Exercise Round 8

The deadline of this exercise round is **Thursday March 13, 2025**. The solutions will be discussed during the exercise session in the T2 lecture hall of Computer Science building starting at 14:15.

The problems should be *solved before the exercise session*. During the session those who have completed the exercises will be asked to present their solutions on the board/screen.

Exercise 1. (Multi-modal posterior)

Consider the following partially observed random-walk model:

$$\begin{aligned} p(x_k \mid x_{k-1}) &= \mathrm{N}(x_k \mid x_{k-1}, 1), \\ p(y_k \mid x_k) &= \mathrm{LogNormal}(y_k \mid \log|x_k| - 1/2, 1), \end{aligned} \tag{1}$$

where $\mathrm{LogNormal}(a, b)$ has the density

$$\mathrm{LogNormal}(x \mid a, b) \propto \exp\left(-\frac{(\log(x) - a)^2}{2b^2}\right). \tag{2}$$

Hint: $\mathrm{LogNormal}(\ln|x_k| - 1/2, 1)$ is defined such that $\mathbb{E}[y_k \mid x_k] = |x_k|$.

Take x_0 to have the standard normal distribution. Generate synthetic data from this model and apply

(a) one of the Gaussian approximation-based filters,

(b) a bootstrap particle filter using 1,000 particles.

Compare the posterior distributions given by the two methods above: for the Gaussian filter, plot the mean and uncertainty (0.95 standard deviation), and for the bootstrap filter, show a scatter plot of the filtering distribution at each time step.

Exercise 2. (Locally Optimal Proposal and Run Time)

Consider the following linear Gaussian state space model:

$$\begin{aligned}
\mathbf{x}_k &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \\
y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + r_k,
\end{aligned} \tag{3}$$

where $\mathbf{x}_k = (x_k, \dot{x}_k)$ is the state, y_k is the measurement, and $\mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathrm{diag}(1/10^2, 1^2))$ and $r_k \sim \mathrm{N}(0, 10^2)$ are white Gaussian noise processes. $\mathbf{x}_0$ has a standard normal distribution.

(a) Derive an expression for the optimal importance distribution for the model:

$$\pi(\mathbf{x}_k) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1}, \mathbf{y}_{1:k}). \tag{4}$$

(b) Simulate some data from the model for $T = 50$ time steps and implement a Kalman filter for it.

(c) Implement a bootstrap particle filter for the model. Implement it using vectorized code such that it is as fast as possible.

(d) Measure the run time taken by each algorithm. Can you get the particle filter to be as fast as the Kalman filter for any number of particles?

Exercise 3

For the same state-space model in equation (3), compute the empirical mean and variance of the weighted particles returned by the particle filter, and compare them against the Kalman filter means and covariances in the following situations (you can look at just the final time step):

1. For number of particles $N \in \{10^2, 10^3, 10^4, 10^5\}$.
2. (a) No resampling (b) Resample at every time step (c) Resample only when the effective sample size is below 75% (choose $N = 10^3$ for this part).