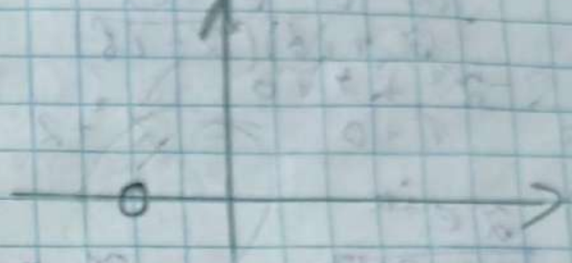
$$\frac{\cos x\cos\frac{1}{x}+\sin x\sin\frac{1}{x}}{1+x^2} \qquad \frac{\frac{1}{4}(e^{ix}+e^{-ix})^2-\frac{1}{4}(e^{2ix}-e^{-2ix})}{1+x^2}$$

$$\operatorname{res}_{i}=\frac{1}{4}\,\frac{e^{2ix}+e^{-2ix}+2e^{ix}e^{-ix}-e^{2ix}+e^{-2ix}}{(x+i)(x-i)}=$$

$$=\frac{2e^{-2ix}+2e^{ix}e^{-ix}}{4(x+i)}=\frac{e^{2}+e^{-1}e^{1}}{4i}$$

$$\oint=\pi i\cdot\frac{e^{2}+1}{2}$$

20.21.



1) $\frac{z}{(z+2)^2}$ $\quad z_0=-2.$

$|z+2|$

$$\frac{z}{(z+2)^2}=\frac{A}{z+2}+\frac{B}{(z+2)^2}=\frac{A(z+2)+B}{(z+2)^2}=\frac{1}{z+2}+\frac{-2}{(z+2)^2}$$

~~1)~~ $\cancel{\frac{1}{z+2}}=$

2) $\frac{e^z+1}{e^z-1}$ $\quad (z_0=0,\ \pm 2\pi i\ \ldots)\quad z_0=2k\pi i$

$$\frac{e^z+e^{2k\pi i}}{e^z-e^{2k\pi i}}$$

$$e^z+\cancel{e^{2k}}\,1=\sum_{k=0}^{n}\frac{z^k}{k!}+1=\left(1+z+\frac{z^2}{2!}\ldots\right)+1=$$

$$=2+\sum_{k=1}^{n}\frac{z^k}{k!}$$

$$\frac{2}{z-2k\pi i}=\frac{2}{e^z-e^{2k\pi i}}$$

3) $\frac{z-1}{\sin^2 z}$ $\quad z_0=0$

$$\sin^2 z=z^2-\left(\frac{z^3}{3!}\right)^2+\ldots$$

$$\frac{z-1}{z^2}=-\frac{1}{z^2}+\frac{1}{z}.$$

4) $\frac{e^{iz}}{z^2+b^2}$ $(z_0 = ib, b>0)$

~~$e^{iz} = 1\cdot i + iz + \ldots$~~

$$\frac{e^{iz}}{z^2+b^2} = \frac{e^{iz}}{(z+ib)(z-ib)} = \frac{e^{-b}}{2ib(z-ib)}$$

5) $\frac{(z^2+1)^2}{z^2+b^2}$ $z_0 = \infty$

$$\frac{(z-i)^2(z+i)^2}{(z+ib)(z-ib)} \quad \frac{z^4+2z^2+1}{z^2+b^2} = \lim_{z=\infty} \frac{z^4+2z^2+1}{z^2+b^2} =$$

$$= \frac{z^2+2+0}{1+0} = z^2+2$$

6) $\frac{ze^{iz}}{(z^2+b^2)^2}$ $z_0 = ib, b>0$

$$\frac{ib\,e^{-b}}{(z+ib)^2(z-ib)^2} = \frac{ib\,e^{-b}}{-4b^2(z-ib)^2} = \frac{e^{-b}}{4b(z-ib)} + \frac{ie^{-b}}{4(z-ib)^2}$$

$A(z-ib)+B = Az - ibA + B$

$A = i$

$B + ibA = 0 \quad B = -b$

7) $\frac{z}{(z^2+b^2)^2}$ $z_0 = ib$

$$\frac{z}{(z+ib)^2(z-ib)^2} = \frac{ib}{-4b^2(z-ib)^2} = \frac{-i}{4b(z-ib)^2}$$

8) $\operatorname{ctg} \pi z$ $(z_0 = 0, \pm 1, \pm 2 \ldots)$ $z_0 = k$

$$\frac{\cos \pi z}{\sin \pi z} = \frac{1 + \left(\sum_{k=1}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}\right)}{\pi \cdot (z-k) + (\ldots)} = \frac{1}{\pi \cdot (z-k)}$$

9) $\frac{1}{\sin \pi z}$ $(z_0 = 0, \pm 1)$ $z_0 = k$

$$\frac{1}{\sin \pi z} = \frac{(-1)^k}{\pi(z-k)}$$

20.32

$$\frac{1-a\cos\varphi}{1-2a\cos\varphi+a^2}$$

$-1<a<1$ $\cancel{\cos\varphi_0\ a}$ $e^{iz}=a;\ \frac{1}{a}$

$$\frac{1-a\cos\varphi}{(\cos\varphi-a)^2}=\cancel{\frac{1-a\cos\varphi}{\cos\varphi}}\ \cancel{\frac{A}{(\cos\varphi-a)}+\frac{B}{(\cos\varphi-a)^2}}$$

$D=4\cos^2\varphi-4=4(\cos^2\varphi-1)=-4\sin^2\varphi$

$$a_1=\frac{2\cos\varphi+2i\sin\varphi}{2}$$

$a_2=\cos\varphi-i\sin\varphi$

$$\frac{1-a\cos\varphi}{(a-\cos\varphi-i\sin\varphi)(a-\cos\varphi+i\sin\varphi)}=\frac{1-a\frac{1}{2}(e^{iz}+e^{-iz})}{(a-e^{iz})(a-e^{-iz})}=$$

$$=\frac{1-a\cos\varphi}{(e^{iz}-a)(e^{iz}-\frac{1}{a})}$$

$Ae^{iz}-\frac{1}{a}A+Be^{iz}-aB$

$\frac{1}{a}A=-aB$ $A=-a^2B$

$A+B=-a\frac{1}{2}-\frac{2}{a}$

$-a^3B=-a^2/2-2$

$B=-\frac{1}{a2}-\frac{2}{a^3}$ $A=1+\frac{2}{a}$???

21.02

1) $\frac{1}{z(z-i)(z+i)}$ $z_1=0$ $z_2=i$ $z_3=-i$

$z_1=0$ $\left.\frac{1}{(z-i)(z+i)}\right|_{z\to0}=1$

$z_3=-i$ $\left.\frac{1}{z(z-i)}\right|_{z\to-i}=-\frac{1}{2}$

$z_2=i$ $\left.\frac{1}{z(z+i)}\right|_{z\to i}=-\frac{1}{2}$

2) $\frac{z^2}{1+z^4} = \frac{z^2}{(z^2-i)(z^2+i)}$ $\cancel{z_1 = \pm i}$ $\cancel{n=2}$.

Замена $t = z^2$

$\cancel{\frac{t}{1+t^2} = \frac{t}{(t-i)(t+i)}}$ $\cancel{t_1 = +i}$ $\cancel{t_2 = -i}$

$\cancel{\underset{i}{res} = \frac{t}{t+i}\Big|_{t\to i} = \frac{1}{2}}$ $\cancel{\underset{-i}{res} = \frac{t}{t-i}\Big|_{t\to i} = \frac{1}{2}}$

$z = e^{i\varphi} = \cancel{\cos\varphi + i\sin\varphi}$ $z = e^{i\varphi}$

$\cos 2\varphi + i\sin 2\varphi$ $e^{2i\varphi}$ $\varphi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

$1 + (\cos 4\varphi + i\sin 4\varphi) = 1 + e^{4i\varphi}$ (1), (2), (3), (4)

$res = \frac{\cancel{z}\, e^{2i\varphi}\, z^2}{(1+e^{4i\varphi})'} = \frac{\cancel{e^{2i\varphi}}\, z^2}{4i\, e^{4i\varphi}\, z^3} = \frac{1}{4i\, e^{2i\varphi} z}$

$\cancel{1}$ $\underset{(1)}{res} = res\ \frac{1-i}{4\sqrt{2}}$ $\underset{(2)}{res} = \frac{1+i}{4\sqrt{2}}$

$\underset{(3)}{res} = -\frac{1+i}{4\sqrt{2}}$ $\underset{(4)}{res} = -\frac{1-i}{4\sqrt{2}}$

3) $\frac{z^2}{(1+z)^3}$ $z_1 = -1$ $n = 3$.

$\underset{-1}{res} = \frac{1}{2!}\frac{d^2}{dz^2}(z^2)\Big|_{z\to -1} = 1.$

4) $\frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ $z_0 = \pm i$ $n = 3$.

$\underset{i}{res} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right) = \frac{1}{2}\frac{d}{dz}\left(-3\cdot\frac{1}{(z+i)^4}\right) =$

$= \frac{1}{2}\left(12\frac{1}{(z+i)^5}\right)\Big|_{z\to i} = 6/i$

$\underset{-i}{res} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{(z-i)^3}\right) = -6/i$

5) $\frac{1}{(z+i)(z-i)(z-1)^2}$ $z_1 = \pm i$ $n=1$ $z_2 = 1$ $n=2$

$\underset{i}{res} = \frac{1}{2i(i-1)^2} = \frac{1}{\cancel{-2i}+4+1} = \frac{1}{\cancel{-2i}+5-2i\cdot 2i} = \frac{1}{4}$

$\underset{-i}{res} = \frac{1}{-2i(-i-1)^2} = \frac{1}{2i+5} = \frac{1}{4}.$

$res = \frac{d}{dz}\left(\frac{1}{(z+i)(z-i)}\right) = -\frac{2z}{(z-i)^2(z+i)^2}\Big|_{z\to 1} = -\frac{2}{(1-i)^2(1+i)^2} =$

$= \frac{1}{2}$

6) $\frac{z^{2n}}{(z-1)^n}$ $\quad n = 1, 2, 3\ldots \quad z = 1 \quad C_{2n}^{n-1}$

7) $\frac{1}{\sin \pi z}$ $\quad z_0 = \cancel{\pi} k$

$\frac{1}{\sin \pi z} = \frac{(-1)^k}{\pi(z-k)} \qquad C_{-1} = \frac{(-1)^k}{\pi}$

8) $ctg\, \pi z \qquad C_{-1} = \frac{1}{\pi} \qquad z_0 = k$

9) $th\, z = \frac{\sin iz}{\cos iz} \qquad z_0 = \frac{i\pi}{2} \pm i\pi k \quad z = \left(k + \frac{1}{2}\right)\pi i$

$i\, \frac{i\left(z - \left(k + \frac{1}{2}\right)\pi i\right)}{i z} \qquad \frac{i}{i} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(-1)^k} \frac{z^{2k+1-2k}}{1!} = \cancel{1} z$

$res = 1.$

10) $cth^2 \pi z = \left(\frac{\cos i\pi z}{\sin i\pi z}\right)^2 \qquad z_0 = \pi i n$

$\sum \left(\frac{1}{z - \pi i n}\right)^2 \quad res = 0.$

11) $\frac{\cos z}{(z-1)^2} = \frac{\cos z}{\cancel{z+i}} \qquad z_0 = 1 \qquad n = 2.$

$\underset{1}{res} = \frac{d}{dz}(\cos z) = -\sin z\Big|_{z\to 1} = -\sin 1$

12) $\frac{1}{e^z + 1} \qquad z_0 = i\pi + 2\pi i k \qquad n = 1$

$\underset{z_0}{res} = \frac{1}{(e^z+1)'} = \frac{1}{e^z}\Big|_{z\to z_0} = -1$

13) $\frac{\sin \pi z}{(z-1)^3} \qquad z_0 = 1 \quad n = 3$

$\underset{z_0}{res} = \frac{d^2}{dz^2}(\sin \pi z) = -\pi^2 \sin \pi z\Big|_{z\to 1} = 0.$

14) $\frac{1}{\sin z^2}$ $\quad z_1 = \pi k_0 \quad n = \frac{1}{2} \quad z_2 = i^k\sqrt{\pi k}$

$\underset{z_2}{res}\ \frac{1}{(z-z_2)^{\frac{1}{2}}} = \frac{2iz}{2(z^2)^{...}-1}\ \underset{z_2}{res} = \frac{1}{2z\cos z^2}\Big|_{z\to z_2} = \frac{\pm i}{2\sqrt{\pi k}}$

$\underset{0}{res}\ \frac{1}{2z\cos z^2}\Big|_{z\to z_1} \Rightarrow = 0.$

21.03

1) $\frac{z^4+1}{z^6-1}$ $\quad \underset{\infty}{res} = \frac{z^4+1}{6z^6} = \frac{1}{6z} + \frac{1}{z^5} = 0.$

2) $\cos\pi\frac{z+2}{2z} = \pi\cdot\sum(-1)^k\left(\frac{z+2}{2z}\right)^{2k}\cdot\frac{1}{(2k)!}$?

3) $\frac{\sin\frac{1}{z}}{z-1} = \frac{1}{z}\left(\frac{1}{z-1}\right) \quad \underset{\infty}{res} = \frac{1}{2z} = 0.$

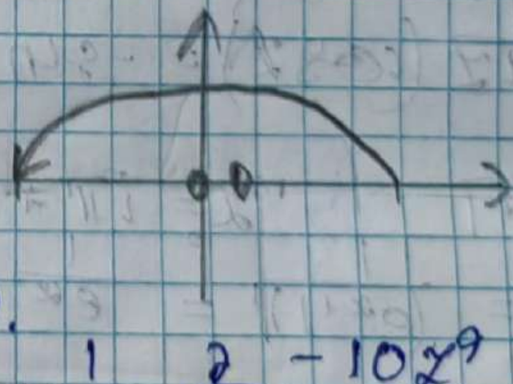
4) $\frac{\cos^2\frac{\pi}{z}}{z+1} = \cos^2\frac{\pi}{z} = 1$?

5) $\frac{(z^{10}+1)\cos\frac{1}{z}}{(z^5+2)(z^6-1)}$

6) $z\cos^2\frac{\pi}{z} = \cancel{z\cdot\frac{\pi^2}{z^2} \Rightarrow c_{-1} = \pi^2}$

22.02

1) $\int\limits_{\partial D}\frac{dz}{z^3(z^{10}-2)}$ $\quad |z|<2$



$z_1 = 0 \quad n = 3$

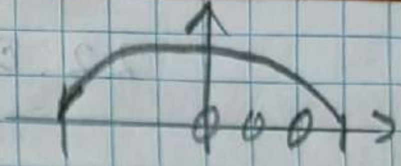
$z_2 = \sqrt[10]{2} \quad n = 10.$

$\underset{0}{res} = \frac{1}{2!}\frac{\partial^2}{\partial z^2}\left(\frac{1}{z^{10}-2}\right) = \frac{1}{2}\frac{\partial}{\partial z}\frac{-10z^9}{(z^{10}-2)^2} =$

$= \frac{1}{2}\cdot(-8) = -4.$

$\underset{z_2}{res} = \cancel{0}.$

3) $\oint\limits_{\partial D} \frac{z^2 \sin^2 \frac{1}{z}}{(z-1)(z-2)} dz \qquad |z| < 3$



$z = 1 \quad z = 2$

$n = 1 \quad n = 1$

$\cancel{\underset{1}{res} = \frac{z^2 \sin^2 \frac{1}{z}}{z-2} = -1}$

$\cancel{\underset{2}{res} = \frac{z^2 \sin^2 \frac{1}{z}}{z-1}}$

$\sin^2 \frac{1}{z} = \cancel{\left(\frac{1}{2i}\left(\frac{1}{z} - z\right)\right)^2} = \left(\frac{1}{z^2}\right) = \frac{z^4 - 2z^2 + 1}{-4z^2}$

$\frac{1}{(z-1)(z-2)}$

$\frac{z^4 - 2z^2 + 1}{-4(z-1)(z-2)} \qquad \underset{1}{res} = \frac{0}{4}$

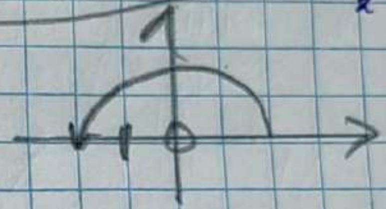
$\underset{2}{res} = \frac{9}{-4} \qquad \oint = \frac{-9\pi i}{2}$

$\underset{1}{res} = -1$

$\underset{2}{res} = 1$

$\oint = 0.$

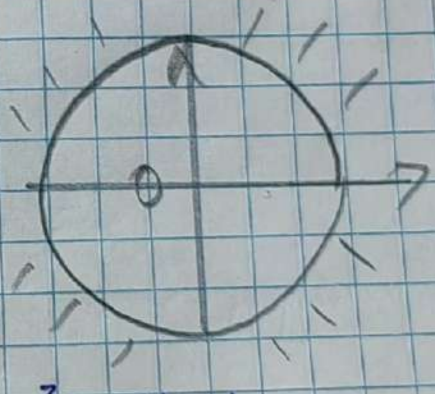
4) $\int \frac{z^3 e^{\frac{1}{z}} dz}{z+1} \qquad z < 2$



$z = -1 \qquad z_1 = 0.$

$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots$

$\underset{-1}{res} = z^3.$



5) $\int \sin \frac{z}{z+1} dz \qquad |z| > 3$

$\sin \frac{z}{z+1} = \frac{\left(\frac{z^2}{(z+1)}\right)^2 + 1}{2i(z/(z+1))} =$

$= \frac{2z^2 + 2z + 1}{(z+1)^2} \cdot \frac{z+1}{2iz} = \frac{2z^2 + 2z + 1}{2iz(z+1)}$

$\underset{0}{res} = \frac{1}{2i} \qquad \underset{-1}{res} = \frac{1}{-2i}$

6) $\int z \sin \frac{z+1}{z-1} \ldots$