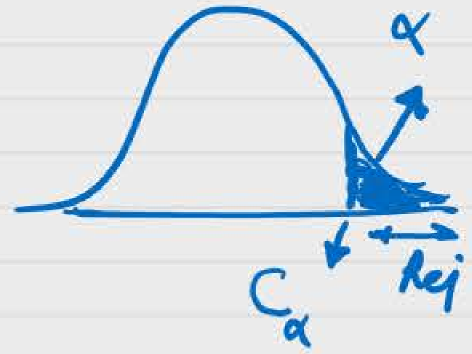
① $H_0: \beta_j \leq 0$

$H_1: \beta_j > 0 \rightarrow$



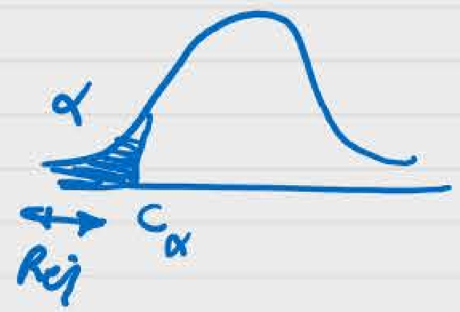
$$t = \frac{\hat{\beta}_j - \beta_{j_{H_0}}}{se(\hat{\beta}_j)} = \frac{\hat{\beta}_j - 0}{se(\hat{\beta}_j)}$$ ✓

$|t| > |c_\alpha| \Rightarrow$ Rej $H_0 \Leftarrow$ Pvalue $< \alpha$

---

② $H_0: \beta_j \geq 0$

$H_1: \beta_j < 0$



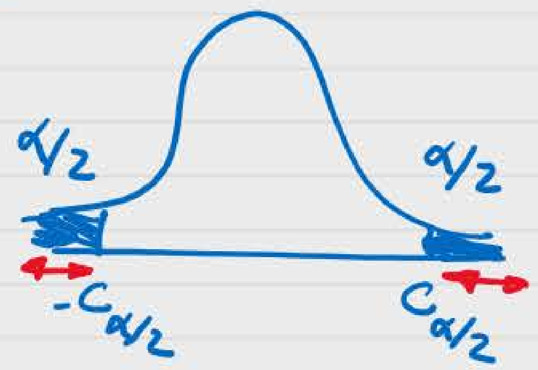
$$t = \frac{\hat{\beta}_j - 0}{se(\hat{\beta}_j)}$$ ✓

$|t| > |c_\alpha| \rightarrow$ Rej $H_0 \Leftarrow$ Pvalue $< \alpha$

---

③ $H_0: \beta_j = 0$

$H_1: \beta_j \neq 0$



$$t = \frac{\hat{\beta}_j - 0}{se(\hat{\beta}_j)}$$

use R

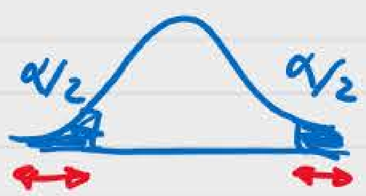
or Manually

$|t| > |c_{\alpha/2}| \rightarrow$ Rej $H_0 \Leftarrow$ Pvalue $< \alpha$

---

④ $H_0: \beta_j = a_j$

$H_1: \beta_j \neq a_j$



$|t| > |c_{\alpha/2}| \rightarrow$ Rej $H_0 \Leftarrow$ ✓ Pvalue $< \alpha$

$$t = \frac{\hat{\beta}_j - a_j}{se(\hat{\beta}_j)}$$

---

⑤ linear Combination of β_j

$H_0: \beta_1 - \beta_2 = 0 \longrightarrow$ use θ Method

$\theta = \beta_1 - \beta_2$ or $\beta_2 = \beta_1 + \theta$

$H_1: \begin{cases} \beta_1 - \beta_2 \neq 0 \\ < 0 \\ > 0 \end{cases}$

Rewrite the model with θ

& find the t_θ

$H_0: \theta = 0$

$H_1: \theta \neq 0$ ($<$ / $>$)

Class 16 – Multiple Regression Model Inference (Part II)

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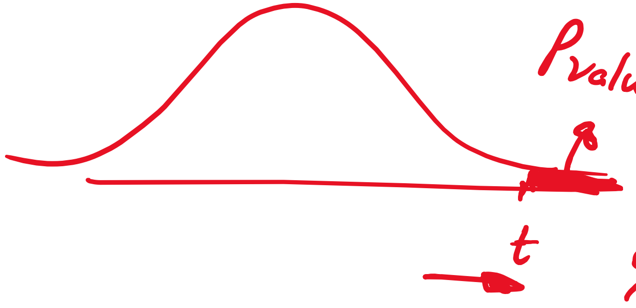
Computing p-values for t-tests



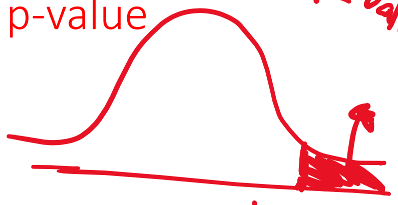
large $t \equiv$ Small Pvalue

α

- Recall 1: The smallest significance level at which the null hypothesis is still rejected, is called the p-value of the hypothesis test

P-value

$t = 2.1 \Rightarrow$



- Recall 2: **p-value** is the corresponding significance level of the test statistic.
- A small p-value is evidence against the null hypothesis **(a good thing!)** because one would reject the null hypothesis even at small significance levels
- A large p-value is evidence in favor of the null hypothesis **(a bad thing!)**
- P-values are more informative than tests at fixed significance levels
- The p-value is the significance level at which one is indifferent between rejecting and not rejecting the null hypothesis.

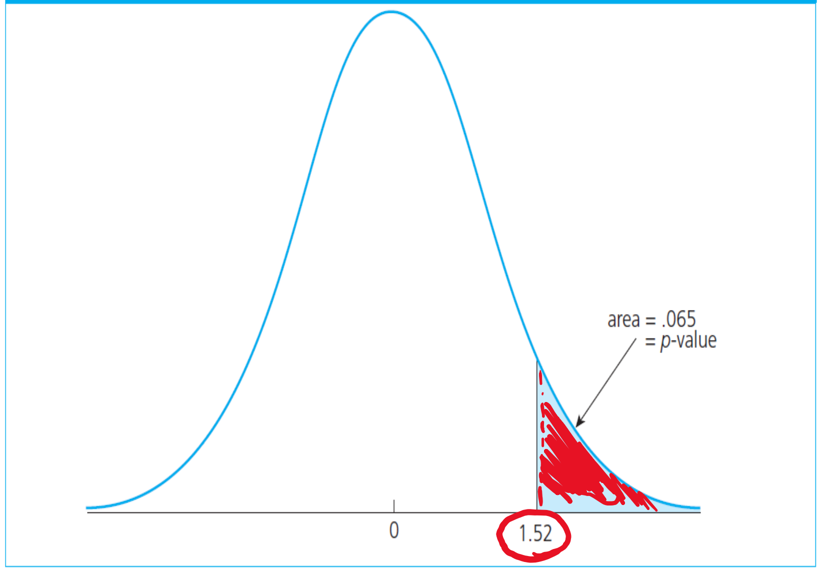
$\frac{\beta_j}{se} \to t \to P_{value}$

Computing and Using p-values (cont'd)

We said that **p-value** is the corresponding significance level of the test statistic.

P-values for one-tailed tests:

FIGURE C.7 The p-value when $t = 1.52$ for the one-sided alternative



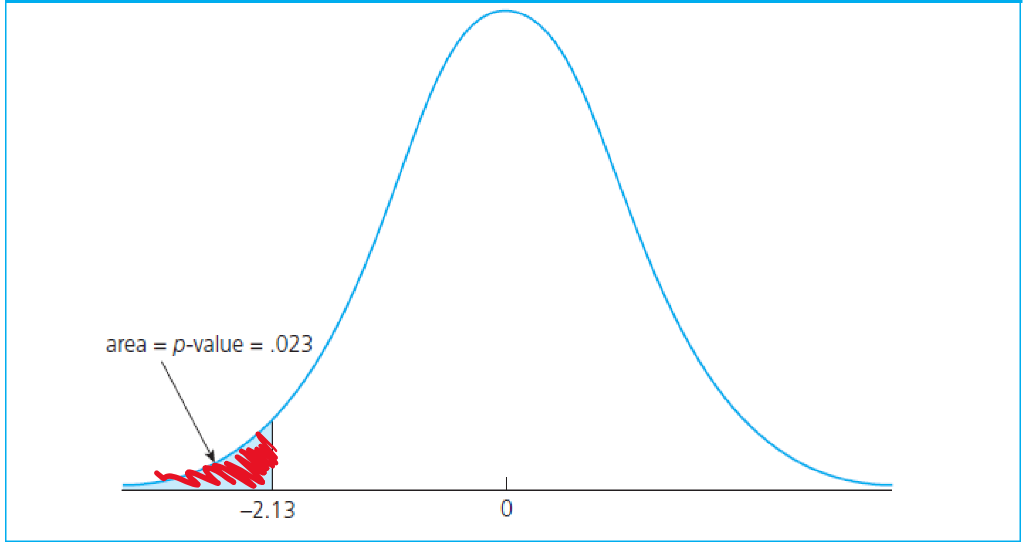
$$p_{value} = P(T > 1.52) = 1 - CDF(1.52) = 0.065$$

n=200

$1 - P_t(1.52, df) =$

198

FIGURE C.8 The p-value when $t = -2.13$ with 19 degrees of freedom for the one-sided alternative $\mu < 0$.



$$p_{value} = P(T < -2.13) = CDF(-2.13) = 0.023$$

n=20

$P_t(-2.13, df) = 0.023$

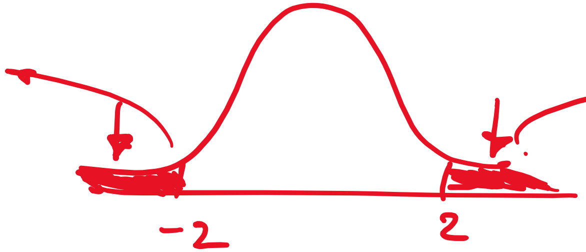
19

Computing and Using p-values (cont'd)

$t = 2 \Longrightarrow$ Pvalue for 2 tailed test

P-values for Two-tailed tests:

$2 \times CDF(-2)$



$(1 - CDF(2)) \times 2$

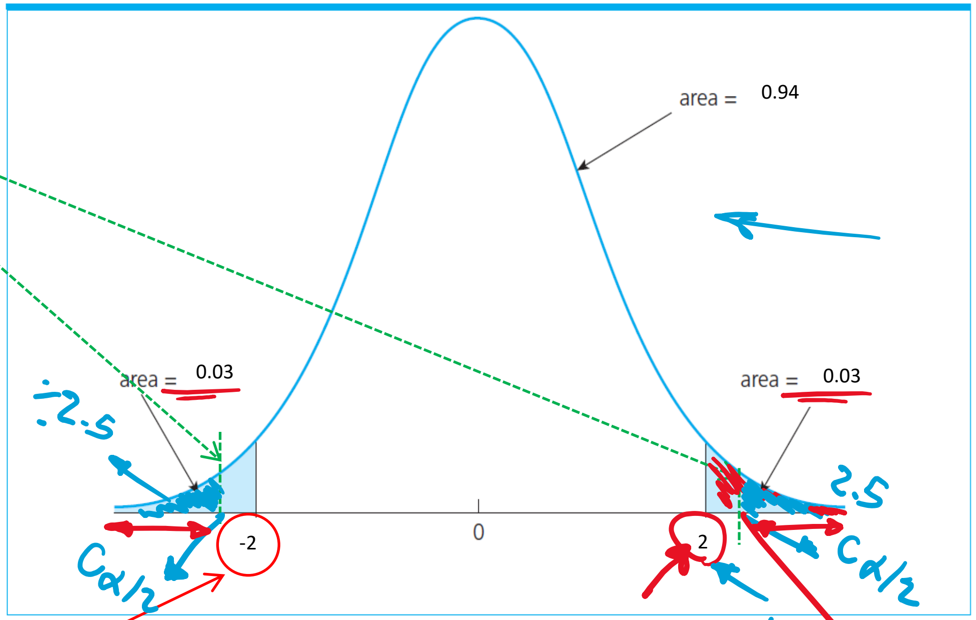
These would be the critical values for a 5% significance level

$|t| > |c| \Longrightarrow$ Rej

$\equiv$

$\boxed{P_{value} < \alpha} \Longrightarrow$ Rej

$\downarrow$ Rej



Value of test statistic

$\therefore\ t < c_{\alpha/2}$

$$p_{value} = P(|T| > |2|) =$$

$$2(1 - CDF(2)) = 0.06\ ,$$

$$df = 20$$

A null hypothesis is rejected if and only if the corresponding p-value is smaller than the significance level.

Do you reject the null here?

$p_{value} = 6\% > \alpha = 5\%$

$\alpha = 5\%$

$int = \beta_0 + \beta_1\, height + \dots + u$

$\Rightarrow R^2 = 90\%$

$\hat{\beta}_1$ was significant

$\boxed{\hat{\beta}_1 = 0.0000001}$

Economic / Practical significance VS. Statistical significance

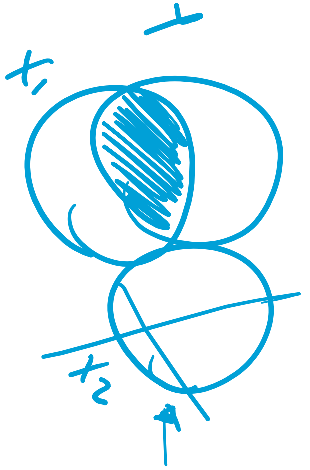
- If a variable is statistically significant, discuss the magnitude of the coefficient to get an idea of its economic or practical importance
- The fact that a coefficient is statistically significant does not necessarily mean it is economically or practically significant!

wage ~ educ

- If a variable is statistically and economically important but has the "wrong" sign, the regression model might be misspecified

wage ~ educ + gender

- If a variable is NOT statistically significant at the usual levels (10%, 5%, or 1%), one may think of dropping it from the regression
- If the sample size is small, effects might be imprecisely estimated so that the case for dropping insignificant variables is less strong



Confidence intervals

$\log(wage) = \beta_0 + \beta_1 educ + u$

$\widehat{\log(wage)} = 0.1 + 0.085\, educ$

β_1

Prob

LB UB

β_1 0.085

Recall: CI is two-sided by nature $\boxed{\hat{\beta}_j \pm C * se(\hat{\beta}_j)}$

Critical value of **two-sided** test

$$P\Big(\underbrace{\hat{\beta}_j - c_{0.05}\cdot se(\hat{\beta}_j)}_{LB} \le \beta_j \le \underbrace{\hat{\beta}_j + c_{0.05}\cdot se(\hat{\beta}_j)}_{UB}\Big) = 0.95$$

$P(0.07 \le \beta_1 \le 0.09) =$ with 95%

Lower bound of the Confidence interval

Upper bound of the Confidence interval

Confidence level

CI will contain β_j with 90% Prob

β_j will lie in CI " 90% Prob

$c_{0.05} \approx 1.96$

0.08 0.09

0.085

%95

%90

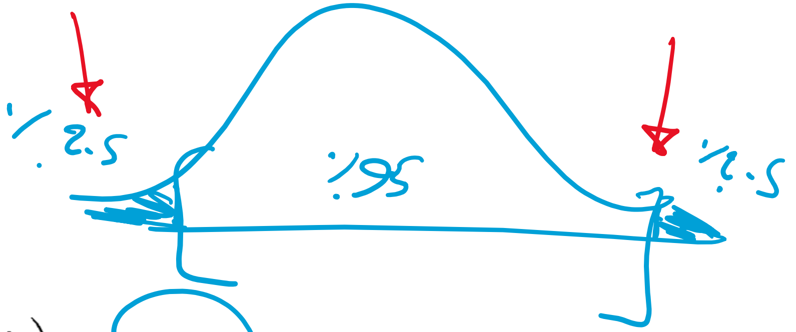
0.085

Interpretation of the confidence interval:

- The bounds of the interval are random
- In repeated samples, the interval will contain the population regression coefficient (β) in $(1-\alpha)\%$ of the cases

two_tailed

Confidence intervals for typical confidence levels



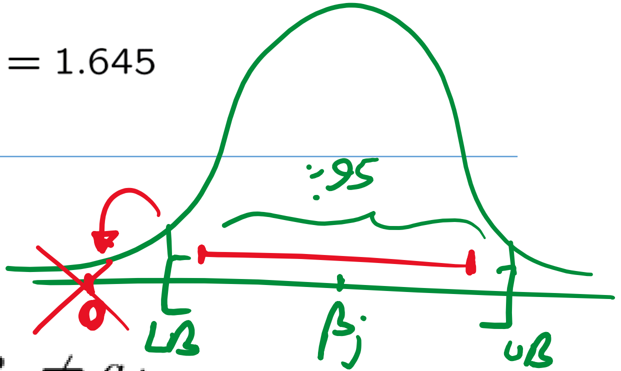
$$P\left(\hat{\beta}_j - c_{0.01} \cdot se(\hat{\beta}_j) \leq \beta_j \leq \hat{\beta}_j + c_{0.01} \cdot se(\hat{\beta}_j)\right) = 0.99$$

$$P\left(\hat{\beta}_j - c_{0.05} \cdot se(\hat{\beta}_j) \leq \beta_j \leq \hat{\beta}_j + c_{0.05} \cdot se(\hat{\beta}_j)\right) = 0.95$$

$$P\left(\hat{\beta}_j - c_{0.10} \cdot se(\hat{\beta}_j) \leq \beta_j \leq \hat{\beta}_j + c_{0.10} \cdot se(\hat{\beta}_j)\right) = 0.90$$

Use rules of thumb $c_{0.01} = 2.576, c_{0.05} = 1.96, c_{0.10} = 1.645$

Relationship between confidence intervals and hypotheses tests



$H_0: \beta_j = a_j$

$$a_j \notin interval \Rightarrow \text{ reject } H_0 : \beta_j = a_j \text{ in favor of } H_1 : \beta_j \neq a_j$$

$\beta_j = 0$

$0 \notin [\quad]$

Example: Model of firms‘ R&D expenditures

① $t_{\hat\beta_1} = \frac{1.084 - 0}{0.06} \approx 18$ $t > 2.4$ Rej

1% 5% → $c_{1\%} = 2.4$

Spending on R&D Annual sales Profits as percentage of sales

$$\widehat{\log(rd)} = -4.38 + 1.084 \log(sales) + .0217\ profmarg$$
$$\quad\ (.47) \quad\ (.060) \qquad\qquad\quad (.0128)$$
$$n = 32,\ R^2 = .918. \quad df = 32-2-1 = 29 \Rightarrow c_{0.05} = 2.045$$

② $t = 18$

-18 18

$Pt(-18, 29) = 1.3 \times 10^{-16}$

Pvalue ≈ 0%

What are the CI for $\hat\beta_1$ and $\hat\beta_2$?

③ 0 0.96 1.21

$1.084 \pm 2.045(.060)$
$= (.961, 1.21)$

$.0217 \pm 2.045(.0218)$
$= (-.0045, .0479)$

0

The effect of sales on R&D is relatively **precisely** estimated as the interval is narrow. Moreover, the effect is significantly different from zero because zero is outside the interval.

This effect is **imprecisely** estimated as the interval is very wide. It is not even statistically significant because zero lies in the interval.

R

```
# chapter 4: MRM, Inference

library(wooldridge)
library(stargazer)


# Example 4-8
MRM <- lm(log(rd)~ log(sales)+ profmarg, rdchem)
summary(MRM)

# finding critical values
df    <- nobs(MRM) - 2-1
alpha <- 0.05
qt(1- alpha/2 , df)


# Look at t_stat
summary(MRM)$coefficients[,"t value"]

# Confidence Interval
confint(MRM, level =  1-alpha)
```

.95

α/2 α/2

2.045

```
> summary(MRM)

Call:
lm(formula = log(rd) ~ log(sales) + profmarg, data = rdchem)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.97681 -0.31502 -0.05828  0.39020  1.21783 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -4.37827    0.46802  -9.355 2.93e-10 ***
log(sales)   1.08422    0.06020  18.012  < 2e-16 ***
profmarg     0.02166    0.01278   1.694    0.101    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5136 on 29 degrees of freedom
Multiple R-squared:  0.918,	Adjusted R-squared:  0.9123 
F-statistic: 162.2 on 2 and 29 DF,  p-value: < 2.2e-16
```

α = 15% Rej

?

α = 5% Fail to Rej

```
> qt(1- alpha/2 , df)
[1] 2.04523
```

```
> # Look at t_stat
> summary(MRM)$coefficients[,"t value"]
(Intercept)  log(sales)    profmarg 
  -9.354916   18.011791    1.694150 
```

```
> # Confidence Interval
> confint(MRM, level =  1-alpha)
                   2.5 %     97.5 %
(Intercept) -5.335478450 -3.4210681
log(sales)   0.961107256  1.2073325
profmarg    -0.004487722  0.0477991
```

Rej

Rej

Fail to Rej

Testing hypotheses about a linear combination of the parameters

Example: Return to education at two-year vs. at four-year colleges

Number of years attending a **2-year** college

Number of years at a **4-year** college

Months in the workforce

$$\log(wage) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 exper + u$$

Data Suggest $\hat{\beta}_{univ} > \hat{\beta}_{jc}$

$$\widehat{\log(wage)} = \underset{(.021)}{1.472} + \underset{(.0068)}{.0667}\,jc + \underset{(.0023)}{.0769}\,univ + \cancel{\underset{(.0002)}{.0049}\,exper}$$

$$n = 6{,}763,\ R^2 = .222.$$

- ✓ The **hypothesis** of interest is whether one year at a junior college is worth one year at a university
- ✓ the **alternative** of interest is one-sided: a year at a junior college is worth less than a year at a university

$\beta_{univ} > \beta_{jc}$

Testing hypotheses about a linear combination of the parameters

- ✓ The **hypothesis** of interest is whether one year at a junior college is worth one year at a university
- ✓ the **alternative** of interest is one-sided: a year at a junior college is worth less than a year at a university

Test $H_0 : \beta_1 - \beta_2 = 0$ against $H_1 : \beta_1 - \beta_2 < 0$

$\beta_{jc} - \beta_{univ} < 0 \Rightarrow \beta_{univ} > \beta_{jc}$

A possible test statistic would be:

$$\frac{\hat{\beta}_j - \beta_{j_{H_0}}}{se(\hat{\beta}_j)}$$

$$t = \frac{\widehat{\beta}_1 - \widehat{\beta}_2}{se(\widehat{\beta}_1 - \widehat{\beta}_2)}$$

$$= \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{se(\hat{\beta}_1 - \hat{\beta}_2)}$$

The difference between the estimates is normalized by the estimated standard deviation of the difference. The null hypothesis would have to be rejected if the statistic is “too negative” to believe that the true difference between the parameters is equal to zero.

Testing hypotheses about a linear combination of the parameters (cont'd)

Impossible to compute with standard regression output because

$t = $

$$se(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{Var(\hat{\beta}_1 - \hat{\beta}_2)} = \sqrt{Var(\hat{\beta}_1) + Var(\hat{\beta}_2) - 2Cov(\hat{\beta}_1, \hat{\beta}_2)}$$

$Var(x+y) = ?$

Usually not available in regression output

Alternative method:

$\beta_1 - \beta_2 < 0$

Define $\theta_1 = \beta_1 - \beta_2$ and test $H_0 : \theta_1 = 0$ against $H_1 : \theta_1 < 0$

w | jc | univ | exp | total

$$\log(wage) = \beta_0 + (\theta_1 + \beta_2)jc + \beta_2 univ + \beta_3 exper + u$$

$\hat{\theta}$ is sig or not

Insert into original regression

$$\log(w) = \beta_0 + \theta_1 jc + \beta_2 \boxed{(jc + univ)} + \beta_3 exper + u$$

a new regressor (= total years of college)

Testing hypotheses about a linear combination of the parameters (cont‘d)

Estimation results

β_1 θ β_2 β_3

Total years of college

$$\widehat{\log(wage)} = \underset{(.021)}{1.472} - \underset{(.0069)}{.0102}\, jc + \underset{(.0023)}{.0769}\, totcoll + \underset{(.0002)}{.0049}\, exper$$

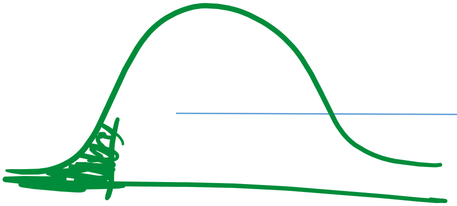
$$n = 6,763, \quad R^2 = .222$$

$P_{value} = 7\%$

%10

%5

%1



-1.48

$Pt(-1.48, 6759)$

$$t = -.0102/.0069 = -1.48$$

$$p-value = P(t-ratio < -1.48) = .070$$

%7

Hypothesis is rejected at 10% level but not at 5% level

Confidence Interval for $\theta_1 = \beta_1 - \beta_2$ $\longrightarrow$ $-.0102 \pm 1.96(.0069) = \underbrace{(-.0237, .0003)}_{0 \in}$

This method works always for single linear hypotheses