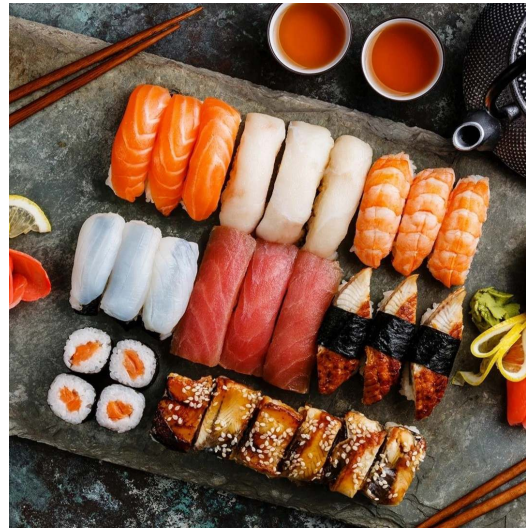


# Analytical Hierarchy Process

Ulink MM Club  
Alan, Frank



# What to take as dinner?



# Constructing the Hierarchy

Goal: choosing the optimal dinner option

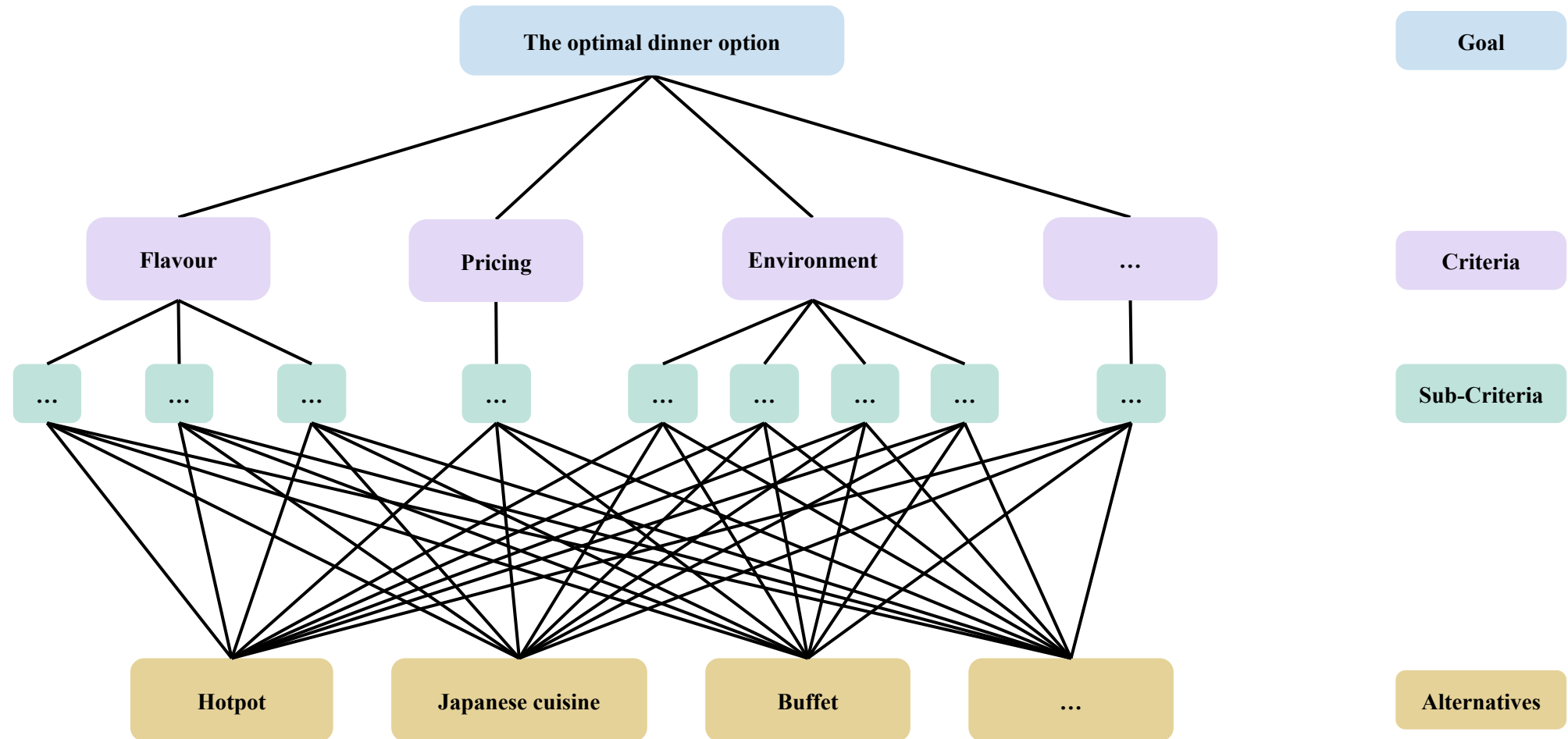
Criteria:

- ◆ Flavour
- ◆ Pricing
- ◆ Restaurant environment
- ◆ ...

Alternatives:

- ◆ Hotpot
- ◆ Japanese cuisine
- ◆ Buffet
- ◆ ...

# Constructing the Hierarchy



# What's next?

Consider the criteria: flavour, pricing, ...

- ◆ Which is more important?
- ◆ How much is the importance difference?
- ◆ How to quantify this relative importance?
- ◆ Pairwise comparisons...

# Saaty's Table

Importance Scales	Definitions of Importance Scale
1	Equally Important Preferred
2	Equally to Moderately Important Preferred
3	Moderately Important Preferred
4	Moderately to Strongly Important Preferred
5	Strongly Important Preferred
6	Strongly to Very Strongly Important Preferred
7	Very Strongly Important Preferred
8	Very Strongly to Extremely Important Preferred
9	Extremely Important Preferred

# Pairwise Comparison Matrix

The pairwise comparison matrix  $A = (a_{ij})_{n \times n}$ :

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} \end{pmatrix}$$

Notes:  $a_{ij}$  denotes the relative importance of  $i$  to  $j$ ;

$a_{ij} = \frac{1}{a_{ji}} \forall i, j$  and all elements on the diagonal are 1.

# Converting to Weights

## Perron-Frobenius Theorem

Let  $A_{n \times n}$  be a positive matrix (i.e.,  $a_{ij} > 0 \forall i, j$ ), then there exists a unique real, positive eigenvalue  $\lambda_{max}$  such that  $\lambda_{max} > |\lambda_i| \forall i$ , and there exists a corresponding eigenvector  $Av = \lambda_{max}v$  with all entries  $v_i > 0$ .

## Eigenvector Method

Therefore, we find the principle right eigenvector for  $A$  and normalize it:

$Aw = \lambda_{max}w$ ,  $w = \frac{w}{\sum_{i=1}^n w_i}$  to obtain the weight vector.



# Converting to Weights

Power Iteration Method (usually how computers find eigenvectors)

Initialize a random vector  $w^{(0)}$ .

Iteratively compute:  $w^{(k+1)} = Aw^{(k)}$ ,

Followed by normalization:  $w^{(k+1)} = \frac{w^{(k+1)}}{\sum_i w_i^{(k+1)}}$ ,

Until convergence:  $w^{(k+1)} \approx w^{(k)}$ .

# Check for Consistency

Ideal case:  $a_{ik} \cdot a_{kj} = a_{ij} \forall i, j, k$  then  $\lambda_{max} = n$

However, in real life that is almost impossible. Hence, we set a “threshold” for acceptable inconsistency:

$$\lambda_{max} = \frac{1}{n} \sum_{i=1}^n \frac{(Aw)_i}{w_i}$$

$$CR = \frac{CI}{RI}, CI = \frac{\lambda_{max} - n}{n - 1}$$

<i>n</i>	<i>RI</i> values	<i>n</i>	<i>RI</i> values
1	0	6	1.2358
2	0	7	1.3322
3	0.5799	8	1.3952
4	0.8921	9	1.4537
5	1.1159	10	1.4882

*Golden, B. L. & Wang, Q. (1990)*

Matrices with  $CR < 0.10$  are considered consistent in acceptable level.

# How to Fix Inconsistency?

- ◆ Re-evaluate scores;
- ◆ Manual adjustments;
- ◆ Make more copies of scores and take their geometric means;
- ◆ Least squares or linear programming adjustments.

# Pros and Cons

## Pros:

- ◆ Handles both qualitative and quantitative criteria;
- ◆ Intuitive pairwise comparisons;

## Cons:

- ◆ Subjective judgements;
- ◆ Assumes independence between criteria (ANP alternative).



The background is a dark blue, almost black, space filled with a complex, three-dimensional grid of lines that recede into the distance, creating a sense of depth. Overlaid on this grid are numerous mathematical formulas and symbols in a light blue or white color. Some of the visible formulas include the binomial theorem  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ , the Taylor series for  $e^x$ , the definition of the gamma function  $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ , and various summation and integral notations. The formulas are scattered across the image, some appearing larger and more prominent than others, contributing to a dense, intellectual atmosphere.

Thank you