Digital Image Processing

Lecture # 07
Frequency Domain Image Analysis

Image Enhancement in Frequency Domain

Joseph Fourier (1768 – 1830)

- Most famous for his work *"La Théorie Analitique de la Chaleur"* published in 1822
- Translated into English in 1878: *"The Analytic Theory of Heat"*

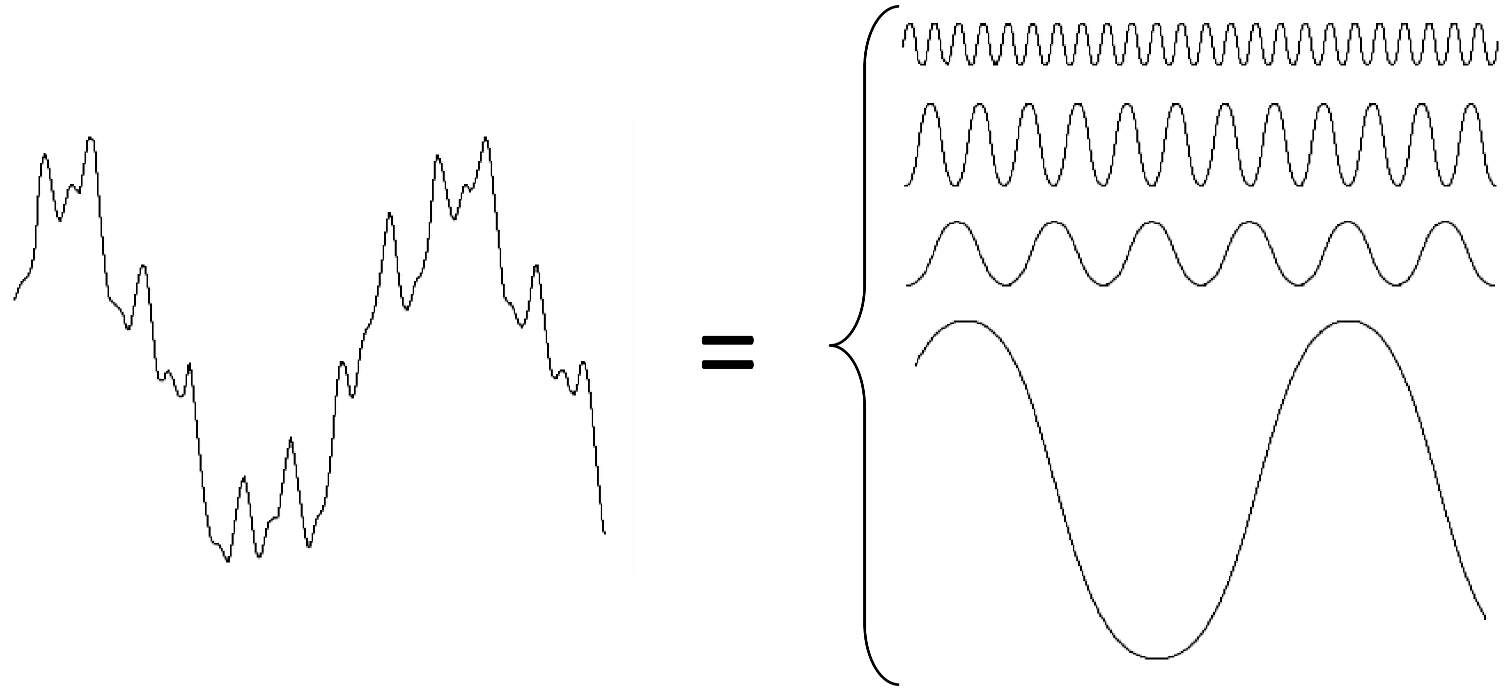
Nobody paid much attention when the work was first published
One of the most important mathematical theories in modern engineering

Background

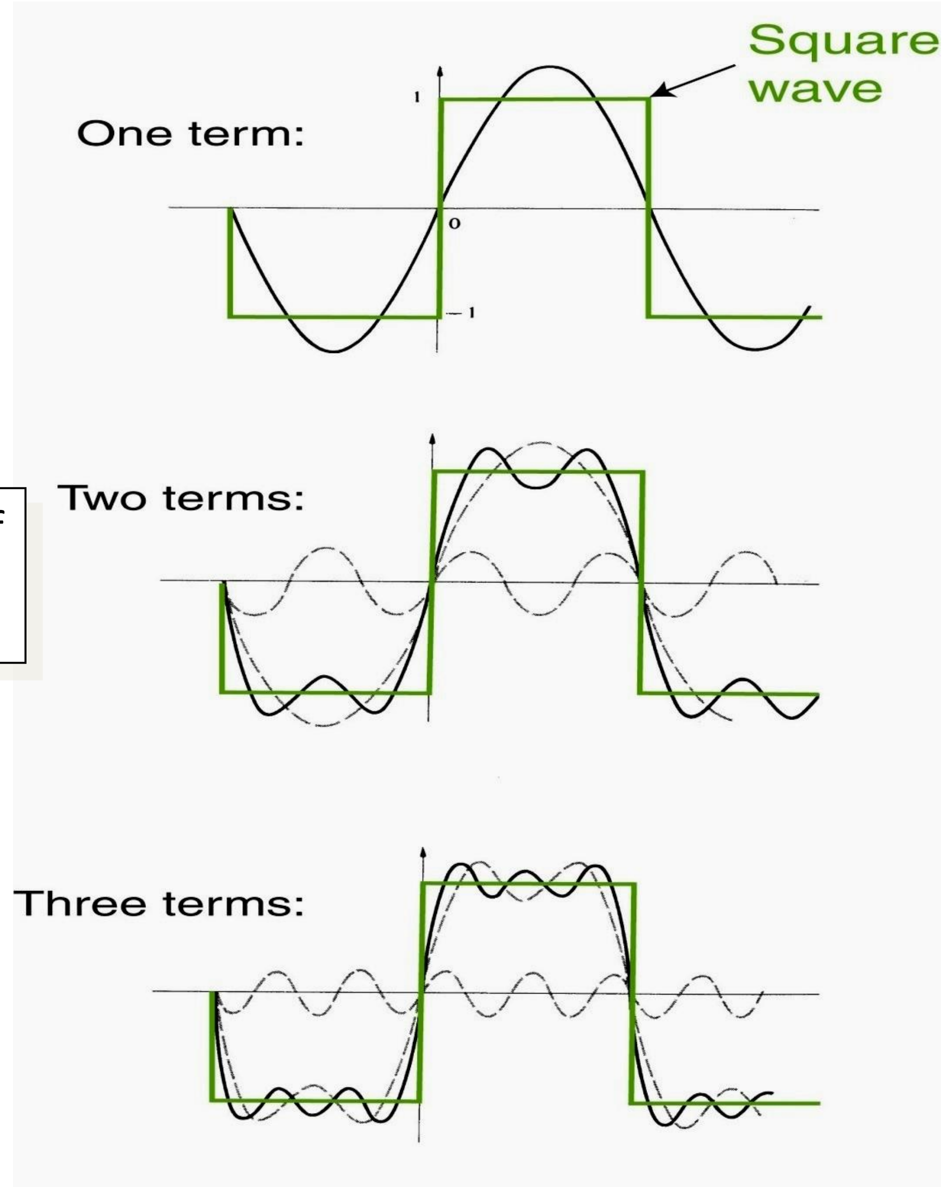
- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).
- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function (Fourier transform).

The big idea …

Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

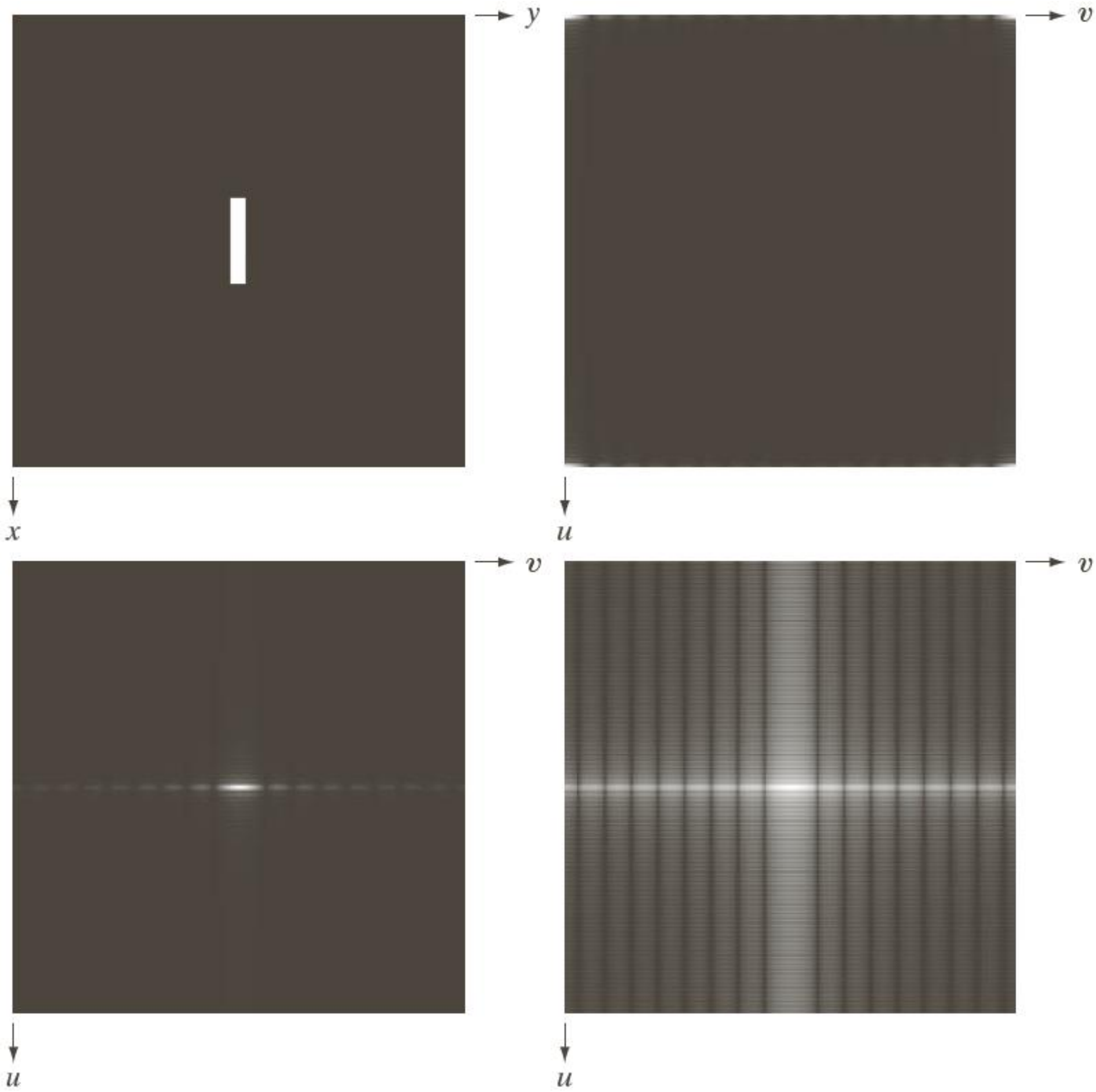


The big idea…



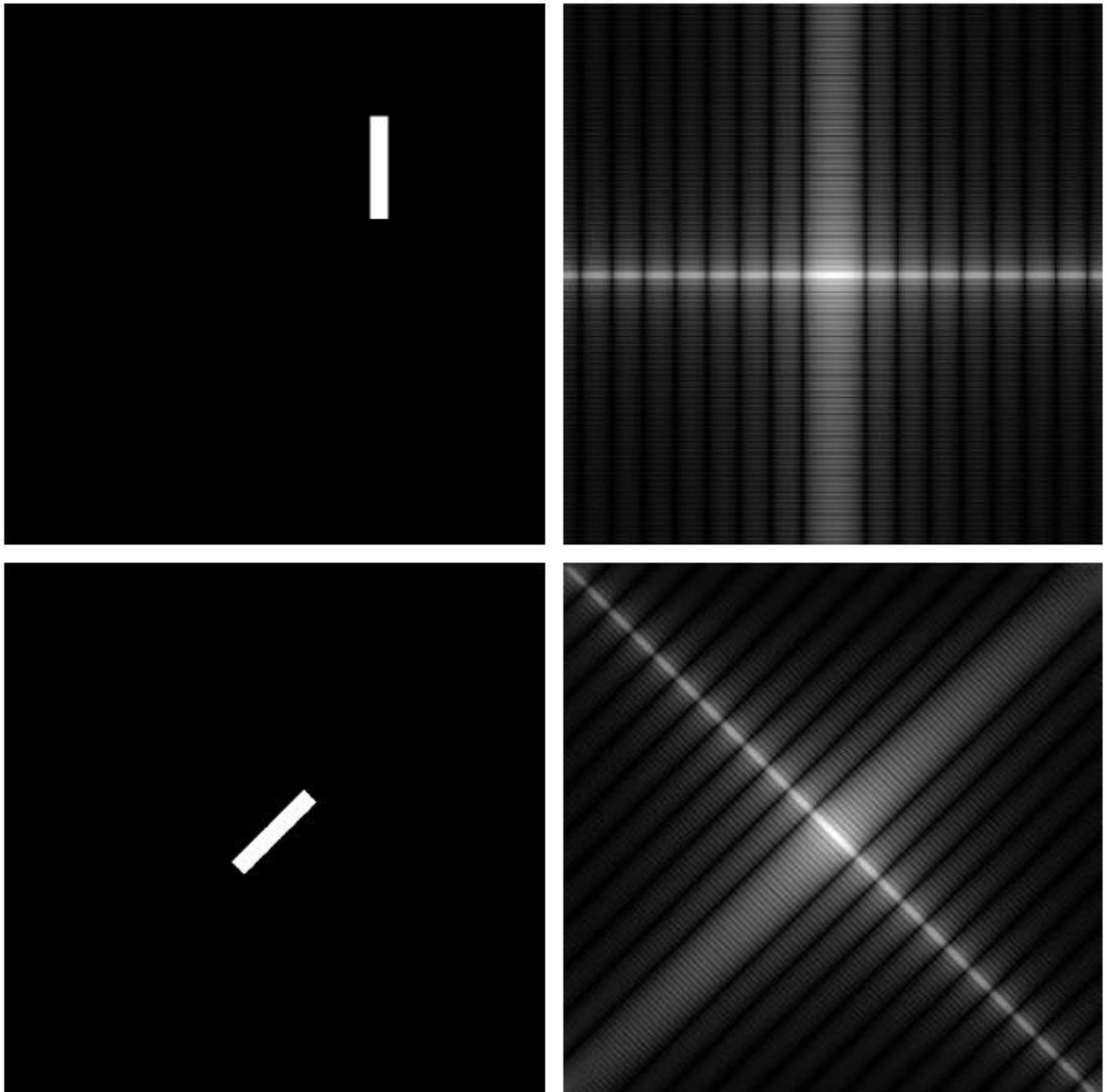
Approximating a square wave as the sum of sine waves

Frequencies in Images



a b
c d

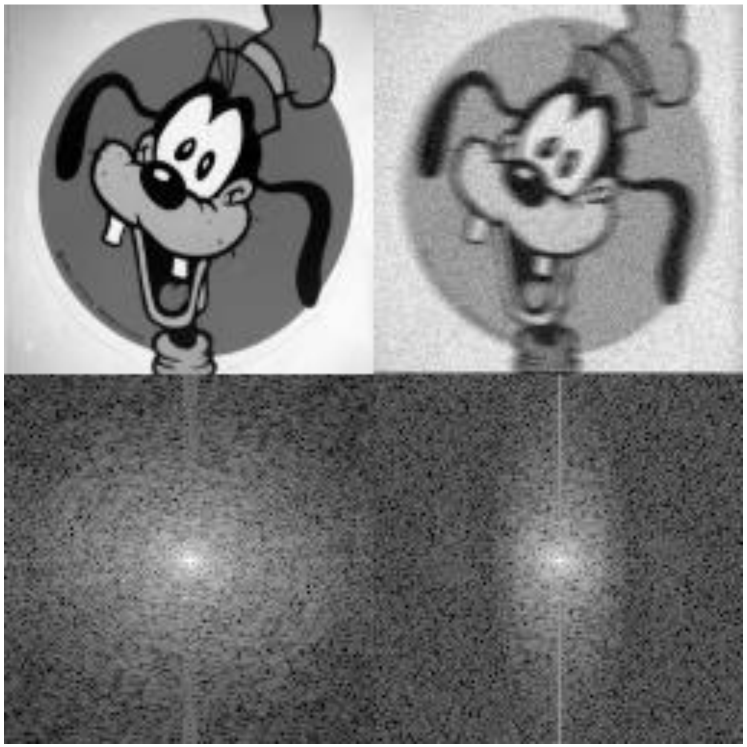
FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

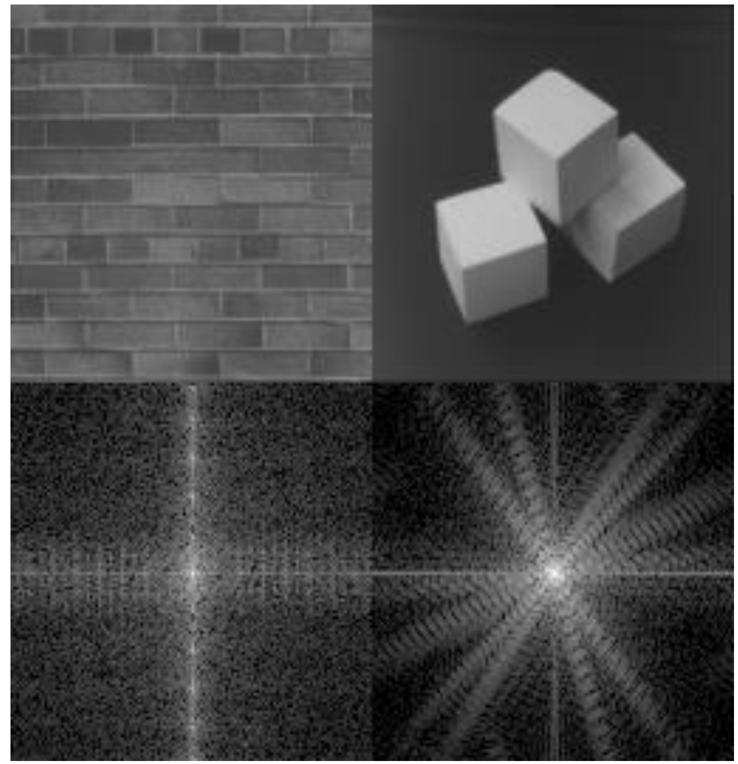


a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

Frequencies in Images





Basic 2D FT

$$x(n_1, n_2) = e^{j(\omega_1' n_1 + \omega_2' n_2)} \longrightarrow \boxed{h(n_1, n_2)} \longrightarrow y(n_1, n_2) = ?$$

(ω_1', ω_2')

$$y(n_1, n_2) = x(n_1, n_2) ** h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j\omega_1'(n_1-k_1)} e^{j\omega_2'(n_2-k_2)} h(k_1, k_2)$$

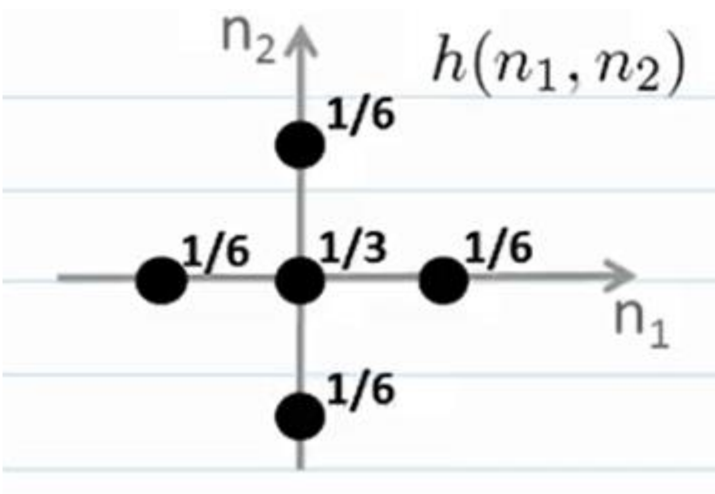
$$= \underbrace{e^{j\omega_1' n_1} e^{j\omega_2' n_2}} \underbrace{\sum_{k_1} \sum_{k_2} h(k_1, k_2) e^{-j\omega_1' k_1} e^{-j\omega_2' k_2}}_{H(\omega_1', \omega_2') \triangleq \text{frequency response}}$$

2D FT

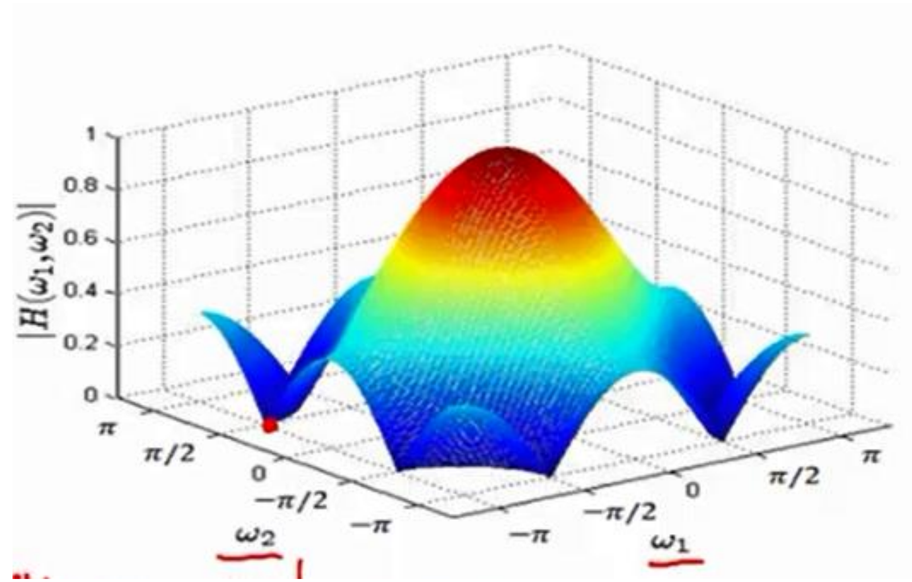
$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

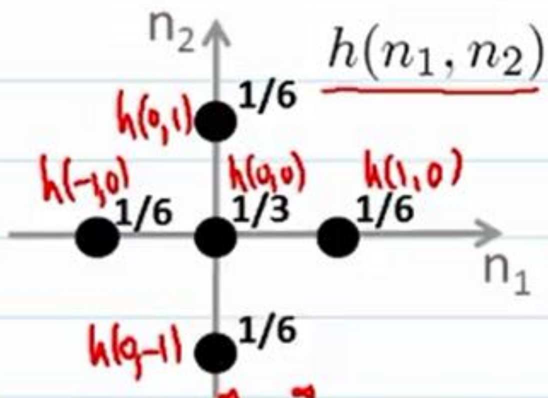
Example

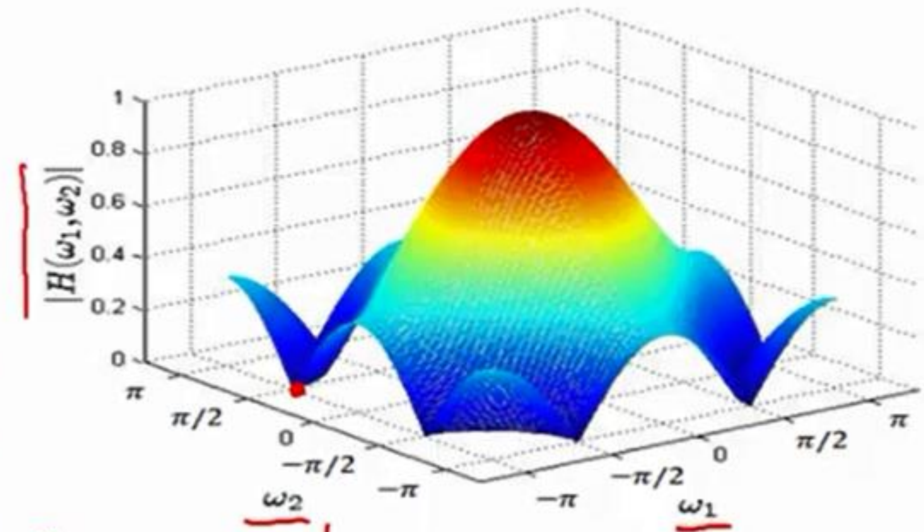


$h(n_1, n_2)$
n_2
n_1
1/6
1/6
1/3
1/6
1/6



$|H(\omega_1, \omega_2)|$
1
0.8
0.6
0.4
0.2
0
π
$\pi/2$
0
$-\pi/2$
$-\pi$
ω_2
$-\pi$
$-\pi/2$
0
$\pi/2$
π
ω_1





$$H(\omega_1,\omega_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} h(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

$$= h(0,0)+h(-1,0)e^{j\omega_1}+h(1,0)e^{-j\omega_1}+h(0,-1)e^{j\omega_2}+h(0,1)e^{-j\omega_2}$$

$$= \frac{1}{3}+\underbrace{\frac{1}{6}e^{j\omega_1}+\frac{1}{6}e^{-j\omega_1}}+\frac{1}{6}e^{+j\omega_2}+\frac{1}{6}e^{-j\omega_2}$$

$$= \frac{1}{3}+\frac{1}{6}2\cos\omega_1+\frac{1}{6}\cdot 2\cos\omega_2$$

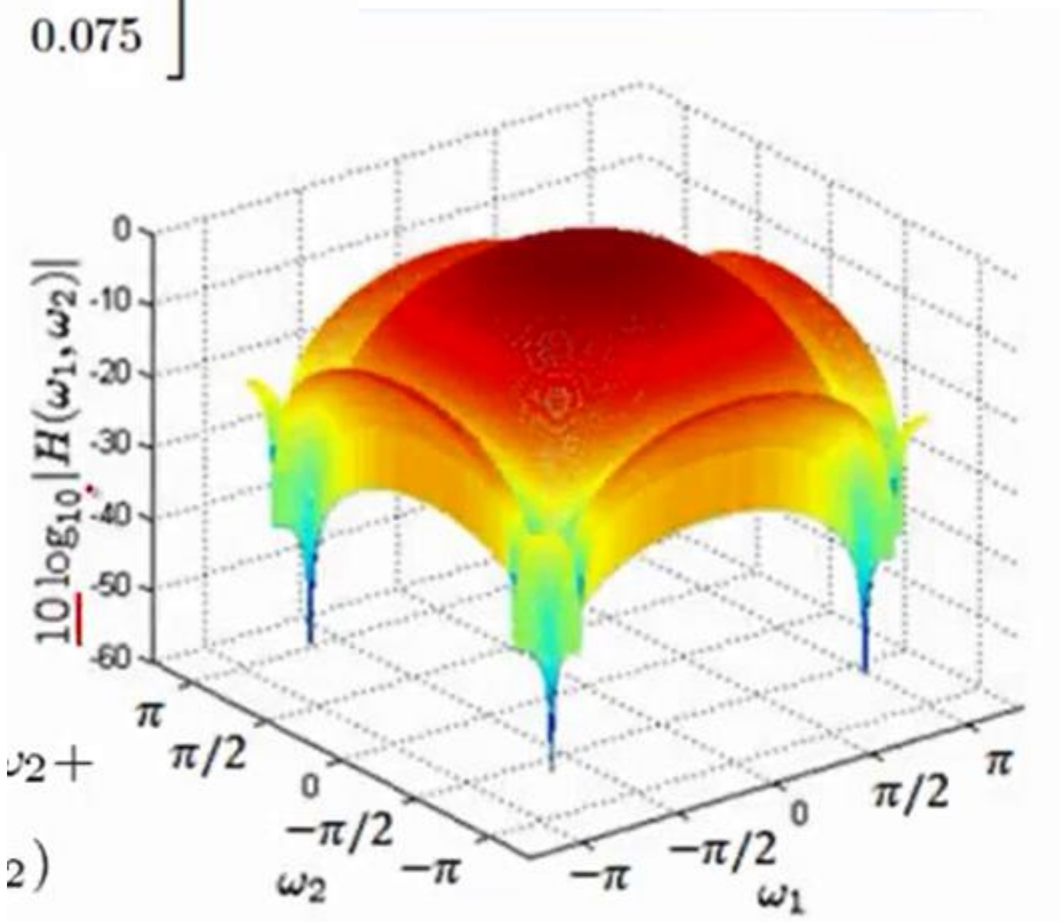
$$= \frac{1}{3}\left(1+\cos\omega_1+\cos\omega_2\right) \qquad -1 = 1\cdot e^{\pm j\pi}$$

$H(0,0)=1$

$H(-\pi,\pi/2)=0$

Example

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$



0.124 · 2 · cos ω_1

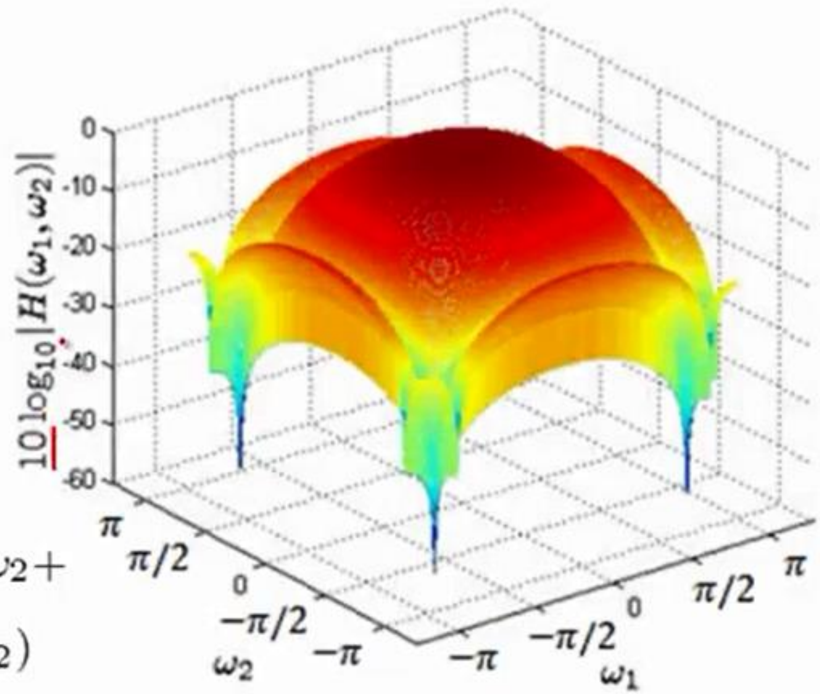
h(-1,1) h(1,1)

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

h(0,0)

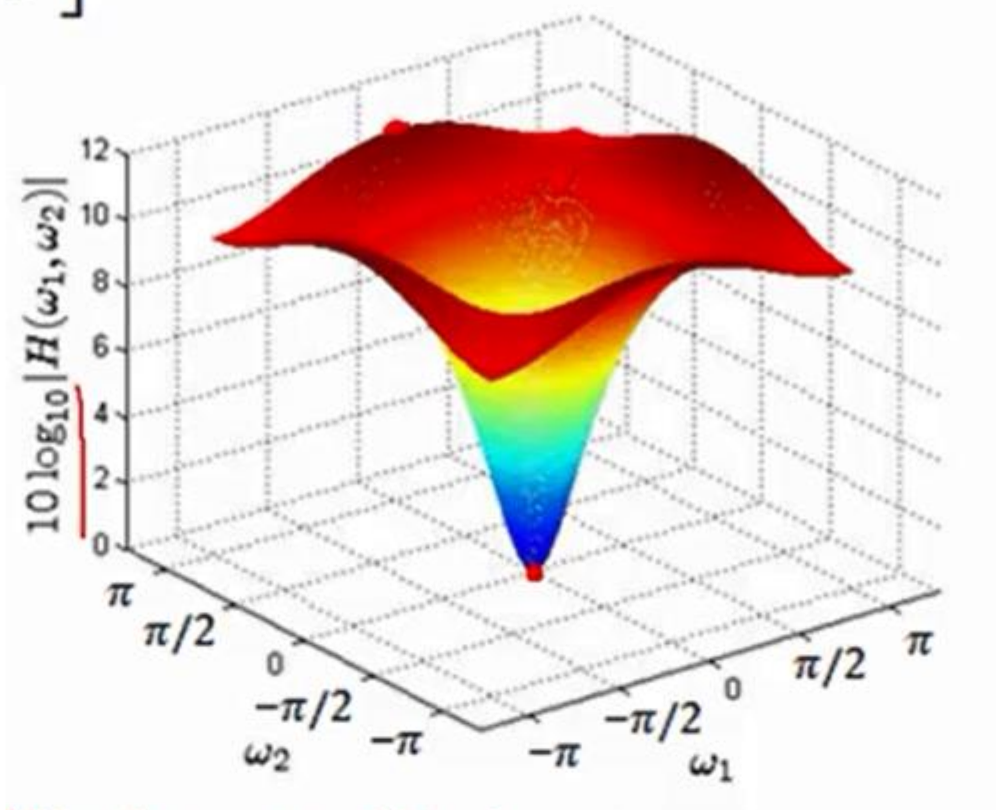
0.075 · 2 · cos(ω_1 + ω_2)

$$H(\omega_1, \omega_2) = 0.204 + 0.124 \cdot 2 \cdot cos\omega_1 + 0.124 \cdot 2 \cdot cos\omega_2 +$$
$$0.075 \cdot 2 \cdot cos(\omega_1 + \omega_2) + 0.075 \cdot 2 \cdot cos(\omega_1 - \omega_2)$$



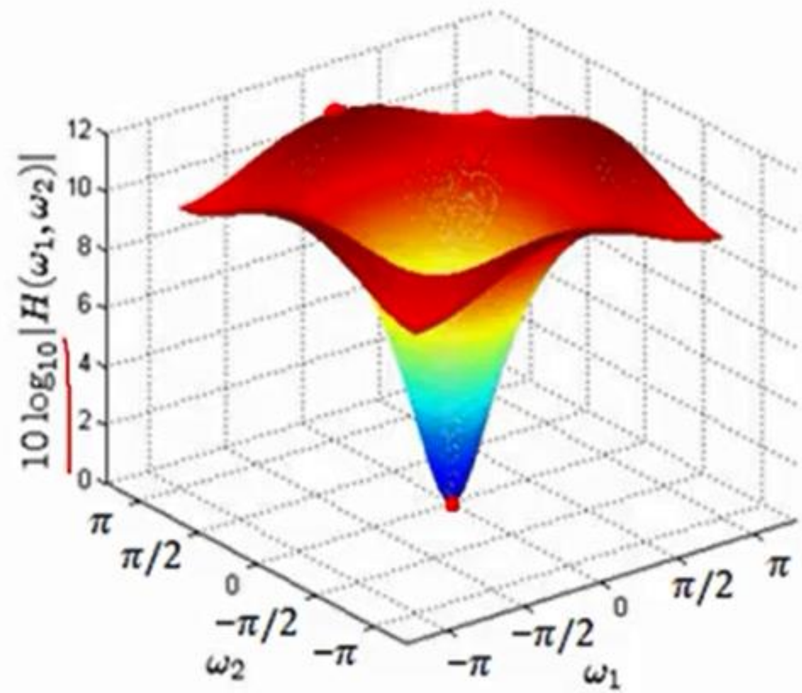
Example

$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$



$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

h(0,0)

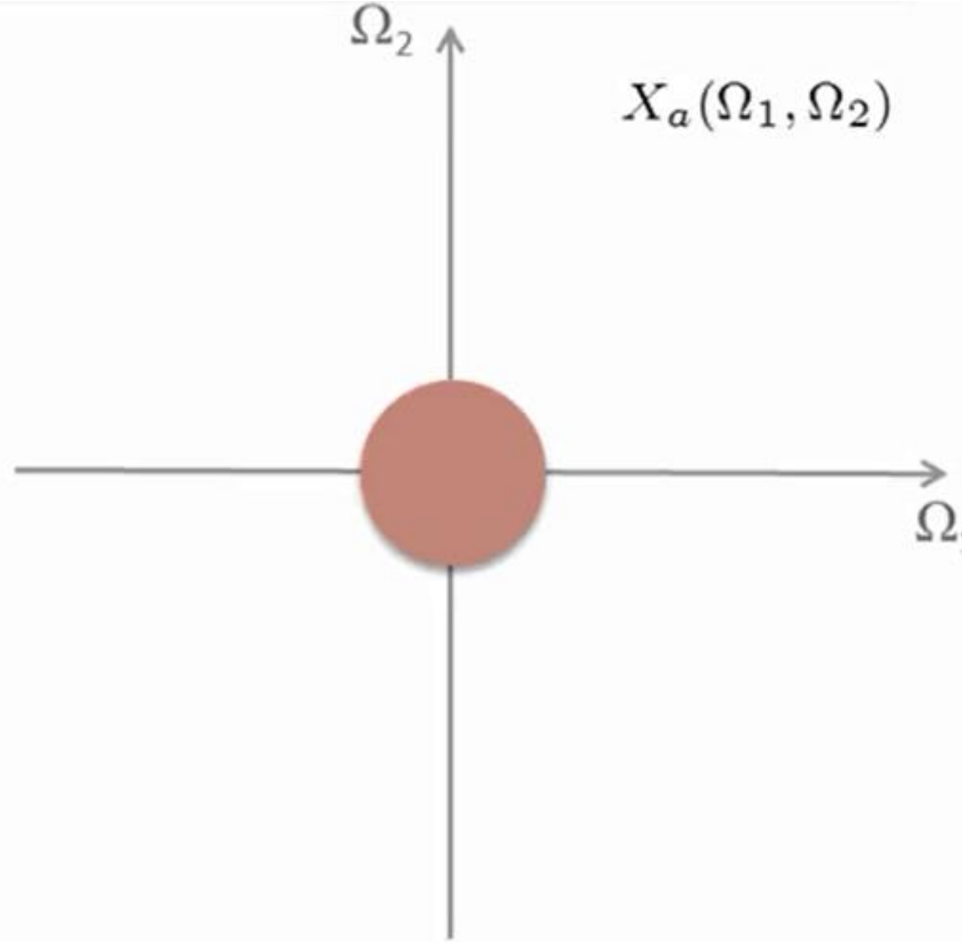


$$H(\omega_1, \omega_2) = 9 - 2\cdot cos\omega_1 - 2\cdot cos\omega_2 - 2\cdot cos(\omega_1 + \omega_2) - 2\cdot cos(\omega_1 - \omega_2)$$

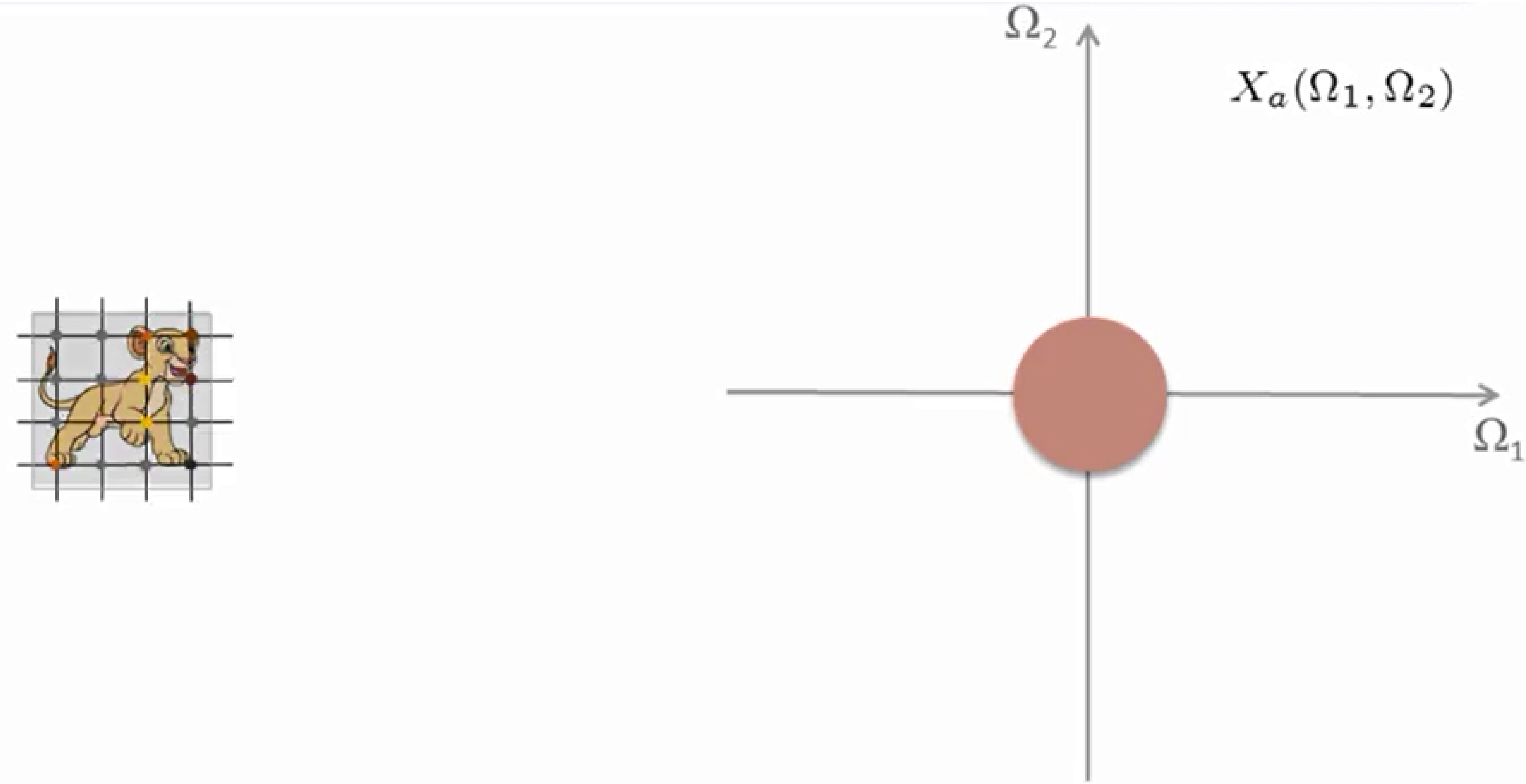
H(0,0)=1 → log H(0,0) = 0

H(0,π) = 13, H(π,π) = 9

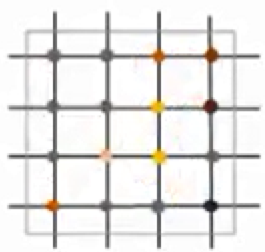
2D Sampling

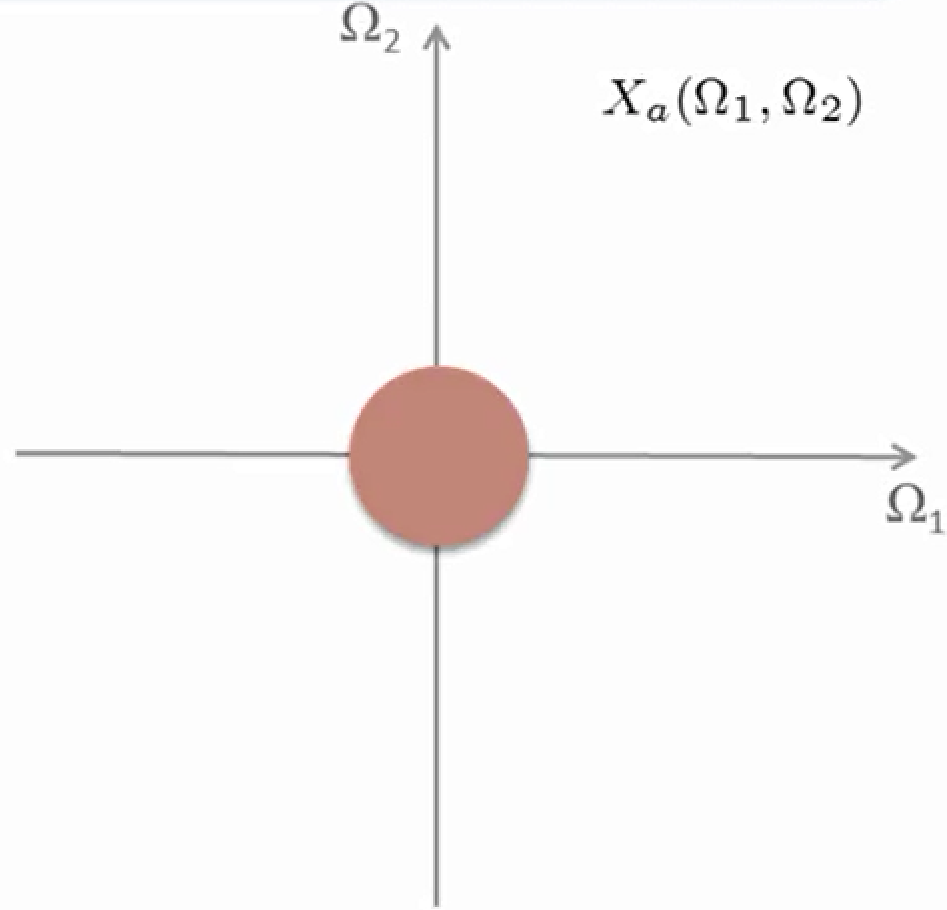


2D Sampling

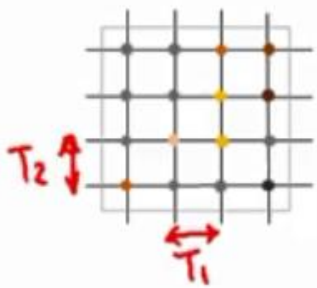


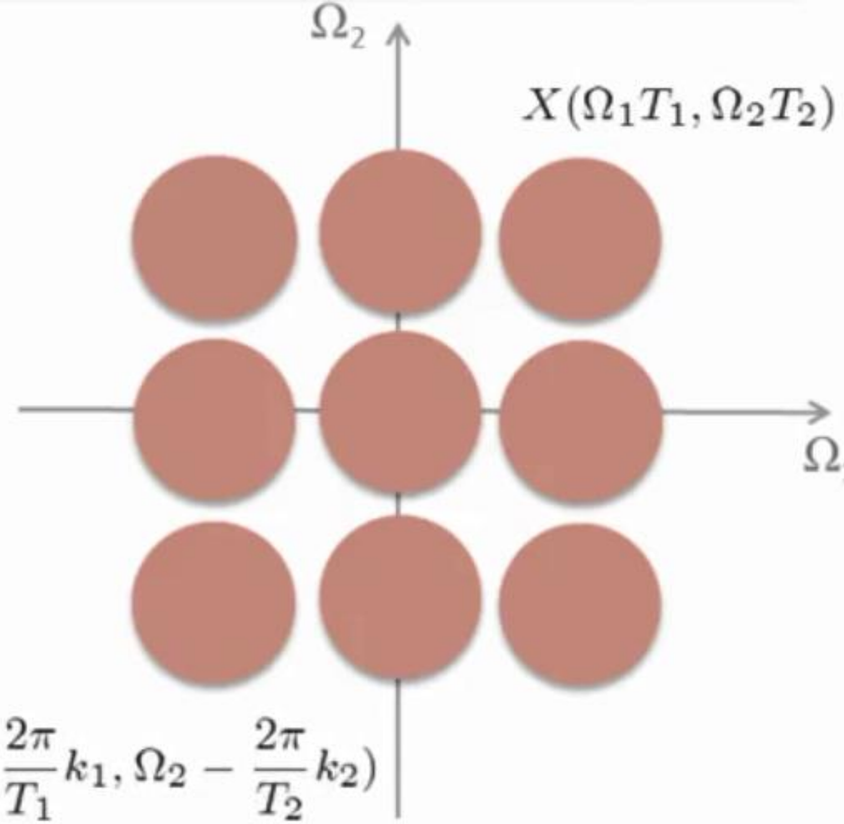
2D Sampling





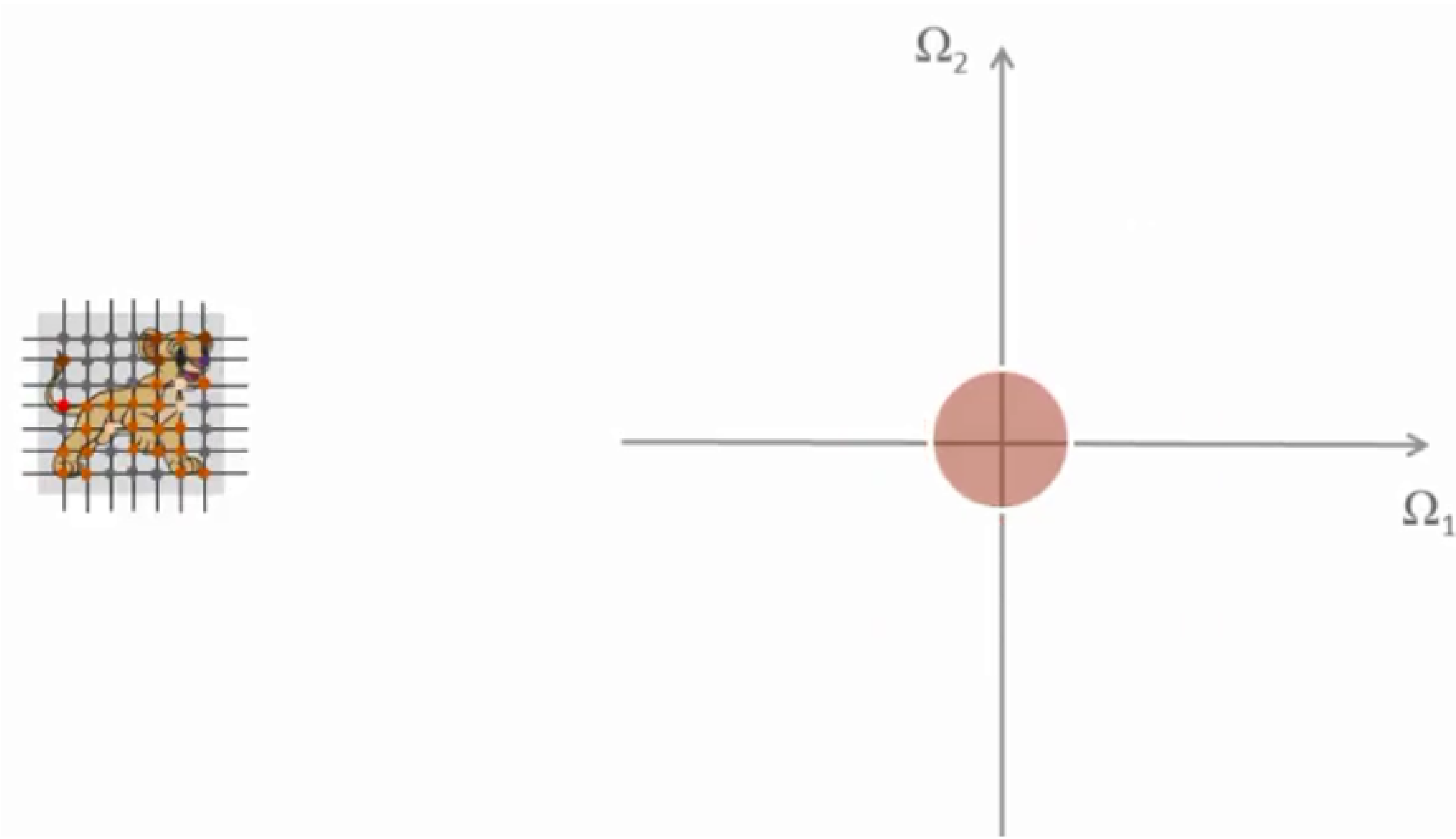
2D Sampling



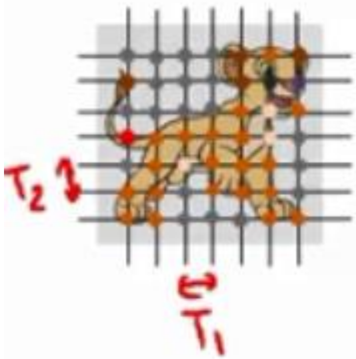


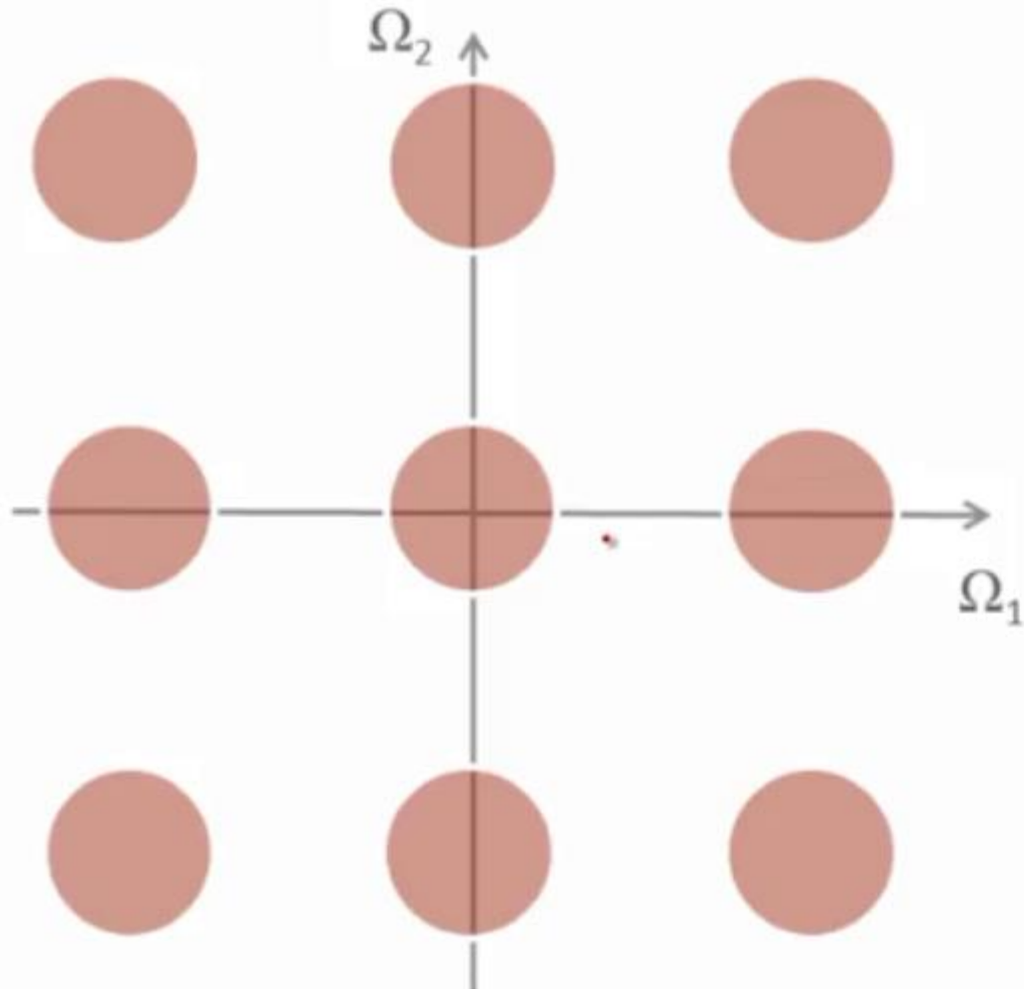
$$X(\Omega_1 T_1, \Omega_2 T_2) = \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a\left(\Omega_1 - \frac{2\pi}{T_1} k_1, \Omega_2 - \frac{2\pi}{T_2} k_2\right)$$

Over Sampling

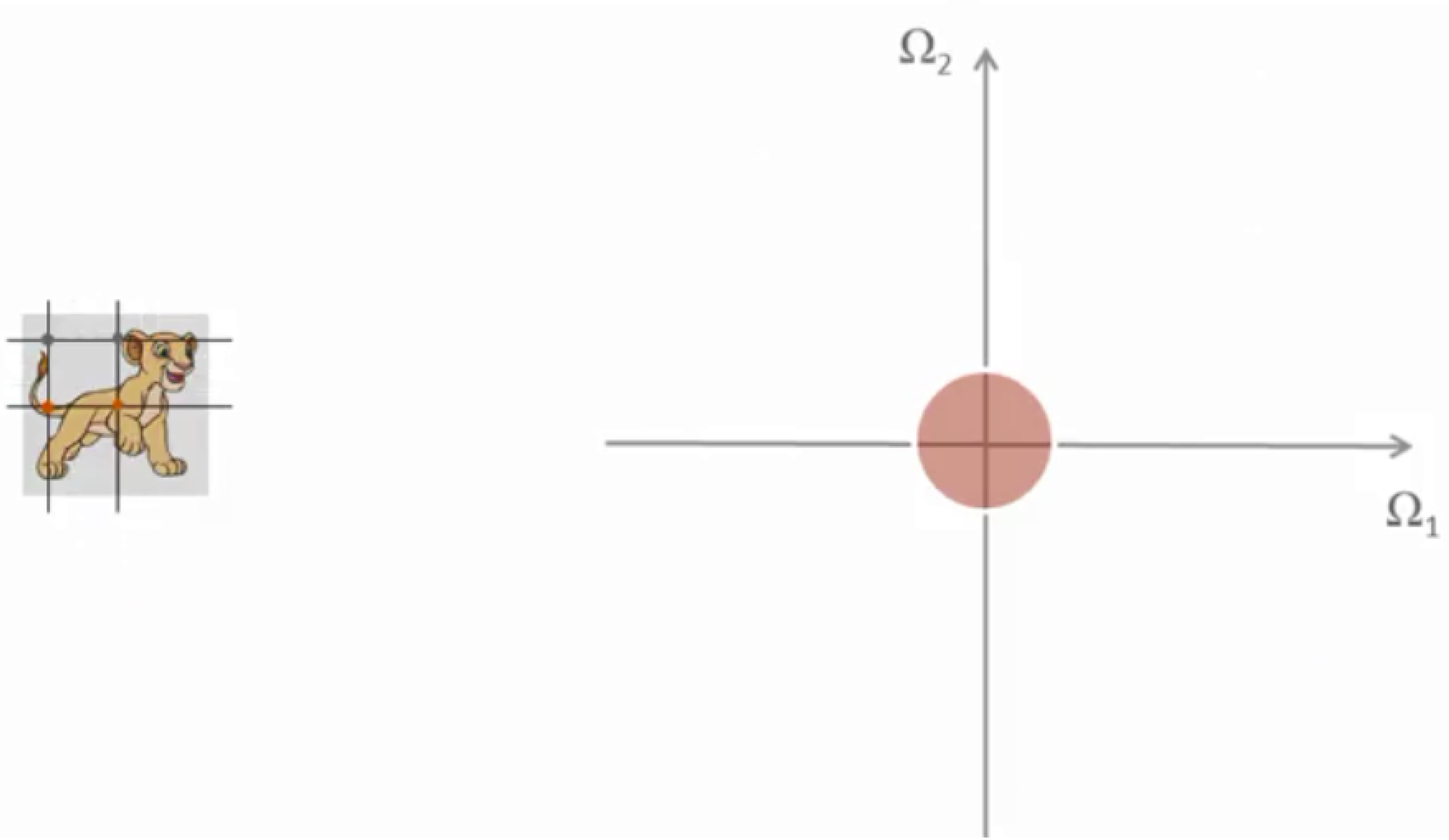


Over Sampling

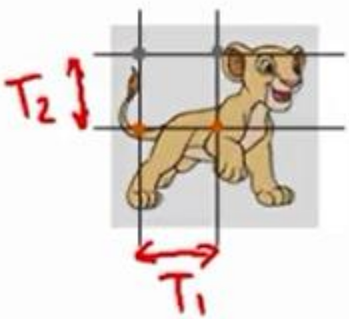


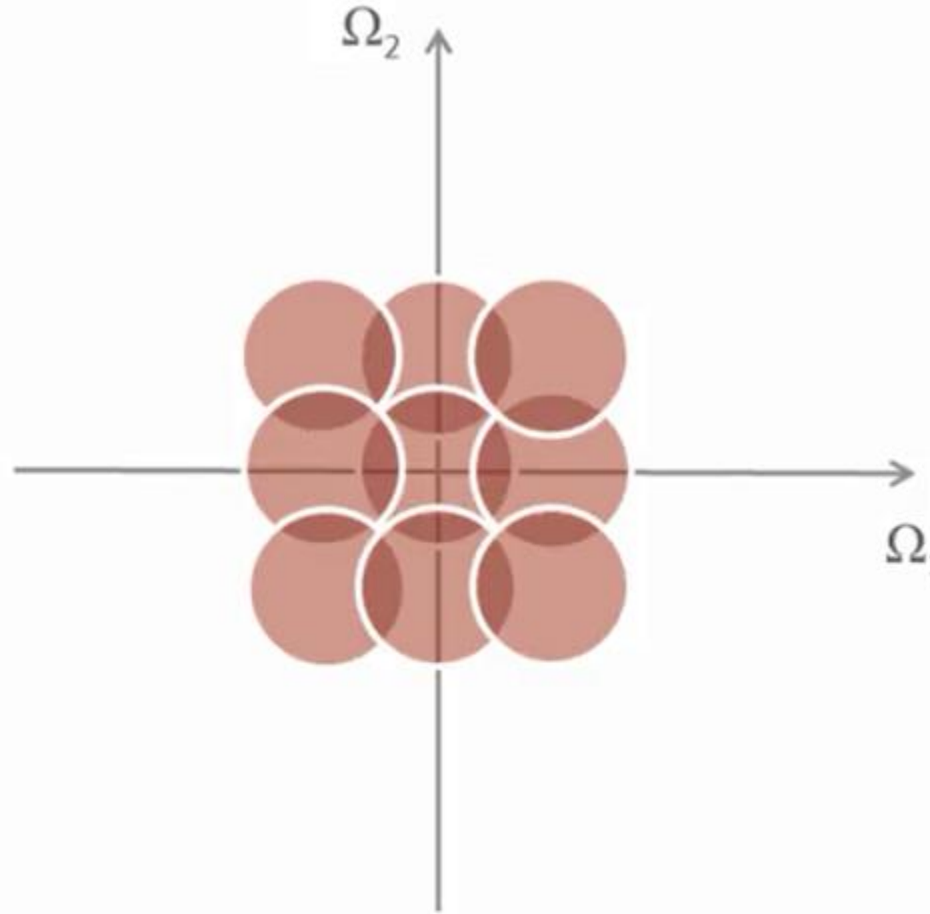


Under Sampling

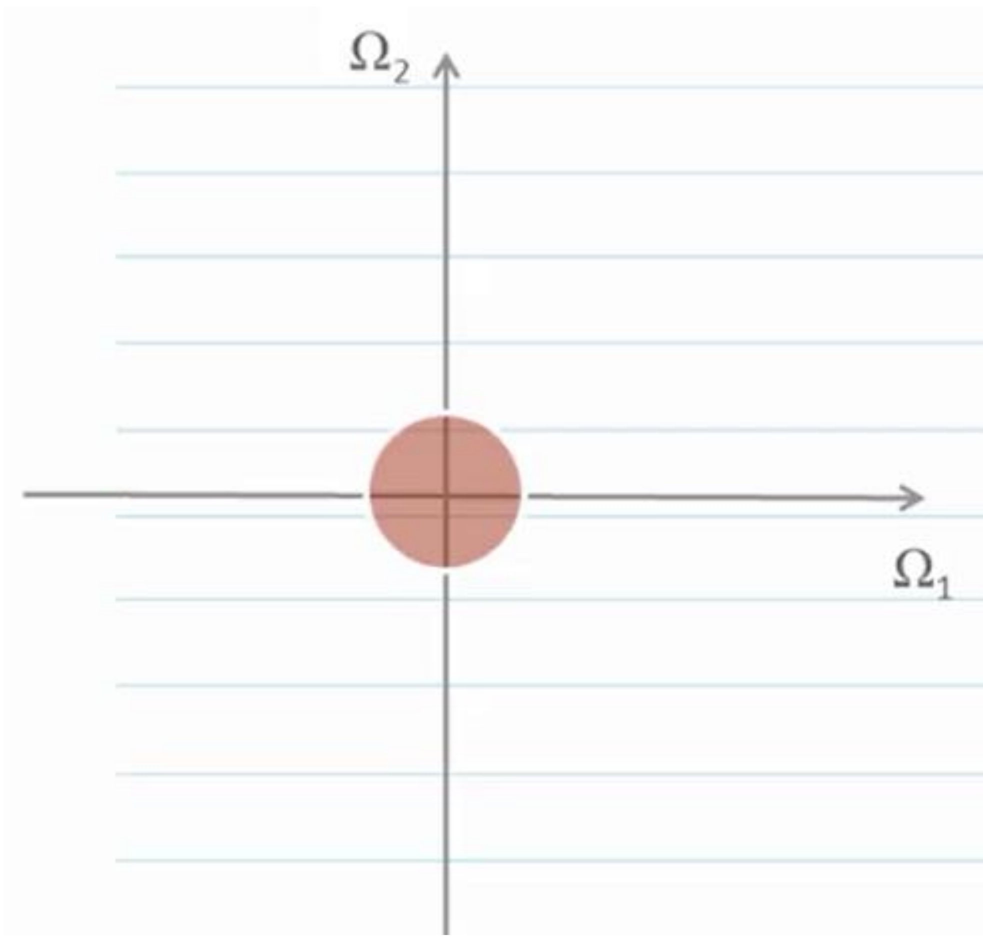


Under Sampling

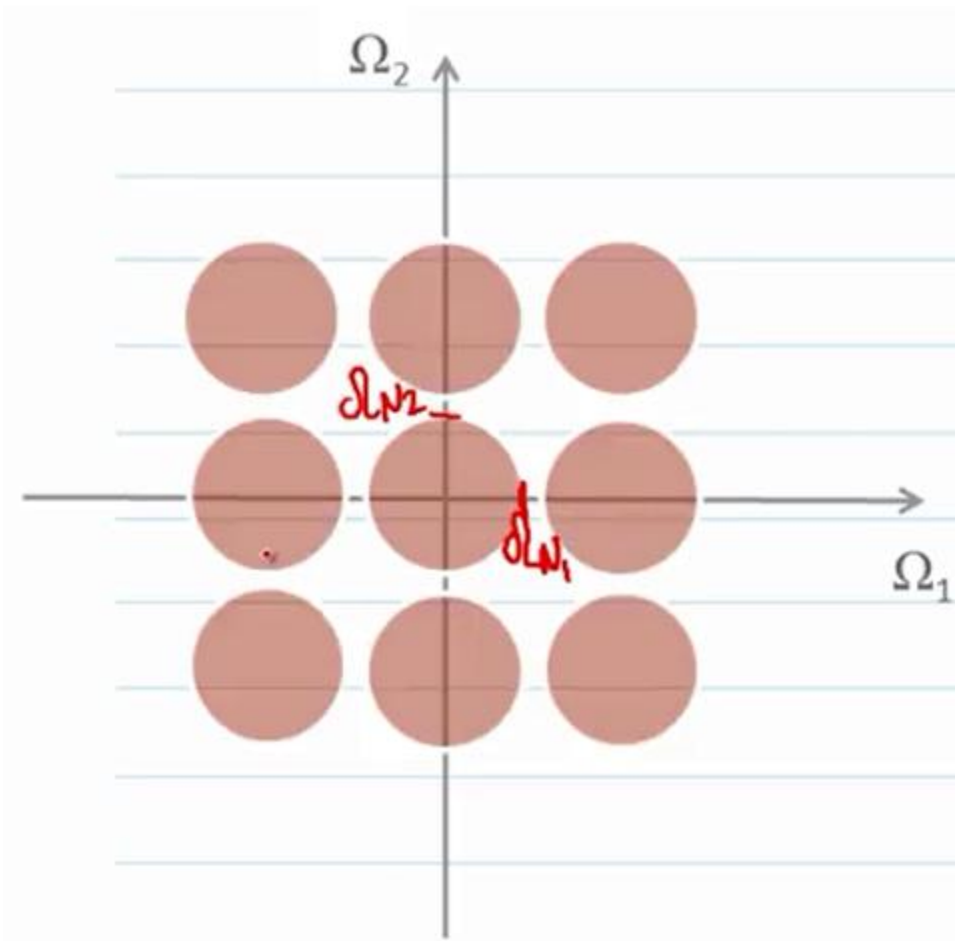




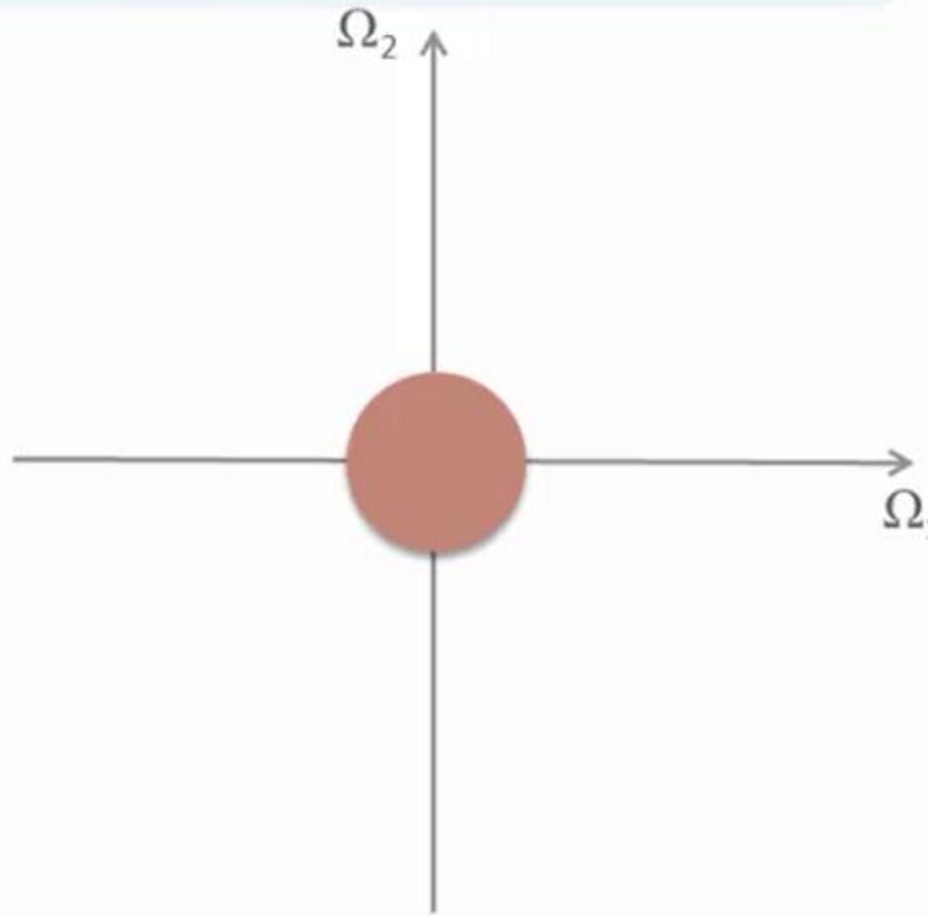
2D Nyquist Theorem



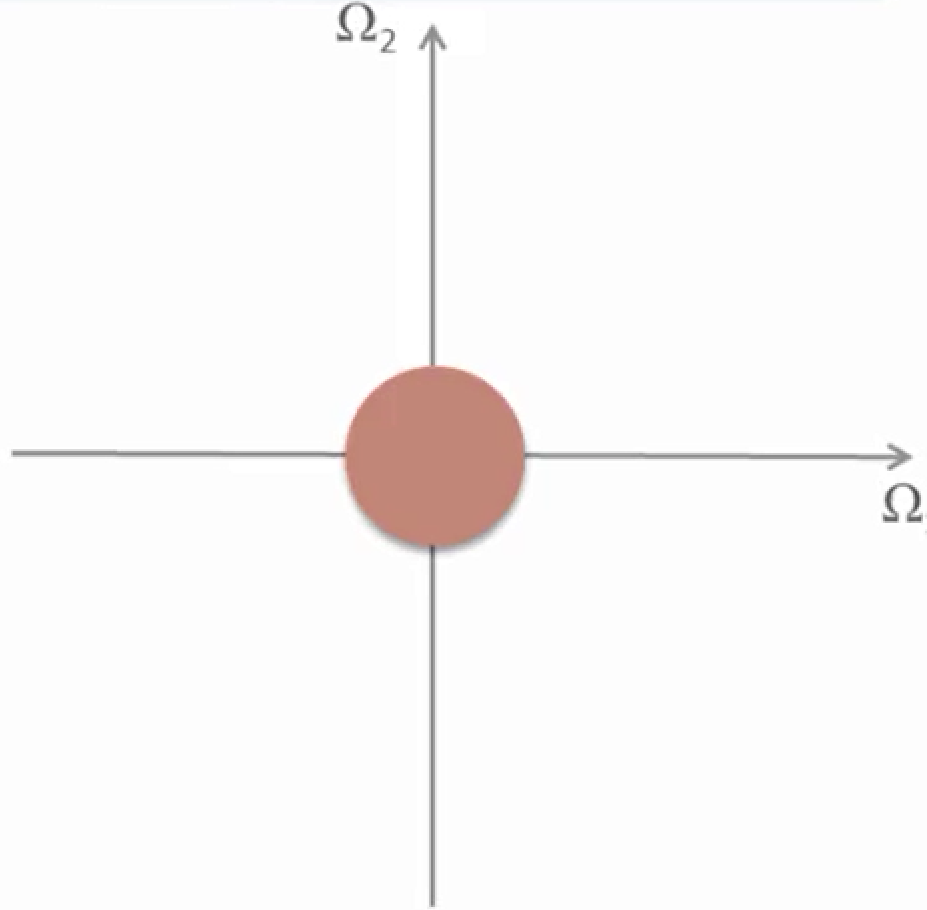
2D Nyquist Theorem



Sampling in the Frequency



Sampling in the Frequency



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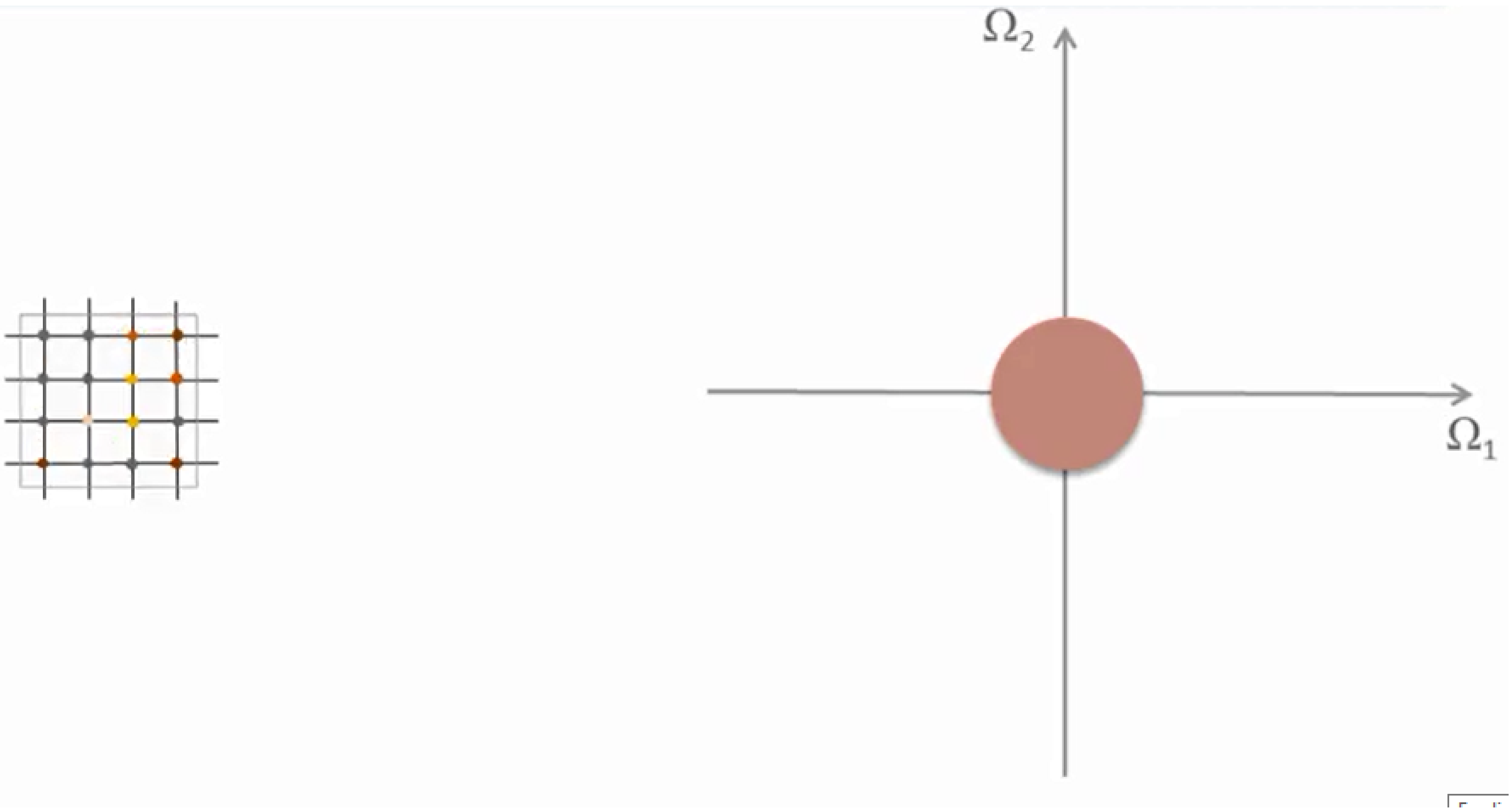

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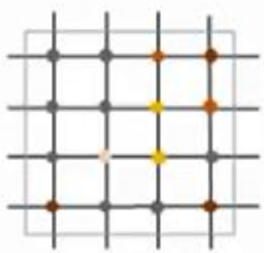
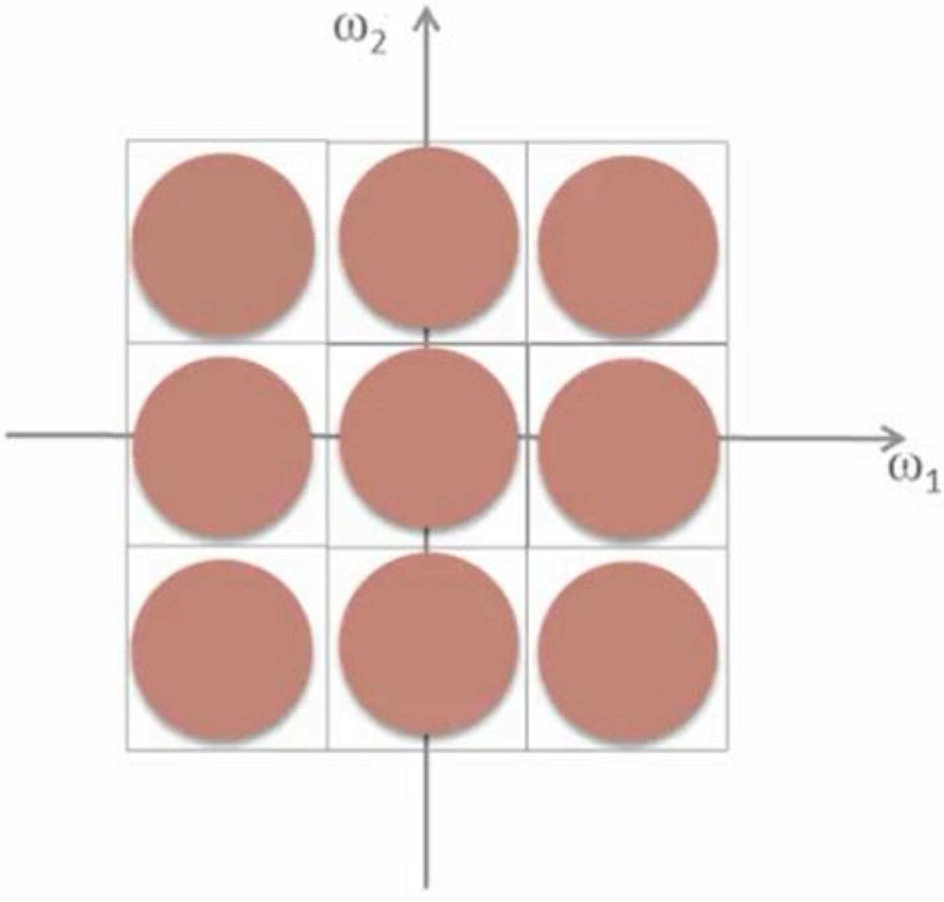
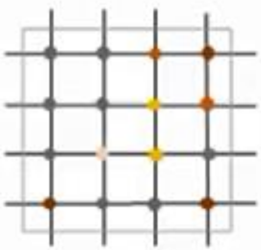
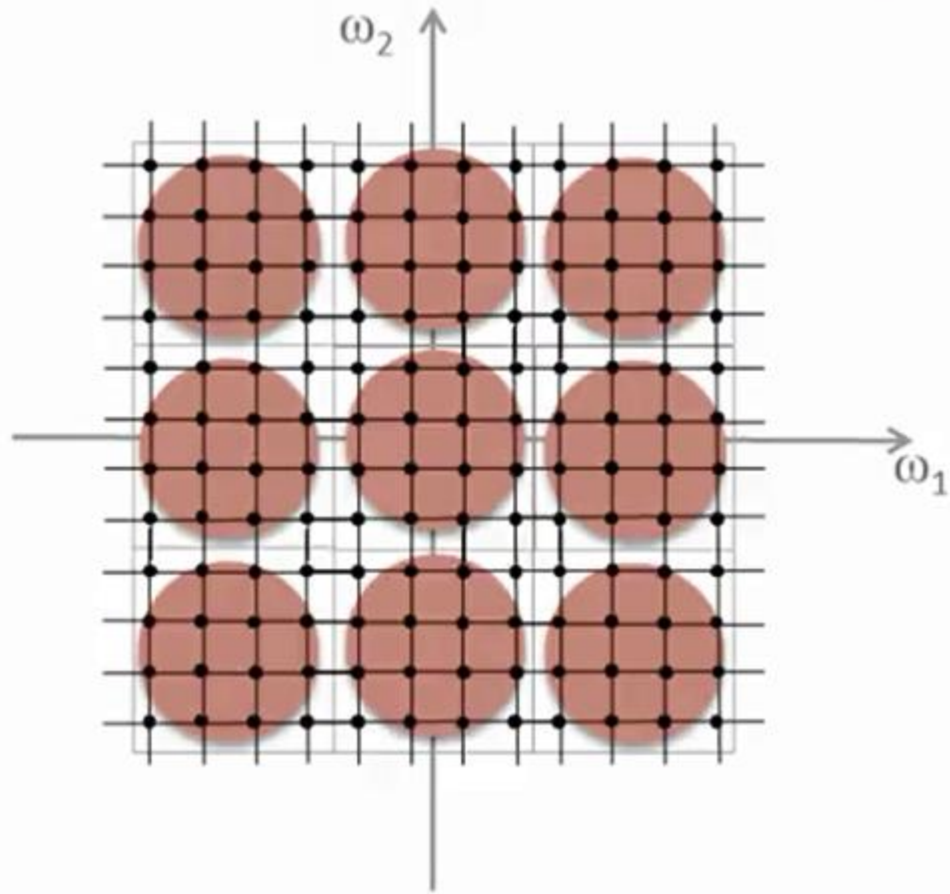


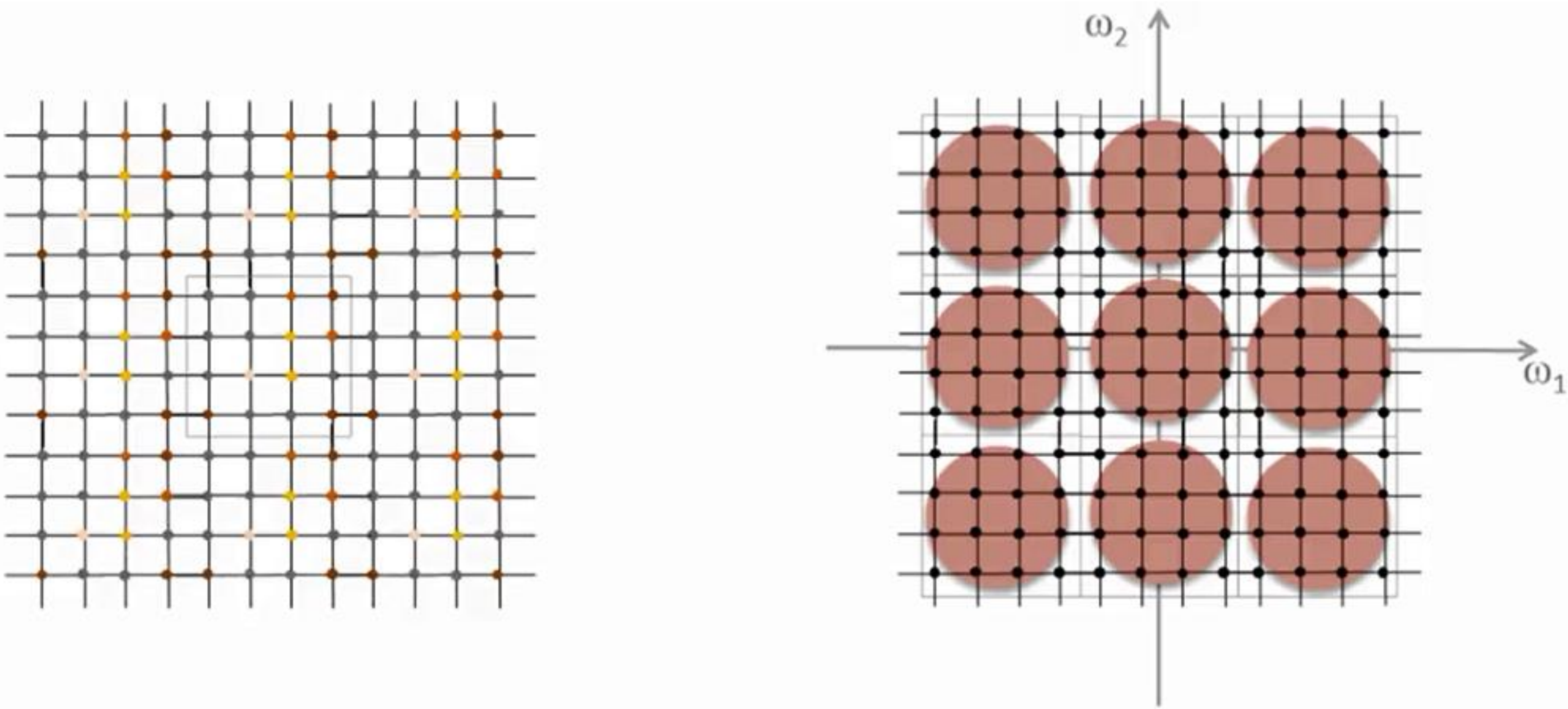


Sampling in the Frequency

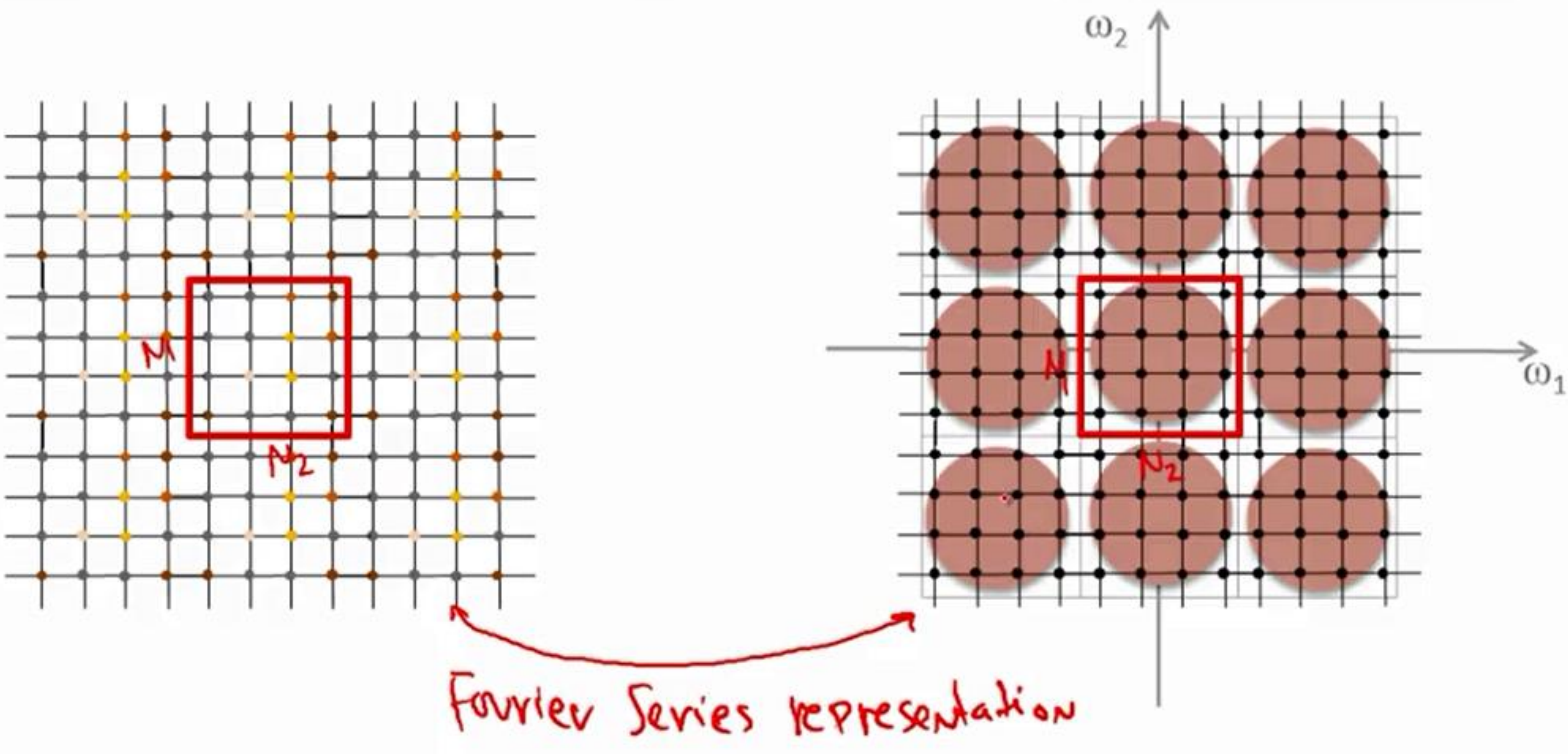




Sampling in the Frequency



Sampling in the Frequency



2D FT to 2D DFT

$$X(\omega_1,\omega_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

$$X(k_1,k_2)=X(\omega_1,\omega_2)\big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2}\qquad \begin{cases} k_1=0,\dots,N_1-1\\ k_2=0,\dots,N_2-1\end{cases}$$

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,n_2)e^{-j\frac{2\pi}{N_1}n_1k_1}e^{-j\frac{2\pi}{N_2}n_2k_2}\qquad \begin{cases} k_1=0,\dots,N_1-1\\ k_2=0,\dots,N_2-1\end{cases}$$

$$x(n_1,n_2)=\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X(k_1,k_2)e^{j\frac{2\pi}{N_1}n_1k_1}e^{j\frac{2\pi}{N_2}n_2k_2}\qquad \begin{cases} n_1=0,\dots,N_1-1\\ n_2=0,\dots,N_2-1\end{cases}$$

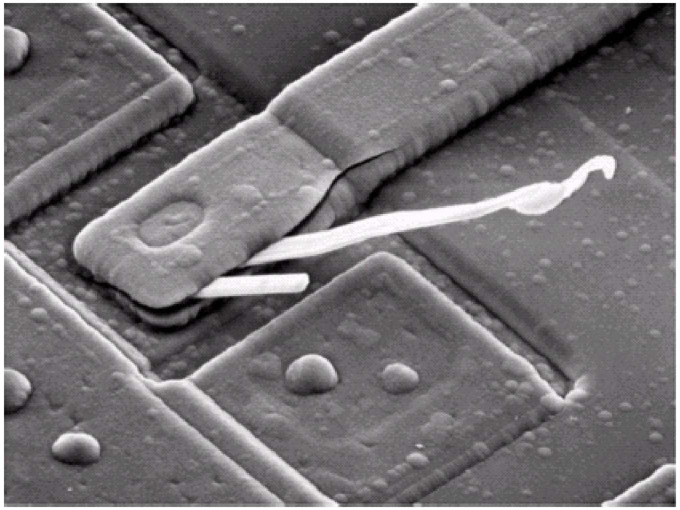
The Discrete Fourier Transform (DFT)

The *Discrete Fourier Transform* of $f(x, y)$, for $x = 0, 1, 2 \ldots M-1$ and $y = 0,1,2 \ldots N-1$, denoted by $F(u, v)$, is given by the equation:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

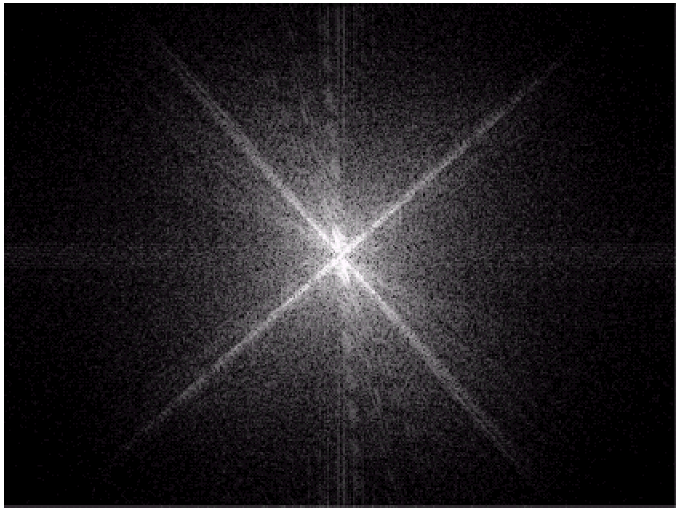
for $u = 0, 1, 2 \ldots M-1$ and $v = 0, 1, 2 \ldots N-1$.

DFT & Images



Scanning electron microscope image of an integrated circuit magnified ~2500 times

DFT



Fourier spectrum of the image

The Inverse DFT

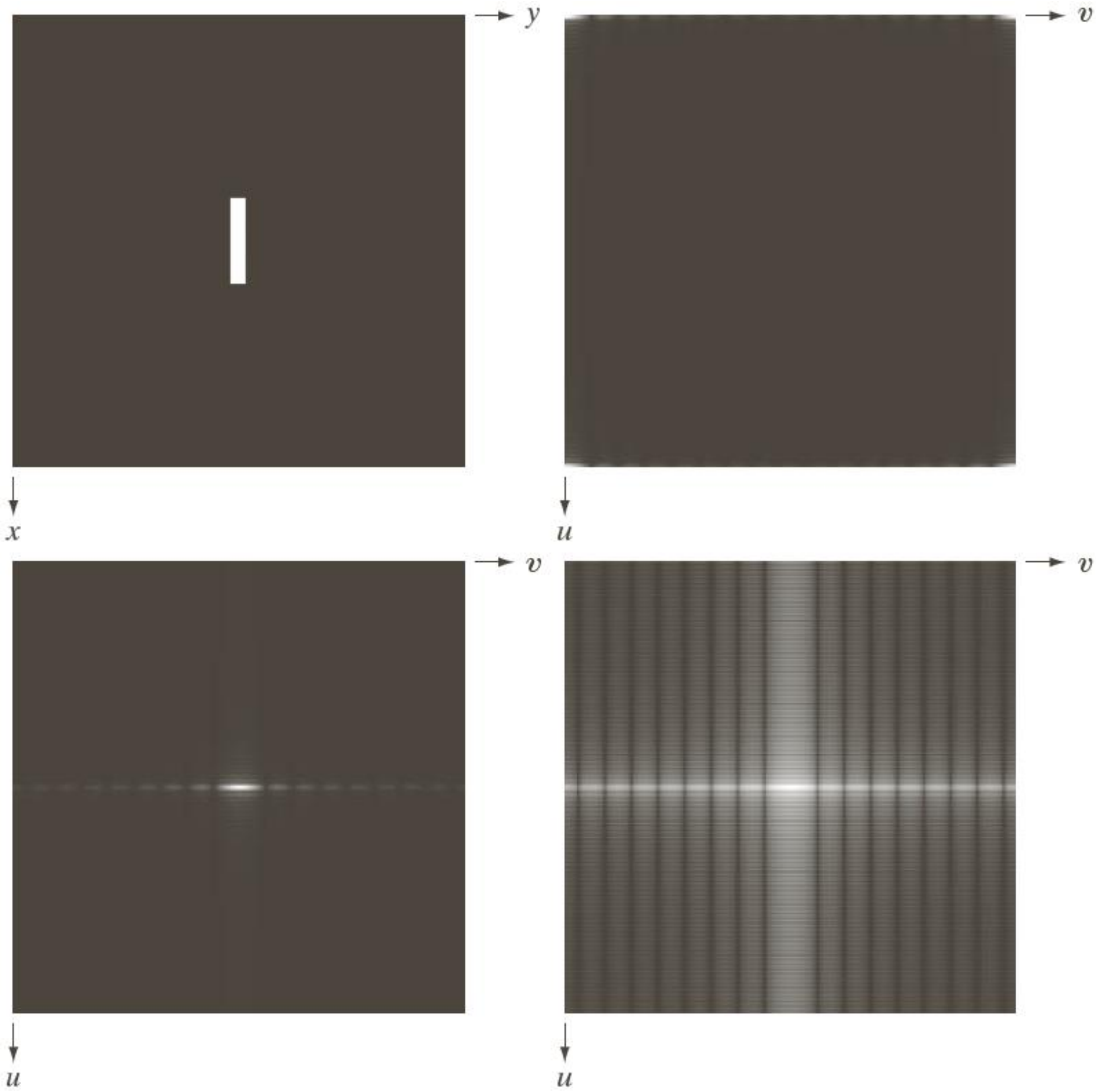
It is really important to note that the Fourier transform is completely **reversible**

The inverse DFT is given by:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

for x = 0, 1, 2…M-1 and y = 0, 1, 2…N-1

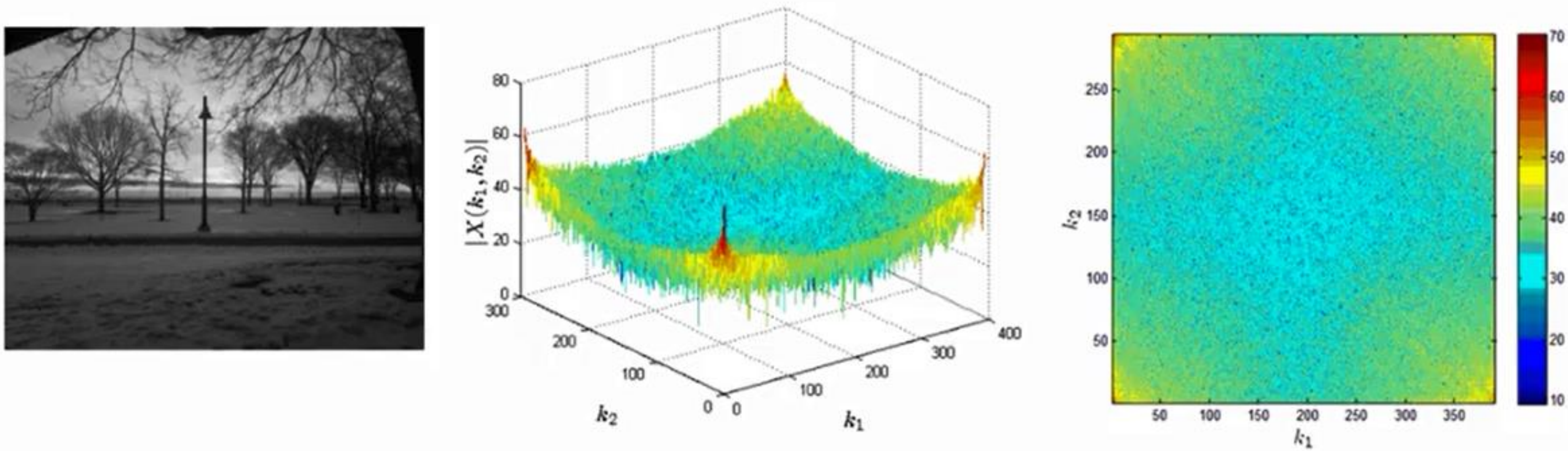
Frequencies in Images



a b
c d

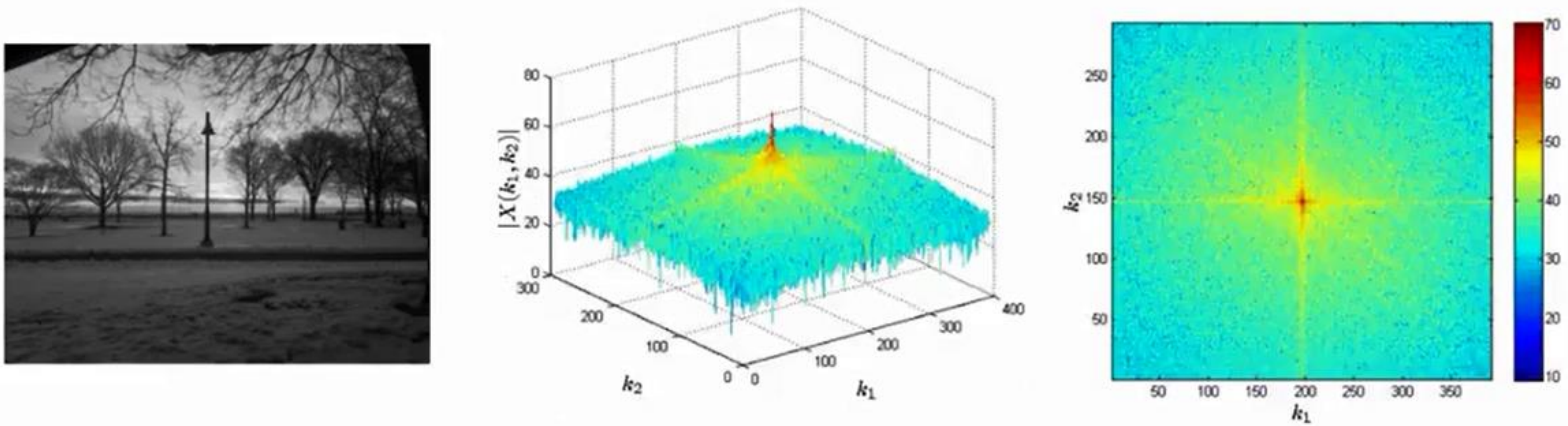
FIGURE 4.24
(a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

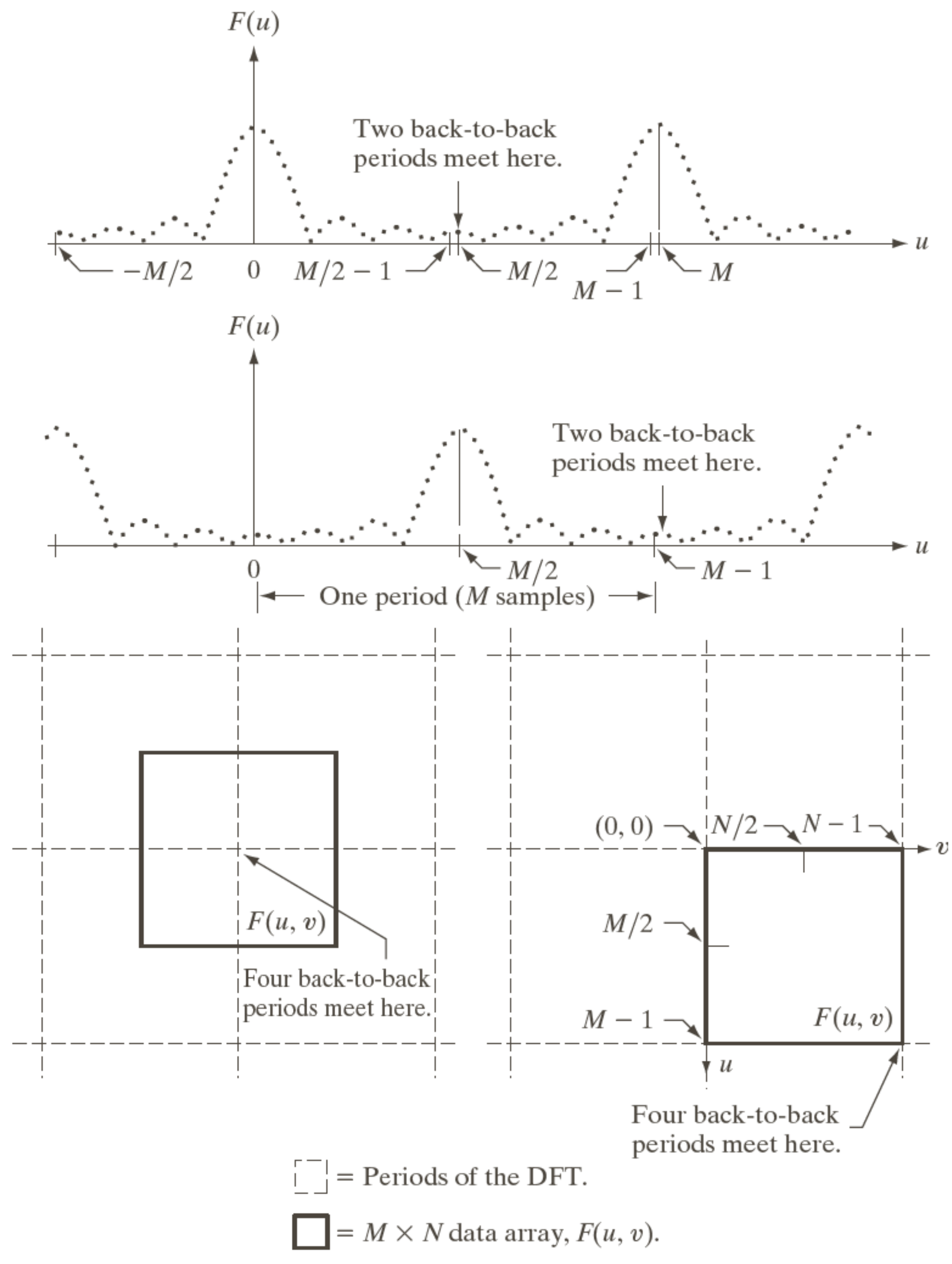
DFT



$|X(k_1, k_2)|$
0 20 40 60 80
300 200 100 0
k_2
0 100 200 300 400
k_1
250 200 150 100 50
k_2
50 100 150 200 250 300 350
k_1
70 60 50 40 30 20 10

Centered DFT



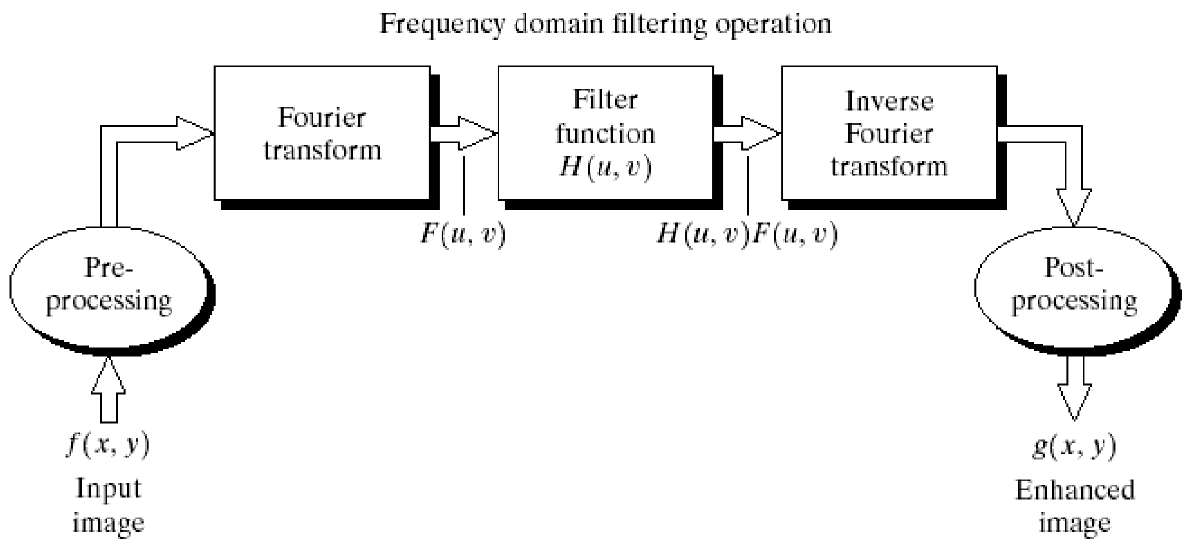


F(u)
Two back-to-back
periods meet here.
u
−M/2
0
M/2 − 1
M/2
M − 1
M
F(u)
Two back-to-back
periods meet here.
u
0
M/2
M − 1
One period (M samples)
F(u, v)
Four back-to-back
periods meet here.
(0, 0)
N/2
N − 1
v
M/2
M − 1
F(u, v)
u
Four back-to-back
periods meet here.
= Periods of the DFT.
= M × N data array, F(u, v).

The DFT and Image Processing

To filter an image in the frequency domain:

1. Compute $F(u,v)$ the DFT of the image
2. Multiply $F(u,v)$ by a filter function $H(u,v)$
3. Compute the inverse DFT of the result



Readings from Book (3rd Edn.)

- Frequency Domain (Chapter-4)

Acknowledgements

- Digital Image Processing”, Rafael C. Gonzalez & Richard E. Woods, Addison-Wesley, 2002
- Brian Mac Namee, Digitial Image Processing, School of Computing, Dublin Institute of Technology
- Digital Image processing Lectures: Coursera

Material in these slides has been taken from, the following resources