Solutions for Exercise Round 5

Exercise 1. (Sigma-point Methods and Linear Functions)

Show that when the function is linear, the spherical cubature rule gives the exact result.

Proof. Let $\mathbf{f}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$ be an affine function ($\mathbf{b} = 0$ gives the linear case) and consider a Gaussian random variable $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$, then we have

$$\int \mathbf{f}(\mathbf{x})\ \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P})\ \mathrm{d}\mathbf{x} = \mathbf{A}\,\mathbf{m} + \mathbf{b}. \tag{1}$$

The cubature rule now gives (assuming $\mathbf{x} \in \mathbb{R}^n$):

$$\begin{aligned}
&\int \mathbf{f}(\mathbf{x})\ \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P})\ \mathrm{d}\mathbf{x} \\
&\approx \frac{1}{2n}\left[\sum_{i=1}^{n} \mathbf{f}(\mathbf{m} + \sqrt{\mathbf{P}}\sqrt{n}\,\mathbf{e}_i) + \sum_{i=1}^{n} \mathbf{f}(\mathbf{m} - \sqrt{\mathbf{P}}\sqrt{n}\,\mathbf{e}_i)\right] \\
&= \frac{1}{2n}\left[\sum_{i=1}^{n} \big(\mathbf{A}\,(\mathbf{m} + \cancel{\sqrt{\mathbf{P}}\sqrt{n}\,\mathbf{e}_i}) + \mathbf{b}\big) + \sum_{i=1}^{n} \big(\mathbf{A}\,(\mathbf{m} - \cancel{\sqrt{\mathbf{P}}\sqrt{n}\,\mathbf{e}_i}) + \mathbf{b}\big)\right] \\
&= \frac{1}{2n}\left[2n\ (\mathbf{A}\,\mathbf{m} + \mathbf{b})\right] \\
&= \mathbf{A}\,\mathbf{m} + \mathbf{b}.
\end{aligned} \tag{2}$$

□

Exercise 2. (Hermite polynomials)

Show that one-dimensional Hermite polynomials $H_0(0) = 1$, $H_1(x) = x$, and $H_2(x) = x^2 - 1$ are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int f(x)\, g(x)\ \mathrm{N}(x \mid 0, 1)\ \mathrm{d}x.$$

Proof. The orthogonality means that

$$\langle H_i, H_j\rangle = 0, \ \text{when } i \neq j. \tag{3}$$

By direct calculation we get

$$
\begin{aligned}
\langle H_0, H_0 \rangle &= \int 1 \cdot 1 \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 1, \\
\langle H_0, H_1 \rangle &= \int 1 \cdot x \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_0, H_2 \rangle &= \int 1 \cdot (x^2 - 1) \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_1, H_0 \rangle &= \int x \cdot 1 \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_1, H_1 \rangle &= \int x \cdot x \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 1, \\
\langle H_1, H_2 \rangle &= \int x \cdot (x^2 - 1) \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_2, H_0 \rangle &= \int (x^2 - 1) \cdot 1 \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_2, H_1 \rangle &= \int (x^2 - 1) \cdot x \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 0, \\
\langle H_2, H_2 \rangle &= \int (x^2 - 1) \cdot (x^2 - 1) \cdot \mathrm{N}(x \mid 0, 1) \; \mathrm{d}x = 2.
\end{aligned} \tag{4}
$$

Hence proved. □

Exercise 3. (Bearings Only Target Tracking with CKF and UKF)

See the example notebook here https://github.com/EEA-sensors/Bayesian-Filtering-and-Smoothing/blob/main/python/example_notebooks/pendulum_ckf.ipynb.