Question 3

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Question 1. We have been given the function:

$$f(c) = z^2 + c$$

which is defines recursively, and that $c \in \{w = x + iy : x \in [-2, 2], y \in [-2, 2]\}$. We want to determine all the points for which the function stays bounded.

Lemma 1. We claim that for all $|z_i| > 2$, the function diverges

Proof. Suppose that $|z_i| > 2$ Suppose that $|z_i| = 2 + \lambda$ where $\lambda > 0$. There are two cases that we must consider.
Case 1 When $|c| \leq 2$. We want to show that $|z_{i+k}| \geq 2 + (k+1)\lambda$ We will proceed by mathematical induction. Here, observe that:

$$|z_{i+1}| = |z^2 + c| \geq |z^2| - |c| \geq |z|^2 - |c| \geq (2 + \lambda)^2 - 2 > 2 + 2\lambda$$

In the above, we have used the fact that $|z^2| = |z|^2$ for all complex numbers. Thus, the lemma holds true for $k = 1$. Assume that the lemma holds true for all $|z_{i+j}| : j \leq k$. We must now prove that it holds for $k + 1$.

$$|z_{i+k+1}| = |z_{i+k}^2 + c| \geq |z_{i+k}^2| - |c| \geq (2 + (k+1)\lambda)^2 - 2 \geq 2 + (k+2)\lambda$$

Then, observe that:

$$\lim_{k \to \infty} |z_{i+k}| = \lim_{k \to \infty} 2 + (k+1)\lambda = \infty$$

Thus, the function diverges for these values of c.
Case 2. Now, we will prove that it diverges for $|c| > 2$. Consider two possible sub-cases:
(A) $2 < |c| < |z_i|$: Here, we claim that $|z_{i+k}| \geq 2 + (k+1)\lambda$

$$|z_{i+1}| = |z_i^2 + c| \geq |z_i^2| - |c| \geq 2 + 2\lambda$$

Following a similar argument as in **Case 1**, we can prove that our claim is true. Then:

$$\lim_{k \to \infty} |z_{i+k}| = \lim_{k \to \infty} 2 + (k+1)\lambda = \infty$$

(B) $2 < |z_i| < |c|$ Here we will again split it into two further sub-cases.
(i) $|z_i^2| > |c|$. Then, we can proceed by a similar argument as in the first case and get that $|z_{i+k}| \to \infty$ as $k \to \infty$.
(ii) $|c| > |z_i^2|$. Assume that $|c| = |z_i^2| + \beta$ for some $\beta > 0$. Then:

$$|z_{i+1}| = |z_i^2 + c| \geq |c| - |z_i^2| > |c|$$

Then, we will end up in case A, so the lemma holds true.
Thus, it has been proven that the function diverges for all $|z_i| > 2$. $\square$

From this, we know that all points that we can set a rough criteria of divergence to be if some $|z_i| > 2$ in the iteration. Therefore, that is the criteria of divergence that we will be using. We will be running roughly a 10 iterations, claiming that if the function doesn't diverge in ten iterations then it doesn't diverge at all. Using this criteria, we find a rough estimate for the points to be:

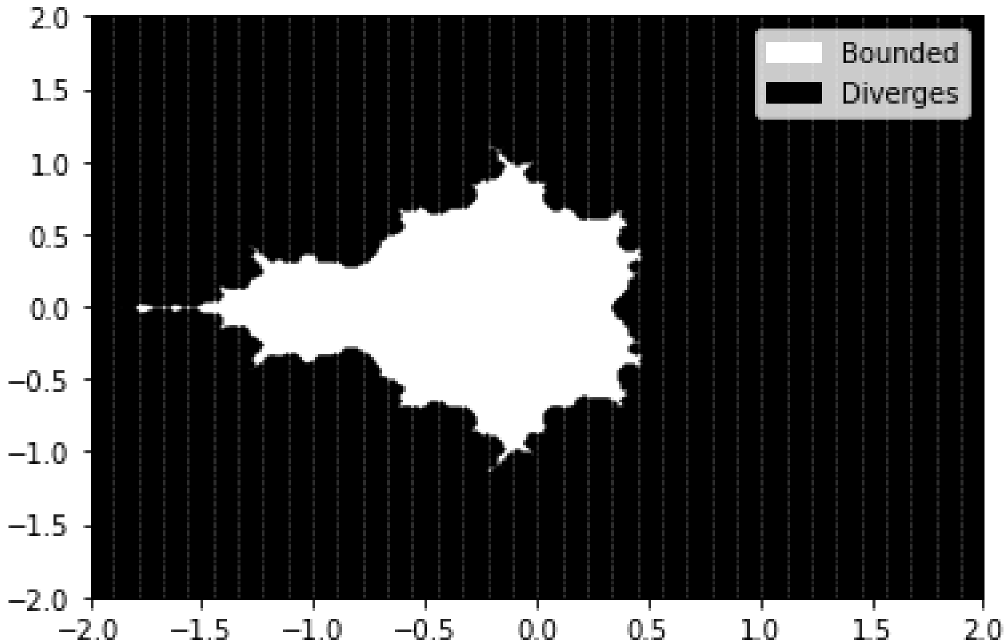

Figure 1: Plot displaying the set of points that remain bounded.

Now, we will find the iteration number for which the function diverges. To do this, we will find $i : |z_i| > 2$.

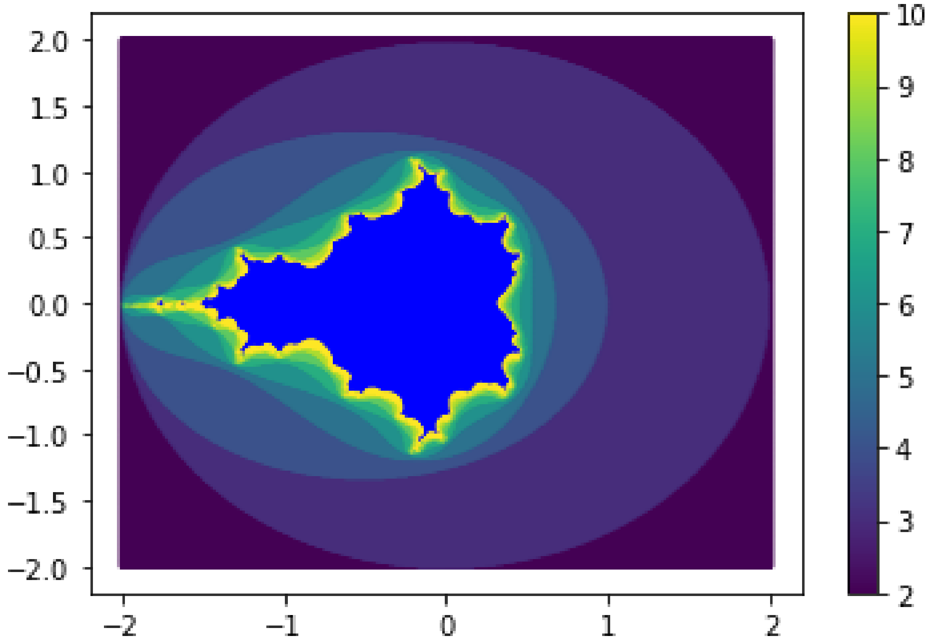

Figure 2: Plot displaying the iteration number at which points diverge.

Question 2. Here, we will be analyzing the three Fourier modes that give the Lorenz equations:

$$\dot{X} = \sigma(X - Y)$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{Z} = -bZ + XY$$

Once a python function is written for $W = (X, Y, Z)$, we can use solve_ivp to solve the equation with the given initial conditions $W_0 = (0, 1, 0)$ and $(\sigma, r, b) = (10, 28, 8/3)$

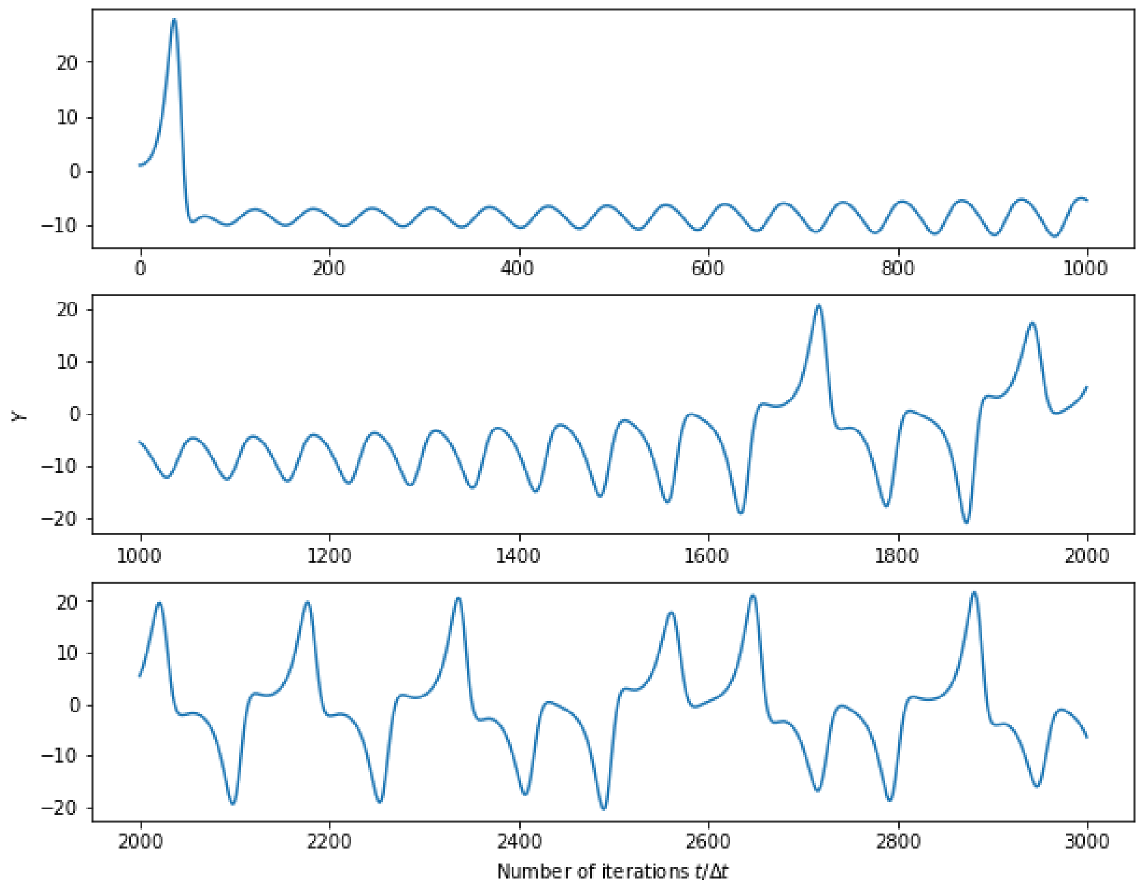

Figure 3: A plot showing the value of the Y Fourier mode against the number of iterations. Here, the number of iterations is represented as $t/\delta t$, where $\delta t = 0.01$

Now, we will perturb the initial conditions slightly, and observe the change that this cause in W

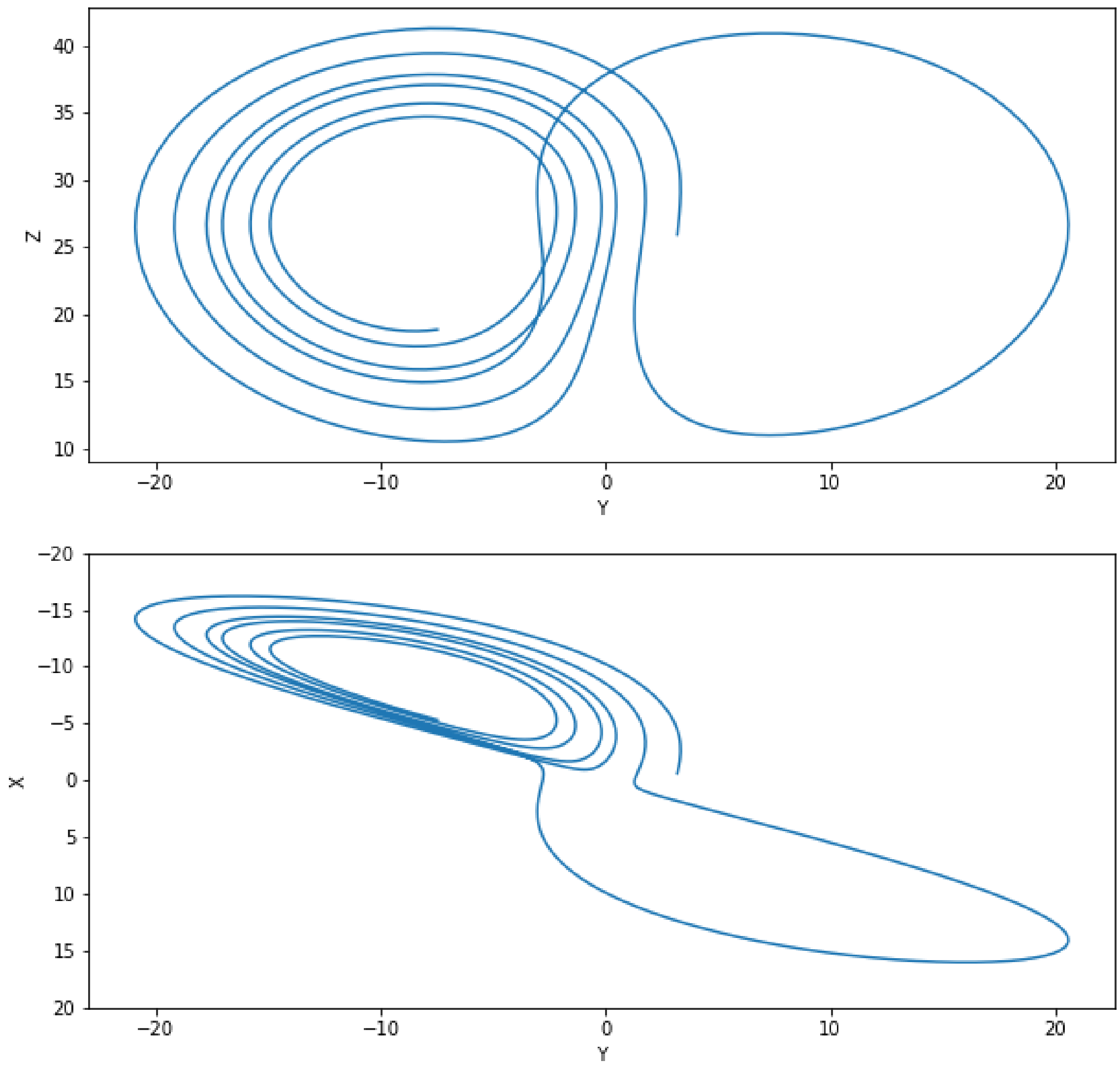

Figure 4: A plot showing the phase portrait of Z against Y AND X against Y.

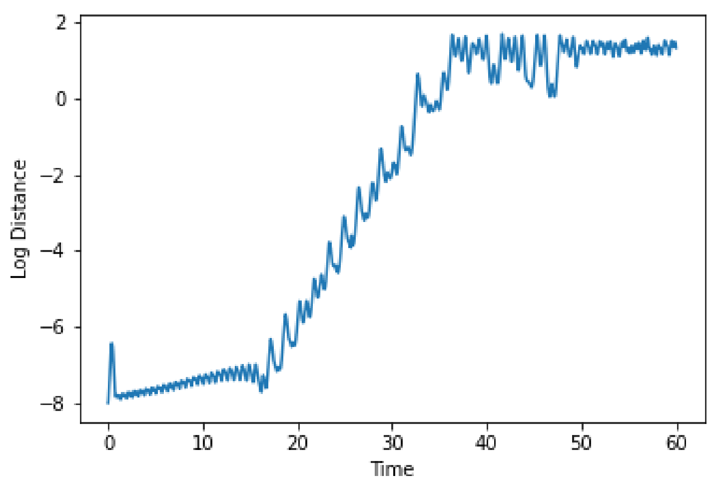

Figure 5: A plot of the logarithmic distance between W and W' against time. W started with initial conditions $W_0 = (0, 1, 0)$ and the initial conditions for W' were $W'_0 = (0, 1{+}1{\times}10^{-8}, 0)$