Numerical Solution of Ordinary Differential Equation

- A first order initial value problem of ODE may be written in the form

$$y'(t) = f(y,t), \qquad y(0) = y_0$$

- Example:

$$y'(t) = 3y + 5, \qquad y(0) = 1$$

$$y'(t) = ty + 1, \qquad y(0) = 0$$

- Numerical methods for ordinary differential equations calculate solution on the points, $t_n = t_{n-1} + h$ where *h* is the steps size

Numerical Methods for ODE

- **Euler Methods**
  - Forward Euler Methods
  - Backward Euler Method
  - Modified Euler Method

- **Runge-Kutta Methods**
  - Second Order
  - Third Order
  - Fourth Order

Forward Euler Method

- Consider the forward difference approximation for first derivative

$$y_n' \cong \frac{y_{n+1} - y_n}{h}, \quad h = t_{n+1} - t_n$$

- Rewriting the above equation we have

$$y_{n+1} = y_n + h y_n', \qquad y_n' = f(y_n, t_n)$$

- So, y_n is recursively calculated as

$$y_1 = y_0 + h y_0' = y_0 + h\, f(y_0, t_0)$$

$$y_2 = y_1 + h\, f(y_1, t_1)$$

$$\vdots$$

$$y_n = y_{n-1} + h\, f(y_{n-1}, t_{n-1})$$

Example: solve

$$y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$$

Solution:

$$\text{for} \quad t_0 = 0, \qquad y_0 = y(0) = 1$$

$$\text{for} \quad t_1 = 0.25, \qquad y_1 = y_0 + hy_0'$$

$$= y_0 + h(t_0 y_0 + 1)$$

$$= 1 + 0.25(0 * 1 + 1) = 1.25$$

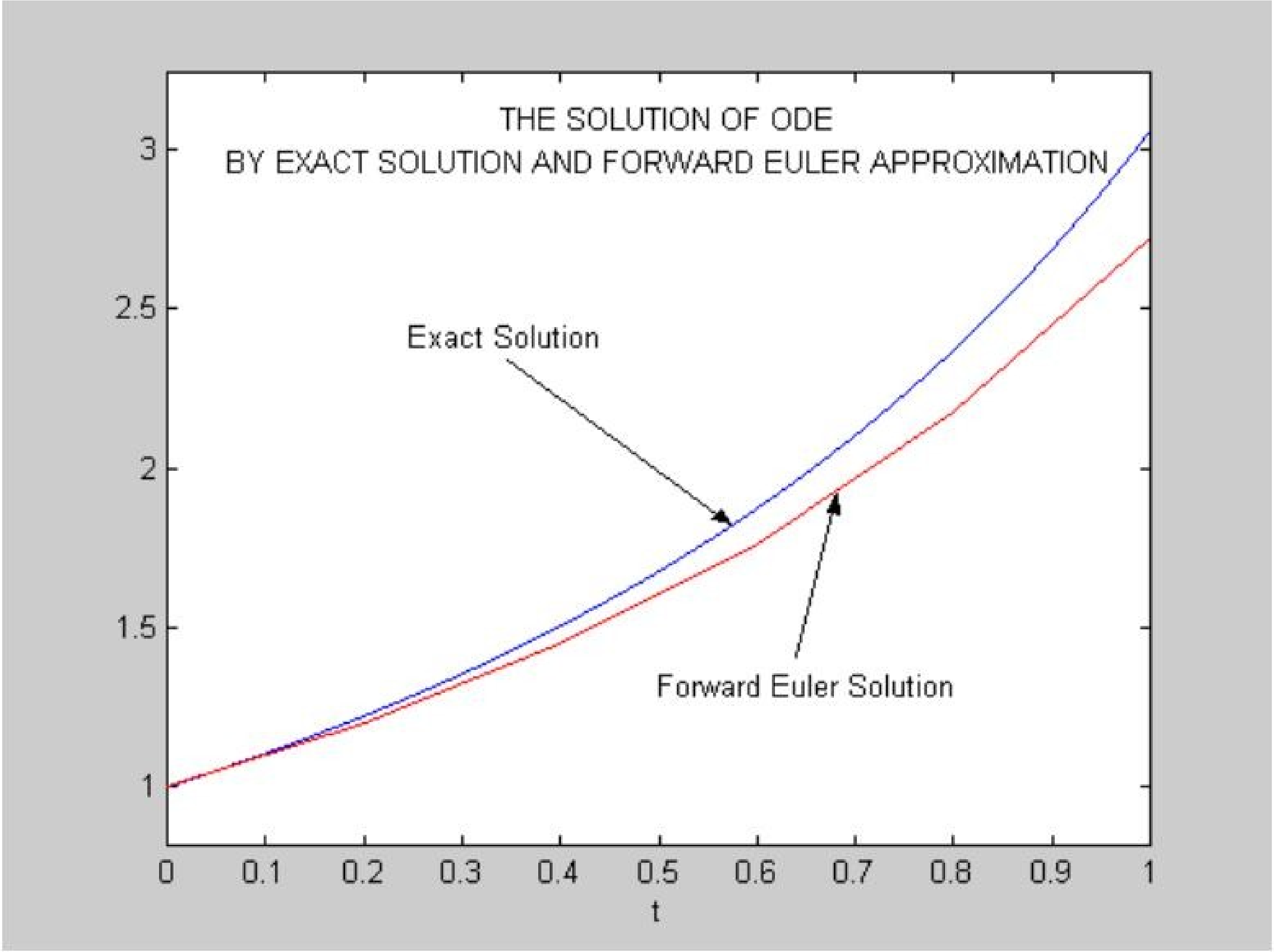
$$\text{for} \quad t_2 = 0.5, \qquad y_2 = y_1 + hy_1'$$

$$= y_1 + h(t_1 y_1 + 1)$$

$$= 1.25 + 0.25(0.25 * 1.25 + 1) = 1.5781$$

etc

Graph the solution



Backward Euler Method

- Consider the backward difference approximation for first derivative

$$y_n' \cong \frac{y_n - y_{n-1}}{h}, \quad h = t_n - t_{n-1}$$

- Rewriting the above equation we have

$$y_n = y_{n-1} + hy_n', \qquad y_n' = f(y_n, t_n)$$

- So, y_n is recursively calculated as

$$y_1 = y_0 + hy_1' = y_0 + h\,f(y_1, t_1)$$
$$y_2 = y_1 + h\,f(y_2, t_2)$$
$$\vdots$$
$$y_n = y_{n-1} + h\,f(y_n, t_n)$$

Example: solve

$$y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$$

Solution:

Solving the problem using backward Euler method for y_n yields

$$y_n = y_{n-1} + hy_n' = y_{n-1} + h(t_n y_n + 1)$$

$$\Leftrightarrow \quad y_n - ht_n y_n = y_{n-1} + h$$

$$\Leftrightarrow \quad y_n = \frac{y_{n-1} + h}{1 - ht_n}$$

So, we have

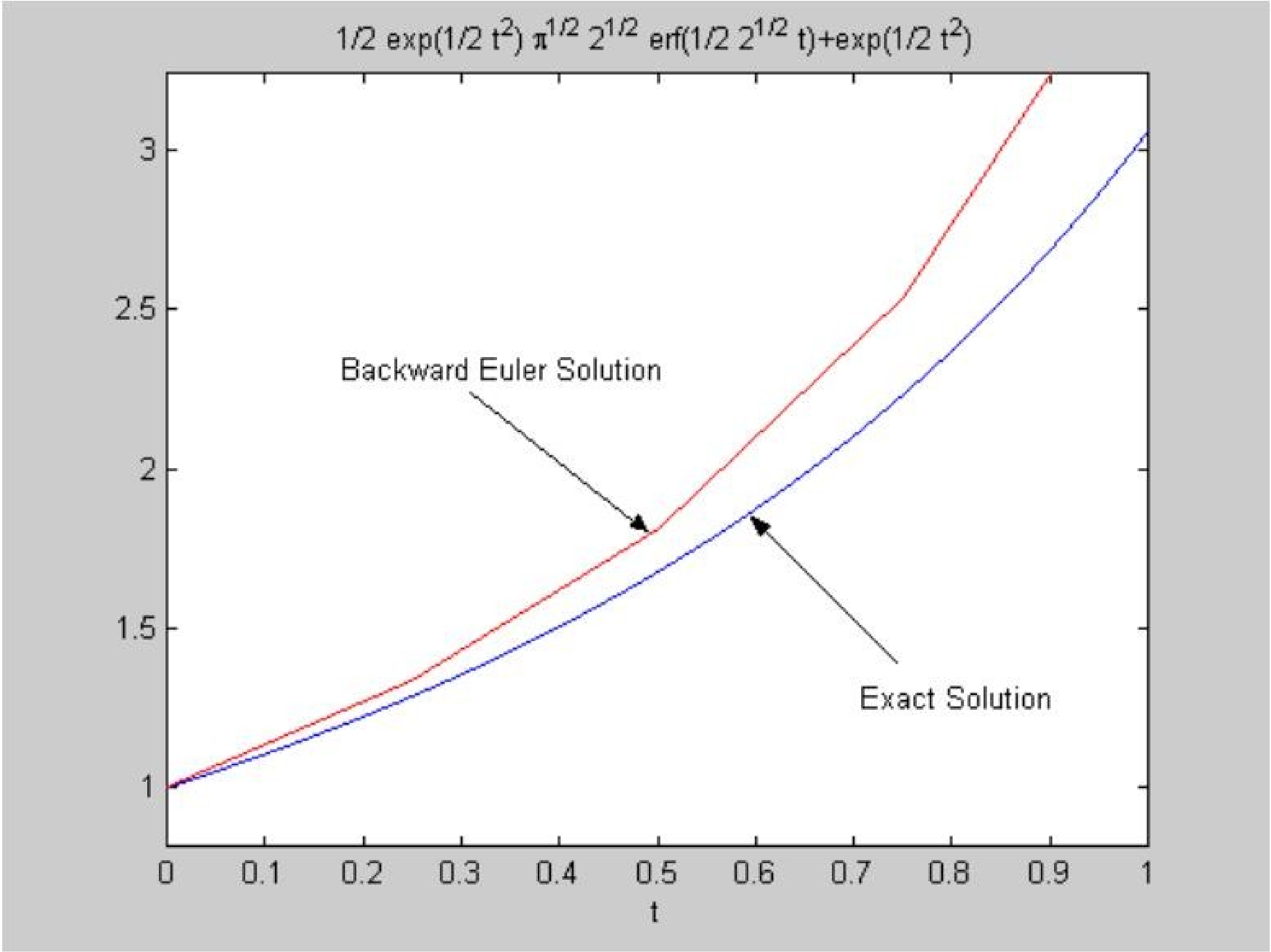
$$\text{for} \quad t_1 = 0.25, \qquad y_1 = \frac{y_0 + h}{1 - ht_1} = \frac{1 + 0.25}{1 - 0.25 * 0.25} = 1.333$$

$$\text{for} \quad t_2 = 0.5, \qquad y_2 = \frac{y_1 + h}{1 - ht_2} = \frac{1.333 + 0.25}{1 - 0.25 * 0.5} = 1.8091$$

$$\text{for} \quad t_3 = 0.75, \qquad y_3 = \frac{y_2 + h}{1 - ht_3} = \frac{1.8091 + 0.25}{1 - 0.25 * 0.75} = 2.5343$$

$$\text{for} \quad t_4 = 1, \qquad y_4 = \frac{y_3 + h}{1 - ht_4} = \frac{2.5343 + 0.25}{1 - 0.25 * 1} = 3.7142$$

Graph the solution



Modified Euler Method

- Modified Euler method is derived by applying the trapezoidal rule to integrating $y_n' = f(y,t)$; So, we have

$$y_{n+1} = y_n + \frac{h}{2}(y_{n+1}' + y_n'), \qquad y_n' = f(y_n, t_n)$$

- If *f* is linear in *y*, we can solved for y_{n+1} similar as backward euler method
- If *f* is nonlinear in *y*, we necessary to used the method for solving nonlinear equations i.e. successive substitution method (*fixed point*)

Example: solve

$$y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$$

Solution:

f is linear in *y*. So, solving the problem using modified Euler method for y_n yields

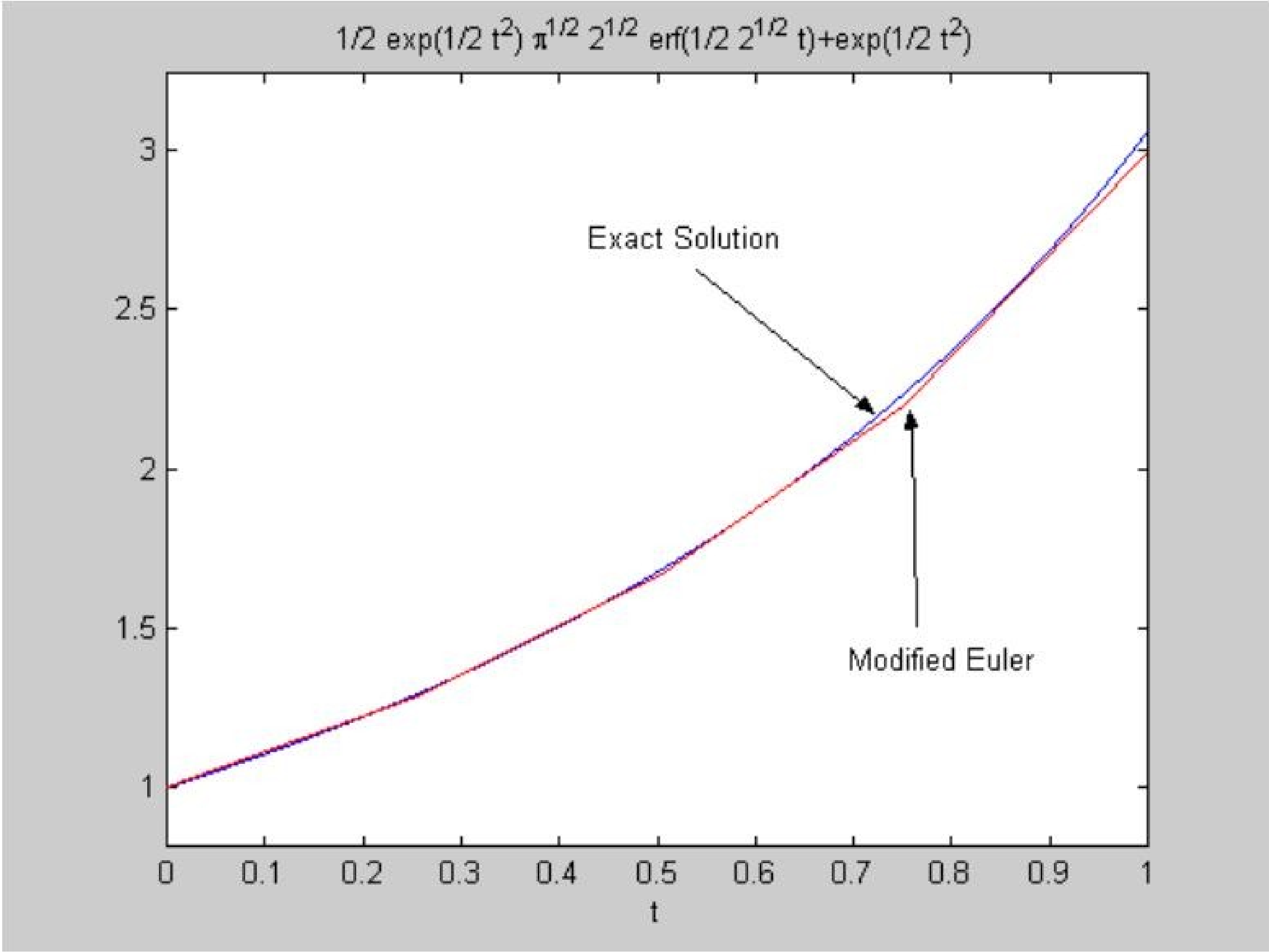
$$y_n = y_{n-1} + \frac{h}{2}(y'_{n-1} + y'_n)$$

$$= y_{n-1} + \frac{h}{2}(t_{n-1}y_{n-1} + 1 + t_n y_n + 1)$$

$$\Leftrightarrow \quad y_n(1 - \frac{h}{2}t_n) = y_{n-1}(1 + \frac{h}{2}t_{n-1}) + h$$

$$\Leftrightarrow \quad y_n = \frac{(1 + \frac{h}{2}t_{n-1})}{(1 - \frac{h}{2}t_n)} y_{n-1} + h$$

Graph the solution



Second Order Runge-Kutta Method

- The second order Runge-Kutta (RK-2) method is derived by applying the trapezoidal rule to integrating $y' = f(y,t)$ over the interval $[t_n, t_{n+1}]$. So, we have

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y,t)dt$$

$$= y_n + \frac{h}{2}\left(f(y_n, t_n) + f(\bar{y}_{n+1}, t_{n+1})\right)$$

We estimate $\bar{y}_{n+1}$ by the forward euler method.

So, we have

$$y_{n+1} = y_n + \frac{h}{2}\left(f(y_n, t_n) + f(y_n + hf(y_n, t_n), t_{n+1})\right)$$

Or in a more standard form as

$$y_{n+1} = y_n + \frac{1}{2}\left(k_1 + k_2\right)$$

where $k_1 = hf(y_n, t_n)$

$$k_2 = hf(y_n + k_1, t_{n+1})$$

Third Order Runge-Kutta Method

- The third order Runge-Kutta (RK-3) method is derived by applying the Simpson's 1/3 rule to integrating $y' = f(y,t)$ over the interval $[t_n, t_{n+1}]$. So, we have

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y,t)dt$$

$$= y_n + \frac{h}{6}\left(f(y_n, t_n) + 4f(\bar{y}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + f(\bar{y}_{n+1}, t_{n+1})\right)$$

We estimate $\bar{y}_{n+\frac{1}{2}}$ by the forward euler method.

The estimate $\bar{y}_{n+1}$ may be obtained by forward difference method, central difference method for h/2, or linear combination both forward and central difference method. One of RK-3 scheme is written as

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right)$$

where $k_1 = hf(y_n, t_n)$

$$k_2 = hf(y_n + \tfrac{1}{2}k_1, t_n + \frac{h}{2})$$

$$k_3 = hf(y_n - k_1 + 2k_2, t_{n+1})$$

Fourth Order Runge-Kutta Method

- The fourth order Runge-Kutta (RK-4) method is derived by applying the Simpson's 1/3 or Simpson's 3/8 rule to integrating $y' = f(y,t)$ over the interval $[t_n, t_{n+1}]$. The formula of RK-4 based on the Simpson's 1/3 is written as

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where $k_1 = hf(y_n, t_n)$

$$k_2 = hf(y_n + \tfrac{1}{2}k_1, t_n + h/2)$$

$$k_3 = hf(y_n + \tfrac{1}{2}k_2, t_n + h/2)$$

$$k_4 = hf(y_n + k_3, t_n + h)$$

- The fourth order Runge-Kutta (RK-4) method is derived based on Simpson's 3/8 rule is written as

$$y_{n+1} = y_n + \frac{1}{8}\left(k_1 + 3k_2 + 3k_3 + k_4\right)$$

where $k_1 = hf(y_n, t_n)$

$$k_2 = hf(y_n + \tfrac{1}{3}k_1, t_n + h/3)$$

$$k_3 = hf(y_n + \tfrac{1}{3}k_1 + \tfrac{1}{3}k_2, t_n + 2h/3)$$

$$k_4 = hf(y_n + 3k_1 - 3k_2 + k_3, t_n + h)$$