Analytical Hierarchy Process

Ulink MM Club

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What to take as dinner?

Constructing the Hierarchy

Goal: choosing the optimal dinner option

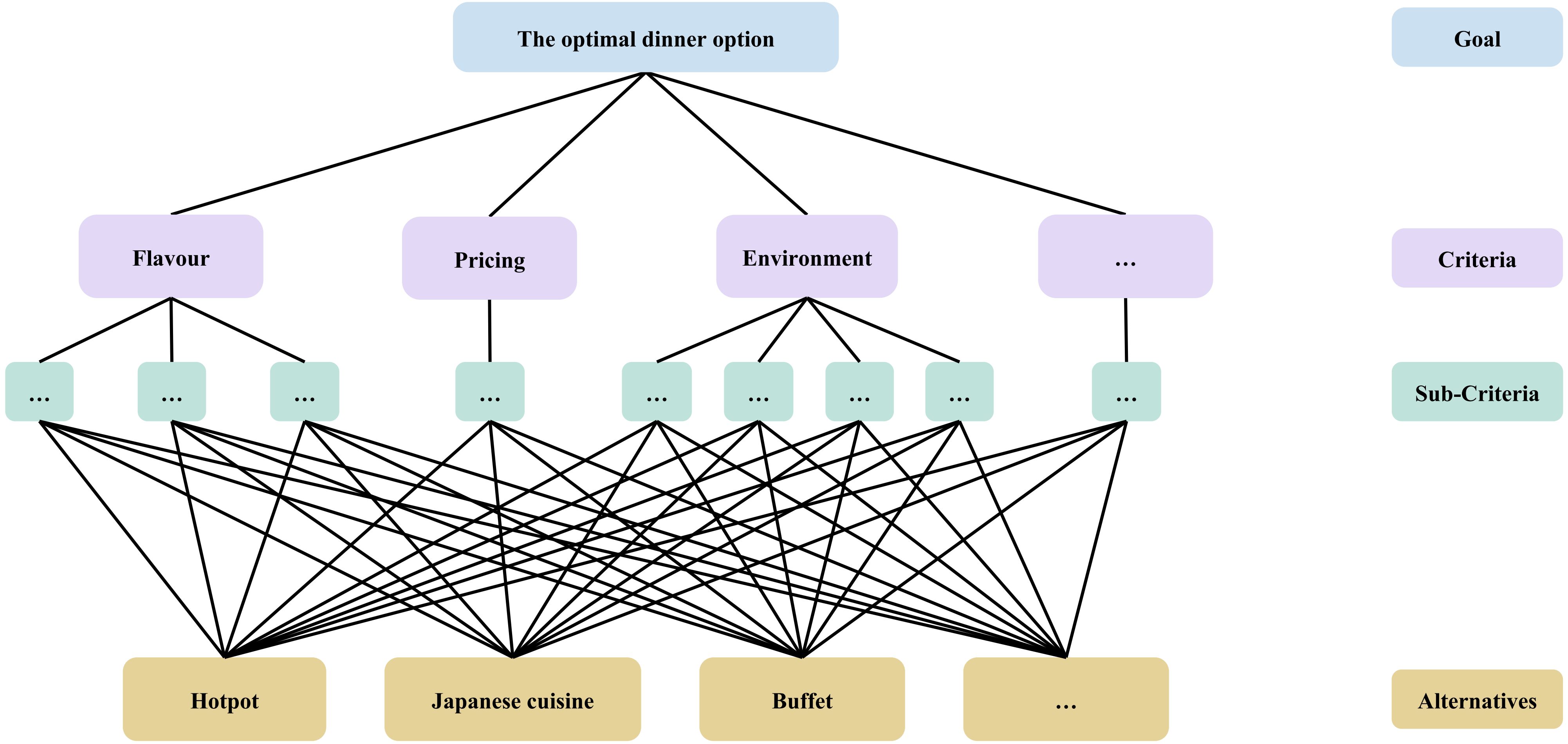
Criteria:

- Flavour
- Pricing
- Restaurant environment
- ...

Alternatives:

- Hotpot
- Japanese cuisine
- Buffet
- ...

Constructing the Hierarchy



What's next?

Consider the criteria: flavour, pricing, ...

- Which is more important?
- How much is the importance difference?
- How to quantify this relative importance?
- Pairwise comparisons...

Saaty's Table

Importance Scales	Definitions of Importance Scale
1	Equally Important Preferred
2	Equally to Moderately Important Preferred
3	Moderately Important Preferred
4	Moderately to Strongly Important Preferred
5	Strongly Important Preferred
6	Strongly to Very Strongly Important Preferred
7	Very Strongly Important Preferred
8	Very Strongly to Extremely Important Preferred
9	Extremely Important Preferred

Pairwise Comparison Matrix

The pairwise comparison matrix $A = (a_{ij})_{n \times n}$:

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} \end{pmatrix}$$

Notes: a_{ij} denotes the relative importance of i to j;

$a_{ij} = \frac{1}{a_{ji}}$ $\forall i, j$ and all elements on the diagonal are 1.

Converting to Weights

Perron-Frobenius Theorem

Let $A_{n\times n}$ be a positive matrix (i.e., $a_{ij} > 0\ \forall i, j$), then there exists a unique real, positive eigenvalue λ_{max} such that $\lambda_{max} > |\lambda_i| \forall i$, and there exists a corresponding eigenvector $Av = \lambda_{max} v$ with all entries $v_i > 0$.

Eigenvector Method

Therefore, we find the principle right eigenvector for A and normalize it:

$Aw = \lambda_{max} w$, $w = \frac{w}{\sum_{i=1}^{n} w_i}$ to obtain the weight vector.

Converting to Weights

Power Iteration Method (usually how computers find eigenvectors)

Initialize a random vector $w^{(0)}$.

Iteratively compute: $w^{(k+1)} = w^{(k)}$,

Followed by normalization: $w^{(k+1)} = \frac{w^{(k+1)}}{\sum_i w_i^{(k+1)}}$,

Until convergence: $w^{(k+1)} \approx w^{(k)}$.

Check for Consistency

Ideal case: $a_{ik} \cdot a_{kj} = a_{ij}\ \forall i, j, k$ then $\lambda_{max} = n$

However, in real life that is almost impossible. Hence, we set a "threshold" for acceptable inconsistency:

$$\lambda_{max} = \frac{1}{n} \sum_{i=1}^{n} \frac{(Aw)_i}{w_i}$$

$$CR = \frac{CI}{RI}, CI = \frac{\lambda_{max} - n}{n - 1}$$

n	***RI*** **values**	***n***	***RI*** **values**
1	0	6	1.2358
2	0	7	1.3322
3	0.5799	8	1.3952
4	0.8921	9	1.4537
5	1.1159	10	1.4882

Golden, B. L. & Wang, Q. (1990)

Matrices with $CR < 0.10$ are considered consistent in acceptable level.

How to Fix Inconsistency?

- Re-evaluate scores;
- Manual adjustments;
- Make more copies of scores and take their geometric means;
- Least squares or linear programming adjustments.

Pros and Cons

Pros:

- Handles both qualitative and quantitative criteria;
- Intuitive pairwise comparisons;

Cons:

- Subjective judgements;
- Assumes independence between criteria (ANP alternative).


Thank you