Laplace Transformation

Part A

Laplace Transformation is an integral transformation which is useful in solving linear ODE such as those arising in the analysis of electronic circuits.

Definition

Let $F(t)$ be a function of t, for $t > 0$. Then the Laplace transformation of $F(t)$ denoted by $\mathcal{L}\{F(t)\}$, where $\boxed{\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt}$ — (i)

Improper integral
= ∞ (Diverges)
or
= numerical value (Converges)

"s" is a complex argument.
→ parameter;
→ initially assume 's' is real
→ later, will consider 's' as complex, which is useful.

At present we will assume the paramer "s" is real.
Later it will be found useful to consider "s" complex.

$F(t)$ exists if integral (i) converges for some real value "s".
Otherwise it does not exist.

It is a useful technique for dealing with linear systems described by ODE.

The table of Laplace transformation of some elementary functions can be found in the corresponding folder of the Google drive.

Find the Laplace Transformation of the following functions:

$$\boxed{\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt}$$

$\boxed{1}$ $\mathcal{L}\{1\} = \frac{1}{s}, \; s > 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\, dt \qquad \text{(improper integral)}$$

$$= \lim_{l \to \infty} \int_0^l e^{-st}\, dt$$

$$= \lim_{l \to \infty} \left[\frac{e^{-st}}{-s}\right]_0^l = \lim_{l \to \infty} \left[\frac{e^{-sl}}{-s} - \frac{e^{-0}}{(-s)}\right]$$

$$= \lim_{l \to \infty} \frac{1 - e^{-sl}}{s}$$

$$= \frac{1}{s} - \frac{e^{-\infty}}{s}$$

$$= \frac{1}{s} - \frac{1}{s e^{\infty}} \qquad \because e^{\infty} = \infty \; ; \; \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$= \frac{1}{s} - 0$$

$$= \frac{1}{s}$$

$\therefore \mathcal{L}\{F(t)\} = f(s)$

$\Rightarrow \mathcal{L}\{1\} = \frac{1}{s}$

$F(t) \longrightarrow f(s)$

$$\int_0^\infty e^{-st} F(t)\, dt$$

$\mathcal{L}\{F(t)\}$

$\boxed{2}$ $\mathcal{L}\{t\} = \lim_{l\to\infty} \int_0^l e^{-st}\, t\, dt, \quad s>0$

$$\int uv = u\int v - \int\left(u' \int v\right)$$

$$\int \underbrace{e^{-st}}_{v} \underbrace{t}_{u}\, dt$$

$$= t\int e^{-st} dt - \int\left\{\frac{d}{dt}(t)\int e^{-st} dt\right\} dt$$

$$= t\left(\frac{e^{-st}}{-s}\right) - \int\left\{(1)\left(\frac{e^{-st}}{-s}\right)\right\} dt$$

$$= -\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st} dt$$

$$= -\frac{t}{s}e^{-st} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)$$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$$\therefore \mathcal{L}\{t\} = \lim_{l\to\infty}\left[-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right]_0^l$$

$$= \lim_{l\to\infty}\left[-\frac{l}{s}e^{-sl} - \frac{e^{-sl}}{s^2} + \frac{0}{s}e^{-0} + \frac{1}{s^2}e^{-0}\right]$$

"0" by L'Hospital's rule

$$= \lim_{l\to\infty}\left[-\frac{l}{s}e^{-sl} - \frac{e^{-sl}}{s^2} + 0 + \frac{1}{s^2}\right]$$

$$= \lim_{l\to\infty}\left[\frac{-l}{se^{sl}} - \frac{1}{s^2 e^{sl}}\right] + \frac{1}{s^2}$$

as "l" approaches to "∞"

$$\frac{1}{e^{sl}} \approx 0$$

$$\lim_{l\to\infty}\frac{-l}{se^{sl}} = \frac{\infty}{\infty}$$

$$= \lim_{l\to\infty}\frac{-1}{se^{sl}\cdot s}$$

$$= \frac{-1}{s^2 e^{\infty}} = \frac{1}{\infty}$$

$$= 0$$

$$= \frac{1}{s^2}$$

$\therefore \mathcal{L}\{F(t)\} = f(s)$

$\Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$

Consider

$\boxed{4}$ $F(t) = e^{at}$

Show $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s>a$ $\because (s-a)>0$

by definition, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$\mathcal{L}\{e^{at}\} = \lim_{l\to\infty} \int_0^l e^{-st} e^{at}\,dt$$

$$= \lim_{l\to\infty} \int_0^l e^{(a-s)t}\,dt$$

$$= \lim_{l\to\infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_0^l$$

$$= \lim_{l\to\infty} \left[\frac{e^{(a-s)l}}{(a-s)} - \frac{e^{(a-s)0}}{(a-s)}\right]$$

$\left\{\begin{array}{l}\because s>a \\ \therefore a-s<0\end{array}\right.$

$$= \frac{e^{-\infty}}{a-s} - \frac{1}{a-s}$$

$$= \frac{1}{e^{\infty}(a-s)} - \frac{1}{a-s}$$

$\dfrac{1}{e^\infty} = \dfrac{1}{\infty} = 0$

$$= 0 - \frac{1}{a-s}$$

$$= -\frac{1}{a-s}$$

$$= \frac{1}{s-a}, \quad s>a$$

$\therefore \mathcal{L}\{F(t)\} = f(s)$

$\Rightarrow \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s>a$

Consider:

$\boxed{5}$ $F(t) = \sin at$

Show: $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = f(s)$

$\Rightarrow \mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt = \frac{a}{s^2+a^2}, \; s>0$

$$\int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{\sin at}_{u}\, dt = \sin at \int e^{-st} dt - \int \{ a\cos at \int e^{-st} dt \} dt$$

$\int uv\, dx$
$= u\int v\,dx - [\int\{u' \int v\,dx\}\,dx]$

$$= \sin at \left(\frac{e^{-st}}{-s}\right) - a\int \left\{ \cos at \left(\frac{e^{-st}}{-s}\right) \right\} dt$$

$$= -\frac{1}{s} \sin at\, e^{-st} + \frac{a}{s} \int \underbrace{\cos at}_{u}\; \underbrace{e^{-st}}_{v}\, dt$$

$$= -\frac{1}{s} \sin at\, e^{-st} + \frac{a}{s}\left[\cos at \left(\frac{e^{-st}}{-s}\right) - \int (-\sin at)(a)\left(\frac{e^{-st}}{-s}\right) dt \right]$$

$$= -\frac{1}{s} \sin at\, e^{-st} - \frac{a}{s^2} \cos at\, e^{-st} - \frac{a^2}{s^2} \int \sin at\, e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at\, dt = -\frac{1}{s} \sin at\, e^{-st} - \frac{a}{s^2} \cos at\, e^{-st} - \frac{a^2}{s^2} \int \sin at\, e^{-st}\, dt$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at\, dt + \frac{a^2}{s^2} \int \sin at\, e^{-st}\, dt = -\frac{1}{s} \sin at\, e^{-st} - \frac{a}{s^2} \cos at\, e^{-st}$$

$$\Rightarrow \left(1 + \frac{a^2}{s^2}\right) \int_0^\infty e^{-st} \sin at\, dt = -\frac{1}{s} \sin at\, e^{-st} - \frac{a}{s^2} \cos at\, e^{-st}$$

$$\Rightarrow \int_0^\infty e^{-st} \sin at\, dt = \frac{1}{1 + \frac{a^2}{s^2}} \left(-\frac{1}{s} \sin at\, e^{-st} - \frac{a}{s^2} \cos at\, e^{-st} \right)$$

$$= \frac{1}{\frac{s^2+a^2}{s^2}} \left(-\frac{1}{s} \sin at e^{-st} - \frac{a}{s^2} \cos at e^{-st}\right)$$

$$= \frac{s^2}{s^2+a^2} e^{-st} \left(\frac{-1}{s} \sin at - \frac{a}{s^2} \cos at\right)$$

$$= \frac{e^{-st}}{s^2+a^2} \left(-s \sin at - a \cos at\right)$$

$$\mathcal{L}\{\sin at\} = \lim_{l \to \infty} \left[\frac{e^{-st}}{s^2+a^2} \left(-s \sin at - a \cos at\right)\right]_0^l$$

$e^{-st}(s \sin at + a \cos at)$

$\int_a^b f(x)dx = F(b)-F(a)$

$$= \frac{-1}{s^2+a^2} \lim_{l \to \infty} \left[e^{-sl}(s \sin al + a \cos al) - e^{-0}(s \sin 0^\circ + a \cos 0^\circ)\right]$$

$$= -\frac{1}{s^2+a^2} \lim_{l \to \infty} \left[\frac{s \sin al}{e^{st}} + \frac{a \cos al}{e^{sl}} - \frac{1}{e^0}(0+a)\right]$$

$$= -\frac{1}{s^2+a^2} \left[\frac{s \left\{\begin{matrix}\text{any no.} \\ \text{b/w } -1 \text{ \& } +1\end{matrix}\right\}}{\infty} + \frac{a \left\{\begin{matrix}\text{any no.} \\ \text{b/w } -1 \\ \text{\& } +1\end{matrix}\right\}}{\infty} - a\right]$$

$$= -\frac{1}{s^2+a^2} [0 + 0 - a] = \frac{a}{s^2+a^2}$$

Consider squeeze Thm:

$g(x) \le f(x) \le h(x)$

$-1 \le \sin x \le 1$

$-1 \le \sin al \le 1$

$\therefore -1$

$\lim_{l \to \infty} (-1) = -1$

$\lim_{l \to \infty} (1) = 1$

$\therefore$ by Squeeze Thm $\lim_{l \to \infty} \sin al =$ any no. between -1 & 1

Similarly by Squeeze Thm

$\lim_{l \to \infty} \cos al =$ any number bewteen -1 & 1

$\therefore -1 \le \cos x \le 1.$

The proof of rest of the functions $\boxed{3}, \boxed{6}, \boxed{7}, \boxed{8}$ are available on the Reference Book "Schaum' outline Laplace Transformation" page 10, 11, 12.

First Translation or Shifting Property

If $\mathcal{L}\{F(t)\} = f(s)$ by definition of equation (i) in page $\boxed{1}$, then $\mathcal{L}\{e^{at} F(t)\} = f(s-a)$ → Refer to exercise no. 2 to Problem Sheet #7

(Exercise Sheet #7)

1. Find the Laplace transformation of each of the following function.

(iii) $7 \sin 2t - 3 \cos 2t$

$\sin at = \frac{a}{s^2+a^2}$, $\cos at = \frac{s}{s^2+a^2}$

$= 7 \mathcal{L}\{\sin 2t\} - 3 \mathcal{L}\{\cos 2t\}$

$= 7 \frac{2}{s^2+2^2} - 3 \frac{s}{s^2+2^2}, \quad s > 0$ {using the properties of L.T. of some elementary function

$= \frac{14}{s^2+4} - \frac{3s}{s^2+4}$

$= \frac{14-3s}{s^2+4}$

$1.$ (iv) $(t^2+1)^2$

$t^n = \frac{n!}{s^{n+1}}$, $n=0,1,2,\cdots$, $s>0$

$= t^4 + 2t^2 + 1$

$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$

$= \frac{4!}{s^{4+1}} + 2\cdot\frac{2!}{s^{2+1}} + \frac{1}{s}$, $s>0$ {by the properties of L. T. of some elementary functions}

$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$

First Translation or Shifting Property

If $\mathcal{L}\{F(t)\} = f(s)$ by definition of equation (i) in page [1], then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$ → Refer to exercise no. 2 to Problem Sheet # 7

(Exercise Sheet # 7)

2. Evaluate each of the following:

shifting property:

(ii) $\mathcal{L}\{5e^{3t}\sin 4t\}$

$\sin at = \frac{a}{s^2+a^2}$

$\mathcal{L}\{F(t)\} = f(s)$

$\Rightarrow \mathcal{L}\{e^{at}F(t)\} = f(s-a)$

$= 5\,\mathcal{L}\{e^{3t}\underbrace{\sin 4t}_{F(t)}\}$

$f(s) = \mathcal{L}\{F(t)\}$

$= \mathcal{L}\{\sin 4t\} = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$, $s>0$ ← $f(s)$

$= 5\left[\frac{4}{s^2-6s+25}\right]$

$\therefore f(s-a) = f(s-3) = \frac{4}{(s-3)^2+16}$, $(s-3)>0$ or $s>3$

$= \frac{20}{s^2-6s+25}$

$\because e^{3t}$ is given

$= \frac{4}{s^2-6s+9+16}$

$= \frac{4}{s^2-6s+25}$

2.(iii) $\mathcal{L}\{(t+2)^2 e^t\}$

$= \mathcal{L}\{(t^2+4t+4)e^t\}$

$= \mathcal{L}\{t^2e^t + 4te^t + 4e^t\}$

$= \mathcal{L}\{e^t \underbrace{t^2}_{F(t)}\} + 4\mathcal{L}\{e^t \underbrace{t}_{F(t)}\} + 4\mathcal{L}\{\underbrace{e^t}\}$

$a=1$ for e^{at} ① $a=1$ ⑪

$e^{at} = \frac{1}{s-a}$

$\therefore e^{1t} = \frac{1}{s-1}$

$= \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1}, \quad s>1$

<u>First Translation or Shifting Property</u>

If $\mathcal{L}\{F(t)\} = f(s)$ by definition of equation ⓘ in page $\boxed{1}$, then $\mathcal{L}\{e^{at}F(t)\} = f(s-a)$ → Refer to exercise no. 2 to Problem Sheet #7

① $f(s) = \mathcal{L}\{t^2\} = \frac{n!}{s^{n+1}}$ (t^n)

$= \frac{2!}{s^3}, \quad s>0$

$= \frac{2}{s^3} = f(s)$

$f(s-a) = f(s-1) = \frac{2}{(s-1)^3}, \quad s>1$

⑪ $f(s) = \mathcal{L}\{t\}$

$f(s) = \frac{1}{s^2}, \quad s>0$

$f(s-a) = f(s-1) = \frac{1}{(s-1)^2}, \quad s>1$

2 (iv) $\mathcal{L}\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

$= \mathcal{L}\{3e^{-t}\sinh 2t - 5e^{-t}\cosh 2t\}$

$= 3\mathcal{L}\{e^{-t}\underbrace{\sinh 2t}_{F(t)}\}^{(1)} - 5\mathcal{L}\{e^{-t}\underbrace{\cosh 2t}_{F(t)}\}^{(11)}$

$a=-1$ for e^{at} $\qquad a=-1$

$= 3 \cdot \dfrac{2}{(s+1)^2-4} - 5\dfrac{s+1}{(s+1)^2-4}$

$= \dfrac{6-5s-5}{(s+1)^2-4}$

$= \dfrac{1-5s}{(s+1)^2-4}$

(1) $f(s) = \mathcal{L}\{F(t)\}$

$= \mathcal{L}\{\sinh 2t\}$

$= \dfrac{2}{s^2-2^2}, \; s>|2|$

$f(s) = \dfrac{2}{s^2-4}$

$f(s-a) = f(s-(-1)) = f(s+1)$

$= \dfrac{2}{(s+1)^2-4}, \; s>-1$

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(11) $f(s) = \mathcal{L}\{\cosh 2t\}$

$= \dfrac{s}{s^2-2^2}, \quad s>|2|$

$f(s) = \dfrac{s}{s^2-4}$

$f(s-a) = f(s-(-1)) = f(s+1)$

$= \dfrac{s+1}{(s+1)^2-4}, \; s>-1$

Inverse Laplace Transformation

$F(t) = \mathcal{L}^{-1}\{f(s)\}$ is the inverse Laplace Transformation of $\mathcal{L}\{F(t)\} = f(s)$.

$\mathcal{L}^{-1}$

$\mathcal{L}^{-1} \rightarrow$ inverse Laplace Transformation operator.

The inverse laplace transformation of a function $f(s)$ is the piecewise continuous real function $F(t)$.

(Exercise Sheet #7)

3. Determine each of the following:

(i) $\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$

$$= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s-4/3}\right\}$$

$$= -4e^{4/3 t}$$

$$\frac{12}{4-3s}$$

$$= \frac{12}{-3s+4}$$

$$= \frac{12}{-3(s-4/3)}$$

$$= \frac{-4}{s-4/3}$$

$$= -4\left(\frac{1}{s-4/3}\right)$$

3. (iii) $\mathcal{L}^{-1}\left\{\dfrac{23s-15}{s^2+8}\right\}$

$$\frac{23s-15}{s^2+8}$$

$$= 23\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2\sqrt{2})^2}\right\} - \frac{15}{2\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$$

$$= \frac{23s-15}{s^2+(2\sqrt{2})^2} = \frac{23s}{s^2+(2\sqrt{2})^2} - \frac{15}{s^2+(2\sqrt{2})^2}$$

$$= 23\left(\frac{s}{s^2+(2\sqrt{2})^2}\right) - \frac{15}{2\sqrt{2}}\left(\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right)$$

$$= 23\cos 2\sqrt{2}\,t - \frac{15}{2\sqrt{2}}\sin 2\sqrt{2}\,t$$

$$\left\{\begin{array}{l} \dfrac{a}{s^2+a^2} = \sin at \\ \dfrac{s}{s^2+a^2} = \cos at \end{array}\right.$$

3. (iv) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{3/2}}\right\}$

$$= \frac{1}{(1/2)!}\,\mathcal{L}^{-1}\left\{\frac{(1/2)!}{s^{1/2+1}}\right\}$$

$$= \frac{1}{(1/2)!}\,t^{1/2}$$

$$= \frac{2}{\sqrt{\pi}}\,t^{1/2}$$

$$\frac{1}{s^{3/2}} = \frac{1}{s^{1/2+1}} = \frac{1}{(1/2)!}\left(\frac{(1/2)!}{s^{1/2+1}}\right)$$

$$\frac{3}{2}$$

$$\frac{1}{2}+1 = n+1$$

$$t^n = \frac{n!}{s^{n+1}}$$

$1! = 1$

$0! = 1$

$(-1/2)! = \sqrt{\pi}$

$(1/2)! = \frac{1}{2}\sqrt{\pi}$

$\Gamma(n) = (n-1)!$

$\Gamma(1/2) = \sqrt{\pi}$

3. vii $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{2s+3}}\right\}$

$$\frac{1}{\sqrt{2s+3}} = \frac{1}{(2s+3)^{1/2}} = \frac{1}{\{2(s+3/2)\}^{1/2}} = \frac{1}{\sqrt{2}\,(s+3/2)^{1/2}}$$

$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{1}{(s+3/2)^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{1}{\underbrace{(s-\underbrace{(-3/2)}_{\text{inner part}})^{1/2}}_{\text{outer part}}}\right\}$$

(chain Rule) {inner part → outer part}

inner part: $\dfrac{1}{s-a} = e^{at}$

$\therefore \dfrac{1}{s-(-3/2)} = e^{-3/2t}$

outer part: $\dfrac{1}{(s)^{1/2}}$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\} \text{ while } s = s+3/2}$$

$$= \frac{1}{\sqrt{2}}e^{-3/2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}}e^{-3/2t}\cdot\frac{2t^{-1/2}}{\sqrt{\pi}}$$

$$= \frac{2}{\sqrt{2\pi}}t^{-1/2}e^{-3/2t}$$

$$= \sqrt{\frac{2}{\pi}}\,t^{-1/2}e^{-3/2t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^{-1/2+1}}\right\} \quad (-1/2 \to n)$$

Cannot take formula ③

Consider: $\dfrac{n!}{s^{n+1}} = t^n,\ n \geq 0$

$\Rightarrow \dfrac{1}{s^{n+1}} = \dfrac{t^n}{n!}$

$$= \frac{t^{-1/2}}{(-1/2+1)!}$$

$$= \frac{t^{-1/2}}{\left(\frac{1}{2}\right)!}$$

$\because n = -1/2$

$\therefore \dfrac{1}{s^{n+1}} = \dfrac{t^n}{(n+1)!}$

$\because n > -1$ & $n < 0$

$\therefore$ using Formula ⑨

$\left(\frac{1}{2}\right)! = \dfrac{\sqrt{\pi}}{2}$

$$= \frac{t^{-1/2}}{\frac{\sqrt{\pi}}{2}} = \frac{2t^{-1/2}}{\sqrt{\pi}}$$

(Exercise Sheet #7)

4. Evaluate each of the following using partial fraction

① $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$s^2-4s+20$

$= s^2-2\cdot s\cdot 2+2^2+16$

$= (s-2)^2+16$

$= \mathcal{L}^{-1}\left\{\frac{6s-4}{\underbrace{(s-2)^2}_{s^2}+\underbrace{16}_{a^2}}\right\}$

$\frac{a}{s^2+a^2} = \sin at$

$= \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+4^2}\right\}$

$\frac{s}{s^2+a^2} = \cos at$

$6s-12+8$

$= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$

$= 6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{\underbrace{(s-2)^2}_{s^2}+4^2}\right\}$

$\begin{cases} s = s-a \\ \therefore e^{at} \\ = e^{2t} \end{cases}$

$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$

$= 2e^{2t}(3\cos 4t + \sin 4t)$

4 iv) $\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)^3} + \frac{C}{(s-2)^2} + \frac{D}{s-2}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1) + C(s+1)(s-2) + D(s+1)(s-2)^2 \quad \text{— (1)}$$

In eqn (1):

if $s=2$: $-21 = 3B \Rightarrow \boxed{B=-7}$

if $s=-1$: $9 = -27A \Rightarrow \boxed{A=-1/3}$

We also need to find C & D, hence expand eqn (1)

$$5s^2-15s-11 = A(s^3-6s^2+12s-8)+Bs+B+C(s^2-s-2)+(Ds+D)(s^2-4s+4)$$

$$= As^3-6As^2+12As-8A+Bs+B+Cs^2-Cs-2C+Ds^3+Ds^2-4Ds^2-4Ds+4Ds+4D$$

$$\Rightarrow 5s^2-15s-11 = s^3(A+D)+s^2(-6A+C+D-4D)+s(12A+B-C-\cancel{4D}+\cancel{4D})+(-8A+B-2C+4D)$$

Equating factors of like terms:

"s^3"

$A+D=0$

$A=-D$

$-1/3 = -D$

$\boxed{D=1/3}$

"s^2"

$-6A+C+D-4D=5$

$-6A+C-3D=5$

$-6(-\frac{1}{3})+C-3(\frac{1}{3})=5$

$2+C-1=5$

$\boxed{C=4}$

"s"

$12A+B-C=15$

"Constant"

$-8A+B-2C+4D = -11$

$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{-\frac{1}{3}}{s+1}+\frac{-7}{(s-2)^3}+\frac{4}{(s-2)^2}+\frac{\frac{1}{3}}{s-2}\right\}$$

$$=-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-7\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s-2)^3}}\right\}+4\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s-2)^2}}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

chain rule ⟷ inner, outer part

$$=-\frac{1}{3}e^{-t}-7\cdot\frac{t^2}{2!}e^{2t}+4\,t\,e^{2t}+\frac{1}{3}e^{2t}$$

$f(s) = \mathcal{L}\{F(t)\}$	$F(t)$
$\frac{1}{s-a}$	e^{at}
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{1}{s^2}$	t

Using shifting law:

$\mathcal{L}\{e^{at}F(t)\} = f(s-a)$

while $\mathcal{L}\{F(t)\} = f(s)$.

4. (vii) $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

s^2+2s+2

$= s^2+2\cdot s\cdot 1+1^2+1$

$=(s+1)^2+1^2$

s^2+2s+5

$= s^2+2\cdot s\cdot 1+1^2+4$

$=(s+1)^2+2^2$

Now

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$\vdots$$

$$\Rightarrow s^2+2s+3 = (A+C)s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + (5B+2D)$$

Equating Factors of like terms:

" s^3 "

$A+C=0$ — (1)

" s^2 "

$2A+B+2C+D=1$

$2(A+C)+B+D=1$

$2(0)+B+D=1$

$B+D=1$ — (2)

" s "

$5A+2B+2C+2D=2$ — (3)

" Constant "

$5B+2D=3$ — (4)

solve (2) & (4) we have $\boxed{B=\frac{1}{3}}$, $\boxed{D=\frac{2}{3}}$

Substitute B & D into (3)

$5A+2C+2\left(\frac{1}{3}\right)+2\left(\frac{2}{3}\right)=2$

$5A+2C=2-\frac{2}{3}-\frac{4}{3}$

$5A+2C=0$ ——— (5)

solve (1) & (5) $\Rightarrow$ $\boxed{A=0}$, $\boxed{C=0}$

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$s^2+2s+2 = s^2+2\cdot s\cdot 1+1^2+1 = (s+1)^2+1^2$$

$$s^2+2s+5 = s^2+2\cdot s\cdot 1+1^2+4 = (s+1)^2+2^2$$

$$= \mathcal{L}^{-1}\left\{\frac{\overset{\nearrow As+B}{1/3}}{s^2+2s+2} + \frac{\overset{\nearrow Cs+D}{2/3}}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + 2/3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\frac{\sin t}{1} + \frac{2}{3}e^{-t}\frac{\sin 2t}{2}$$

$$e^{at} = \frac{1}{s-a}$$

$$\sin at = \frac{a}{s^2+a^2}$$

$$\Rightarrow \frac{\sin at}{a} = \frac{1}{s^2+a^2}$$

$$= \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

Applications to Ordinary Differential Equations (ODE) with constant coefficients:

In mathematics, an ODE is a differential equation containing one or more functions of one independent variable and its derivatives.

Consider

$$Y'' + \alpha Y' + \beta Y = Y(t) \quad \text{—— (1)}$$

initial values:

$$Y(0) = A, \quad Y'(0) = B \quad \text{—— (2)}$$

A, B are given constants (arbitrary values)

Taking the Laplace transformation of both sides of (1) and using (2), we obtain an algebric equation for determination of $\mathcal{L}\{Y(t)\} = y(s)$.

$\because \mathcal{L}\{F(t)\} = f(s)$

$\therefore \mathcal{L}\{Y(t)\} = y(s)$

Hence $\mathcal{L}\{Y'' + \alpha Y' + \beta Y\} = \mathcal{L}\{Y(t)\} = y(s) = y$

$\mathcal{L}\{Y''\} + \alpha \mathcal{L}\{Y'\} + \beta \mathcal{L}\{Y\} = \text{''}$ → taken L.T. in eqn (1)

Formula: $\mathcal{L}\{Y^{(n)}\} = s^n \mathcal{L}\{Y\} - s^{n-1} Y(0) - s^{n-2} Y'(0) - s^{n-3} Y''(0) \cdots - s^1 Y^{(n-2)}(0) - s^0 Y^{(n-1)}(0).$

[while $Y^{(n)} = \dfrac{d^{(n)} Y}{dx^n}$ & $\mathcal{L}\{Y\} = y$]

(Exercise Sheet #7)

5. Solve the given differential equation:

(i) $y'' - 3y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$; $Y = ?$

$\Rightarrow \mathcal{L}\{Y'' - 3Y' + 2Y\} = \mathcal{L}\{4e^{2t}\}$ (taking L.T. on both sides)

$\Rightarrow \mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$

$\Rightarrow [s^2\mathcal{L}\{Y\} - s^1 Y(0) - s^0 Y'(0)] - 3[s^1\mathcal{L}\{Y\} - s^0 Y(0)]$

$+ 2[\mathcal{L}\{Y\}] = 4\left(\frac{1}{s-2}\right)$

$\Rightarrow [s^2 y - s(-3) - 5] - 3[sy - (-3)] + 2y = \frac{4}{s-2}$ $\{\mathcal{L}\{Y\} = y\}$

$\Rightarrow s^2y + 3s - 5 - 3sy - 9 + 2y = \frac{4}{s-2}$

$\Rightarrow s^2y - 3sy + 3s + 2y - 14 = \frac{4}{s-2}$

$\Rightarrow s^2y - 3sy + 2y = \frac{4}{s-2} - 3s + 14$

$\Rightarrow y(s^2 - 3s + 2) = \frac{4}{s-2} - 3s + 14$

$\Rightarrow y = \left(\frac{1}{s^2 - 3s + 2}\right)\left(\frac{4}{s-2} - 3s + 14\right)$

$\Rightarrow \mathcal{L}\{Y\} = \left(\frac{1}{s^2 - 3s + 2}\right)\left(\frac{4}{s-2} - 3s + 14\right)$ $\because \mathcal{L}\{Y\} = y$

$$Y = \mathcal{L}^{-1}\left\{ \frac{1}{s^2-3s+2}\left(\frac{4}{s-2} - 3s + 14\right)\right\}$$

s^2-3s+2
$= s^2-2s-s+2$
$= (s-2)(s-1)$

$$= \mathcal{L}^{-1}\left\{ \frac{4}{(s-2)(s^2-3s+2)} - \frac{3s}{s^2-3s+2} + \frac{14}{s^2-3s+2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{4}{(s-2)(s-2)(s-1)} - \frac{3s}{(s-2)(s-1)} + \frac{14}{(s-2)(s-1)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{4}{(s-2)^2(s-1)} - \frac{3s}{(s-2)(s-1)} + \frac{14}{(s-2)(s-1)} \right\}$$

$$= 4\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)^2(s-1)} \right\} - 3\mathcal{L}^{-1}\left\{ \frac{s}{(s-2)(s-1)} \right\} + 14\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)(s-1)} \right\}$$

$\downarrow$ 1 $\qquad \downarrow$ 2 $\qquad \downarrow$ 3

1 $\quad \frac{1}{(s-2)^2(s-1)} = \frac{A}{(s-2)^2} + \frac{B}{s-2} + \frac{C}{s-1}$

$$1 = A(s-1) + B(s-2)(s-1) + C(s-2)^2 \quad \text{— (i)}$$

$$= As - A + Bs^2 - 2Bs - Bs + 2B + Cs^2 - 4Cs + 4C$$

$$= s^2(B+C) + s(A-2B-B-4C) + (-A+2B+4C) \quad \text{— (ii)}$$

$$1 = s^2(B+C) + s(A-2B-B-4C)$$

Equating factors of like terms

"s^2"

$B+C=0$ — (*)

"s"

$A-3B-4C=0$ — (**)

"constant"

$-A+2B+4C=1$ $\to$ (***)

in eqn (i)

if $s=1 \Rightarrow 1 = C(-1)^2 \Rightarrow \boxed{C=1}$

if $s=2 \Rightarrow 1 = A(1) \Rightarrow \boxed{A=1}$

substitute A & C into (***)

$-(1)+2B+4(1)=1$

$2B = 1-4+1$

$2B = -2$

$\boxed{B=-1}$

$$\therefore \frac{1}{(s-2)^2(s-1)} = \frac{1}{(s-2)^2} + \frac{(-1)}{s-2} + \frac{1}{s-1}$$

② $$\frac{S}{(S-2)(S-1)} = \frac{A}{S-2} + \frac{B}{S-1}$$

$$\Rightarrow S = A(S-1) + B(S-2) \quad \text{--- (i)}$$

in eqn (i)

if $S=1 \Rightarrow 1 = B(-1) \Rightarrow \boxed{B=-1}$

if $S=2 \Rightarrow 2 = A(1) \Rightarrow \boxed{A=2}$

$$\therefore \frac{S}{(S-2)(S-1)} = \frac{2}{S-2} + \frac{(-1)}{S-1}$$

③ $$\frac{1}{(S-2)(S-1)} = \frac{A}{S-2} + \frac{B}{S-1}$$

$$1 = A(S-1) + B(S-2) \quad \text{--- (i)}$$

in eqn (i)

if $S=1 \Rightarrow 1 = B(-1) \Rightarrow \boxed{B=-1}$

if $S=2 \Rightarrow 1 = A(1) \Rightarrow \boxed{A=1}$

$$\therefore \frac{1}{(S-2)(S-1)} = \frac{1}{S-2} + \frac{-1}{S-1}$$

$$Y = 4\mathcal{L}^{-1}\left\{\frac{1}{(S-2)^2(S-1)}\right\} - 3\mathcal{L}^{-1}\left\{\frac{S}{(S-2)(S-1)}\right\} + 14\mathcal{L}^{-1}\left\{\frac{1}{(S-2)(S-1)}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{(S-2)^2} - \frac{1}{S-2} + \frac{1}{S-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{S-2} - \frac{1}{S-1}\right\} + 14\mathcal{L}^{-1}\left\{\frac{1}{S-2} - \frac{1}{S-1}\right\}$$

$$= 4\left[\mathcal{L}^{-1}\left\{\frac{1}{\underbrace{(S-2)^2}_{\text{chain rule}}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{S-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\}\right] - 3\left[2\mathcal{L}^{-1}\left\{\frac{1}{S-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\}\right]$$

$$+ 14\left[\mathcal{L}^{-1}\left\{\frac{1}{S-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{S-1}\right\}\right]$$

$$= 4\left[e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{S^2}\right\} - e^{2t} + e^t\right] - 3\left[2e^{2t} - e^t\right] + 14\left[e^{2t} - e^t\right]$$

$$Y = 4\left[e^{2t} \cdot t - e^{2t} + e^{t}\right] - 6e^{2t} + 3e^{t} + 14e^{2t} - 14e^{t}$$

$$= 4te^{2t} - 4e^{2t} + 4e^{t} + 8e^{2t} - 11e^{t}$$

$$= 4te^{2t} - 7e^{t} + 4e^{2t}$$

5. (ii) $Y'' + 9Y = \cos 2t$, $Y(0) = 1$, $Y(\pi/2) = -1$

$\because$ $Y'(0)$ is unknown, assume $Y'(0) = C$

$$\mathcal{L}\{Y'' + 9Y\} = \mathcal{L}\{\cos 2t\}$$

$$\mathcal{L}\{Y''\} + 9\mathcal{L}\{Y\} = \frac{s}{s^2 + 2^2}$$

$$\left[s^2 \mathcal{L}\{Y\} - s^1 Y(0) - s^0 Y'(0)\right] + 9y = \frac{s}{s^2 + 4}$$

$$s^2 y - s - C + 9y = \frac{s}{s^2 + 4}$$

$$y(s^2 + 9) = \frac{s}{s^2 + 4} + s + C$$

$$y = \frac{1}{s^2 + 9}\left(\frac{s}{s^2 + 4} + s + C\right)$$

$$\mathcal{L}\{Y\} = \frac{s}{(s^2 + 9)(s^2 + 4)} + \frac{s}{s^2 + 9} + \frac{C}{s^2 + 9}$$

$$Y = \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 9)(s^2 + 4)} + \frac{s}{s^2 + 9} + \frac{C}{s^2 + 9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 9)(s^2 + 4)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 9}\right\} + C\,\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 9}\right\}$$

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} = \frac{(As+B)(s^2+4)+(Cs+D)(s^2+9)}{(s^2+9)(s^2+4)}$$

$$s = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 9Cs + 9D$$

$$s = s^3(A+C) + s^2(B+D) + s(4A+9C) + (4B+9D)$$

Equating factor of like terms:

"s^3" $A+C=0$ ⟶ (1)

"s^2" $B+D=0$ ⟶ (2)

"s" $4A+9C=1$ ⟶ (3)

"Constant" $4B+9D=0$ ⟶ (4)

solve (1) & (3)

$$\begin{aligned} (1)\times 4 &\rightarrow 4A+4C=0 \\ (3) &\rightarrow 4A+9C=1 \\ & \quad (-)\quad (-) \quad (-) \\ \hline & -5C = -1 \end{aligned}$$

$$\boxed{C = \frac{1}{5}}$$

(1) ⟶ $A+C=0$

$A+\frac{1}{5}=0$

$$\boxed{A = -1/5}$$

Solve (2) & (4)

$B = D = 0$

$$\therefore \frac{s}{(s^2+9)(s^2+4)} = \frac{-1/5\, s+0}{s^2+9} + \frac{\frac{1}{5}s+0}{s^2+4}$$

$$= \frac{-\frac{1}{5}s}{s^2+9} + \frac{\frac{1}{5}s}{s^2+4}$$

from page $\boxed{23}$

$$Y = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)(s^2+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + c\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{5}s}{s^2+9} + \frac{\frac{1}{5}s}{s^2+4}\right\} + \cos 3t + c\cdot\frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \cos 3t + \frac{c}{3}\sin 3t$$

$$Y = -\frac{1}{5}\cos 3t + \frac{1}{5}\cos 2t + \cos 3t + \frac{c}{3}\sin 3t$$

$$Y = \cos 3t\left(1 - \frac{1}{5}\right) + \frac{1}{5}\cos 2t + \frac{c}{3}\sin 3t$$

$$Y = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$$ → "t" is the variable

$$\therefore Y(t) = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t \quad \text{—} (*)$$

$$Y\left(\frac{\pi}{2}\right) = \frac{4}{5}\cos 3\left(\frac{\pi}{2}\right) + \frac{c}{3}\sin 3\left(\frac{\pi}{2}\right) + \frac{1}{5}\cos 2\left(\frac{\pi}{2}\right) = -1$$

(Given $Y\left(\frac{\pi}{2}\right) = -1$) t Y

$$\Rightarrow \frac{4}{5}(0) + \frac{c}{3}(-1) + \frac{1}{5}(-1) = -1$$

$$\Rightarrow -\frac{c}{3} - \frac{1}{5} = -1$$

$$\Rightarrow \frac{c}{3} + \frac{1}{5} = 1 \Rightarrow \frac{c}{3} = 1 - \frac{1}{5}$$

$$\Rightarrow \frac{c}{3} = \frac{4}{5}$$

$$\Rightarrow c = \frac{3\times 4}{5}$$

$$\Rightarrow \boxed{c = \frac{12}{5}}$$

Substitute "c" into eqn (*)

$$Y = \frac{4}{5}\cos 3t + \frac{12/5}{3}\sin 3t + \frac{1}{5}\cos 2t \Rightarrow Y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$$

Ordinary Differential Equations with Variable Coefficients

The Laplace transform can be used in solving ODE in which the coefficients are variable.
A particular differential equation where the method proves useful is one in which the terms have the form

$$t^m y^{(n)}(t)$$

the Laplace transform of which is:

$$\mathcal{L}\{t^m y^{(n)}(t)\} = (-1)^m \frac{d^m}{ds^m} \mathcal{L}\{y^{(n)}(t)\}$$

Examples on ODE with variable coefficients:

$\boxed{1}$ $tY'' + (1-2t)Y' - 2Y = 0$; $Y(0) = 1$, $Y'(0) = 2$

$$\Rightarrow tY'' + Y' - 2tY' - 2Y = 0$$

$$\Rightarrow \mathcal{L}\{tY'' + Y' - 2tY' - 2Y\} = \mathcal{L}\{0\}$$

$$\Rightarrow \mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} - 2\mathcal{L}\{tY'\} - 2\mathcal{L}\{Y\} = 0$$

$$\Rightarrow (-1)^1 \frac{d}{ds} \mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} - 2(-1)^1 \frac{d}{ds} \mathcal{L}\{Y'\} - 2y = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 \mathcal{L}\{Y\} - s^1 Y(0) - s^0 Y'(0)\right] + \left[s^1 \mathcal{L}\{Y\} - s^0 Y(0)\right]$$

$$+2\frac{d}{ds}\left[s^1 \mathcal{L}\{Y\} - s^0 Y(0)\right] - 2y = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 y - s - 2\right] + \left[sy - 1\right] + 2\frac{d}{ds}\left[sy - 1\right] - 2y = 0$$

$$\Rightarrow \frac{d}{ds}\left(-s^2y+s+2\right)+sy-1+2\frac{d}{ds}(sy-1)-2y=0$$

$$\Rightarrow \frac{d}{ds}(-s^2y)+\frac{d}{ds}(s)+\frac{d}{ds}(2)+sy-1+2\left[\frac{d}{ds}(sy)-\frac{d}{ds}(1)\right]-2y=0$$

$\frac{d}{dx}(uv)=uv'+vu'$

$$\Rightarrow -s^2\frac{dy}{ds}+(-2s)y+1+0+sy-1+2\left[y+s\frac{dy}{ds}-0\right]-2y=0$$

$$\Rightarrow -s^2\frac{dy}{ds}-2sy+sy+2y+2s\frac{dy}{ds}-2y=0$$

$$\Rightarrow s^2\frac{dy}{ds}+2sy-sy-2s\frac{dy}{ds}=0 \quad [\div \text{ by } -1]$$

$$\Rightarrow s^2\frac{dy}{ds}-2s\frac{dy}{ds}+sy=0$$

$(\div$ by $s)$

$$\Rightarrow s\frac{dy}{ds}-2\frac{dy}{ds}+y=0$$

$$\Rightarrow (s-2)\frac{dy}{ds}+y=0$$

$$\Rightarrow (s-2)\frac{dy}{ds}=-y \Rightarrow (s-2)\,dy=-y\,ds$$ [separate variable]

$$\Rightarrow dy/-y=\frac{ds}{s-2}$$

$$\Rightarrow \frac{dy}{y}+\frac{ds}{s-2}=0$$

$\log_e x = y \rightarrow x=e^y$

$$\Rightarrow \int\frac{1}{y}\cdot dy+\int\frac{1}{s-2}\,ds=\int 0$$

$$\Rightarrow \ln y+\ln(s-2)=c$$ $\rightarrow$ (integrate both sides)

$$\Rightarrow \ln(y(s-2))=c$$

$$\Rightarrow \log_e[y(s-2)]=c$$ $\rightarrow$ $y(s-2)=e^c=c$

Relebel the constant $e^c=c$

$$y = \frac{c}{s-2}$$

$$\mathcal{L}\{Y\} = \frac{c}{s-2}$$

$$Y = c\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$Y = ce^{2t}$$

$$\Rightarrow Y(t) = ce^{2t} \quad \text{—— (i)}$$

t Y

Given $Y(0) = 1$

Substitute $t = 0$, $Y = 1$ into (i)

$$Y(0) = ce^{2(0)} = 1 \quad \text{Given}$$

$$\Rightarrow ce^{0} = 1$$

$$\Rightarrow c = 1$$

Substitute "c" into (i)

$$Y = e^{2t}$$

$\boxed{2}$ $Y'' - tY' + Y = 1$; $Y(0) = 1$, $Y'(0) = 2$

Ans: $Y = 1 + 2t$

TRY

Examples on Variable Coefficient

$$\mathcal{L}\{t^m y^{(n)}t\} = (-1)^m \frac{d^m}{ds^m} \underbrace{\mathcal{L}\{y^{(n)}(t)\}}_{f(s)} \quad \because \mathcal{L}\{y(t)\} = f(s)$$

1. $\mathcal{L}\{t^2 \sin 2t\}$

$$= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin 2t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{2}{s^2+2^2}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+4)0 - 2(2s+0)}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2(-4) - (-4s)2(s^2+4)2s}{(s^2+4)^4}$$

$$= \frac{(s^2+4)(-4s^2-16+16s^2)}{(s^2+4)^4}$$

$$= \frac{8s^2-16}{(s^2+4)^3}$$

2. $\mathcal{L}\{t e^{3t} \cos 2t\}$

$$= (-1)^1 \frac{d}{ds} \mathcal{L}\{e^{3t} \cos 2t\}$$

$$\mathcal{L}\{F(t)\} = f(s)$$

$$\mathcal{L}\{e^{at} F(t) = f(s-a)$$

$$= -\frac{d}{ds} \mathcal{L}\{\underbrace{e^{3t}}_{e^{at}} \underbrace{\cos 2t}_{F(t)}\}$$

$$F(t) = \cos 2t$$

$$f(s) = \frac{s}{s^2+2^2}$$

$$f(s-a) = f(s-3) = \frac{s-3}{(s-3)^2+4}$$

$$= -\frac{d}{ds}\left[\frac{s-3}{(s-3)^2+4}\right]$$

$$= -\left[\frac{\{(s-3)^2+4\}\{1\} - (s-3)\{2(s-3)\}}{\{(s-3)^2+4\}^2}\right]$$

$$= -\left[\frac{(s-3)^2+4-2(s-3)^2}{((s-3)^2+4)^2}\right]$$

$$= \frac{(s-3)^2-4}{((s-3)^2+4)^2}$$

$\boxed{3}$ Solve $tY'' + Y' + 4tY = 0,\ Y(0) = 3,\ Y'(0) = 0.$

take L. T. on both sides

$t^m = (-1)^m \frac{d^{(m)}}{ds^{(m)}}$

$$\mathcal{L}\{tY''\} + \mathcal{L}\{Y'\} + \mathcal{L}\{4tY\} = \mathcal{L}\{0\}$$

$$\Rightarrow (-1)^1 \frac{d}{ds}\mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} + 4(-1)^1 \frac{d}{ds}\mathcal{L}\{Y\} = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 \mathcal{L}\{Y\} - s^1 Y(0) - s^0 Y'(0)\right] + \left[s^1 \mathcal{L}\{Y\} - s^0 Y(0)\right]$$

$$-4\frac{d}{ds}[y] = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2 y - 3s - (1)(0)\right] + \left[sy - 1(3)\right] - 4\frac{dy}{ds} = 0$$

$$\Rightarrow \frac{d}{ds}\left(-s^2 y + 3s\right) + sy - 3 - 4\frac{dy}{ds} = 0$$

$$\Rightarrow \frac{d}{ds}s^2 y - \frac{d}{ds}3s - sy + 3 + 4\frac{dy}{ds} = 0 \quad \left(\div \text{ by } -1\right)$$

$$\Rightarrow s^2\frac{dy}{ds} + 2sy - 3 - sy + 3 + 4\frac{dy}{ds} = 0$$

$$\Rightarrow (s^2+4)\frac{dy}{ds} + sy = 0$$

$$\Rightarrow (s^2+4)\frac{dy}{ds} = -sy \Rightarrow (s^2+4)dy = -sy\,ds$$

$$\Rightarrow dy/y = -s\,ds/(s^2+4) \quad \text{(separate variable)}$$

$$\Rightarrow \frac{dy}{y} + \frac{s}{s^2+4}ds = 0$$

$$\Rightarrow \int \frac{1}{y}dy + \int \frac{s}{s^2+4}ds = \int 0 = c$$

$$\Rightarrow \ln y + \int \frac{s}{s^2+4}\,ds = c$$

$$\Rightarrow \ln y + \int \frac{\frac{1}{2}\,du}{u} = c$$

let $s^2+4=u$

$2s\,ds = du$

$s\,ds = \frac{1}{2}du$

$$\Rightarrow \ln y + \frac{1}{2}\ln u = c$$

$$\Rightarrow \ln y + \frac{1}{2}\ln(s^2+4) = c$$

$$\Rightarrow \ln y + \ln\sqrt{s^2+4} = c$$

$$\Rightarrow \ln\left(y\sqrt{s^2+4}\right) = c$$

$$\Rightarrow \log_e\left(y\sqrt{s^2+4}\right) = c$$

$$\Rightarrow y\sqrt{s^2+4} = e^c = C$$

Relabel the constant $e^c = C$

$$\Rightarrow y = \frac{C}{\sqrt{s^2+4}}$$

$$\Rightarrow \mathcal{L}\{Y\} = \frac{C}{\sqrt{s^2+4}}$$

$Y(0) = C\,\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+4}}\right\} = 3 \cdot 1$ $f(s)$ t

Given $Y(0)=3$ $\hookrightarrow t$

$\Rightarrow C = 3$ **by equating factors of like terms**

$$\Rightarrow Y = C\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^{1/2}}\right\}$$

$\frac{(n+1)!}{s^{n+1}} = t^n$

$n > -1$

$\frac{1}{s^{1/2}} = \frac{1}{s^{-\frac{1}{2}+1}}$, $n = -\frac{1}{2}$

$= \frac{1}{s^{n+1}}$ use formula 9

$\hookrightarrow \frac{t^n}{(n+1)!}$

$$\Rightarrow Y = 3\,\frac{\sin 2t}{2}\,\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{3\sin 2t}{2} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^{-\frac{1}{2}+1}}\right\}$$

$$= \frac{3\sin 2t}{2} \cdot \frac{t^{-1/2}}{\left(-\frac{1}{2}+1\right)!}$$

$$= \frac{3\sin 2t}{2} \frac{1}{(+\frac{1}{2})!} \cdot t^{-1/2}$$

$$Y = \frac{3\sin 2t}{2} \cdot \frac{1}{\frac{\sqrt{\pi}}{2}} \cdot \frac{1}{t^{1/2}} \longrightarrow Y(t)$$

t is the variable

$$Y = \frac{3\sin 2t}{\sqrt{\pi t}}$$

Examples on IVP (initial value problem)

Reading

$\boxed{1}$ $Y''' - Y'' - Y' + Y = 8te^{-t}$

while $Y(0)=0$, $Y'(0)=1$, $Y''(0)=0$

$F(t)$, $a=-1$

$$\mathcal{L}\{Y'''\} - \mathcal{L}\{Y''\} - \mathcal{L}\{Y'\} + \mathcal{L}\{Y\} = 8\,\mathcal{L}\{te^{-t}\}$$

$$\Rightarrow [s^3\mathcal{L}\{Y\} - s^2Y(0) - s^1Y'(0) - s^0y''(0)] - [s^2\mathcal{L}\{Y\} - s^1Y(0) - s^0Y'(0)]$$

$$- [s^1\mathcal{L}\{Y\} - s^0Y(0)] + y = 8 \cdot \frac{1}{(s+1)^2}$$

$f(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$

$f(s-a)$
$= f(s-(-1))$
$= f(s+1) = \frac{1}{(s+1)^2}$

$$\Rightarrow [s^3y - s^2(0) - s(1) - 1(0)] - [s^2y - s(0) - 1] - [sy - 0] + y = \frac{8}{(s+1)^2}$$

$$\Rightarrow s^3y - s - s^2y + 1 - sy + y = \frac{8}{(s+1)^2}$$

$$\Rightarrow y(s^3 - s^2 - s + 1) = \frac{8}{(s+1)^2} + s - 1$$

$$\Rightarrow y(s^2(s-1) - 1(s-1)) = \frac{8}{(s+1)^2} + s - 1$$

$$y(s^2-1)(s-1) = \frac{8}{(s+1)^2} + s - 1$$

$$y = \frac{8}{(s+1)^2\underbrace{(s^2-1)}_{(s+1)(s-1)}(s-1)} + \frac{s}{\underbrace{(s^2-1)}_{(s+1)(s-1)}(s-1)} - \frac{1}{\underbrace{(s^2-1)}_{(s+1)(s-1)}(s-1)}$$

$$= \underbrace{\frac{8}{(s+1)^3(s-1)^2}}_{a} + \underbrace{\frac{s}{(s+1)(s-1)^2}}_{b} - \underbrace{\frac{1}{(s+1)(s-1)^2}}_{c}$$

a:

$$\frac{8}{(s+1)^3(s-1)^2} = \frac{A}{(s+1)^3} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{D}{(s-1)^2} + \frac{E}{s-1}$$

$$8 = A(s-1)^2 + B(s+1)(s-1)^2 + C(s+1)^2(s-1)^2 + D(s+1)^3 + E(s+1)^3(s-1) \quad \text{(i)}$$

if $s = 1$ then $8 = D(2)^3 \Rightarrow \boxed{D = 1}$

if $s = -1$ then $8 = A(-2)^2 \Rightarrow \boxed{A = 2}$

Substitute $A = 2$, $D = 1$ into (i)

$$\therefore 8 = 2(s-1)^2 + B(s+1)(s-1)^2 + C(s+1)^2(s-1)^2 + 1(s+1)^3 + E(s+1)^3(s-1)$$

$$8 = 2s^2 - 4s + 1 + (Bs+B)(s^2-2s+1) + (s^2+2s+1)(s^2-2s+1)C + s^3 + 3s^2 + 3s + 1 + (Es^3 + 3Es^2 + 3Es + E)(s-1)$$

$$8 = 5s^2 - s + Bs^3 - Bs - Bs^2 + B + s^3 + s^4E + 2s^3E - 2Es - E + s^4C - 2s^2C + C + 2$$

"s^4"

$E + C = 0$

$\boxed{E = -C}$

"s^3"

$B + 1 + 2E = 0$

"s^2"

$5 - B - 2C = 0$

"s"

$-1 - B - 2E = 0$

$-B - 2(-C) = 1$

$-B + 2C = 1$ (iv)

"constant"

$B - E + C + 2 = 8$

$B - E + C = 6$

$B - (-C) + C = 6$

$B + 2C = 6$ (v)

IV + V $\Rightarrow 4C = 7$

$$\boxed{C = 7/4}$$

$\therefore E = -C$

$$\therefore \boxed{E = -7/4}$$

$\therefore B + 2C = 6$ — V

$$B = 6 - 2C = 6 - 2\left(7/4\right) = 6 - \frac{7}{2} = \frac{5}{2}$$

$$\therefore \boxed{B = 5/2}$$

$$\frac{8}{(s+1)^3(s-1)^2} = \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{1}{(s-1)^2} + \frac{-7/4}{s-1}$$

b:

$$\frac{s}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$$

$$s = A(s-1)^2 + B(s+1) + C(s+1)(s-1)$$

$s = 1: \Rightarrow 1 = 2B \Rightarrow \boxed{B = 1/2}$

$s = -1: \Rightarrow -1 = A(-2)^2 \Rightarrow 4A = -1 \Rightarrow \boxed{A = -1/4}$

$$\therefore s = -\frac{1}{4}(s-1)^2 + \frac{1}{2}(s+1) + C(s^2-1)$$ → substitute $B = \frac{1}{2}, A = -\frac{1}{4}$

$$s = -\frac{s^2}{4} - C + s + \frac{1}{4} + s^2 C$$

"s^2"

$-\frac{1}{4} + C = 0$

$\boxed{C = \frac{1}{4}}$

"s"

$1 = 1$

"Constant"

$0 = -C + \frac{1}{4}$ → $\therefore C = \frac{1}{4}$

$$\therefore \frac{s}{(s+1)(s-1)^2} = \frac{-1/4}{s+1} + \frac{1/2}{(s-1)^2} + \frac{1/4}{s-1}$$

(c):

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{(s-1)^2} + \frac{C}{s-1}$$

$$1 = A(s-1)^2 + B(s+1) + C(s-1)(s+1)$$

$$\Rightarrow 1 = A(s-1)^2 + B(s+1) + C(s^2-1) = \frac{1}{4}(s-1)^2 + \frac{1}{2}(s+1) + Cs^2 - C$$

$$= \frac{1}{4}(s^2-2s+1) + \frac{s}{2} + \frac{1}{2} + Cs^2 - C$$

$$= \frac{s^2}{4} - \frac{s}{2} + \frac{1}{4} + \frac{s}{2} + \frac{1}{2} + Cs^2 - C$$

$$= s^2\left(\frac{1}{4} + C\right) + \frac{3}{4} - C$$

if $s = 1$: $1 = 2B \Rightarrow \boxed{B = 1/2}$

if $s = -1$: $1 = A(-2)^2 \Rightarrow \boxed{A = 1/4}$

$$\therefore 1 = s^2\left(\frac{1}{4} + C\right) + \frac{3}{4} - C$$

"s^2"

$0 = +\frac{1}{4} + C$

$C = 0 - 1/4 \Rightarrow \boxed{C = \frac{-1}{4}}$

"Constant"

$1 = \frac{3}{4} - C$

$C = \frac{3}{4} - 1 = -\frac{1}{4}$

$$\therefore \frac{1}{(s+1)(s-1)^2} = \frac{1/4}{s+1} + \frac{1/2}{(s-1)^2} + \frac{-1/4}{s-1}$$

$$y = \frac{8}{(s+1)^3(s-1)^2} + \frac{s}{(s+1)(s-1)^2} - \frac{1}{(s+1)(s-1)^2}$$

ie. a + b + c

$$= \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{1}{(s-1)^2} + \frac{-7/4}{s-1} + \frac{-1/4}{s+1} + \frac{1/2}{(s-1)^2} + \frac{1/4}{(s-1)}$$

$$+ \frac{1/4}{s+1} + \frac{1/2}{(s-1)^2} - \frac{1/4}{s-1}$$

$$= \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5\ 7/4}{s-1}$$

$$\mathcal{L}\{Y\} = \frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5\ 7/4}{s-1}$$

$$Y = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3} + \frac{5/2}{(s+1)^2} + \frac{7/4}{s+1} + \frac{2}{(s-1)^2} - \frac{5\ 7/4}{s-1}\right\}$$

$$Y = 2\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(\underbrace{S+1}_{e^{-t}})^3}}_{e^{-t}}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(\underbrace{S+1}_{e^{-t}})^2}}_{S^2}\right\} + \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{S+1}\right\}$$

$$S^3 \to S^{2+1}$$

$$+2\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(\underbrace{S-1}_{e^{t}})^2}}_{S^2}\right\} - \frac{7}{4}\mathcal{L}^{-1}\left\{\frac{51}{S-1}\right\}$$

$$= 2e^{-t}\frac{t^2}{2!} + \frac{5}{2}e^{-t}t + \frac{7}{4}e^{-t} + 2e^{t}t - \frac{7}{4}e^{t}$$

$$= t^2e^{-t} + \frac{5t}{2}te^{-t} + \frac{7}{4}e^{-t} + 2te^{t} - \frac{5}{4}e^{t}$$

Examples on IVP

Reading

$\boxed{2}$ $Y''' + 4Y'' + 5Y' + 2Y = 10\cos t$; $Y(0) = Y'(0) = 0$, $Y''(0) = 0$

$$\mathcal{L}\{Y'''\} + 4\mathcal{L}\{Y''\} + 5\{Y'\} + 2\mathcal{L}\{Y\} = 10\mathcal{L}\{\cos t\}$$

$$\Rightarrow [S^3\mathcal{L}\{Y\} - S^2Y(0) - S^1Y'(0) - S^0Y''(0)] + 4[S^2\mathcal{L}\{Y\} - S^1Y(0) - S^0Y'(0)]$$

$$+5[S^1\mathcal{L}\{Y\} - S^0Y(0)] + 2y = 10\cdot\frac{S}{S^2+1^2} = \frac{10S}{S^2+1}$$

$$\Rightarrow S^3y + 4S^2y + 5Sy + 2y = \frac{10S}{S^2+1}$$

$$\Rightarrow y(S^3 + 4S^2 + 5S + 2) = \frac{10S}{S^2+1}$$

Rational Roots:

$S^3 + 4S^2 + 5S + 2$:

$S = -1$, $S = -2$

Polynomial eqn:

$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0x^0 = 0$

if a_0 & a_n are integers, then if there is a rational solution, it could be found by checking all the numbers produced for $\frac{\pm \text{divider of } a_0}{\text{dividers of } a_n}$

$a_0 = 2, \quad a_n = 1$

The dividers of a_0: 1, 2 The dividers of a_n: 1

The following rational numbers are candidate roots: $\frac{[1, 2]}{1}$

$s+1=0$; $s+2=0$

$s=-1$ $\quad$ $s=-2$

$$s^3+4s^2+5s+2$$

$$=(s+1)\frac{s^3+4s^2+5s+2}{s+1}$$

$$=(s+1)(s^2+3s+2)$$

$$=(s+1)(s+1)(s+2)$$

$$=(s+1)^2(s+2)$$

$$\Rightarrow y(s^3+4s^2+5s+2) = \frac{10s}{s^2+1}$$

$$\Rightarrow y(s+1)^2(s+2) = \frac{10s}{s^2+1}$$

$$\Rightarrow y = \frac{10s}{(s^2+1)(s+1)^2(s+2)}$$

$$\frac{10s}{(s^2+1)(s+1)^2(s+2)} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)} + \frac{E}{s+2}$$

$$10s = (As+B)(s+1)^2(s+2) + C(s^2+1)(s+2) + D(s^2+1)(s+1)(s+2)$$
$$+ E(s^2+1)(s+1)$$

if $s = -1$: $-10 = C(2) \Rightarrow \boxed{C = -5}$

if $s = -2$: $-20 = E(5)(-1) \Rightarrow \boxed{E = 4}$

$$\therefore 10s = (As+B)(s+1)^2(s+2) - 5(s^2+1)(s+2) + 4(s^2+1)(s+1)$$
$$+ D(s^2+1)(s+1)(s+2)$$
$$= -s^3 + As^4 + s^3B - 6s^2 - s + s^4D + 5As^2 + 5Bs + 4As^3 + 3s^3D + 3s^2D + 3Ds + 2D - 6$$
$$+ 4s^2B + 2As + 2B$$

"s^4"

$A + D = 0$

$A = -D$

"s^3"

$-1 + B + 4A + 3D = 0$

$B - 4D + 3D = 1$

$B - D = 1$ ②

"s^2"

$-6 + 5A + 4B + 3D = 0$

"s"

$-1 + 5B + 2A + 3D = 10$

"constant"

$2B + 2D - 6 = 0$

$B + D = 3$ ⑤

② + ⑤ $\Rightarrow 2D = 4$

$\boxed{D = 2}$

$\therefore \boxed{A = -2}$

from ②

$B - D = 1$

$B - 2 = 1$

$\boxed{B = 3}$

$$y = \frac{10s}{(s^2+1)(s+1)^2(s+2)}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{(s+1)^2} + \frac{D}{s+1} + \frac{E}{s+2}$$

$$\mathcal{L}\{Y\} = \frac{-2s+3}{s^2+1} + \frac{-5}{(s+1)^2} + \frac{2}{s+1} + \frac{4}{s+2}$$

$$Y = -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$+ 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -2\cos t + 3\sin t - 5e^{-t}t + 2e^{-t} + 4e^{-2t}$$

Example: solve the IVP

Reading

$\boxed{3}$ $Y'' - 4Y' + 4Y = 4\cos 2t;\ Y(0) = 2,\ Y'(0) = 5$

$$\mathcal{L}\{Y''\} - 4\mathcal{L}\{Y'\} + 4\mathcal{L}\{Y\} = 4\mathcal{L}\{\cos 2t\}$$

$$\Rightarrow [s^2\mathcal{L}\{Y\} - s^1 Y(0) - s^0 Y'(0)] - 4[s^1\mathcal{L}\{Y\} - s^0 Y(0)]$$

$$+ 4y = 4\left(\frac{s}{s^2+4}\right)$$

$$\Rightarrow s^2y - 2s - 5 - 4sy + 8 + 4y = \frac{4s}{s^2+4}$$

$$\Rightarrow y(s^2 - 4s + 4) = \frac{4s}{s^2+4} + 2s - 3 = \frac{4s + 2s^3 + 8s - 3s^2 - 12}{s^2+4}$$

$$y(s-2)^2 = \frac{4s+2s^3+8s-3s^2-12}{s^2+4}$$

$$y = \frac{2s^3-3s^2+12s-12}{(s^2+4)(s-2)^2}$$

$$\therefore y = \frac{2s^3-3s^2+12s-12}{(s^2+4)(s-2)^2} = \frac{As+B}{s^2+4} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}$$

$$2s^3-3s^2+12s-12 = (As+B)(s-2)^2 + C(s^2+4) + D(s^2+4)(s-2)$$

if $s=2$: $16 = C(8) \Rightarrow \boxed{C=2}$

$$\therefore 2s^3-3s^2+12s-12 = (As+B)(s-2)^2 + 2(s^2+4) + D(s^2+4)(s-2)$$

$$2s^3-3s^2+12s-12 = As^3+Bs^2-4As^2-4Bs+4As+4B+2s^2+8$$
$$+Ds^3-2Ds^2+4sD-8D$$

"s^3"

$2 = A + D$

$A = 2 - D$

"s^2"

$-3 = B - 4A + 2 - 2D$

$B - 4A - 2D = -5$

$B - 4(2-D) - 2D = -5$

$B - 8 + 4D - 2D = -5$

$B + 2D = 3$ — (II)

"s"

$12 = -4B + 4A + 4D$

$4A - 4B + 4D = 12$

$A - B + D = 3$

$A - (-1) + 2 = 3$

$\boxed{A = 0}$

"constant"

$-12 = 4B + 8 - 8D$

$4B - 8D = -20$

$B - 2D = -5$ — (IV)

Solve (II) & (IV)

$\boxed{B=-1}$ $\boxed{D=2}$

$$\therefore y = \frac{0(s)-1}{s^2+4} + \frac{2}{(s-2)^2} + \frac{2}{(s-2)}$$

$$\Rightarrow \mathcal{L}\{Y\} = -\frac{1}{s^2+4} + \frac{2}{(s-2)^2} + \frac{2}{s-2}$$

$$\Rightarrow Y = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\}$$

$$Y = -\frac{1}{2}\sin 2t + 2e^{-2t}t + 2e^{-2t}$$

(Example)

Solve the following system:

$$\left.\begin{aligned} \frac{dx}{dt} + y &= \sin t \\ \frac{dy}{dt} + x &= \cos t \end{aligned}\right\} \text{— (1)} \qquad x(0)=2, \quad y(0)=0$$

let $\frac{dX}{dt} + Y = \sin t$ — (i) $\qquad \frac{dY}{dt} + X = \cos t$ — (ii)

let $\mathcal{L}\{X(t)\} = x(s)$; $\mathcal{L}\{Y(t)\} = y(s) \Rightarrow \mathcal{L}\{Y\} = y$

$\Rightarrow \mathcal{L}\{X\} = x$

(i)

$$X' + Y = \sin t$$

$$\mathcal{L}\{X'\} + \mathcal{L}\{Y\} = \mathcal{L}\{\sin t\}$$

$$[s^1 \mathcal{L}\{X\} - s^0 X(0)] + y = \frac{1}{s^2+1}$$

$$sx - 2 + y = \frac{1}{s^2+1}$$

$$sx + y = \frac{1}{s^2+1} + 2$$

$$sx + y = \frac{1+2s^2+2}{s^2+1} = \frac{2s^2+3}{s^2+1} \quad \text{— (a)}$$

(ii) $Y' + X = \cos t$

$$\mathcal{L}\{Y'\} + \mathcal{L}\{X\} = \mathcal{L}\{\cos t\}$$

$$[s^1 \mathcal{L}\{Y\} - s^0 Y(0)] + x = \frac{s}{s^2+1}$$

$$sy - 0 + x = \frac{s}{s^2+1}$$

$$sy + x = \frac{s}{s^2+1} \quad \text{— (b)}$$

$$\Rightarrow x + sy = \frac{s}{s^2+1}$$

(a) $\times 1 \Rightarrow sx + y = \frac{2s^2+3}{s^2+1}$

(b) $\times s \Rightarrow sx + s^2 y = \frac{s^2}{s^2+1}$

$(-) \quad (-) \quad (-)$

Subtract

$$y(1-s^2) = \frac{2s^2+3-s^2}{s^2+1} = \frac{s^2+3}{s^2+1}$$

$$y = \frac{-(s^2+3)}{(s^2+1)(s^2-1)} \quad , \quad Y = ?$$

$$ⓐ \times s \Rightarrow \quad s^2 x + s y = \frac{2s^3+3s}{s^2+1}$$

$$ⓑ \times 1 \Rightarrow \quad x + s y = \frac{s}{s^2+1}$$

$$(-) \quad (-) \quad (-)$$

Subtract

$$x(s^2-1) = \frac{2s^3+3s-s}{s^2+1}$$

$$x = \frac{2s^3+2s}{(s^2+1)(s^2-1)}$$

$$= \frac{2s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$x = \frac{2s}{s^2-1}, \quad X = ?$$

$\boxed{a:}$

$$y = \frac{-s^2-3}{(s^2+1)(s^2-1)} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)} + \frac{D}{(s-1)} \qquad s^2-1=(s+1)(s-1)$$

$$\Rightarrow -s^2-3 = (As+B)(s^2-1) + C(s^2+1)(s-1) + D(s^2+1)(s+1) \quad \text{(iii)}$$

if $s=1 \Rightarrow -4 = 4D \Rightarrow \boxed{D=-1}$

if $s=-1 \Rightarrow -4 = -4C \Rightarrow \boxed{C=1}$

In eqn (iii)

$$\therefore -s^2-3 = (As+B)(s^2-1) + \overset{C}{1}(s^2+1)(s-1) + \overset{D}{(-1)}(s^2+1)(s+1)$$

$$-s^2-3 = -2s^2 + As^3 - As + s^2B - B - 2$$

"s^2"

$-1 = -2 + B$

$\boxed{B=1}$

"s"

$0 = -A$

$\boxed{A=0}$

$$\therefore y = \frac{1}{s^2+1} + \frac{1}{s+1} - \frac{1}{s-1}$$

$\therefore Y = ?$

$\because\ y = \frac{1}{s^2+1} + \frac{1}{s+1} - \frac{1}{s-1}$ from pg 42

$\therefore\ \mathcal{L}\{Y\} = \mathcal{L}\{\frac{1}{s^2+1} + \frac{1}{s+1} - \frac{1}{s-1}$ $\quad \because\ \mathcal{L}\{Y\} = y$

$$Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$Y = \sin t + e^{-t} - e^{t}$$

b:

$$x = \frac{2s}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow 2s = A(s-1) + B(s+1)$$

if $s = 1 \Rightarrow 2 = B(2) \Rightarrow \boxed{B = 1}$

if $s = -1 \Rightarrow -2 = -2A \Rightarrow \boxed{A = 1}$

$$\therefore\ x = \frac{1}{s+1} + \frac{1}{s-1}$$

$\because\ \mathcal{L}\{X\} = x$

$$\mathcal{L}\{X\} = \frac{1}{s+1} + \frac{1}{s-1}$$

$$X = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^{-t} + e^{t}$$

$\therefore\ X(t) = e^{-t} + e^{t}$

$Y(t) = \sin t + e^{-t} - e^{-t}$