Exercise Round 7

The deadline of this exercise round is **Thursday March 6, 2025**. The solutions will be discussed during the exercise session in the T2 lecture hall of Computer Science building starting at 14:15.

The problems should be *solved before the exercise session*. During the session those who have completed the exercises will be asked to present their solutions on the board/screen.

Exercise 1. (Optimal importance distributions)

Recall that in order to compute $\int f(x)q(x)\mathrm{d}x = \int f(x)\frac{q(x)}{\pi(x)}\pi(x)\mathrm{d}x$ you can use the empirical approximation $\int f(x)\mathrm{d}x \approx \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)})\frac{q(x^{(i)})}{\pi(x^{(i)})}$, where the $x^{(i)}$'s are i.i.d. sampled from π.

(a) Suppose that $f(x) > 0$ for all x. Find the distribution π that minimises the variance of $\frac{1}{N}\sum_{i=1}^{N} f(x^{(i)})\frac{q(x^{(i)})}{\pi(x^{(i)})}$. *Hint:* The optimal variance is zero.

(b) Derive the optimal π in the general case when f can be positive or negative.

(c) Is the goal of designing such an importance density realistic? What would you do in practice?

Exercise 2. (Rejection Sampling for Beta distribution)

We want to sample from the Beta distribution with the density

$$q(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}\,(1-x)^{\beta-1}, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function and $\alpha = 2$, $\beta = 2$.

(a) We aim to use rejection sampling with uniform distribution $\pi(x) = \mathrm{U}(x \mid 0, 1)$ as the proposal distribution. Compute the corresponding bound M.

(b) Implement the rejection sampling algorithm using the above proposal distribution and bound. Check numerically that the expected number of rounds for acceptance is M. Also check that the histogram of the samples matches the true density.

Exercise 3. (Rejection Sampling Failure)

Consider the model

$$\begin{aligned} p(x) &= \mathrm{N}(x \mid 0, 1), \\ p(y \mid x) &= \mathrm{N}(y \mid x, 1). \end{aligned} \tag{2}$$

(a) Compute the posterior distribution $p(x \mid y)$.

(b) Let us now assume that we use a distribution $\pi(x) = \mathrm{N}(x \mid 0, v)$ with some variance v as the proposal distribution for rejection sampling. Show that there exists no bound M such that $p(x \mid y)/\pi(x) \leq M$. *Hint:* What is the maximum of $-(x - y/2)^2 + \frac{1}{2v}x^2$ over x and y?

(c) What does this result imply on the weights of importance sampling if we use $\pi(x)$ as the importance distribution?