Exercise Round 3

The deadline of this exercise round is **Thursday January 30, 2025**. The solutions will be discussed during the exercise session in the T2 lecture hall of Computer Science building starting at 14:15.

The problems should be *solved before the exercise session*. During the session those who have completed the exercises will be asked to present their solutions on the board/screen.

Exercise 1. (Kalman Filter with Non-Zero Mean Noises)

Derive the Kalman filter equations for the following linear-Gaussian filtering model with non-zero-mean noises:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \\ \mathbf{y}_k &= \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k, \end{aligned} \tag{1}$$

where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{m}_q, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{m}_r, \mathbf{R})$.

Exercise 2. (Filtering of Finite-State HMMs)

Write down the Bayesian filtering equations for finite-state hidden Markov models (HMM). That is, write down the prediction and update equations for the system where $x_k \in \{1, \ldots, N_x\}$, the dynamic model is defined by the discrete probability distribution

$$p(x_k = i \mid x_{k-1} = j),\ \ i, j \in \{1, \ldots, N_x\}, \tag{2}$$

and the measurement model is

$$p(y_k \mid x_k = i),\ \ i \in \{1, \ldots, N_x\}. \tag{3}$$

Exercise 3. (Kalman Filter for Noisy Resonator)

Consider the following dynamic model:

$$
\begin{aligned}
\mathbf{x}_k &= \begin{pmatrix} \cos(\omega) & \frac{\sin(\omega)}{\omega} \\ -\omega\,\sin(\omega) & \cos(\omega) \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \\
y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + r_k,
\end{aligned}
$$

where $\mathbf{x}_k \in \mathbb{R}^2$ is the state, y_k is the measurement, $r_k \sim \mathrm{N}(0, 0.1)$ is a white Gaussian measurement noise, and $\mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$, where

$$
\mathbf{Q} = \begin{pmatrix} \frac{q^c\,\omega - q^c\,\cos(\omega)\,\sin(\omega)}{2\omega^3} & \frac{q^c\,\sin^2(\omega)}{2\omega^2} \\ \frac{q^c\,\sin^2(\omega)}{2\omega^2} & \frac{q^c\,\omega + q^c\,\cos(\omega)\,\sin(\omega)}{2\omega} \end{pmatrix}. \tag{4}
$$

The angular velocity is $\omega = 1/2$ and the spectral density is $q^c = 0.01$. The model is a discretized version of the noisy resonator model with a given angular velocity ω.

In the file `exercise6_4.m` (MATLAB) or `Exercise6_4.ipynb` (Python) of the book's companion code repository[1] there is simulation of the dynamic model together with a baseline solution, where the measurement is directly used as the estimate of the state component x_1, and the second component x_2 is computed as a weighted average of the measurement differences. Implement the Kalman filter for the model and compare its performance (in RMSE sense) to the baseline solution. Plot figures of the solutions.

[1] https://github.com/EEA-sensors/Bayesian-Filtering-and-Smoothing