AER210 Midterm Field Guide — Multiple Integrals & Vector Calculus

1. Multiple Integrals

1.1 Regions & Setup (Type I/II) — Stewart 15.1–15.2

Type I (vertical slices):

$$R = \{(x,y) : a \le x \le b,\, g_1(x) \le y \le g_2(x)\}, \qquad \iint_R f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II (horizontal slices):

$$R = \{(x,y) : c \le y \le d,\, h_1(y) \le x \le h_2(y)\}, \qquad \iint_R f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Midterm pattern: reversing order/sketching region (2024 Q1b).
Checklist:
- Sketch axes, curves, intercepts.
- Decide vertical/horizontal order.
- Write new bounds before touching integrand.

1.2 Double Integrals in Polar — Stewart 15.3

Polar: $x = r\cos\theta,\ y = r\sin\theta,\ dA = r\,dr\,d\theta$. "Circle/annulus/rotational symmetry → polar." Paraboloids intersection → footprint curve in xy-plane gives r-limits.

$$V = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} [z_{\text{top}}(r) - z_{\text{bot}}(r)]\,r\,dr\,d\theta$$

1.3 Applications of $\iint$: Mass, COM, MOI — Stewart 15.4

$$m = \iint_R \rho\,dA, \qquad \bar{x} = \frac{1}{m}\iint_R x\rho\,dA, \quad \bar{y} = \frac{1}{m}\iint_R y\rho\,dA$$
$$I_O = \iint_R (x^2+y^2)\rho\,dA$$

Change of variables often simplifies these (2022 Q6).

1.4 Surface Area — Stewart 15.5 & 16.6

Graph $z = f(x,y)$: $dS = \sqrt{1 + f_x^2 + f_y^2}\,dA$. Parametric $\mathbf{r}(u,v)$: $dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv$.
- Compute $\mathbf{r}_u, \mathbf{r}_v$
- Cross product, then integrate magnitude over parameter box

1.5 Triple Integrals — Stewart 15.6

Switch order to simplify; choose suitable coordinates. Cylindrical: $dV = r\,dr\,d\theta\,dz$; Spherical: $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

1.6 Cylindrical & Spherical Coords — Stewart 15.7–15.8

$$\begin{aligned}
&\text{Cylindrical: } (r,\theta,z),\ x = r\cos\theta,\ y = r\sin\theta,\ dV = r\,dr\,d\theta\,dz \\
&\text{Spherical: } (\rho,\phi,\theta),\ x = \rho\sin\phi\cos\theta,\, y = \rho\sin\phi\sin\theta,\, z = \rho\cos\phi, \\
&dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta
\end{aligned}$$

1.7 Taylor in Two Variables (2$^{\text{nd}}$ order)

$$f(x,y) \approx f_0 + f_x\Delta x + f_y\Delta y + \tfrac{1}{2}(f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2)$$

Used for local quadratic approximations; Hessian signs → shape info.

1.8 Change of Variables & Jacobian — Stewart 15.9

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v))\Big|\frac{\partial(x,y)}{\partial(u,v)}\Big|\,du\,dv$$

Choose new variables to straighten level curves. Checklist:

- Guess u, v from bounding curves.
- Compute Jacobian J.
- Substitute and replace $dx\,dy$ with $|J|\,du\,dv$.

2. Vector Calculus

2.1 Line Integrals — Stewart 16.2

Scalar: $\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$. Vector (work): $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$.

- Parameterize C; compute $\mathbf{r}'(t)$.
- Plug into correct formula.
- If field looks like gradient, jump to FTLI.

2.2 Fundamental Theorem for Line Integrals (FTLI) — Stewart 16.3

If $\mathbf{F} = \nabla f$, then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A)$. Conservative test: $\nabla\times\mathbf{F} = \mathbf{0} \Rightarrow$ path independent.

2.3 Green's Theorem — Stewart 16.4

$$\oint_C P\,dx + Q\,dy = \iint_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dA$$

Use for positively oriented closed C.

- Check CCW orientation
- Compute integrand
- Integrate over region

2.4 Parametric Surfaces & Surface Area — Stewart 16.6

$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$, Area $= \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv$.

2.5 Surface Integrals — Stewart 16.7

Scalar: $\iint_S f\,dS$; Flux: $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$. Graph $z = f(x,y)$:

$$\mathbf{n} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1 + f_x^2 + f_y^2}}, \quad \mathbf{F}\cdot\mathbf{n}\,dS = \mathbf{F}\cdot\langle -f_x, -f_y, 1\rangle\,dx\,dy$$

2.6 Divergence & Curl — Stewart 16.5

$\nabla\cdot\mathbf{F}$: sources/sinks; $\nabla\times\mathbf{F}$: rotation. $\nabla\times(\nabla f) = \mathbf{0}$. If curl $= 0$ on simply connected domain $\to$ conservative.

2.7 Divergence Theorem — Stewart 16.9

$$\iint_{\partial V} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V (\nabla\cdot\mathbf{F})\,dV$$

2.8 Stokes' Theorem — Stewart 16.8

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

Boundary orientation : right-hand rule. Checklist:

- Compute curl
- Choose S with easy dS
- Dot, convert to polar if disk footprint

3. Past-Midterm Style Playbook

- Reverse order → sketch and flip (2024 Q1b)
- Volume between paraboloids → polar $z_t - z_b$ (2022/2024)
- Param curve line integral → param + plug (2024 Q2); if curl = 0 → FTLI
- Green's on polygon → double integral (2022 Q3; 2024 Q3)
- Conservative → potential → $f(B) - f(A)$
- Surface patch → param; flux via $\mathbf{r}_u \times \mathbf{r}_v$
- Stokes on cap → curl, upward normal, polar disk (2024 Q8)
- Change of vars → rectangularize (2022 Q6; 2024 Q6)

4. Little Tricks & Assumptions

- Orientation matters (Green/Stokes/flux)
- Always attach Jacobian: polar r; cylindrical r; spherical $\rho^2 \sin\phi$
- Conservative → closed loop integral = 0
- Green's prefers triangles/rectangles/disks
- Switch order when integrand separates
- Symmetry: odd -¿ 0; even -¿ double half-region
- Piecewise curves: keep orientation or use Green's

5. Harder Integrals & Trig Identities

Basic Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\int \sin\theta\cos\theta\, d\theta = \tfrac{1}{2}\sin^2\theta + C$$

Power Reduction

$$\sin^2\theta = \tfrac{1-\cos 2\theta}{2}, \qquad \cos^2\theta = \tfrac{1+\cos 2\theta}{2}$$

Double-Angle Formulas

$$\begin{aligned} \sin 2A &= 2\sin A\cos A, \\ \cos 2A &= \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1, \\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A} \end{aligned}$$

Sum and Difference Formulas

$$\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B, \\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B, \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \end{aligned}$$

Product-to-Sum Formulas

$$\begin{aligned} \sin A\sin B &= \tfrac{1}{2}\big[\cos(A-B) - \cos(A+B)\big], \\ \cos A\cos B &= \tfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big], \\ \sin A\cos B &= \tfrac{1}{2}\big[\sin(A+B) + \sin(A-B)\big] \end{aligned}$$

Polar, Cylindrical, and Spherical Volume Elements

$$\begin{aligned} dA &= r\,dr\,d\theta, \\ dV_{\text{cyl}} &= r\,dr\,d\theta\,dz, \\ dV_{\text{sph}} &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

6. Quadric Surfaces (Recognition Cheat)

$$
\begin{aligned}
&\text{Ellipsoid: } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\\
&\text{Cone: } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0\\
&\text{Elliptic paraboloid: } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}\\
&\text{Hyperboloid (1 sheet): } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\\
&\text{Hyperboloid (2 sheets): } -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\\
&\text{Hyperbolic paraboloid: } \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}
\end{aligned}
$$

Intersect with coordinate planes to identify quickly.

7. One-Page Checklists

A) Volume between two surfaces: intersect $\to$ polar $\to$ integrate r.
B) Reverse order: sketch, rewrite as Type I/II.
C) Line integral: decide scalar/vector $\to$ FTLI if curl $= 0$.
D) Green's: CCW $\to$ compute $\to$ integrate.
E) Conservative: curl 0 $\to$ find $f \to f(B) - f(A)$.
F) Param surface: $\mathbf{r}_u \times \mathbf{r}_v$.
G) Stokes cap: upward normal $\to$ polar disk.
H) Change of vars: pick $u, v \to$ compute $|J|$.

8. Quick Definitions

Type I/II regions: vertical vs horizontal.
Piecewise smooth curve: $\mathbf{r}'(t)$ continuous/non-zero.
Conservative field: path independent; $\mathbf{F} = \nabla f$.
Flux = flow across surface ($\mathbf{F}\cdot\mathbf{n}$).
Source/Sink : sign of divergence.
Laplacian $\Delta f = \nabla\cdot\nabla f$.

9. Mini Practice Prompts

1. Reverse order of $\int_0^1\int_x^1 e^{x/y}\,dy\,dx$
2. Volume between $z = 4 - r^2$ and $z = 3r^2$
3. Check conservativity of $\mathbf{F} = (3 + 2xy^2, 2x^2y)$; find f
4. Use Green's on triangle $(0, 0), (2, 1), (0, 1)$
5. Verify Stokes for paraboloid cap (upward normal)