Senior Year Project

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Contents

1 The Black-Scholes Model

Black-Scholes is a pricing model used to determine the fair price or theoretical value for a call or a put option based on six variables such as volatility, type of option, underlying stock price, time, strike price, and risk-free rate. The quantum of speculation is more in case of stock market derivatives, and hence proper pricing of options eliminates the opportunity for any arbitrage. There are two important models for option pricing – Binomial Model and Black-Scholes Model. The model is used to determine the price of a European call option, which simply means that the option can only be exercised on the expiration date.

The Black-Scholes model allows us the calculate the price of European call and put options.The value of a call option, f_c, and a put option, f_p, are given by the following formulae;

$$f_c = S\Phi(d_1) - K\mathrm{e}^{-rT}\Phi(d_2), \tag{1}$$

$$f_p = K\mathrm{e}^{-rT}\Phi(-d_2) - S\Phi(-d_1). \tag{2}$$

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{3}$$

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma.\sqrt{T}. \tag{4}$$

C = Call option price

S = Current stock price

K = Strike price of the option

r = risk-free interest rate (a number between 0 and 1)

σ = volatility of the stocks return (a number between 0 and 1)

t = time to option maturity (in years)

N = normal cumulative distribution function

1.1 The Black-Scholes Assumptions

Lognormal distribution: The Black-Scholes-Merton model assumes that stock prices follow a lognormal distribution based on the principle that asset prices cannot take a negative value; they are bounded by zero.

No dividends: The BSM model assumes that the stocks do not pay any dividends or returns.

Expiration date: The model assumes that the options can only be exercised on its expiration or maturity date. Hence, it does not accurately price American options. It is extensively used in the European options market.

Random walk: The stock market is a highly volatile one, and hence, a state of random walk is assumed as the market direction can never truly be predicted.

Frictionless market: No transaction costs, including commission and brokerage, is assumed in the BSM model.

Risk-free interest rate: The interest rates are assumed to be constant, hence making the underlying asset a risk-free one.

Normal distribution: Stock returns are normally distributed. It implies that the volatility of the market is constant over time.

No arbitrage: There is no arbitrage. It avoids the opportunity of making a riskless profit.

1.2 Example

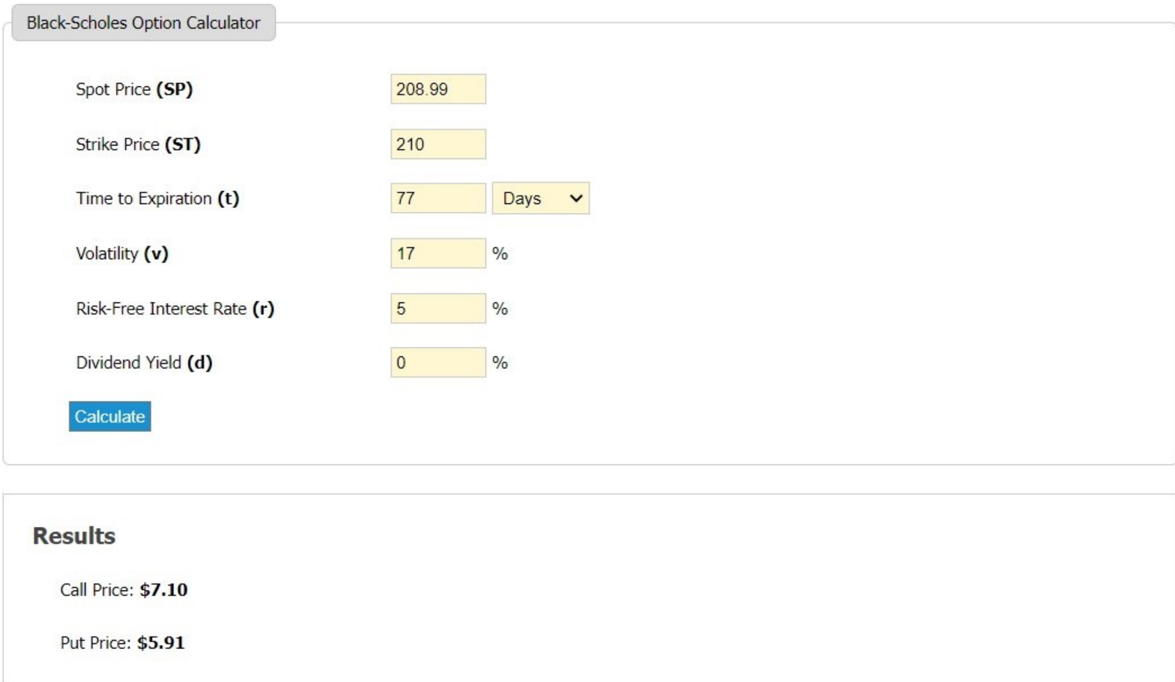
You want to buy an IBM European call option with a strike price of \$210. The stock is currently trading at a price of \$208.99 . You calculate the volatility of the stock to be 17%. The rate at which you can borrow and lend money is 5% (this is the risk-free interest rate). The time to maturity of the option is 77 days. What price should you pay (per share) for the option contract?

$$\mathrm{d}_1 = \frac{1}{0.17\sqrt{0.21095}}\left[\ln\left(\frac{208.99}{210}\right)+0.21095\left(0.05+\frac{0.17^2}{2}\right)\right] = 0.052$$
$$\mathrm{d}_2 = \frac{1}{0.17\sqrt{0.21095}}\left[\ln\left(\frac{208.99}{210}\right)+0.21095\left(0.05-\frac{0.17^2}{2}\right)\right] = -0.027$$

Now that we've calculated d_1 and d_2 we will calculate, f_c, the value of European call option. Now we plug these in equation (1) to calculate the price of the call option.

$$\mathrm{f}_c = 208.99\Phi(0.052) - 210\mathrm{e}^{0.7*0.21095}\Phi(-0.027) = 7.10$$

Then value of call-option after 77 days comes out to be \$7.10. The value is verified using online Black-Scholes calculator.



1.3 Greeks

"Greeks" is a term used in the options market to describe the different dimensions of risk involved in taking an options position. These variables are called Greeks because they are typically associated with Greek symbols. Each "Greek" variable is a result of an imperfect assumption or relationship of the option with another underlying variable. Traders use different Greek values, such as delta, theta, and others, to assess options risk and manage option portfolios.

2 Numerical Solution of Black-Scholes model

The main purpose for making Black-Scholes Calculator is to check our analytical solution obtained from Black-Scholes model against the numerical solution obtained after solving Black-Scholes Equation.The implementation of Black-Scholes Calculator required two things mainly the analytical and numerical solution of the Black-Scholes equation.Analytical solution is straightforward and it can be obtained from the formula given in (2). However Obtaining Numerical Solution is not an easy task. A numerical scheme is required which ensures a stable solution for the Black-Scholes equations.

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} = rC \tag{5}$$

An important thing to consider is the choice of our numerical scheme. Not all Numerical Schemes ensure a stable solution for the Black-Scholes Equation.Some common schemes used to solve BS equations are Implicit Method, Explicit Method and Crank–Nicolson method. Usually the Crank–Nicolson scheme is the most accurate scheme for small time steps. For larger time steps, the implicit scheme works better since it is less computationally demanding. The explicit scheme is the least accurate and can be unstable, but is also the easiest to implement and the least numerically intensive. Implicit Method was used for making this calculator.

2.1 Finite Difference Method

We may use finite difference methods to solve the Black-Scholes equation and therefore price options. We first recall a few basic results about Taylor series and finite difference methods. Let $f(x)$ be a function that is twice differentiable, we know using Taylors theorem that,

$$f''(x) = \frac{1}{(\Delta x)^2}(f(x+\Delta x) + f(x-\Delta x) - 2f(x)) + \mathscr{O}((\Delta x)^2), \tag{6}$$

$$f'(x) = \frac{1}{2\Delta x}(f(x+\Delta x) + f(x-\Delta x)) + \mathscr{O}((\Delta x)^2). \tag{7}$$

These are well known approximation to us, for the single variable case. We know that Equation (7) is known as the *central difference approximation*, we are also familiar with the *forward difference approximation* and *backward difference approximation*, which respectively are,

$$f'(x) = \frac{1}{\Delta x}(f(x+\Delta x) - f(x)),$$
$$f'(x) = \frac{1}{\Delta x}(f(x) - f(x-\Delta x)).$$

[1]

2.2 Terminal and Boundary Condition

Say that we truncate the plane so that $S \in [S_{\min}, S_{\max}]$, we still consider $t \in [0, T]$. Then we have the boundary and terminal conditions for a European option, are given in Table 1, where $V(S,t)$ is the value of the option at stock price S and at time t. These boundary and side conditions

Boundary	European Call	European Put
$t = T$	$V(S,T) = \max(S_{\min} + j\Delta S - K, 0)$	$V(S,T) = \max(K - S_{\min} - j\Delta S, 0)$
$S = S_{\min}$	$V(S_{\min},t) = \max(S_{\min} - K, 0)$	$V(S_{\min},t) = K\mathrm{e}^{-r(T-t)}$
$S = S_{\max}$	$V(S_{\max},t) = \max(S_{\max} - K, 0)$	$V(S_{\max},t) = \max(K - S_{\max}, 0)$

Table 1: The conditions at each boundary of the gird for a European option

are found by using the payoff function at the necessary points. The points of interest are at the termination date and the sides as this is where we have truncated our plane to, hence we arrive at the above. [1]

2.3 Implicit-Method

When approximating the Black-Scholes differential equation we use a backwards difference approximation for the $\frac{\partial f}{\partial S}$ derivative, a forward difference approximation for the $\frac{\partial f}{\partial t}$ and a central difference approximation for the $\frac{\partial^2 f}{\partial S^2}$ derivative. Hence, the Black-Scholes equation becomes,

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+rj\Delta S\frac{f_{i,j+1}-f_{i,j-1}}{2\Delta S}+\frac{1}{2}\sigma^2(j\Delta S)^2\frac{f_{i,j+1}+f_{i,j-1}-2f_{i,j}}{\Delta S^2}=rf_{i,j}. \qquad (8)$$

We may then rearrange this to find coefficients for the $f_{i,j-1}$, $f_{i,j}$ and $f_{i,j+1}$. This yields that (8) is now,

$$a_j f_{i,j-1}+b_j f_{i,j}+c_j f_{i,j+1}=f_{i+1,j} \qquad (9)$$

where,

$$\begin{aligned}
a_j &= \frac{1}{2}(r-q)j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t,\\
b_j &= 1+\sigma^2 j^2\Delta t+r\Delta t,\\
c_j &= -\frac{1}{2}(r-q)j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t.
\end{aligned}$$

Note that these constants are independent of S. This is because the ΔS and ΔS^2 introduced by the derivatives are the same ΔS that we are considering the option over in any given triangle. Thus once we discretize the plane we have cancellation resulting in the above relationships.

Now we need to solve this so that we can calculate the value of our option at $t=0$ i.e. $i=0$. For a European option we begin by considering $i=N-1$ in (9), for $j=1,\dots,M-1$. Thus we have $M-1$ simultaneous equations to solve. If we write out a few of these,

$$
\begin{aligned}
a_1 f_{N-1,0} + b_1 f_{N-1,1} + c_1 f_{N-1,2} &= f_{N,1} \\
a_2 f_{N-1,1} + b_2 f_{N-1,2} + c_2 f_{N-1,3} &= f_{N,2} \\
&\vdots \\
a_{M-2} f_{N-1,M-3} + b_{M-2} f_{N-1,M-2} + c_{M-2} fN-1, M-1 &= f_{N,M-1} \\
a_{M-1} f_{N-1,M-2} + b_{M-1} f_{N-1,M-1} + c_{M-1} fN-1, M &= f_{N,M}.
\end{aligned}
$$

Note that all the $f_{i,N}$ are known from the boundary conditions. Furthermore the terms $a_1 f_{N-1,0}$ and $c_{M-1} f_{N-1,M}$ are known from the boundary conditions. We may then express our system with unknowns on the left and knowns on the right.

$$
\begin{pmatrix}
b_1 & c_1 & & & & \\
a_2 & b_2 & c_2 & & & \\
 & a_3 & b_3 & c_3 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & a_{M-2} & b_{M-2} & c_{M-2} \\
 & & & & a_{M-1} & b_{M-1}
\end{pmatrix}
\begin{pmatrix}
f_{N-1,1} \\ f_{N-1,2} \\ f_{N-1,3} \\ \vdots \\ f_{N-1,M-2} \\ f_{N-1,M-1}
\end{pmatrix}
=
\begin{pmatrix}
f_{N,1} \\ f_{N,2} \\ f_{N,3} \\ \vdots \\ f_{N,M-2} \\ f_{N,M-1}
\end{pmatrix}
+
\begin{pmatrix}
-a_1 f_{N-1,0} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -c_{M-1} f_{N-1,M}
\end{pmatrix}.
$$

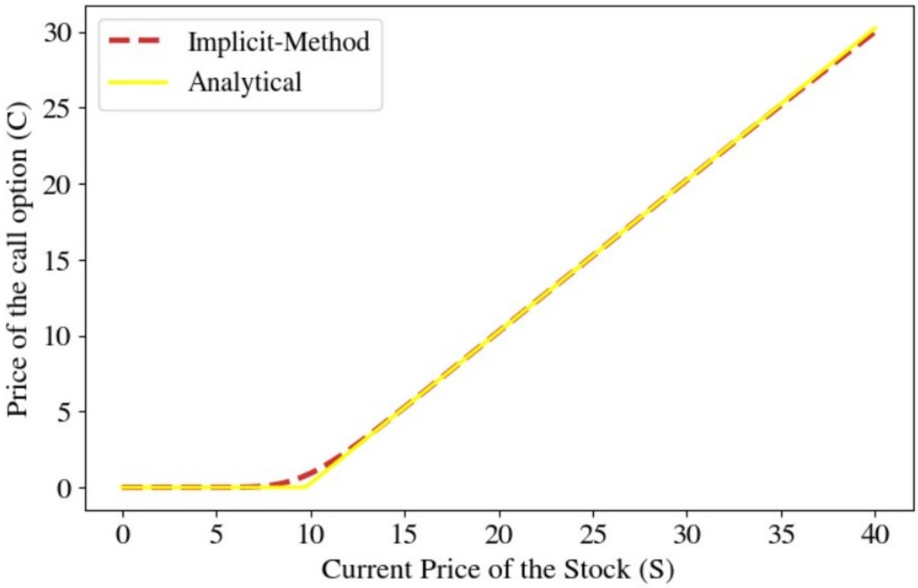
Allowing A to be the leftmost matrix and F_i to be the leftmost vector so that F_{i+1} is the second leftmost vector and B_i to be the rightmost vector we have that,

$$
AF_i = F_{i+1} + B_i. \tag{10}
$$

In the matrix form above we have taken $i = N-1$ however we note that this is true for arbitrary i. We may solve these using the tridiagonal matrix algorithm for $N-1$, then the F_{N-1} we have found is used in (10) with $i = N-2$ to find F_{N-2}. We may continue doing this until we reach $i = 0$ and we may find the value of our option at this point. [1]

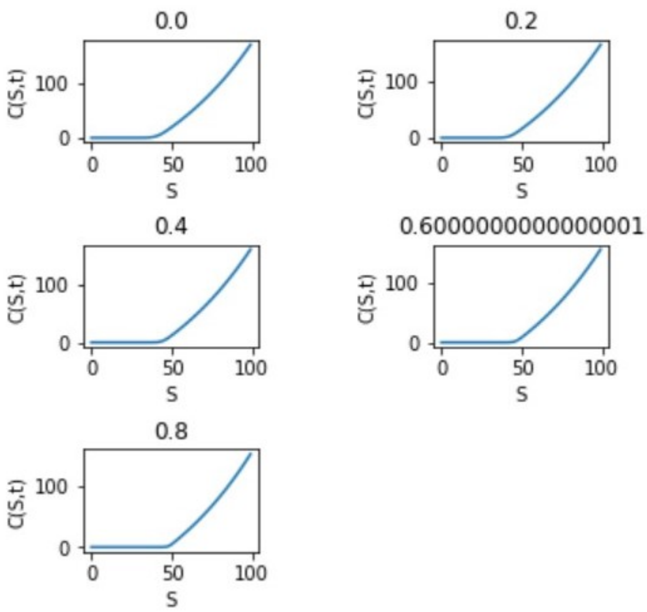
2.4 Results

After solving the Black-Scholes equation by analytical and numerical schemes we obtain the following results which tells us that both are approximately the same. One can always change other parameters like volatility, risk-free rate, Time of maturity e.t.c to observe the changes in the solution curve.

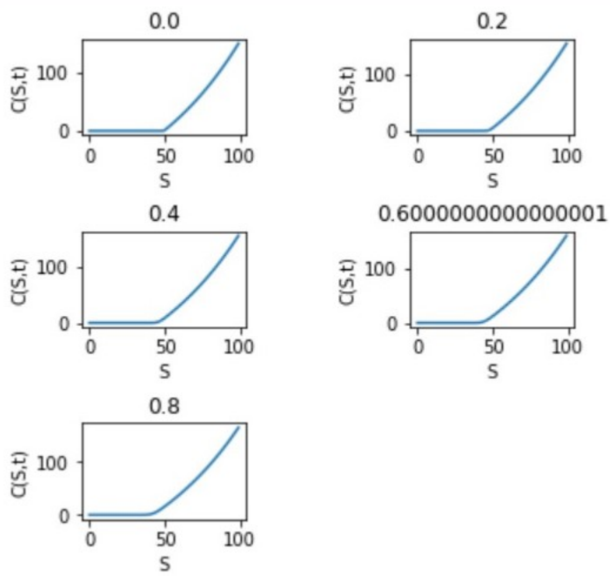


2.5 Benchmarks

The results were quite close if we look at the graph in the result section. But we need to see results over a long period at different time steps to see how our numerical and analytical method perform. The picture below shows the options price at different time-steps obtained using the implicit method.



The same parameters were used for analytical method and the following results were obtained.



The results are quite close but the parameters used were hypothetical. So in order to get a better sense of the results introducing some non-linearity in the parameters like volatility will help us identify the stability of these methods.

2.6 Limitations of the Black-Scholes Model

There are limitations on the Black-Scholes model, which is one of the most popular models for options pricing. Some of the standard limitations of the Black-Scholes model are:

- Assumes constant values for the risk-free rate of return and volatility over the option duration. None of those will necessarily remain constant in the real world.
- Assumes continuous and costless trading—ignoring the impact of liquidity risk and brokerage charges.
- Assumes stock prices to follow a lognormal pattern, e.g., a random walk (or geometric Brownian motion pattern), thus ignoring large price swings that are observed more frequently in the real world.
- Assumes no dividend payout—ignoring its impact on the change in valuations.
- Assumes no early exercise (e.g., fits only European options). That makes the model unsuitable for American options. [6]

2.7 Heston Model

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One of the major limitations of Black-Scholes model was that is assumed constant values for the volatility and risk-free rate over the option duration.Heston model solves this problem by incorporating stochastic volatility.The Heston Model is a closed-form solution for pricing options that seeks to overcome some of the shortcomings presented in the Black-Scholes option pricing model.

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t}, \tag{11}$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2t}. \tag{12}$$

Where:

S_t = asset price at time t.

r = risk-free interest rate.

$\sqrt{V_t}$ =volatility (standard deviation) of the asset price.

σ = volatility of the $\sqrt{V_t}$.

θ=long-term price variance.

k=rate of reversion to θ.

dt=indefinitely small positive time increment.

W_{1t} = Brownian motion of the asset price.

W_{2t} =Brownian motion of the asset's price variance.

Both the basic Black-Scholes model and the Heston Model still only provide option pricing estimates for a European option, which is an option that can only be exercised on its expiration date. [2]

2.8 Comparison of Heston and Black-Schole model

Till now we have a decent understanding of both models and how they work. To compare these models we will test them by computing an option of APPL stock with expiry of 1 year. The detail of the option are as follow.

Let us consider a European call option for AAPL with a strike price of 130 maturing on 15th Jan, 2023. Let the spot price be 127.62. The volatility of the underlying stock is know to be 20% and has a dividend yield of 1.63%. Lets value this option as of 18th May, 2022.

```
engine = ql.AnalyticHestonEngine(ql.HestonModel(heston_process),0.01, 1000)
european_option.setPricingEngine(engine)
h_price = european_option.NPV()
print ("The Heston model price is",h_price)
```

```
The Heston model price is 9.874999039942828
```

```
flat_vol_ts = ql.BlackVolTermStructureHandle(
    ql.BlackConstantVol(calculation_date, calendar, volatility, day_count)
)
bsm_process = ql.BlackScholesMertonProcess(spot_handle,
                                           dividend_yield,
                                           flat_ts,
                                           flat_vol_ts)
european_option.setPricingEngine(ql.AnalyticEuropeanEngine(bsm_process))
bs_price = european_option.NPV()
print ("The Black-Scholes-Merton model price is ", bs_price)
```

```
The Black-Scholes-Merton model price is  10.509100772000188
```

There is not a big difference in the pricing done by both models. But heston model is more complex as it requires the recalibration of various parameters like k , θ. So in general Black-scholes model is more easy to use.

3 Understanding Volatility Smile

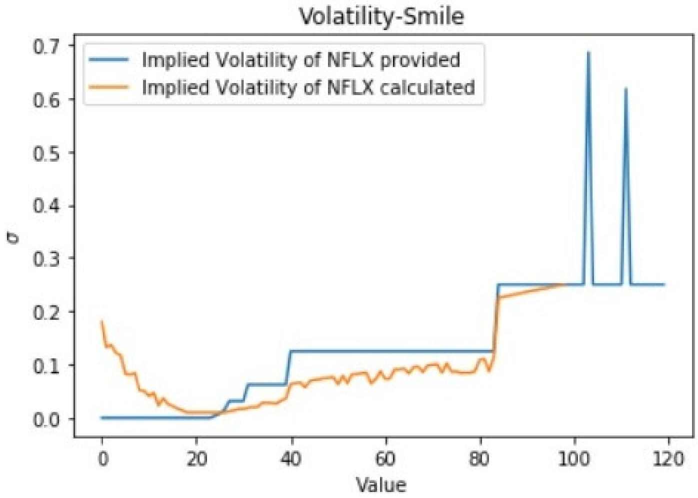
3.1 Implied Volatility

Volatility is a major factor in the price of an option. When thinking of volatility it is useful to think of it as the range of potential future stock prices.Implied volatility is an estimate of the future variability for the asset underlying the options contract. The Black-Scholes model is used to price options. The model assumes the price of the underlying asset follows a geometric Brownian motion with constant drift and volatility.

As with any equation, Black-Scholes can be used to determine any single variable when all the other variables are known.To calculate implied volatility i took options data of NFLX(Netflix) to get a varying volatility. We can look at calculating implied-volatility as a minimization problem. Below we minimize the absolute difference between the market price and the Black-Scholes price.We have bounded this minimization such that the volatility is less than 600 %.

$$impliedvolatility = argmin_{\sigma \in (0,6]} |Vmarket - VBS(S,K,T,r,\sigma)| \quad (13)$$

Now lets calculate the implied volatility for NFLX. The below mentioned image shows the implied-volatility from the real data and implied-volatility calculated using Newton-Raphson method. [4]

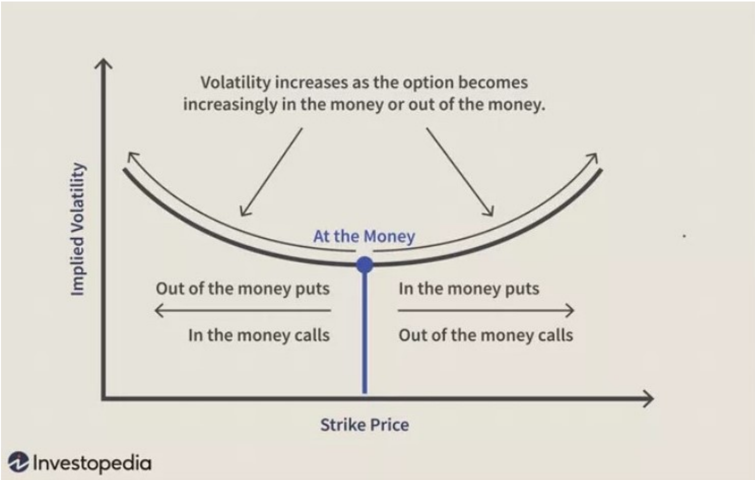


There is a significant difference at the start but the calculated Implied Volatility follows the general trend of original volatility. Our Target was to get a non-linear profile of the volatility and we have now got one. Now we have obtained a volatility which looks like a smile. Lets see how should one interpret this option smile.

3.2 Interpretation of volatility smile

Volatility smiles can be seen when comparing various options with the same underlying asset and same expiration date but different strike prices. If the implied volatility is plotted for each of the different strike prices, then there may be a U-shape. The U-shape is not always as perfectly formed as depicted in the graph above.

If the option has a U-shape, then options that are ITM and OTM by an equal amount should have roughly the same implied volatility. The further ITM or OTM, the greater the implied volatility, with the lowest implied volatilities near the ATM options. If this is not the case, then the option does not align with a volatility smile. The chart below will help you better understand this concept. [5]



3.3 Limitations of volatility smile

As for every model that we have discussed there are always some limitations to the method that we use. One of the limitation is visible in our implied volatility of NFLX that we calculated. The smile that we obtained is not smooth which results in certain options showing more or less implied volatility as compared to the actual volatility. Its a tool for traders to look for options that have a high or low implied volatility, but this alone cannot help in deciding which options to select. [5]

4 Trading Strategies

4.1 Introduction

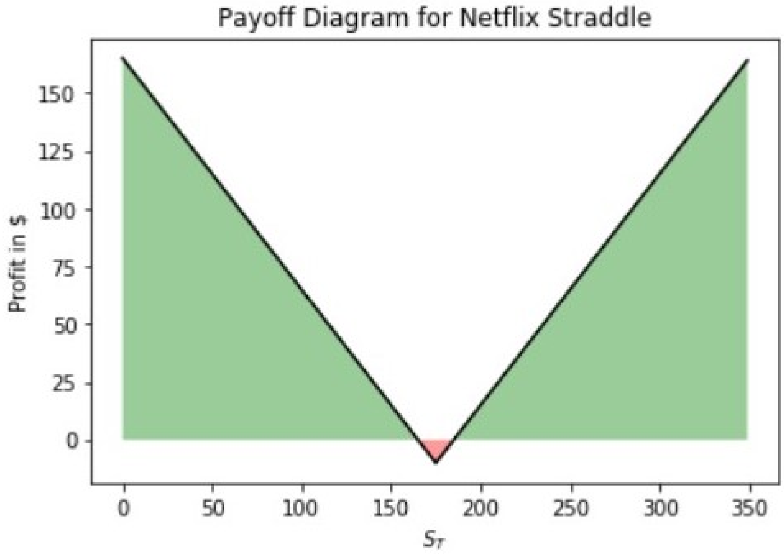
After having a basic understanding of how the option pricing works we can use this limited knowledge to learn about the trading strategies used by the retail traders. A trading strategy is a set of rules that a trader follow to profit from the markets while keeping the risk to a minimum. It's not necessary that the strategies discussed here will work in the real market conditions because most of the strategies that are written in literature don't work because if they did everyone would have made a fortune in the stock market. Below are some of the famous used strategies in the financial markets.

4.2 Straddle Strategy

A straddle is a strategy in which a trader buys both a call option and a put option at the same strike price. The idea here is that the trader will benefit from a large move in the price of the underlying stock regardless of the direction of the move. This is an especially useful strategy when a trader predicts a large increase in the uncertainty of an asset price, and the likelihood of a large move, but he is unsure about the direction of the move.

We will now use this information on a real life stock. At the time of writing this article there is an option available for Netflix trading at a strike price of 175. Lets see how our strategy works on the Netflix options. [3]

```
obj = OptionStrat('Netflix Straddle', 175)
obj.long_put(175,5)
obj.long_call(175, 5)
obj.plot(color='black')
```

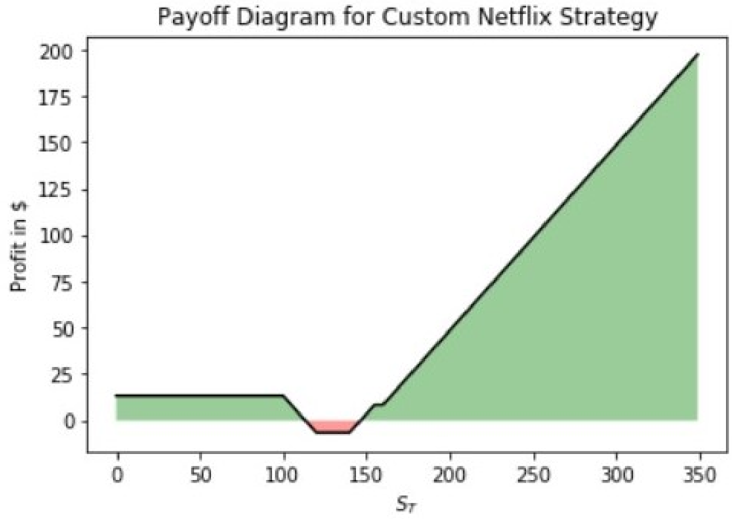


If there is a huge market move to be expected in Netflix stock the payoff for our options would look like this. And if there is no significant movement in the market we will only lose the premium we paid for our options. So the profit is unlimited as long as the market keeps going up. So this strategy works when there is huge volatility expected in the markets. Similarly we can implement a short straddle as well which is the opposite of we did here.

4.3 Custom Strategy

In straddle we used a combination of call and put options to benefit from an expected market move. However we can use different combinations of call and put options to see what payoff we will get in the market. Building a custom strategy requires an in depth knowledge of option pricing models. Here we will use our limited knowledge to see what we can achieve in the market. We will again use the Netflix stock and use a combination of options to reduce our risk.

```
obj = OptionStrat('Custom Netflix Strategy', 175)
obj.long_call(160,2, 1)
obj.long_call(140,4 , 1)
obj.short_call(155,0.75,1)
obj.long_put(120, 2, 1)
obj.short_put(100, 0.65, 1)
obj.plot(color='black')
```



So we made a decent strategy with a very low risk using a combination of call and put options. One can always experiment different strategies on different stocks to see what payoff they will get.

So the parameter for call and put options are as follow

longcall (Strikeprice,Cost, Number of options)

longput (strikeprice, Premium,Number of options)

This will help understand the inputs in the code snippet added above.

5 Conclusion

Working on these concepts was a challenging task. This project gave me mathematical understanding of Quantitative Finance. Although most of the techniques discussed were outdated but they help in understanding the basic option pricing theory. This project covers basics of Black

Scholes model, numerical technique to solve BS model, comparison between numerical and analytical solutions of Black-Scholes, an overview of implied volatility along with limitations of Black-Scholes and an overview of trading strategies along-with a custom trading strategy for Netflix.

6 Bibliography

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