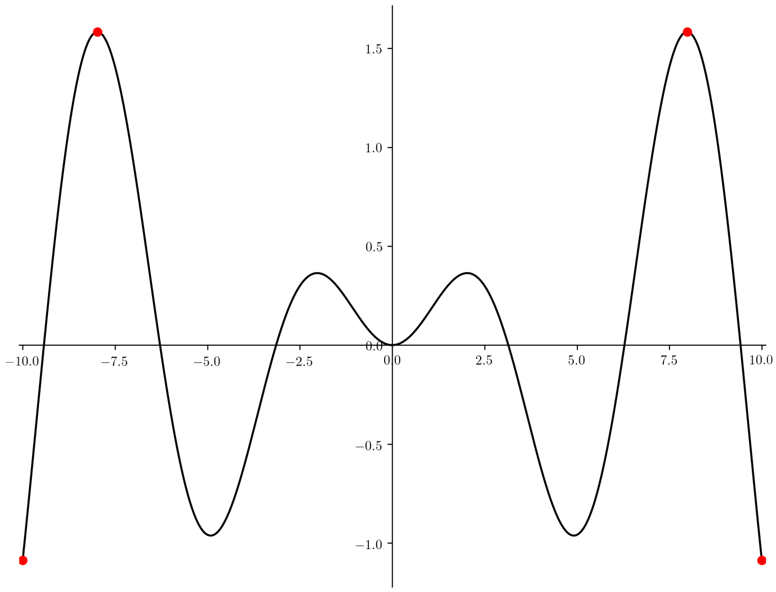


Figure 13: Wiggly function with local maxima and minima

Note that the two key features for the theorem to hold is that the function is continuous (so, no jumps to infinity or discontinuities) and that the interval considered is closed. In the graph, we are considering $f(x)$ in the interval $x \in [-10, 10]$, if we considered instead $x \in (-10, 10)$ there would exist no minima because it is not possible to find x_m such that $f(x_m) \leq f(x)\ \forall\ x \in (-10, 10)$.

Definition 14. Let a function $f : \mathcal{D} \to \mathbb{R}$ defined in a domain $\mathcal{D} \subseteq \mathbb{R}$. Then:

1. $f(x)$ achieves a local maximum in $x_1 \in \mathcal{D}$ if:

$$f(x_1) \geq f(x)\ \forall\ x \in \mathcal{D} \cap (x_1 - \varepsilon, x_1 + \varepsilon) \text{ for some } \varepsilon > 0$$

2. $f(x)$ achieves a local minimum in $x_2 \in \mathcal{D}$ if:

$$f(x_2) \leq f(x)\ \forall\ x \in \mathcal{D} \cap (x_2 - \varepsilon, x_2 + \varepsilon) \text{ for some } \varepsilon > 0$$

So the function $f(x)$ achieves a local maximum (minimum) if the function at a certain point is greater (lower) or equal than the rest of points in a neighborhood denoted by ε.

3 Exercises

Exercise 1 Compute the equation of the lines, their slopes, a Euclidean vector, cutting points with the axes, and its graph.

1. Line that goes through $(1,2)$ and $(-1,3)$.
2. Line that goes through $(0,2)$ and $(3,2)$.
3. Line that goes through $(1,1)$ and $(1,5)$.
4. Line that goes through $(5,1)$ and is a parallel to $x+y=2$.
5. Line that goes through $(1,0)$ and is perpendicular to $x+2y=1$.
6. Line that goes through $(1,1)$ and cuts the lines $y=x+1$ and $x+y=1$ in the same point.

Exercise 2 Compute the domain of the following functions.

1. $f(x)=\log(x^2+x-6)$
2. $f(x)=\sqrt{\frac{x^2-2x+1}{|x+5|}}$
3. $f(x)=\sqrt{\log(x+5)}$
4. $f(x)=\sqrt{x^2+x-2}$

Exercise 3 Prove the fundamental relationship between the sine and cosine, i.e., $\sin^2(\alpha)+\cos^2(\alpha)=1$. *Hint:* use the Pythagorean theorem.

Exercise 4 Is $(f\circ g)=(g\circ f)$ generally fulfilled? Try out a simple example and reason the answer.

Exercise 5 Given the functions $f(x)=x-1$ and $g(x)=\log(x)$

1. Compute $f\circ g$, $g\circ f$ and their domains.
2. Compute f^{-1} and g^{-1}.

Exercise 6 Assuming that the following functions are linear, give an economic interpretation of the slope of the function:

1. $F(q)$ is the revenue from producing q units of output.
2. $G(x)$ is the cost of purchasing x units of some commodity.
3. $H(p)$ is the amount of the commodity consumed when its price is p.
4. $C(Y)$ is the total national consumption when national income is Y.
5. $S(Y)$ is the total national savings when national income is Y.

Exercise 7 Prove that $f(x) = x^5 - 3x - 5$ has **at least** one real root in the interval $[0,2]$. Indicate the results you use and argue that the function fulfills the necessary requirements to apply those results.

Exercise 8 Prove that any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

Exercise 9 Analyse the continuity of

1. $$f(x) = \frac{1}{x}$$

2. $$f(x) = |x|$$

3. $$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$$

Exercise 10 Download the Penn World Tables 9.1 from here and compute GDP per capita (let it be called Y^c, where c stands for the country) for the U.S., South Korea, Singapore, and India. Transform it into $y^c = \log(Y^c)$ and plot the data in the same graph. If you would have to guess a functional form for y^c for each country except the U.S. which would it be? How does it compare to the U.S.? Read about *balanced growth*, can you relate it to the data?