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Solutions for Exercise Round 6

Let us again consider the following non-linear state space model from Exercise 1 of Round 4:

$$\begin{aligned} x_k &= x_{k-1} - 0.01\,\sin(x_{k-1}) + q_{k-1}, \\ y_k &= 0.5\,\sin(2\,x_k) + r_k, \end{aligned} \tag{1}$$

where q_{k-1} has a variance of 0.01^2 and r_k has a variance of 0.02.

(a) Write down the prediction equations for the Gaussian filter for this model. Note that technically you can compute the expectation integrals in closed form, but you don't necessarily need to here.

Proof. We get

$$\begin{aligned} m_k^- &= m_{k-1} - 0.01 \int \sin(x_{k-1})\,\mathrm{N}(x_{k-1}|m_{k-1}, P_{k-1})dx_{k-1}, \\ P_k^- &= \int [x_{k-1} - 0.01\,\sin(x_{k-1}) - m^-]^2\,\mathrm{N}(x_{k-1}|m_{k-1}, P_{k-1})dx_{k-1} + Q \end{aligned} \tag{2}$$

with $Q = 0.01^2$. □

(b) Write down the equations of the (prior) linearization of the dynamic model and the prediction step of the statistical linear regression filter for this model.

Proof. We get the following, which are actually all scalar equations (the transposes are redundant):

$$\begin{aligned} \mu_k^- &= m_{k-1} - 0.01 \int \sin(x_{k-1})\,\mathrm{N}(x_{k-1} \mid m_{k-1}, P_{k-1})dx_{k-1}, \\ P_k^x &= \int [x_{k-1} - 0.01\,\sin(x_{k-1}) - \mu^-]^2\,\mathrm{N}(x_{k-1} \mid m_{k-1}, P_{k-1})dx_{k-1} + Q, \\ P_k^{xx} &= \int [x_{k-1} - m_{k-1}][x_{k-1} - 0.01\,\sin(x_{k-1}) - \mu^-]\,\mathrm{N}(x_{k-1}|m_{k-1}, P_{k-1})dx_{k-1}, \\ A_{k-1} &= [P_k^{xx}]^\top P_k^{-1}, \\ a_{k-1} &= \mu_k^- - A_{k-1} m_{k-1}, \\ L_{k-1} &= P_k^x - A_{k-1} P_{k-1} A_{k-1}^\top, \\ m_k^- &= A_{k-1} m_{k-1} + a_{k-1}, \\ P_k^- &= A_{k-1} P_{k-1} A_{k-1}^\top + L_{k-1}. \end{aligned} \tag{3}$$

□

(c) Show that the equations from (a) and (b) are equivalent.

Proof. Note the m_k^- and P_k^- in (a) are equal to μ_k^- and P_k^x in (b). For the SLRF we then get indeed

$$\begin{aligned} [m_k^-]_{\text{SLRF}} &= A_{k-1} m_{k-1} + a_{k-1} \\ &= A_{k-1} m_{k-1} + \mu_k^- - A_{k-1} m_{k-1} \\ &= \mu_k^- \end{aligned} \tag{4}$$

and

$$\begin{aligned} [P_k^-]_{\text{SLRF}} &= A_{k-1} P_{k-1} A_{k-1}^T + L_{k-1} \\ &= A_{k-1} P_{k-1} A_{k-1}^T + P_k^x - A_{k-1} P_{k-1} A_{k-1}^\top \\ &= P_k^x. \end{aligned} \tag{5}$$

□

Exercise 3. (Conditional moments)

Write down the conditional moments for the model in Equation (1).

Proof. The conditional moments are simply

$$\begin{aligned} \mu_k^-(x_{k-1}) &= \mathrm{E}[x_k \mid x_{k-1}] = x_{k-1} - 0.01\, \sin(x_{k-1}) \\ P_k^x(x_{k-1}) &= \mathrm{Cov}[x_k \mid x_{k-1}] = 0.01^2 \\ \mu_k(x_{k-1}) &= \mathrm{E}[y_k \mid x_k] = 0.5\, \sin(2\, x_k) \\ P_k^y(x_{k-1}) &= \mathrm{Cov}[y_k \mid x_k] = 0.02. \end{aligned} \tag{6}$$

□