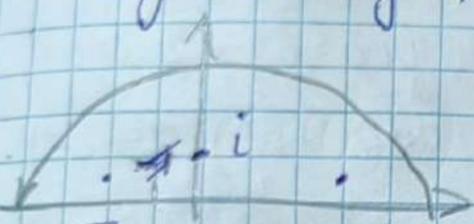
$f(z) = \frac{1}{z^4+1}$ $h(z_0) = 1$ $g(z) = z^4+1$

$\operatorname{res} f(z) = \frac{1}{4z^3}$

3.1

1) $\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx$ $f(z) \sim \frac{1}{\cancel{z^2}}$ $\pm i$ $z \to 0$ – полюс III п.

$\operatorname{res} f(z) = \frac{h(z)}{g'(z)} = \frac{\cancel{x^4}}{(1+x^6)'} = \frac{\cancel{x^4}}{6x^5} = \frac{1}{5}$

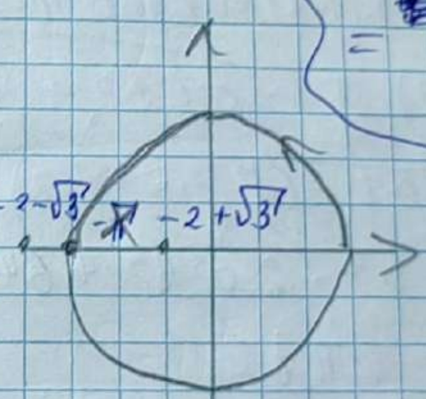


$\oint = 2\pi i \cdot \frac{1}{5} \operatorname{res}$

$\operatorname{res}_{i} = \frac{1}{2!} \lim_{i} \frac{d^2}{dz^2} \frac{x^4}{x^3+i} =$

$= \frac{4x^3(x^3+i) - 3x^4}{(x^3+i)^2}$

2) $\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\,d\theta$



$z = e^{i\varphi}$

$dz = e^{i\varphi}\,i\,d\varphi$, $d\varphi = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\cos 2\theta = \frac{1}{2}\left(\cancel{\frac{1}{2}} z^2 + \cancel{\frac{1}{2}} z^{-2}\right)$

$I = \oint f(z)\,dz$ $\cancel{f(z)} = \frac{z^4+1}{iz(z^2+4z+1)}\,dz$

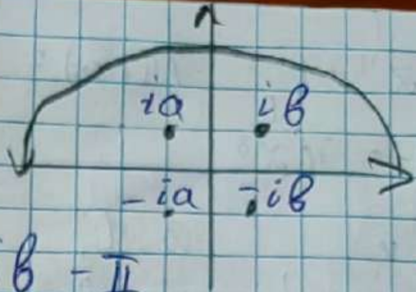
z^2+4z+1 $z = -2 \pm \sqrt{3}$ – I п. $f(z) = \frac{z^4+1}{\cancel{z}(z^2+4z+1)}$

$f(z) = \frac{h(z)}{g'(z)} = \frac{z^4+1}{2z^{\cancel{2}}+4z} = \frac{98 - 56\sqrt{3}}{4\sqrt{3}}$

~~$\operatorname{res} f(z) = f(z)(z-a) = \frac{z^4+1}{iz(z^2+4z+1)} \cdot (z+2-\sqrt{3}) =$~~

~~$= \frac{z^4+1}{iz(z-2+\sqrt{3})}$~~ $I = \oint = 2\pi i \cdot \frac{98 - 56\sqrt{3}}{\sqrt{3}}$

3) $\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$



$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)^2}$

$z_1 = ia \quad z_2 = -ia$ — I порядка $\quad z_3 = \pm ib$ — II порядка.

1) $\frac{1}{(z^2+a^2)'(z^2+b^2)^2} = \frac{1}{2z(z^2+b^2)^2 + 2z(z^2+b^2)(z^2+a^2)}$

$= \frac{1}{2 \cdot ia(-a^2+b^2)^2} = \frac{1}{2ia^5 - 4ia^3b^2 + 2iab^4}$

2) $\frac{d}{dz}\left((z-ib)^2 \cdot \frac{1}{(z^2+a^2)(z^2+b^2)^2}\right) = \frac{d}{dz}\frac{1}{(z^2+a^2)(z+ib)^2} =$

$= \frac{1}{2z(z+ib)^2 + (2ib)(z^2+a^2)} = \frac{1}{2ib(2ib)^2 + (2ib)}$

$\cdot(-b^2+a^2) = -8ib^3 + 4b^4 - 4b^2a^2 = \frac{1}{4b^4} \cdot \frac{1}{-2ib+b^2}$

$= \frac{1}{4b^2} \cdot \frac{1}{(ai-b)^2}$

1) $\frac{1}{2ia} \cdot \frac{1}{a^4 - 2a^2b^2 + b^4} = \frac{1}{2ia} \cdot \frac{1}{(a^2-b)^2(ai+b)^2}$

$I = \oint = 2\pi i$

$(ai+b)^2 \cdot ia + 2b^2 = -a^3 - 2a^2b + iab^2 + 2b^2 =$

$= a^2(a+b)$

3.2.

$f(z) = z^3 \cos\frac{1}{z-2} \qquad z = \infty$

$f(z) = z^3 \cdot \left(1 - \frac{1}{2(z-2)^2} + \frac{1}{4!(z-2)^4}\right)$

$\frac{z^3}{24(z^4 - 8z^3 + 24z^2 - 24z + 16)} = \frac{1}{24z^4 + 384}$

$((z-2)^3 + 8) \cdot \left(1 + \frac{1}{2(z-2)^2} + \frac{1}{4!(z-2)^4}\right)$

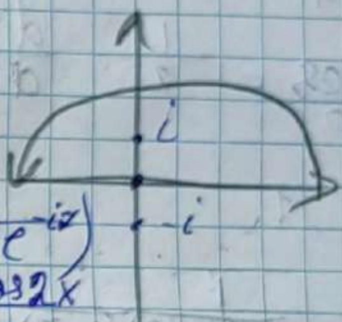
$((z-2)+2)^3 = (z-2)^3 + 6(z-2)^2 + 12(z-2) + 8$

$\cos\left(\frac{1}{z-2}\right) = \sum \frac{(-1)^n}{(2n)!\,(z-2)^{2n}}$

$f(z) = \sum a_n (z-2)^n \Rightarrow a_{-1} =$

3.5.

$\int_{-\infty}^{\infty} \frac{\sin^2 x\, dx}{x^2(x^2+1)}$



$x_1 = 0 \quad n=2$ $\qquad \sin^2 x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right)$

$x_2 = \pm i \quad n=1 \qquad \sin^2(x) = \frac{1-\cos 2x}{2}$

$\cancel{\frac{\frac{1}{2} - \left(e^{2ix} + e^{-2ix}\right)}{2\,x^2(x^2+1)}} \qquad \frac{-z^2 + 1 + 2z}{2z}$

$\frac{1 - \frac{1}{2}\left(z + \frac{1}{z}\right)}{2\,z^2(z+i)(z-i)} = \frac{-z^2+2z-1}{4z^3(z+i)(z-i)}$

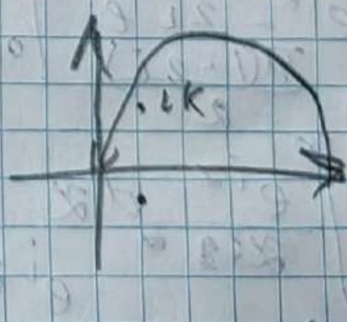
$\operatorname{res}_i \frac{-z^2+2z-1}{12z^2(z+i)(z-i) + \ldots + 4z^3(z+i)} = \frac{-z^2+2z-1}{4z^3(z+i)} =$

$= \frac{1+2i-1}{-4i \cdot 2i} = \frac{2i}{8} = \frac{i}{4}$

$\operatorname{res}_0 = \lim \frac{d}{dz} \frac{-z^2+2z-1}{4z(z+i)} = 0$

3.8.

$\int_0^{\infty} \frac{x \sin ax}{x^2+k^2}\, dx \qquad x = \pm ik$



$\operatorname{res}_{ik} \frac{x \sin ax}{2x} = \frac{\sin ax}{2} =$

$= \frac{e^{iax} - e^{-ixa}}{4i} = \frac{e^{-ka} - e^{ka}}{4i}$

$\oint =$

$$\int_0^{\infty} \frac{x^{a-1}}{1-x^{b}}\,dx$$

3.3

$$\frac{1}{z^3(1-z^2)} = \cancel{\frac{1}{z^3(-1+z)(-i+z)}}$$

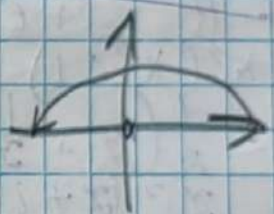
$$\underset{1}{res} = \frac{1}{z^3(1-z^2)} = \frac{1}{3z^2(1-z^2)+(-2z^4)} = -1$$

$$\underset{0}{res} = \frac{1}{2}\,\frac{d^2}{dz^2}\left(\frac{1}{1-z^2}\right) = \frac{1}{2}\left(-\frac{6x^2+2}{x^6-3x^4+3x^2-1}\right)\Bigg|_{x\to 0} =$$

$= 1$

$$\frac{1}{z^3(1+z)(1-z)} = -\frac{1}{(z^3+z^4)}$$

$$\int_0^{\infty} \frac{x-\sin x}{x^3}\,dx \qquad x=0 \quad n=3.$$



$$\sin x = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$x = e^{ix}$$

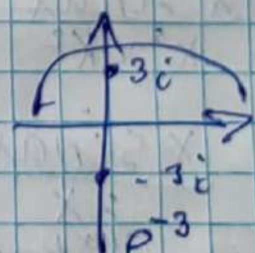
$$\frac{1}{4i}\int_0^{\infty} \frac{2ie^{iz}-e^{iz}+e^{-iz}}{2i\,z^3}\,dz \qquad \frac{-1}{2}+\frac{1}{4i}$$

$$\underset{0}{res} = \frac{1}{4i}\,\frac{d^2}{dz^2}\left(2ie^{iz}-e^{iz}\right) = \frac{d}{dz}\left(-2e^{iz}-ie^{iz}\right)$$

$$= e^{iz}-2ie^{iz}\Big|_0 = \frac{1-2i}{4i}$$

$$\frac{\pi(1-2i)}{2}$$

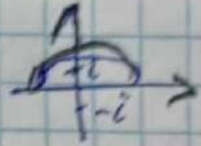
$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+9}\,dz \qquad z = \pm i3$$



$$\underset{3i}{res}\ \frac{e^{iz}}{(z+3i)(z-3i)} = \frac{e^{iz}}{z+3i} = \frac{e^{-3}}{6i}$$

$$\oint = \pi\cdot e^{-3}/3$$

$$\int_{-\infty}^{\infty} \frac{\cos(x-\frac{1}{x})}{1+x^2}\,dx \qquad x=\pm i$$



$$\frac{\cos x \cos\frac{1}{x} + \sin x \sin\frac{1}{x}}{1+x^2} \qquad \frac{\frac{1}{4}(e^{ix}+e^{-ix})^2 - \frac{1}{4}(e^{2ix}-e^{-2ix}}{1+x^2} =$$

$$\underset{i}{res} = \frac{1}{4}\,\frac{\cancel{e^{2ix}} + e^{-2ix} + 2e^{ix}e^{-ix} - \cancel{e^{2ix}} + e^{-2ix}}{(x+i)(x-i)} =$$

$$= \frac{2e^{-2ix} + 2e^{ix}e^{-ix}}{4(x+i)} = \frac{e^2 + e^{-1}e^{1} + 1}{4i}$$

$$\oint = \pi i \cdot \frac{e^2+1}{2}$$