1 Reviewing Contradiction Proofs

Proving $P \Rightarrow Q$ By Contradiction. We know that

$$\neg(P \Rightarrow Q) \equiv p \wedge \neg Q$$

Then our proof could follow the structure:
Proof Structure:

Assume $P \wedge \neg Q$
...
Therefore, Contradict. ■

Alternatively we can start it directly:
Proof Structure:

Assume P
WTS Q by Contradict.
Assume $\neg P$
...
Therefore, Contradict. ■

Recall the proof of $\forall n, k, d \in \mathbb{Z}^+, n = kd \Rightarrow k \leq n \wedge d \leq n$
Proof:

Let $n, kd \in \mathbb{Z}^+ . Assumen = dk$
Assume for contra, $k > n \vee d > n$
Case $k > n$ *Continuing solving this, we note that this case is unused.*

$$\begin{aligned} n &= kd \\ \therefore k &= \frac{n}{d} \\ &\leq n && \therefore d \geq 1 \end{aligned}$$

■

2 Induction Proofs

Consider the question:

For which $n \in \mathbb{N}$ is $n + 2 < n^{2-1}$?

We can examine the the table

n	$n+2$	2^{n-1}	$n+2<n^{2-1}$
0	2	$\frac{1}{2}$	`False`
1	3	1	`False`
2	4	2	`False`
3	5	4	`False`
4	6	8	`True`
5	7	16	`True`

We come up with the conjecture:

$$\forall n \in \mathbb{N}, n \geq 4 \Rightarrow n + 2M2^{n-1}$$

We can examine this conjecture by extending the table:

$$\begin{aligned} 7 &> 16 \\ 8 = 7 + 1 &< 7 + 16 && \text{Since } 2^{n-1} \text{ always increases by at least 16} \\ &< 16 + 16 \\ &< 32 \end{aligned}$$

We also note:
Let $P(n)$ be $n + 2 < 2^{n-1}$, then $(n+1)+2 = (n+2)+1$
Therefore, $2^{(n+1)-1} = 2 \cdot 2^{n-1} = 2^{n-1} \cdot 2^{n-1}$

We notice that we can prove $P(4) \Rightarrow P(5) \wedge P(5) \Rightarrow P(6) \wedge P(6) \Rightarrow P(7) \ldots$

Example

let $c \in \mathbb{N}$
Suppose $P(c) \wedge \forall n \in \mathbb{N}, n \geq c \Rightarrow \big(P(n) \Rightarrow P(n+1)\big)$
Then $\forall n \in \mathbb{N}, n \geq c \Rightarrow P(n)$

Proof Structure:

Base Case: Prove $P(c)$
Inductive Step: Let $n \in \mathbb{N}$
Assume $n \geq c$ and $P(n)$ – this is by the induction hypothesis, *explicitely state it in proofs*
...
Therefore $P(n+1)$ ■

Proving the statement from above,

$$\begin{aligned} Pn &: n + 2 < 2^{n-1} \\ Q(n) &: n \geq 4 \Rightarrow n + 2 < 2^{n-1} \qquad \forall n \in \mathbb{N}, Q(n) \end{aligned}$$

Proof:

Try induction from 0.
Base Case: $0 \geq 4 \Rightarrow \ldots$ Vacuously True!
Inductive Step: Let $n \in \mathbb{N}$ Assume $n \geq 4 \Rightarrow n + 2 < 2^{n-1}$
WTS: $n + 1 \geq 4 \Rightarrow n + 1 + 2 < 2^{n+1-1}$ Assume $n + 1 \geq 4$
Then $n \geq 3$

Case $\boldsymbol{n = 3}$: WTS $n + 1 + 2 < 2$
Case $\boldsymbol{n \geq 4}$: WTS $n + 2 < n^{n-1}$ By the induction hypothesis

■