Solutions for Exercise Round 3

Exercise 1. (Kalman Filter with Non-Zero Mean Noises)

Derive the Kalman filter equations for the following linear-Gaussian filtering model with non-zero-mean noises:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \\ \mathbf{y}_k &= \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k, \end{aligned} \tag{1}$$

where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{m}_q, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{m}_r, \mathbf{R})$.

Proof. We use the solution of Exercise 3 from the previous round. Having noise with a non-zero mean is the same thing as having a bias (constant offset) in the relationship. We know from a) that if $\mathbf{x}_{k-1} \sim \mathrm{N}(\mathbf{m}_{k-1}, \mathbf{P}_{k-1})$,

$$\mathbf{x}_k \sim \mathrm{N}(\mathbf{A}\,\mathbf{m}_{k-1} + \mathbf{m}_q, \mathbf{A}\,\mathbf{P}_{k-1}\mathbf{A}^\mathsf{T} + \mathbf{Q}) \tag{2}$$

and therefore that, writing $\mathbf{m}_k^- = \mathbf{A}\,\mathbf{m}_{k-1} + \mathbf{m}_q$ and $\mathbf{P}_k^- = \mathbf{Q} + \mathbf{A}\,\mathbf{P}_{k-1}\mathbf{A}^\mathsf{T}$

$$\begin{pmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{m}_k^- \\ \mathbf{H}\,\mathbf{m}_k^- + \mathbf{m}_r \end{pmatrix}, \begin{pmatrix} \mathbf{P}_k^- & \mathbf{P}_k^-\,\mathbf{H}^\mathsf{T} \\ \mathbf{H}\,\mathbf{P}_k^- & \mathbf{H}\,\mathbf{P}_k^-\mathbf{H} + \mathbf{R} \end{pmatrix} \right),$$

Now using c), we get

$$\mathbf{x}_k \mid \mathbf{y}_k \sim \mathrm{N}\left(\mathbf{m}_k, \mathbf{P}_k\right).$$

with $\mathbf{m}_k = \mathbf{m}_k^- + \mathbf{P}_k^-\,\mathbf{H}^\mathsf{T}\left(\mathbf{H}\,\mathbf{P}_k^-\mathbf{H} + \mathbf{R}\right)^{-1}(\mathbf{y}_k - \mathbf{H}\mathbf{m}_k^- - \mathbf{m}_r)$ and $\mathbf{P}_k = \mathbf{P}_k^- - \mathbf{P}_k^-\,\mathbf{H}^\mathsf{T}\left(\mathbf{H}\,\mathbf{P}_k^-\mathbf{H} + \mathbf{R}\right)^{-1}\mathbf{H}\,\mathbf{P}_k^-$.

□

Exercise 2. (Filtering of Finite-State HMMs)

Write down the Bayesian filtering equations for finite-state hidden Markov models (HMM). That is, write down the prediction and update equations for the system where $x_k \in \{1, \dots, N_x\}$, the dynamic model is defined by the discrete probability distribution

$$p(x_k = i \mid x_{k-1} = j), \; i, j \in \{1, \dots, N_x\}, \tag{3}$$

and the measurement model is

$$p(y_k \mid x_k = i), \; i \in \{1, \dots, N_x\}. \tag{4}$$

Proof. The prediction takes the form

$$p(x_k = i \mid y_{1:k-1}) = \sum_{j=1}^{N_x} p(x_k = i \mid x_{k-1} = j)\, p(x_k = j \mid y_{1:k-1}) \tag{5}$$

and the update is

$$p(x_k = i \mid y_{1:k}) = \frac{p(y_k \mid x_k = i)\, p(x_k = i \mid y_{1:k-1})}{\sum_{j=1}^{N_x} p(y_k \mid x_k = j)\, p(x_k = j \mid y_{1:k-1})}. \tag{6}$$

Please also see how these are presented in the course book. □

Exercise 3. (Kalman Filter for Noisy Resonator)

See the notebook.