Differential Equations Model

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Abstract

In this article we model three population problem with differential equations model, using the Malthusian growth model, the Logistic model, the Lotka-Volterra model and its variants respectively.

Keywords

Differential equations model, Logistic model, Lotka-Volterra model

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1 Population Growth Model

1.1 Problem Restatement

In two centuries, there is a huge increase of the population in America, so the government wants to predict the population in 30 years, with the population records of each year we need to use the model to predict the growth.

The problem provides us with the population size of America from year 1790 through year 2000 by decades. We need to use the given data to find the best estimate of the parameters and perform predictions for the future population in 3 decades.

1.2 Assumptions and Justifications

- Assumption 1: Let $x(t)$ denote the population size at time t, meanwhile $x(t)$ is continuous and differentiable for all time t.
- Justification 1: We use a ordinary differential equation to define the growth rate of the population, hence the function $x(t)$ itself has to be continuous and differentiable.
- Assumption 2: The growth rate of the population is set to be a constant r, where growth rate = birth rate − death rate.
- Justification 2: To simplify our model, we use a constant growth rate for the population size.
- Assumption 3: The increase and decrease in population are only based on individuals' birth and death, with equal fertility ability and death rate in every individual.
- Justification 3: We assume equality for all individuals to simplify our model and calculation.

1.3 Model Input

Year	1790	1800	1810	1820	1830	1840	1850	1860
Population	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4
Year	1870	1880	1890	1900	1910	1920	1930	1940
Population	38.6	50.2	62.9	76.0	92.0	106.5	123.2	131.7
Year	1950	1960	1970	1980	1990	2000		
Population	150.7	179.3	204.0	226.5	251.4	281.4		

Table 1: The American population size recorded each decade (1970-2000).

1.4 Malthusian Growth Model

The Malthusian growth model, named after Thomas Robert Malthus, is essentially exponential growth based on the idea of the function being proportional to the speed to which the function grows [1].

We define the model using a differential equation, that is:

$$\frac{dx}{dt} = rx \tag{1}$$

With initial condition $x(0) = x_0$, the constant growth rate r is estimated from the given data (table 1).

1.5 Logistic Model

However, the Malthusian growth model brings up a severe problem, that is the population size is impossible to show a permanent growth, therefore we turn to an alternative, which adds a restriction from natural resources on the growth rate. The larger the population, the more restrictive effect on its growth rate.

$$\frac{dx}{dt} = r(x) \tag{2}$$

Where the function for growth rate, $r(x)$, is given by:

$$r(x) = r - sx$$

We assume the population limit to be x_m, that is:

$$r(x_m) = 0$$

From above we can obtain the o.d.e. for the model:

$$\frac{dx}{dt} = r\left(1 - \frac{x}{x_m}\right)x \tag{3}$$

With initial condition $x(t_0) = x_0$, the constant growth rate r and the population limit x_m are estimated from the given data (table 1).

1.6 Model Output

We use the smallest residual sum of squares to find the best estimate for the parameters: growth rate r for the Malthusian growth model and an extra population limit x_m for the Logistic model.

Our estimates for the parameters are as follows:

Model	Parameter	Estimate
Malthusian growth model	growth rate r	0.0212
Logistic model	growth rate r	0.0274
	population limit x_m	342.452

Table 2: The estimates for parameters of two models.

From the model we get the predictions for American population size in the following 3 decades.

Year	Malthusian growth model prediction	Logistic model prediction
2010	413.1	282.7
2020	510.6	295.0
2030	631.2	305.1

Table 3: The prediction of population sizes using two models.

It's not hard to observe that the Logistic model provides a much better fit to the given data and hence a better prediction (see figure 1).

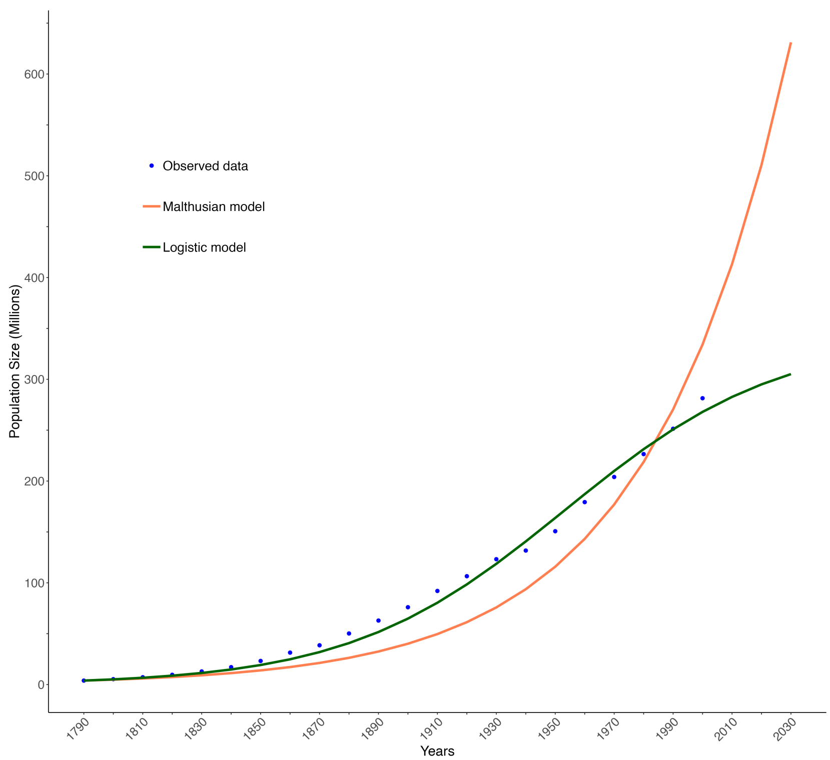

Figure 1: The prediction of population sizes based on given data.

2 Species Population Model-Problem A

2.1 Problem Restatement

The question provide us with a situation with three species A, B and C living together, within which species A and species B has a predator-prey relationship while species A and species C has a competitive relationship. We aim to construct a system of differential equations to model the population sizes of these species.

2.2 Assumptions and Justifications

- Assumption 1: The prey population grows exponentially in the absence of predators.
- Justification 1: This assumes that the prey has unlimited resources (food, space) and no other limiting factors, such as disease or competition. Thus, the growth rate is proportional to the prey population size.
- Assumption 2: In the absence of prey, the predator population declines exponentially.
- Justification 2: Without the prey as a food source, predators cannot sustain their population, leading to a natural decline due to starvation or migration.
- Assumption 3: The rate at which predators consume prey is proportional to the frequency of encounters between them, which is assumed to be proportional to the product of their population sizes.
- Justification 3: This follows from the idea that more predators and more prey lead to more encounters. The predating rate is also considered constant, assuming homogeneous spatial distribution and no changes in behavior over time.
- Assumption 4: The reproduction rate of predators is directly proportional to the amount of prey consumed.
- Justification 4: More prey captured translates into more energy and resources for the predators, allowing them to reproduce at a rate proportional to the prey intake.
- Assumption 5: There are no time delays in responses to changes in population sizes, and the model assumes homogeneous mixing of predators and prey (no spatial structure).
- Justification 5: This simplifies the model to only consider immediate effects of population changes on the rates of birth, death, and predating.

- Assumption 6: The coefficients representing growth, death, and interaction rates are constants.
- Justification 6: These coefficients are used to simplify the analysis and are assumed not to vary over time or with population densities.

2.3 Lotka-Volterra Model & Species Competition Model

2.3.1 Lotka-Volterra Predator-prey Model

The Lotka-Volterra model, formulated in the early 1900s, provides a mathematical framework for understanding the dynamics of predator-prey interactions within ecological systems [2]. This model employs a set of coupled differential equations that delineate population changes over time, illustrating the reciprocal influence of predator and prey populations. In the absence of predators, prey populations exhibit exponential growth, while the predator population responds to the availability of prey. However, as predator populations increase, prey numbers decline, leading to subsequent reductions in predator populations. This cyclical dynamic underscores the intricate balance of ecological relationships and highlights the ultimate dependencies inherent in natural ecosystems [3].

For the given species A and species B following such a model, we derive the following o.d.e. system:

$$\begin{cases} \frac{d}{dt}A\left(t\right) = A\left(r_a - \lambda_a B - e\right) \\ \frac{d}{dt}B\left(t\right) = B\left(-r_b + \lambda_b A - e\right) \end{cases} \tag{4}$$

2.3.2 Species Competition Model

The species competition model, particularly the Lotka-Volterra competition equations, helps us understand how species compete for limited resources. This model looks at factors like population growth rates and resource availability, showing how one species can hinder the growth of another [4]. Possible outcomes include competitive exclusion, where one species out-competes the other, or coexistence, where both species thrive together. This model is essential for understanding species interactions and their effects on community structure and biodiversity in ecosystems [5].

For species A and C given such relationship, we obtain o.d.e.s as follows:

$$\begin{cases} \frac{d}{dt}A\left(t\right) = r_a A\left(1 - \frac{A}{N_a} - \sigma_a \frac{C}{N_c}\right) \\ \frac{d}{dt}C\left(t\right) = r_c C\left(1 - \frac{C}{N_c} - \sigma_c \frac{A}{N_a}\right) \end{cases} \tag{5}$$

2.3.3 Two Models Combined

Therefore we combine the two models (Eq.4 and Eq.5) together and obtain the following equation system (note that we combine both models in species

A, which involves both the predator-prey model and the species competition model):

$$\begin{cases} \frac{d}{dt}A\left(t\right) = A\left(r_a - \lambda_a B - e\right) - r_a A\left(\frac{A}{N_a} + \sigma_a \frac{C}{N_c}\right) \\ \frac{d}{dt}B\left(t\right) = B\left(-r_b + \lambda_b A - e\right) \\ \frac{d}{dt}C\left(t\right) = r_c C\left(1 - \frac{C}{N_c} - \sigma_c \frac{A}{N_a}\right) \end{cases} \tag{6}$$

2.4 Model output

Our model gives the output as follows:

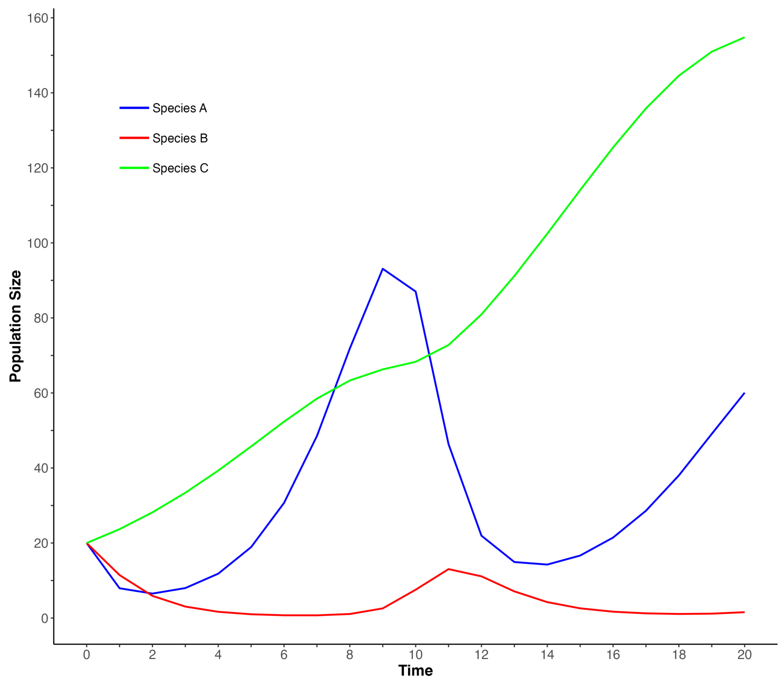

Figure 2: The prediction of three species based on their relationship.

3 Species Population Model-Problem B

3.1 Problem Restatement

The question provide us with a situation with three species A, B and C living together, following a food chain relationship (species A at the top and species C at the bottom). We aim to construct a system of differential equations to model the population sizes of these species.

3.2 Assumptions and Justifications

The assumptions are identical to the ones of the Lotka-Volterra predator-prey model (see section 2.2).

3.3 Double Lotka-Volterra Model

Therefore we combine two Lotka-Volterra predator-prey models together and obtain the following equation system for the food chain:

$$\begin{cases} \frac{d}{dt}A\left(t\right) = A\left(r_a - \lambda_a B - e\right) \\ \frac{d}{dt}B\left(t\right) = B\left(-r_b + \lambda_b A + r_b - \lambda_b C - e\right) = B\left(\lambda_b A - \lambda_b C - e\right) \\ \frac{d}{dt}C\left(t\right) = C\left(-r_c + \lambda_c B - e\right) \end{cases} \tag{7}$$

Note that there are two identical Lotka-Volterra predator-prey models applied to species A & species B and species B & species C.

3.4 Model output

Our model gives the output as follows:

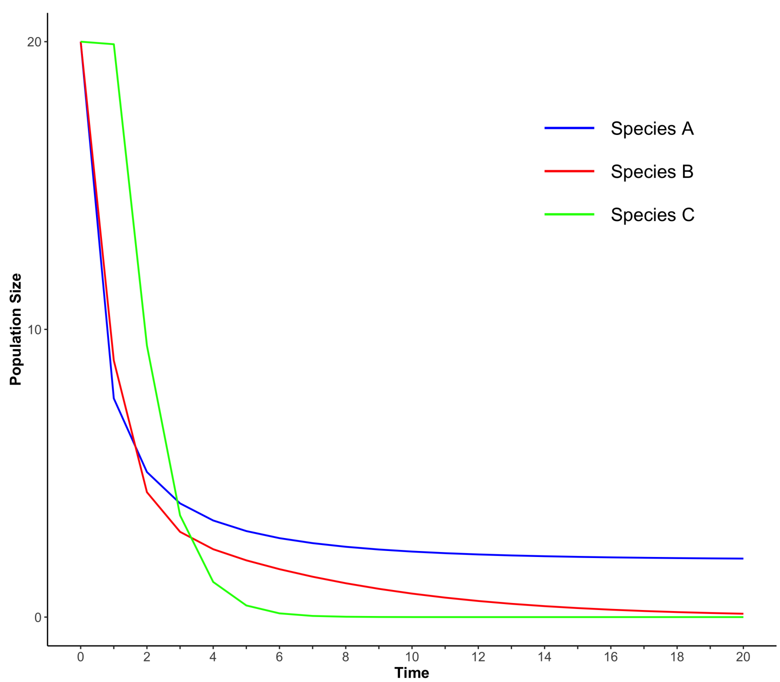

Figure 3: The prediction of three species based on their relationship.

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