Heat Transfer: Mechanisms and Steady Conduction
Lecture Notes

Contents

1 Chapter 16: Mechanisms of Heat Transfer **3**

1.1 16.1 Introduction to Heat Transfer 3

1.2 16.2 Conduction and Fourier's Law 3

1.3 16.3 Convection 6

1.4 16.4 Thermal Radiation 7

1.5 16.5 Simultaneous Heat Transfer Mechanisms 10

2 Chapter 17: Steady Heat Conduction **11**

2.1 17.1 Steady Heat Conduction in Plane Walls 11

2.2 17.2 Convection Resistance and Basic Thermal Resistance Network 12

2.3 17.3 Composite Walls and Generalized Resistance Networks 13

2.4 17.4 Thermal Contact Resistance 14

2.5 17.5 Heat Conduction in Cylinders and Spheres 14

2.6 17.6 Critical Radius of Insulation 16

2.7 17.7 Heat Transfer from Finned Surfaces 17

3 Chapter 18: Transient Heat Conduction **20**

3.1 18.1 Transient Heat Conduction & Lumped System Analysis 20

3.2 18.2 One- and Multi-Dimensional Non-Lumped Transient Conduction 22

3.3 18.3 Semi-Infinite Body 23

4 Chapter 19: Forced Convection and Boundary Layers **24**

4.1 19.1 Forced Convection and the Velocity Boundary Layer 24

4.2 19.2 Thermal Boundary Layer 24

4.3 19.3 Reynolds, Prandtl, and Nusselt Numbers 25

4.4 19.4 Flow over Flat Plates, Cylinders, and Spheres 25

5 Chapter 21: Thermal Radiation **27**

5.1 21.1 Fundamentals of Thermal Radiation 27

5.2 21.4 Radiative Properties of Surfaces 28
5.3 21.5 View Factors 29
5.4 21.6 Radiation Exchange Between Black Surfaces 29
5.5 21.7 Radiation Networks and Resistances (Black Surfaces) 30
5.6 21.8 Radiation Exchange Between Gray, Diffuse Surfaces 30

1 Chapter 16: Mechanisms of Heat Transfer

1.1 16.1 Introduction to Heat Transfer

Heat transfer is the movement of energy due to a temperature difference. It is distinct from thermodynamics:

- **Thermodynamics:** Deals with initial and final equilibrium states.
- **Heat transfer:** Deals with the *rate* and *mechanism* of energy transfer between locations at different temperatures.

There are three basic mechanisms of heat transfer:

1. **Conduction** – is the transfer of heat from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles.
2. **Convection** – is the transfer of heat between a solid surface and an adjacent moving fluid.
3. **Thermal radiation** – energy transfer by electromagnetic waves due to the temperature of the bodies.

In many practical situations, more than one mechanism acts simultaneously (e.g. hot object in air loses heat by both convection and radiation).

1.2 16.2 Conduction and Fourier's Law

Consider a plane wall of thickness Δx and area A, with its two faces held at temperatures T_1 and T_2 ($T_1 > T_2$). Heat flows in the x-direction from the hot side to the cold side by conduction.

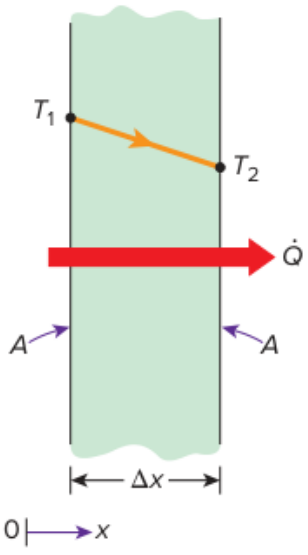

FIGURE 16–1
Heat conduction through a large plane wall of thickness Δx and area A.

Figure 1: Illustration of conduction through a plane wall.

Macroscopic form of Fourier's law

The rate of heat transfer by conduction through the wall is

$$\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{\Delta x} \tag{1}$$

Where:

- $\dot{Q}_{\text{cond}}$ = rate of heat transfer by conduction (W)
- k = thermal conductivity of the material (W/(m K))
- A = cross-sectional area normal to heat flow (m^2)
- T_1, T_2 = temperatures on the two faces of the wall (K or °C)
- Δx = wall thickness (m)

Purpose: Eq. (1) gives the *overall rate* of heat conduction through a plane layer when the temperature difference and thickness are known.

Differential form (one-dimensional steady conduction)

In the limit as $\Delta x \to 0$, we obtain the differential form

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx} \tag{2}$$

Where:

- $\frac{dT}{dx}$ = temperature gradient in the direction of heat flow (K/m).

The minus sign indicates that heat flows in the direction of *decreasing* temperature (i.e. from hot to cold).

Purpose: Eq. (2) is used when the temperature varies continuously through the material and we want the local heat flux and differential equations for temperature distribution.

Thermal conductivity

Thermal conductivity k is the rate of heat transfer through a unit thickness of the material per unit area per unit temperature difference (the ability of a material to conduct heat).

- High k (e.g. metals) → good conductors.
- Low k (e.g. foam, air) → good insulators.

Thermal conductivity generally depends on temperature and material structure (pure metals, alloys, gases, liquids, etc.).

Microscopic interpretation. Temperature represents the **average kinetic energy** of the molecules or atoms of a substance. In liquids and gases, molecular motion includes translational, vibrational, and rotational components. When faster (hotter) and slower (colder) molecules collide, part of the kinetic energy of the faster molecule is transferred to the slower one — much like elastic balls colliding at different velocities. The greater the temperature difference, the more energetic these collisions, and thus the higher the rate of energy transfer by conduction.

Kinetic theory of gases. From the kinetic theory, the thermal conductivity of gases is predicted — and experimentally confirmed — to vary as:

$$k \propto \frac{\sqrt{T}}{\sqrt{M}} \tag{3}$$

Where:

- T = thermodynamic temperature (K)
- M = molar mass of the gas (kg/mol)

Thus, for a particular gas (fixed M), k increases with temperature because higher molecular speeds increase the frequency of energy-exchanging collisions. At a fixed temperature, gases with higher molar mass M conduct heat less effectively because heavier molecules move more slowly.

Example: At $T = 1000$ K:

$$k_{\text{He}} = 0.343 \text{ W/(m K)}, \quad k_{\text{air}} = 0.0667 \text{ W/(m K)} \tag{4}$$

showing that helium ($M = 4$) conducts heat much better than air ($M = 29$).

TABLE 16–1

The thermal conductivities of some materials at room temperature

Material	k, W/m·K*
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminum	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.607
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Glass fiber	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

*Multiply by 0.5778 to convert to Btu/h·ft·°F.

Figure 2: Representative thermal conductivity values for various materials.

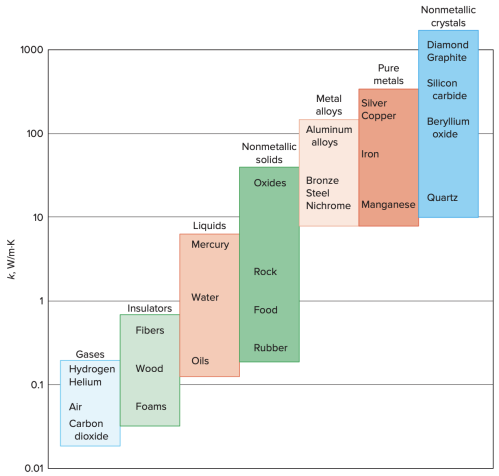

FIGURE 16–6
The range of thermal conductivity of various materials at room temperature.

Figure 3: Range of thermal conductivity of various materials at room temperature.

Thermal Diffusivity

Thermal diffusivity α represents how quickly heat diffuses through a material. It indicates how fast a material responds to changes in temperature — that is, how fast heat spreads through it compared to how much energy it can store.

$$\boxed{\alpha = \frac{k}{\rho c_p}} \quad [\text{m}^2/\text{s}] \tag{5}$$

Where:

- α = thermal diffusivity (m^2/s)

- k = thermal conductivity (W/(m K))
- ρ = density (kg/m^3)
- c_p = specific heat capacity (J/(kg K))

Concept:

- k measures how well a material *conducts* heat.
- ρc_p measures how much heat the material can *store*.
- Thus, α compares *heat conduction* to *heat storage*.

Interpretation:

- High $\alpha \rightarrow$ heat diffuses quickly; the material reacts fast to temperature changes.
- Low $\alpha \rightarrow$ heat moves slowly; the material absorbs heat and warms up gradually.

Heat flux

The **heat flux** (heat transfer rate per unit area) is

$$q''_x = \frac{\dot{Q}_{\text{cond}}}{A} = -k\frac{dT}{dx} \tag{6}$$

Where:

- q''_x = heat flux in the x-direction (W/m^2).

Purpose: q''_x measures the intensity of conduction at a surface or inside a material.

1.3 16.3 Convection

Convection is heat transfer between a solid surface and a moving fluid in contact with that surface. It combines:

- **Conduction** within the fluid layer adjacent to the surface, and
- **Bulk fluid motion** carrying energy away or toward the surface.

Convection is classified as:

- **Forced convection:** Fluid motion is caused by external means (fan, pump, wind, etc.).
- **Natural (free) convection:** Fluid motion is due to density differences caused by temperature gradients (buoyancy-driven).

Newton's law of cooling

The convective heat transfer rate from a surface at temperature T_s to a fluid of temperature T_∞ is

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty) \tag{7}$$

Where:

- $\dot{Q}_{\text{conv}}$ = convective heat transfer rate (W)
- h = convection heat transfer coefficient (W/(m^2 K))
- A_s = surface area where convection occurs (m^2)
- T_s = surface temperature (K or °C)
- T_∞ = temperature of the fluid far from the surface (K or °C)

Purpose: Eq. (7) is the basic correlation used to compute convective heat loss or gain from surfaces when h is known (typically from correlations in later chapters).

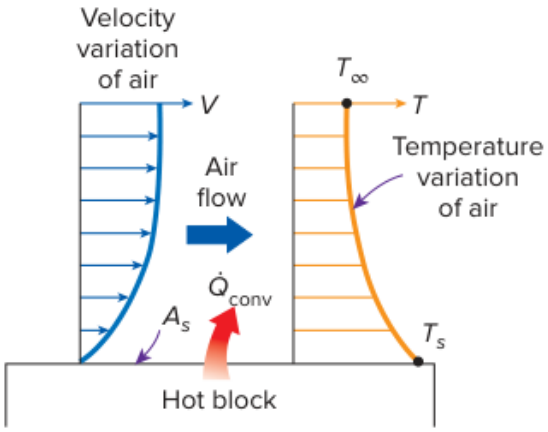

FIGURE 16–11
Heat transfer from a hot surface to air by convection.

Figure 4: Illustration of convection from a surface to a moving fluid.

1.4 16.4 Thermal Radiation

Thermal radiation is the transfer of energy by electromagnetic waves (photons) emitted by matter because of its temperature.

Key characteristics:

- Does not require a medium (can occur in a vacuum).
- Travels at the speed of light.
- Is characterized by wavelength and direction.
- All bodies at a temperature above absolute zero emit thermal radiation.

In heat transfer, we are primarily interested in **thermal radiation**, i.e. the portion of electromagnetic radiation emitted due to the temperature of a body (not x-rays, radio waves, etc. that may be unrelated to temperature).

Radiation is fundamentally a **volumetric** phenomenon (emission occurs within the medium), but for opaque solids it is usually treated as a **surface** phenomenon, since radiation emitted from deeper layers is absorbed within a few micrometres of the surface.

Blackbody and Stefan–Boltzmann law

An ideal surface that emits the *maximum possible* radiation at a given temperature is called a **blackbody**. The radiation it emits is **blackbody radiation**.

The maximum rate of radiation that can be emitted from a surface of area A_s at thermodynamic temperature T_s is given by the **Stefan–Boltzmann law**:

$$\dot{Q}_{\text{emit,max}} = \sigma A_s T_s^4 \tag{8}$$

Where:

- $\dot{Q}_{\text{emit,max}}$ = maximum radiative heat emission rate from the surface (W)
- σ = Stefan–Boltzmann constant, $\sigma = 5.670 \times 10^{-8}$ W/(m^2 K^4)
- A_s = surface area (m^2)
- T_s = surface absolute temperature (K)

Sometimes it is convenient to define the **blackbody emissive power** (radiation per unit area):

$$E_b = \sigma T_s^4 \tag{9}$$

so that $\dot{Q}_{\text{emit,max}} = A_s E_b$.

Purpose: Eq. (8) gives the theoretical upper limit on radiation emission from a surface at temperature T_s (blackbody).

Real surfaces and emissivity

Real surfaces emit less radiation than a blackbody at the same temperature. This deviation is quantified by the **emissivity** ε:

$$\dot{Q}_{\text{emit}} = \varepsilon \sigma A_s T_s^4 \tag{10}$$

Where:

- $\dot{Q}_{\text{emit}}$ = radiative heat emission rate from the real surface (W)
- ε = emissivity of the surface, $0 \le \varepsilon \le 1$ (dimensionless)

An ideal blackbody has $\varepsilon = 1$. Typical emissivities for common materials at $T \approx 300$ K are listed in Table 16–6.

Interpretation:

- ε close to 1 $\rightarrow$ surface is a good emitter and good absorber.
- ε small $\rightarrow$ surface is a poor emitter/absorber (often a good reflector).

Absorptivity and Kirchhoff's law

In addition to emitting radiation, a surface can **absorb** or **reflect** incident radiation. For opaque surfaces (no transmission),

$$\alpha + \rho = 1 \tag{11}$$

where

- α = **absorptivity**: fraction of incident radiation absorbed,
- ρ = reflectivity: fraction reflected.

The rate at which a surface absorbs radiation is

$$\dot{Q}_{\text{absorbed}} = \alpha \, \dot{Q}_{\text{incident}} \tag{12}$$

Where:

- $\dot{Q}_{\text{absorbed}}$ = rate of radiation energy absorbed by the surface (W)
- $\dot{Q}_{\text{incident}}$ = rate of incident radiation on the surface (W)
- α = absorptivity (dimensionless, $0 \le \alpha \le 1$)

A blackbody has $\alpha = 1$ and absorbs all incident radiation.

Kirchhoff's law of radiation. For a surface in thermal equilibrium at a given temperature and wavelength,

$$\varepsilon = \alpha \tag{13}$$

That is, *good emitters are also good absorbers* (and poor emitters are poor absorbers).

Net radiation exchange with large surroundings

Consider a small surface (A_s) of emissivity ε at temperature T_s, completely enclosed by a much larger isothermal surface at temperature T_{sur} that behaves as a blackbody. The net radiation heat transfer from the small surface to the surroundings is:

$$\dot{Q}_{\text{rad}} = \varepsilon \sigma A_s \left(T_s^4 - T_{\text{sur}}^4\right) \tag{14}$$

Where:

- $\dot{Q}_{\text{rad}}$ = net radiative heat transfer from the surface (W)
- T_{sur} = absolute temperature of the surrounding surfaces (K)

Purpose: Eq. (14) is used when a surface exchanges radiation with large, approximately isothermal surroundings (e.g. a small object in a large room or the sky).

Combined convection and radiation

In many practical situations (e.g. a hot surface in air), heat transfer from a surface occurs *simultaneously* by convection and radiation.

The total heat transfer rate is obtained by adding the convective and radiative contributions:

$$\dot{Q}_{\text{total}} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad}} \tag{15}$$

With

$$\dot{Q}_{\text{conv}} = h_{\text{conv}} A_s (T_s - T_\infty), \qquad \dot{Q}_{\text{rad}} = \varepsilon \sigma A_s (T_s^4 - T_{\text{sur}}^4), \tag{16}$$

we can define a **combined heat transfer coefficient** h_{combined} such that

$$\dot{Q}_{\text{total}} = h_{\text{combined}} A_s (T_s - T_\infty) \tag{17}$$

For many engineering applications it is reasonable to take $T_{\text{sur}} \approx T_\infty$, which leads to

$$h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \tag{18}$$

with the **radiation heat transfer coefficient**

$$h_{\text{rad}} = \varepsilon \sigma (T_s + T_{\text{sur}})(T_s^2 + T_{\text{sur}}^2) \tag{19}$$

Where:

- h_{conv} = convection heat transfer coefficient ($\text{W}/(\text{m}^2\,\text{K})$)
- h_{rad} = equivalent radiation heat transfer coefficient ($\text{W}/(\text{m}^2\,\text{K})$)
- h_{combined} = effective combined convection–radiation coefficient ($\text{W}/(\text{m}^2\,\text{K})$)
- T_∞ = fluid (air) temperature far from the surface (K)

Purpose: Using h_{combined} allows us to treat convection and radiation together as one effective surface heat transfer mechanism when computing overall energy balances.

1.5 16.5 Simultaneous Heat Transfer Mechanisms

Although there are three fundamental mechanisms of heat transfer — **conduction**, **convection**, and **radiation** — not all three occur simultaneously in every medium. Depending on the material and environment, one or two modes may dominate.

Summary of Heat Transfer Modes by Medium

Medium	Dominant Mechanisms of Heat Transfer
Opaque Solid	Heat transfer occurs by **conduction only** within the solid since radiation cannot penetrate deeply. The surface may also exchange heat by convection and radiation with its surroundings.
Semitransparent Solid	Both **conduction and radiation** occur inside the material (e.g., glass, some ceramics).
Still Fluid (no motion)	Heat transfer occurs by **conduction** and possibly **radiation**.
Flowing Fluid	Heat transfer occurs by **convection** (combined conduction + bulk fluid motion) and **radiation**.
Vacuum	Only **radiation** can occur since conduction and convection require a material medium.

Examples:

- A cold rock in warm air gains heat at its surface by **convection** (air) and **radiation** (from surroundings), and transfers it inward by **conduction**.
- Inside a gas, heat transfer is either by **conduction** (no motion) or **convection** (with bulk motion) — not both.
- Gases are mostly **transparent to radiation**, except certain gases like ozone, which absorb specific wavelengths (e.g., ultraviolet).
- Liquids are typically **strong absorbers** of radiation.

Key Takeaways:

- Conduction dominates in solids; convection dominates in fluids.
- Radiation becomes significant when temperature differences are large or when a vacuum is present.
- In most systems, two modes act together (e.g., convection + radiation from a hot surface).

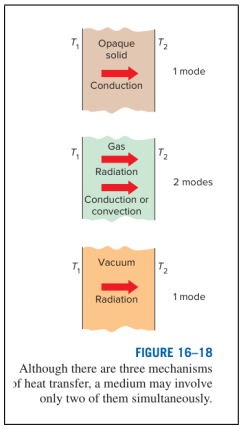

Figure 5: Typical combinations of heat transfer modes in solids, fluids, and vacuum.

2 Chapter 17: Steady Heat Conduction

2.1 17.1 Steady Heat Conduction in Plane Walls

We analyze one-dimensional, steady conduction in a plane wall of thickness L, with no heat generation, and constant properties.

Assumptions:

- Steady state ($\partial T/\partial t = 0$).
- One-dimensional conduction in the x-direction.
- No internal heat generation.
- Constant thermal conductivity k.

Temperature distribution

The governing differential equation for 1-D steady conduction without heat generation is

$$\frac{d^2T}{dx^2} = 0 \tag{20}$$

Integrating twice:

$$T(x) = C_1 x + C_2$$

Constants are found from boundary conditions:

$$T(0) = T_1, \quad T(L) = T_2$$

which yields a linear temperature distribution

$$T(x) = T_1 + (T_2 - T_1)\frac{x}{L} \tag{21}$$

Where:

- $T(x)$ = temperature at position x in the wall (K)
- T_1 = temperature at $x = 0$ (K)
- T_2 = temperature at $x = L$ (K)
- L = wall thickness (m)

Purpose: Eq. (21) gives the steady-state temperature profile through a plane wall under these conditions.

Heat transfer rate and thermal resistance of a plane layer

Using Fourier's law, for steady state the heat transfer rate is constant:

$$\dot{Q} = kA\frac{T_1 - T_2}{L} \tag{22}$$

This can be written in a **resistance form**:

$$\dot{Q} = \frac{T_1 - T_2}{R_{\text{cond}}} \quad \text{with} \quad R_{\text{cond}} = \frac{L}{kA} \tag{23}$$

Where:

- R_{cond} = thermal resistance of the wall layer (K/W)

Purpose: This form is analogous to electrical circuits and allows us to build thermal resistance networks.

2.2 17.2 Convection Resistance and Basic Thermal Resistance Network

For convection at a surface of area A_s, Newton's law of cooling can be written as

$$\dot{Q} = hA_s(T_s - T_\infty) = \frac{T_s - T_\infty}{R_{\text{conv}}} \quad \text{with} \quad R_{\text{conv}} = \frac{1}{hA_s} \tag{24}$$

Where:

- R_{conv} = convection thermal resistance (K/W)
- T_s = surface temperature (K)
- T_∞ = fluid temperature far from surface (K)

Purpose: Convection is modeled as a surface resistance, analogous to conduction resistance, enabling combined conduction–convection networks.

Wall between two fluids: series resistances

For a plane wall of thickness L separating two fluids at temperatures $T_{\infty 1}$ and $T_{\infty 2}$:

- Convection from hot fluid to surface 1: $R_{\text{conv},1} = 1/(h_1 A)$
- Conduction through the wall: $R_{\text{cond}} = L/(kA)$
- Convection from surface 2 to cold fluid: $R_{\text{conv},2} = 1/(h_2 A)$

Total resistance (in series) is

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cond}} + R_{\text{conv},2} \tag{25}$$

and

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} \tag{26}$$

Purpose: Eqs. (25) and (26) provide a compact way to compute the heat transfer rate through composite plane systems.

Radiation resistance and combined convection–radiation

A surface exposed to a gas usually exchanges heat by both **convection** and **thermal radiation**.

Net radiation heat transfer between a surface (area A_s, temperature T_s) and large surroundings at T_{surr} can be written as

$$\dot{Q}_{\text{rad}} = \varepsilon\sigma A_s\left(T_s^4 - T_{\text{surr}}^4\right) = h_{\text{rad}} A_s(T_s - T_{\text{surr}}) = \frac{T_s - T_{\text{surr}}}{R_{\text{rad}}} \tag{27}$$

Where:

- ε = surface emissivity (dimensionless),
- σ = Stefan–Boltzmann constant ($5.67 \times 10^{-8}\,\text{W}/(\text{m}^2\,\text{K}^4)$),
- h_{rad} = *radiation* heat transfer coefficient ($\text{W}/(\text{m}^2\,\text{K})$),

- $R_{\text{rad}} = 1/(h_{\text{rad}} A_s)$ = radiation resistance.

An effective radiation coefficient is defined by linearizing:

$$h_{\text{rad}} = \varepsilon\sigma\left(T_s^2 + T_{\text{surr}}^2\right)(T_s + T_{\text{surr}}) \tag{28}$$

When the surrounding air and surfaces are at about the same temperature, $T_{\text{surr}} \approx T_\infty$, radiation and convection from the same surface may be lumped into a **combined coefficient**:

$$h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \quad \Rightarrow \quad \dot{Q} = h_{\text{combined}} A_s (T_s - T_\infty) \tag{29}$$

Purpose: This lets us treat convection + radiation as a single surface resistance,

$$R_{\text{surf,combined}} = \frac{1}{h_{\text{combined}} A_s},$$

so the resistance network stays simple (only "convection-like" nodes).

TABLE 17–2

Thermal contact conductance of some metal surfaces in air (from various sources)

Material	Surface condition	Roughness, μm	Temperature, °C	Pressure, MPa	h_c*, W/m²·K
Identical Metal Pairs					
416 Stainless steel	Ground	2.54	90–200	0.17–2.5	3800
304 Stainless steel	Ground	1.14	20	4–7	1900
Aluminum	Ground	2.54	150	1.2–2.5	11,400
Copper	Ground	1.27	20	1.2–20	143,000
Copper	Milled	3.81	20	1–5	55,500
Copper (vacuum)	Milled	0.25	30	0.17–7	11,400
Dissimilar Metal Pairs					
Stainless steel–Aluminum		20–30	20	10 20	2900 3600
Stainless steel–Aluminum		1.0–2.0	20	10 20	16,400 20,800
Steel Ct-30–Aluminum	Ground	1.4–2.0	20	10 15–35	50,000 59,000
Steel Ct-30–Aluminum	Milled	4.5–7.2	20	10 30	4800 8300
Aluminum–Copper	Ground	1.17–1.4	20	5 15	42,000 56,000
Aluminum–Copper	Milled	4.4–4.5	20	10 20–35	12,000 22,000

Figure 6: Plane wall conduction and temperature profile (Chapter 17).

2.3 17.3 Composite Walls and Generalized Resistance Networks

For multiple layers in series (each with thickness L_i, conductivity k_i, and area A):

$$R_{\text{cond,total}} = \sum_i \frac{L_i}{k_i A} \tag{30}$$

Total resistance for a system with convection on both sides becomes:

$$R_{\text{total}} = \frac{1}{h_1 A} + \sum_i \frac{L_i}{k_i A} + \frac{1}{h_2 A} \tag{31}$$

and the steady heat transfer rate is

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} \tag{32}$$

Purpose: These equations generalize the thermal resistance idea to multi-layer systems.

More complex systems may have **parallel thermal paths** (analogous to parallel resistors in circuits). For such cases:

$$\frac{1}{R_{\text{eq}}} = \sum_j \frac{1}{R_j} \tag{33}$$

Where:

- R_j = individual thermal resistances for each parallel path,
- R_{eq} = equivalent resistance seen between the same two nodes.

Purpose: Parallel networks are used when heat can flow through more than one path simultaneously (e.g. through a stud and insulation in a wall).

2.4 17.4 Thermal Contact Resistance

At the interface between two solids, microscopic roughness leads to actual contact only at small spots, with thin gaps filled with air or other fluid. This causes an additional **thermal contact resistance** between the solids.

The heat transfer across the interface can be expressed as

$$\dot{Q} = h_c A(T_{s1} - T_{s2}) = \frac{T_{s1} - T_{s2}}{R_c} \quad \text{with} \quad R_c = \frac{1}{h_c A} \tag{34}$$

Where:

- h_c = thermal contact conductance ($\text{W}/(\text{m}^2\,\text{K})$)
- A = apparent contact area (m^2)
- T_{s1}, T_{s2} = temperatures of the two solid surfaces at the interface (K)
- R_c = thermal contact resistance (K/W)

Purpose: Eq. (34) allows us to model the imperfect heat transfer at solid–solid interfaces as an extra resistance in series.

Thermal contact resistance depends on:

- Contact pressure,
- Surface roughness and hardness,
- Interstitial material (air, grease, filler, etc.).

2.5 17.5 Heat Conduction in Cylinders and Spheres

For cylindrical or spherical geometries with radial conduction, the area normal to heat flow varies with radius, so the thermal resistance takes a different form.

Radial conduction in a cylinder

For a hollow cylinder with inner radius r_1, outer radius r_2, length L, and constant conductivity k, under 1-D steady radial conduction and no heat generation:

$$\dot{Q} = \frac{2\pi k L(T_1 - T_2)}{\ln(r_2/r_1)} \tag{35}$$

Where:

- T_1 = temperature at $r = r_1$ (K)

- T_2 = temperature at $r = r_2$ (K)
- r_1, r_2 = inner and outer radii (m)
- L = cylinder length (m)

The associated **cylindrical conduction resistance** is

$$R_{\text{cond,cyl}} = \frac{\ln(r_2/r_1)}{2\pi kL} \tag{36}$$

Purpose: These expressions are used for pipes, insulation on pipes, and other cylindrical geometries.

Similar expressions exist for spherical shells, with different area dependence, but the same idea: conduction resistance depends on geometry.

Cylinders and spheres with convection on both sides

For a **cylindrical shell** with inner radius r_1, outer radius r_2, length L, conductivity k, convection on the inner and outer surfaces with h_1, h_2, and bulk fluid temperatures $T_{\infty 1}$, $T_{\infty 2}$:

$$R_{\text{conv},1} = \frac{1}{h_1 A_1} = \frac{1}{h_1(2\pi r_1 L)} \tag{37}$$

$$R_{\text{cond,cyl}} = \frac{\ln(r_2/r_1)}{2\pi kL} \tag{38}$$

$$R_{\text{conv},2} = \frac{1}{h_2 A_2} = \frac{1}{h_2(2\pi r_2 L)} \tag{39}$$

Total resistance and heat transfer rate:

$$R_{\text{total,cyl}} = R_{\text{conv},1} + R_{\text{cond,cyl}} + R_{\text{conv},2} \tag{40}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total,cyl}}} \tag{41}$$

For a **spherical shell** with radii r_1, r_2:

$$A_1 = 4\pi r_1^2, \quad A_2 = 4\pi r_2^2 \tag{42}$$

$$R_{\text{cond,sph}} = \frac{r_2 - r_1}{4\pi k r_1 r_2} \tag{43}$$

$$R_{\text{total,sph}} = \frac{1}{h_1 A_1} + R_{\text{cond,sph}} + \frac{1}{h_2 A_2} \tag{44}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total,sph}}} \tag{45}$$

Once $\dot{Q}$ is known, any surface or interface temperature T_j is found from

$$\Delta T_{i\to j} = \dot{Q}\, R_{i\to j}$$

just as in the plane-wall case.

Multilayer cylindrical and spherical shells

For a **multilayer cylinder** with n concentric layers (conductivities k_i, radii r_i):

$$R_{\text{cond,total,cyl}} = \sum_{i=1}^{n} \frac{\ln(r_{i+1}/r_i)}{2\pi k_i L} \tag{46}$$

Total resistance with convection on both sides:

$$R_{\text{total}} = \frac{1}{h_1 2\pi r_1 L} + \sum_{i=1}^{n} \frac{\ln(r_{i+1}/r_i)}{2\pi k_i L} + \frac{1}{h_2 2\pi r_{n+1} L} \tag{47}$$

An analogous expression holds for **multilayer spheres**, replacing each cylindrical conduction term with the spherical form

$$R_{\text{cond,sph},i} = \frac{r_{i+1} - r_i}{4\pi k_i r_i r_{i+1}}.$$

Purpose: These formulas extend the resistance-network method to pipes with insulation, cladding, or multiple solid layers, and to spherical tanks.

2.6 17.6 Critical Radius of Insulation

For **curved** geometries (cylinders, spheres), adding insulation has two competing effects:

- Increases conduction resistance (good for reducing heat transfer),
- Increases outer surface area → decreases convection resistance (could increase heat transfer).

For a cylinder of outer radius r_1 covered by insulation of conductivity k up to radius r_2, exposed to convection coefficient h:

$$R_{\text{ins}} = \frac{\ln(r_2/r_1)}{2\pi k L}, \qquad R_{\text{conv}} = \frac{1}{h(2\pi r_2 L)}. \tag{48}$$

Total resistance:

$$R_{\text{total}}(r_2) = R_{\text{ins}} + R_{\text{conv}}.$$

Differentiating $\dot{Q} = (T_1 - T_\infty)/R_{\text{total}}$ with respect to r_2 and setting $d\dot{Q}/dr_2 = 0$ gives the **critical radius**.

Critical radius for a cylinder

$$r_{\text{cr,cyl}} = \frac{k}{h} \tag{49}$$

Interpretation:

- If $r_2 < r_{\text{cr,cyl}}$: adding insulation *increases* $\dot{Q}$ (heat loss goes up).
- If $r_2 = r_{\text{cr,cyl}}$: $\dot{Q}$ is maximized.
- If $r_2 > r_{\text{cr,cyl}}$: adding insulation behaves as expected and *reduces* $\dot{Q}$.

For typical insulating materials ($k \approx 0.04$–$0.06\,\mathrm{W/(m\,K)}$) and natural convection in air ($h \gtrsim 5\,\mathrm{W/(m^2\,K)}$), $r_{\mathrm{cr,cyl}}$ is on the order of $\sim 1\,\mathrm{cm}$ or less, so most pipe insulation is safely beyond the critical radius.

Critical radius for a sphere

A similar derivation for a spherical shell yields

$$r_{\mathrm{cr,sph}} = \frac{2k}{h} \tag{50}$$

Purpose: Critical radius analysis tells us whether adding insulation to small cylinders/spheres (e.g. thin wires) might actually *increase* heat transfer (useful for electrical wires where we want better cooling), and assures us that for most practical pipe insulation the usual intuition "more insulation = less heat loss" holds.

2.7 17.7 Heat Transfer from Finned Surfaces

Fins (extended surfaces) increase the area available for convection/radiation and can greatly enhance heat transfer from a base surface at T_b to a fluid at T_∞.

Governing differential equation for a fin

Consider a straight fin of constant cross-section:

- Cross-sectional area A_c,
- Perimeter p,
- Thermal conductivity k,
- Convection coefficient h to surroundings at T_∞,
- Temperature excess $u(x) = T(x) - T_\infty$.

Energy balance on a differential element gives the **fin equation**:

$$\frac{d^2u}{dx^2} - m^2 u = 0, \qquad m^2 = \frac{hp}{kA_c} \tag{51}$$

General solution:

$$u(x) = C_1 e^{mx} + C_2 e^{-mx}.$$

Constants are set by boundary conditions at the fin base and tip.

Boundary condition at the base

At $x = 0$ the fin is attached to a wall at known temperature T_b:

$$u(0) = u_b = T_b - T_\infty \tag{52}$$

Common fin-tip conditions and solutions

1. Infinitely long fin ($T(L) \to T_\infty$, or $u(L) \to 0$ as $L \to \infty$):

$$\frac{T(x)-T_\infty}{T_b-T_\infty} = e^{-mx} \tag{53}$$

$$\dot{Q}_{\text{long fin}} = \sqrt{hpkA_c}\,(T_b-T_\infty) \tag{54}$$

2. Negligible heat loss at tip (adiabatic tip, $dT/dx|_{x=L}=0$):

$$\frac{T(x)-T_\infty}{T_b-T_\infty} = \frac{\cosh\big(m(L-x)\big)}{\cosh(mL)} \tag{55}$$

$$\dot{Q}_{\text{adiabatic tip}} = \sqrt{hpkA_c}\,(T_b-T_\infty)\,\tanh(mL) \tag{56}$$

3. Specified tip temperature $T(L)=T_L$:

$$\frac{T(x)-T_\infty}{T_b-T_\infty} = \frac{\dfrac{T_L-T_\infty}{T_b-T_\infty}\sinh(mx)+\sinh\big(m(L-x)\big)}{\sinh(mL)} \tag{57}$$

4. Convection at tip (with same h):

Tip condition from energy balance:

$$-kA_c\frac{dT}{dx}\bigg|_{x=L} = hA_c\big(T(L)-T_\infty\big).$$

Solution for temperature distribution:

$$\frac{T(x)-T_\infty}{T_b-T_\infty} = \frac{\cosh\big(m(L-x)\big)+\frac{h}{mk}\sinh\big(m(L-x)\big)}{\cosh(mL)+\frac{h}{mk}\sinh(mL)} \tag{58}$$

Corresponding heat transfer rate:

$$\dot{Q}_{\text{conv tip}} = \sqrt{hpkA_c}\,(T_b-T_\infty)\,\frac{\sinh(mL)+\frac{h}{mk}\cosh(mL)}{\cosh(mL)+\frac{h}{mk}\sinh(mL)} \tag{59}$$

Corrected fin length

In practice, fins with convection at the tip are often approximated as *adiabatic-tip* fins by using a **corrected length** L_c:

$$L_c = L + \frac{A_c}{p} \tag{60}$$

Common cases:

$$\text{Rectangular fin (thickness } t): \quad L_c = L + \frac{t}{2} \tag{61}$$

$$\text{Cylindrical pin fin (diameter } D): \quad L_c = L + \frac{D}{4} \tag{62}$$

Then use the adiabatic-tip formulas (55)–(56) with L replaced by L_c.

Purpose: Simplifies analysis while giving accurate results when $mL \gtrsim 1$.

Fin efficiency

Define the *fin efficiency* η_{fin} as

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin,max}}} = \frac{\text{actual heat transfer from fin}}{hA_{\text{fin}}(T_b - T_\infty)} \tag{63}$$

For a straight fin of constant cross section:

- **Very long fin:**
$$\eta_{\text{long}} = \frac{1}{mL} \tag{64}$$
- **Adiabatic-tip fin (or convective tip with corrected L_c):**
$$\eta_{\text{adiabatic}} = \frac{\tanh(mL)}{mL} \tag{65}$$

Fin heat transfer in terms of efficiency:

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\, hA_{\text{fin}}(T_b - T_\infty) \tag{66}$$

Rule of thumb: fins with $\eta_{\text{fin}} \lesssim 0.6$ are usually not economical (too long / too much material for little extra heat transfer).

Fin effectiveness

Fin effectiveness measures how worthwhile it is to add the fin compared to leaving just the base area A_b exposed:

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\,\eta_{\text{fin}} \tag{67}$$

Interpretation:

- $\varepsilon_{\text{fin}} < 1$: fin acts like *insulation* (bad design).
- $\varepsilon_{\text{fin}} \approx 1$: fin pointless (little benefit).
- $\varepsilon_{\text{fin}} \gg 1$: fin significantly enhances heat transfer (desirable).

Design implications:

- Want high k (good conductor: aluminum, copper),
- Want large p/A_c (thin, slender fins),
- Fins are most effective at low h (gases, natural convection) and on the gas side of heat exchangers.

Overall effectiveness of a finned surface

For a surface with unfinned area A_{unfin} and total fin area A_{fin} (sum of all fins), and base temperature T_b:

$$\dot{Q}_{\text{total, finned}} = hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty) \tag{68}$$
$$= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty) \tag{69}$$

If the same geometric area with no fins is $A_{\text{no fin}}$, define **overall surface effectiveness**:

$$\varepsilon_{\text{overall}} = \frac{\dot{Q}_{\text{total, finned}}}{\dot{Q}_{\text{total, no fin}}} = \frac{A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}}}{A_{\text{no fin}}} \tag{70}$$

This accounts for both the individual fin performance and how many fins are packed onto the surface.

Thermal resistance of a heat sink

Commercial heat sinks (for electronics) are often characterized by an overall thermal resistance between the *base* (temperature T_b) and ambient (T_∞):

$$R_{\text{hs}} = \frac{T_b - T_\infty}{\dot{Q}_{\text{hs}}} \quad [\text{K/W}] \tag{71}$$

Purpose: This single number captures the combined effect of fin geometry, material, and natural/forced convection, and is used directly to check whether a heat sink keeps an electronic component below its maximum safe temperature.

3 Chapter 18: Transient Heat Conduction

3.1 18.1 Transient Heat Conduction & Lumped System Analysis

In transient (unsteady) conduction, the temperature depends on both position and time:

$$T = T(x, y, z, t).$$

We are usually interested in how a solid heats up or cools down when its surroundings change suddenly (step change in T_∞ or h).

Energy balance for a solid

For a solid of volume V and surface area A_s, with uniform temperature $T(t)$ (no internal generation), exchanging heat with a fluid at T_∞ by convection:

$$\dot{Q}_{\text{conv}} = hA_s \left[T(t) - T_\infty\right],$$

and the rate of change of internal energy is

$$\dot{E}_{\text{stored}} = \rho V c_p \frac{\mathrm{d}T}{\mathrm{d}t}.$$

Energy conservation (cooling case, $T > T_\infty$):

$$-\dot{Q}_{\text{conv}} = \dot{E}_{\text{stored}} \quad \Rightarrow \quad -hA_s \left[T(t) - T_\infty\right] = \rho V c_p \frac{\mathrm{d}T}{\mathrm{d}t}.$$

This first-order ODE has the solution

$$T(t) - T_\infty = (T_i - T_\infty) \exp\left(-\frac{hA_s}{\rho V c_p} t\right),$$

or in dimensionless form

$$\theta(t) \equiv \frac{T(t) - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{hA_s}{\rho V c_p} t\right).$$

Variables.

- $T(t)$: uniform temperature of the solid at time t.
- T_∞: ambient (free-stream) fluid temperature.
- T_i: initial temperature of the solid at $t = 0$.
- h: convection heat transfer coefficient.
- A_s: surface area of the solid.
- ρ: density of the solid.
- c_p: specific heat of the solid.

Characteristic length, Biot number, Fourier number

For conduction problems we define a characteristic length

$$L_c = \frac{V}{A_s},$$

and the **thermal diffusivity**

$$\alpha = \frac{k}{\rho c_p}.$$

Two important dimensionless groups:

$$\mathrm{Bi} = \frac{hL_c}{k} \quad \text{(Biot number)}$$

$$\mathrm{Fo} = \frac{\alpha t}{L_c^2} \quad \text{(Fourier number)}.$$

Physical meaning.

- Bi compares *internal conduction resistance* in the solid to *external convection resistance*.
  - $\mathrm{Bi} \ll 1$ (typically $\mathrm{Bi} \leq 0.1$): conduction inside is very fast compared to convection; the solid tends to remain nearly uniform in temperature (lumped system is valid).
  - $\mathrm{Bi} \gtrsim 0.1$: significant temperature gradients inside the solid; lumped analysis is no longer accurate.
- Fo measures *how long* heat has had to diffuse relative to the size of the object. Larger Fo means more time for temperature to "even out".

Lumped-system solution in dimensionless form

Using $L_c = V/A_s$ and $\alpha = k/(\rho c_p)$, the lumped solution can be written as

$$\theta(t) = \frac{T(t) - T_\infty}{T_i - T_\infty} = \exp(-\mathrm{Bi}\,\mathrm{Fo}),$$

with

$$\mathrm{Bi} = \frac{hL_c}{k}, \qquad \mathrm{Fo} = \frac{\alpha t}{L_c^2}.$$

When is lumped analysis valid?

- Compute $\mathrm{Bi} = hL_c/k$ using the *solid*'s properties and characteristic length.
- If $\mathrm{Bi} \leq 0.1$, the lumped model is usually acceptable.
- If $\mathrm{Bi} > 0.1$, internal gradients are important and a non-lumped (spatially varying) solution is needed.

3.2 18.2 One- and Multi-Dimensional Non-Lumped Transient Conduction

When Bi is not small, temperature varies inside the solid. The governing equation (no generation) for 1-D transient conduction in a plane wall is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t},$$

with appropriate boundary and initial conditions (e.g. symmetry at the center, convection at the surface, and uniform initial temperature).

Non-dimensional formulation (plane wall example)

Define

$$X = \frac{x}{L}, \qquad u(X,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}, \qquad t = \mathrm{Fo} = \frac{\alpha t_{\mathrm{phys}}}{L^2}.$$

Then the 1-D transient conduction problem in a plane wall can be written in dimensionless form as

$$\begin{aligned} \frac{\partial^2 u}{\partial X^2} &= \frac{\partial u}{\partial t}, \\ \left.\frac{\partial u}{\partial X}\right|_{X=0} &= 0, \\ \left.\frac{\partial u}{\partial X}\right|_{X=1} &= -\mathrm{Bi}\, u(1,t), \\ u(X,0) &= 1. \end{aligned}$$

The solution for $u(X,t)$ is an infinite series involving eigenvalues that depend on Bi. In practice, we usually:

- Use **Heisler charts** or tables for $u(X,t)$ as a function of X, Bi, and Fo.
- Or use a **one-term approximation** (first term of the series) once time is not too small, e.g. for $\mathrm{Fo} \gtrsim 0.2$.

General form of the 1-D solution. For a plane wall with convection at the surface, the dimensionless temperature often has the form

$$u(X,t) \approx A_1 \exp\!\left(-\lambda_1^2 t\right)\, \phi_1(X),$$

where λ_1 and A_1 depend on Bi (tabulated), and $\phi_1(X)$ is a spatial eigenfunction (e.g. cosine or Bessel function depending on geometry).

Similar formulations exist for:

- **Long cylinder**: conduction in r only.
- **Sphere**: conduction in r only.

Multidimensional problems and product solution

Near corners and edges, heat can flow in more than one direction (e.g. x and y). For some geometries (rectangular plates, etc.) with simple boundary conditions, the **product solution** approach can be used:

$$u(x, y, t) = u_x(x, t)\, u_y(y, t),$$

or more generally as a sum of such products. Each factor satisfies a 1-D transient conduction problem in its coordinate direction.

Key ideas:

- Multidimensional solutions are built from 1-D eigenfunctions in each coordinate direction.
- In practice, engineers often use charts/tables or numerical methods (software) for true 2-D/3-D transients.

3.3 18.3 Semi-Infinite Body

A **semi-infinite medium** occupies $x \geq 0$ and is so large that the disturbance at the surface never reaches the "far" end (we treat it as infinite).

Typical setup:

- Initial condition: $T(x, 0) = T_i$ for $x \geq 0$.
- Boundary condition at the surface: $T(0, t) = T_s$ (sudden change) or convection at $x = 0$.
- As $x \to \infty$: $T(x, t) \to T_i$.

Similarity variable

For the constant surface-temperature case, we introduce the **similarity variable**

$$\eta = \frac{x}{2\sqrt{\alpha t}},$$

which collapses x and t into a single variable. The dimensionless temperature

$$\theta(x, t) = \frac{T(x, t) - T_s}{T_i - T_s}$$

depends only on η, not on x and t separately:

$$\theta = \theta(\eta).$$

The analytical solution is expressed in terms of the (Gaussian) **error function**:

$$\theta(\eta) = \operatorname{erf}(\eta) \quad \Rightarrow \quad \frac{T(x, t) - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right).$$

Physical interpretation.

- For small t, only a thin layer near the surface has responded to the temperature change.
- The **thermal penetration depth** grows roughly like

$$\delta \sim \mathcal{O}\big(\sqrt{\alpha t}\big).$$

- For $x \gg \sqrt{\alpha t}$, the temperature is still essentially T_i.

4 Chapter 19: Forced Convection and Boundary Layers

4.1 19.1 Forced Convection and the Velocity Boundary Layer

Forced convection occurs when fluid motion is driven by a fan, pump, blower, or external flow (e.g. air flowing over a plate). The convective heat transfer rate is given by Newton's law of cooling:

$$\dot{Q} = hA_s(T_s - T_\infty).$$

When a fluid flows over a solid surface:

- At the surface: due to the no-slip condition, $u = 0$.
- Away from the surface: $u \to U_\infty$ (free-stream velocity).

The region where the velocity changes from 0 to U_∞ is the **velocity boundary layer**.

Velocity boundary layer thickness. The **boundary layer thickness** $\delta(x)$ is typically defined where $u \approx 0.99\,U_\infty$. It:

- Starts at zero at the leading edge ($x = 0$).
- Grows downstream as the fluid is slowed by viscous effects.
- Is thinner in high-speed or low-viscosity flows (large Reynolds number).

4.2 19.2 Thermal Boundary Layer

Similarly, there is a **thermal boundary layer**, where the fluid temperature changes from T_s at the surface toward T_∞ in the free stream.

- The thickness of the thermal boundary layer is denoted $\delta_t(x)$.
- The relative thicknesses of δ_t and δ depend on the **Prandtl number**.

Prandtl number.

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k},$$

where ν is kinematic viscosity and α is thermal diffusivity.

- $\mathrm{Pr} \ll 1$ (liquid metals): thermal diffusion is very fast; δ_t is *thicker* than the velocity boundary layer.
- $\mathrm{Pr} \gg 1$ (oils): thermal diffusion is slow; δ_t is *thinner* than the velocity boundary layer.
- $\mathrm{Pr} \approx 1$ (many gases, water at some conditions): δ_t and δ are of similar thickness.

4.3 19.3 Reynolds, Prandtl, and Nusselt Numbers

Reynolds number (external flow over flat plate). For a position x from the leading edge:

$$\mathrm{Re}_x = \frac{\rho U_\infty x}{\mu} = \frac{U_\infty x}{\nu}.$$

- Small Re_x: viscous forces dominate, flow is laminar.
- Large Re_x: inertial forces dominate, flow becomes turbulent.
- For a flat plate, transition typically begins around $\mathrm{Re}_x \sim 10^5$ and is often taken as fully turbulent after $\mathrm{Re}_x \sim 3 \times 10^6$. A commonly used *engineering* critical value is

$$\mathrm{Re}_{\mathrm{cr}} \approx 5 \times 10^5.$$

Prandtl number (again). As above,

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}.$$

It compares momentum and thermal diffusivities and controls the relative thickness of velocity and thermal boundary layers.

Nusselt number. The **Nusselt number** is a dimensionless heat transfer coefficient:

$$\mathrm{Nu} = \frac{hL_c}{k}.$$

For local values over a flat plate we use

$$\mathrm{Nu}_x = \frac{h_x x}{k},$$

and for average over a plate of length L,

$$\mathrm{Nu}_L = \frac{h_L L}{k}.$$

Physical meaning:

- Nu measures the enhancement of heat transfer by convection relative to pure conduction across the same distance.
- Larger Nu means stronger convective transport and a larger h.

4.4 19.4 Flow over Flat Plates, Cylinders, and Spheres

Flow over a flat plate (isothermal surface)

For a flat plate at uniform surface temperature T_s in a fluid at T_∞, with free-stream velocity U_∞:

Local Nusselt number (laminar).

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}, \quad \mathrm{Pr} > 0.6,\ \mathrm{Re}_x < 5 \times 10^5.$$

Local Nusselt number (turbulent).

$$\mathrm{Nu}_x = 0.0296\,\mathrm{Re}_x^{0.8}\,\mathrm{Pr}^{1/3}, \quad 0.6 \le \mathrm{Pr} \le 60,\ 5\times10^5 \lesssim \mathrm{Re}_x \lesssim 10^7.$$

Average Nusselt number over the plate. For a plate of length L:

$$\mathrm{Nu}_L = \frac{h_L L}{k}.$$

If the flow is laminar over the entire plate:

$$\mathrm{Nu}_L = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}, \quad \mathrm{Re}_L < 5\times10^5.$$

If the flow is turbulent over the entire plate:

$$\mathrm{Nu}_L = 0.037\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{1/3}, \quad 0.6 \le \mathrm{Pr} \le 60,\ 5\times10^5 \lesssim \mathrm{Re}_L \lesssim 10^7.$$

For cases where the flow is laminar near the leading edge and turbulent further downstream, a combined-correlation often used in practice is

$$\mathrm{Nu}_L = \left(0.037\,\mathrm{Re}_L^{0.8} - 871\right)\mathrm{Pr}^{1/3}, \quad 0.6 \le \mathrm{Pr} \le 60.$$

Cross-flow over cylinders and spheres

For external flow over a cylinder or sphere (diameter D), we use D as the characteristic length, so

$$\mathrm{Re} = \frac{\rho U_\infty D}{\mu}, \qquad \mathrm{Nu} = \frac{hD}{k}.$$

Cylinder (cross-flow). A widely used correlation:

$$\mathrm{Nu}_{\mathrm{cyl}} = 0.3 + 0.62\,\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}\left[1+\left(\frac{0.4}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}\left[1+\left(\frac{\mathrm{Re}}{282{,}000}\right)^{5/8}\right]^{4/5},$$

valid for $\mathrm{Re}\,\mathrm{Pr} \gtrsim 0.2$ (typical external flows).

Sphere (cross-flow). A commonly used correlation:

$$\mathrm{Nu}_{\mathrm{sph}} = 2 + \left[0.4\,\mathrm{Re}^{1/2} + 0.06\,\mathrm{Re}^{2/3}\right]\mathrm{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4},$$

for about $3.5 \le \mathrm{Re} \le 8\times10^4$ and $0.7 \le \mathrm{Pr} \le 380$. Here μ_∞ is evaluated at the free-stream temperature, and μ_s at the surface temperature.

How to use these correlations in practice.

1. Choose the characteristic length:
   - Flat plate: L in flow direction (for Nu_L).
   - Cylinder or sphere: diameter D.

2. Compute Re and Pr using fluid properties at the *film temperature* (typically $T_f = (T_s + T_\infty)/2$ for external flow).

3. Select the correct regime (laminar / turbulent / combined).

4. Compute Nu, then solve for h from

$$h = \frac{\mathrm{Nu}\, k}{L_c}.$$

5. Finally, compute $\dot{Q} = hA_s(T_s - T_\infty)$ as needed.

5 Chapter 21: Thermal Radiation

Thermal radiation is energy emitted by matter due to its temperature, in the form of electromagnetic waves. Unlike conduction and convection, radiation:

- requires no medium (can occur in a vacuum),
- travels at the speed of light,
- depends strongly on surface properties and temperature,
- is governed by electromagnetic wave physics rather than fluid mechanics.

Radiation becomes especially significant at high temperatures because radiative heat flux scales with T^4.

5.1 21.1 Fundamentals of Thermal Radiation

All matter at $T > 0\,\mathrm{K}$ emits thermal radiation. The emission originates from oscillations of electrons in atoms/molecules.

The key quantity is the **spectral emissive power**, $E_\lambda(T)$:

$$E_\lambda(T) = \text{energy emitted per unit area per unit wavelength per unit time.}$$

The **total emissive power** (all wavelengths) is:

$$E(T) = \int_0^\infty E_\lambda(T)\, \mathrm{d}\lambda.$$

A theoretical surface that emits the maximum possible radiation at any wavelength and temperature is a **blackbody**. It serves as the reference for all real surfaces.

Stefan–Boltzmann law

The total radiation emitted by a blackbody is

$$E_b = \sigma T^4, \qquad \sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4}.$$

Real surfaces emit less than a blackbody. Their emissive power is

$$E = \varepsilon \sigma T^4,$$

where $0 \le \varepsilon \le 1$ is the **emissivity**.

Kirchhoff's law (important)

For a surface in thermal equilibrium:

$$\varepsilon_\lambda = \alpha_\lambda.$$

That is, a good absorber is a good emitter at the same wavelength.

This becomes essential when studying radiation exchange between surfaces.

(Very Brief) Essentials from 21.2–21.3

Although not covered in lecture, we need two basic ideas to understand later sections:

1. Spectral radiative properties. Surfaces interact with radiation via:

$$\alpha_\lambda + \rho_\lambda + \tau_\lambda = 1$$

(absorption + reflection + transmission).

Opaque surfaces: $\tau_\lambda = 0$.

2. Directional radiative behavior. Real surfaces may emit or reflect differently depending on direction. A **diffuse** surface emits uniformly in all directions. Most engineering surfaces are assumed diffuse for simplicity.

These basics enable the radiation network methods used in Sections 21.5–21.8.

5.2 21.4 Radiative Properties of Surfaces

A surface interacting with incident radiation has:

- **Absorptivity** α: fraction absorbed.
- **Reflectivity** ρ: fraction reflected.
- **Transmissivity** τ: fraction transmitted (usually 0 for solids).

For opaque solids:

$$\alpha + \rho = 1.$$

Emissivity

$$\varepsilon = \frac{\text{actual emitted radiation}}{\text{blackbody emission at same } T}.$$

A "black" surface has $\varepsilon = 1$. Polished metals have very low emissivity (0.02–0.1), and rough/dark surfaces have high emissivity (0.8–0.95).

Gray surface approximation

If ε and α do not vary significantly with wavelength, we treat the surface as **gray**:

$$\varepsilon = \text{constant over all wavelengths.}$$

This dramatically simplifies radiation heat transfer.

5.3 21.5 View Factors

When two surfaces exchange radiation, not all energy leaving surface 1 reaches surface 2. The geometric fraction that does is the **view factor** (or configuration factor):

$$F_{1\to 2} = \text{fraction of radiation leaving surface 1 that strikes surface 2 directly.}$$

It depends only on geometry, not temperature or material.

View factor definition

$$F_{1\to 2} = \frac{1}{A_1}\int_{A_1}\int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\pi r^2}\,\mathrm{d}A_1\,\mathrm{d}A_2.$$

In practice we rarely evaluate this integral; instead we use known formulas or view factor algebra.

Important view factor rules

1. Reciprocity relation

$$A_1 F_{1\to 2} = A_2 F_{2\to 1}.$$

2. Summation rule For any enclosure:

$$\sum_{j=1}^{N} F_{i\to j} = 1.$$

3. Surface to itself

$$F_{i\to i} = 0 \quad \text{for planar surfaces.}$$

Common special cases

- A large plane facing a small object: $F_{obj\to wall} = 1$ but $F_{wall\to obj} \ll 1$ from reciprocity.
- Two infinite parallel plates: $F_{1\to 2} = 1$.
- Small surface inside a large enclosure: $F_{1\to surroundings} = 1$.

These will be crucial for black and gray surface radiation exchange.

5.4 21.6 Radiation Exchange Between Black Surfaces

A black surface emits radiation as σT^4 and absorbs *all* incident radiation ($\alpha = 1$). Therefore blackbody radiation exchange between two surfaces is the simplest case.

For two diffuse, black surfaces 1 and 2:

$$\dot{Q}_{1\to 2} = A_1 F_{1\to 2}\,\sigma\left(T_1^4 - T_2^4\right).$$

This result is extremely useful:

- No radiosity or iterative calculation.
- No emissivity terms.
- Only geometry (via view factors) and temperatures matter.

Enclosures with more surfaces

In an enclosure with multiple black surfaces, net heat loss from surface i is:

$$\dot{Q}_i = \sum_{j=1}^{N} A_i F_{i\to j} \sigma \left(T_i^4 - T_j^4\right).$$

Black-surface analysis forms the conceptual foundation for gray-surface networks.

5.5 21.7 Radiation Networks and Resistances (Black Surfaces)

Although black surfaces have the simplest exchange relationships, the **network method** helps visualize and generalize to gray surfaces.

Radiation network concept

We define a "radiation potential"

$$E_b = \sigma T^4,$$

analogous to voltage. The resistance between two black surfaces is:

$$R_{1\to 2} = \frac{1}{A_1 F_{1\to 2}}.$$

Thus:

$$\dot{Q}_{1\to 2} = \frac{E_{b1} - E_{b2}}{R_{1\to 2}}.$$

This matches your convection-conduction electrical analogy from earlier chapters.

The radiation network becomes essential when emissivity is not 1.

5.6 21.8 Radiation Exchange Between Gray, Diffuse Surfaces

Real surfaces have emissivity $\varepsilon < 1$ and may reflect radiation. We use **radiosity** and **irradiation**:

- Radiosity J: total radiation *leaving* a surface (emitted + reflected).
- Irradiation G: total radiation *arriving at* a surface.

21.8.1 Surface relations

For a gray, diffuse surface:

$$J = \varepsilon E_b + (1 - \varepsilon) G.$$

Net heat leaving the surface:

$$\dot{q} = J - G.$$

Convective-like radiation coefficients

We define a **surface radiation resistance**:

$$R_{\text{surf},i} = \frac{1-\varepsilon_i}{\varepsilon_i A_i}.$$

And an **exchange resistance** (from view factor):

$$R_{i\to j} = \frac{1}{A_i F_{i\to j}}.$$

Together these form a thermal-resistance-like network similar to conduction and convection.

21.8.2 Two gray surfaces

For two diffuse, gray surfaces exchanging radiation:

$$\dot{Q} = \frac{E_{b1} - E_{b2}}{R_{\text{surf},1} + R_{1\to 2} + R_{\text{surf},2}}.$$

Explicitly:

$$\dot{Q} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{1\to 2}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}}.$$

If the two surfaces form an enclosure (common case) Then $F_{1\to 2} = 1$ and $A_1 = A_2$:

$$\dot{Q} = \frac{\sigma A \left(T_1^4 - T_2^4\right)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}.$$

This is one of the most commonly used gray-surface formulas in engineering.

21.8.3 Radiation Exchange With Many Gray Surfaces

The full multidimensional problem uses:

- A radiosity node at each surface,
- A surface resistance $R_{\text{surf},i}$,
- A geometry resistance $R_{i\to j}$,
- Energy conservation at each node.

This is analogous to solving a linear circuit with multiple nodes. However, for most practical textbook problems, only two-surface gray radiation is required.

Summary of radiation exchange steps (gray surfaces):

1. Compute $E_{bi} = \sigma T_i^4$.
2. Compute $R_{\text{surf},i} = \dfrac{1-\varepsilon_i}{\varepsilon_i A_i}$.
3. Compute $R_{1\to 2} = \dfrac{1}{A_1 F_{1\to 2}}$.
4. Use
$$\dot{Q} = \frac{E_{b1} - E_{b2}}{R_{\text{surf},1} + R_{1\to 2} + R_{\text{surf},2}}.$$

This parallels conduction and convection resistance networks and matches the style of earlier chapters.