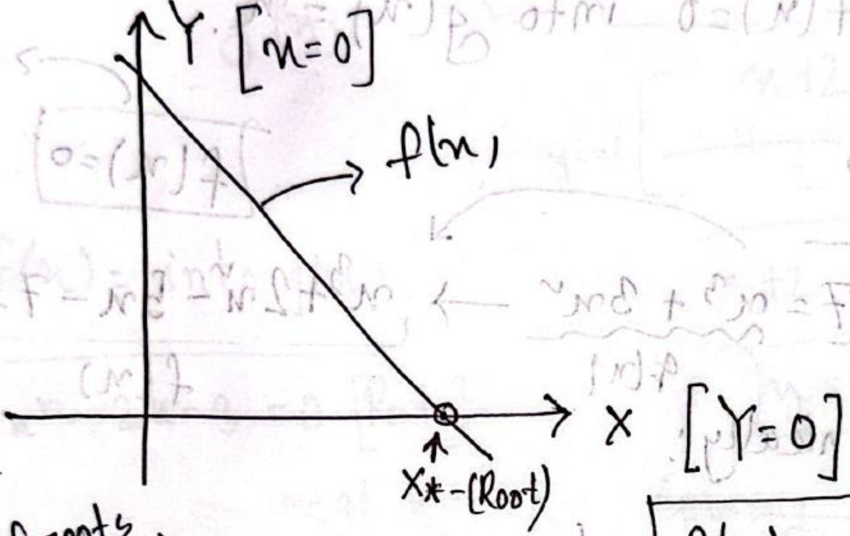
CH-4 → "Non-linear Equations"

4.2 → Fixed Point Iteration

→ finding the Roots of Non-linear Equations



$\boxed{f(x) = 0}$

No. of roots → degree (n)

degree ≡ no. of roots.

$f(x) = x^{2} - 2x = 0$ ← dg.

$\Rightarrow x(x-2) = 0$

$x = 0$ or $x = 2$

→ What if we're dealing with non-linear eqns / with high powers (>4) / Series / Polynomial / Rational functions.

↳ We apply some iterative Algos → Fix point iterations
→ Newton's method

Fixed Point Iteration

→ We want to solve $f(x)=0$

→ Need to transform $f(x)=0$ into a new form

→ Convert $f(x)=0$ into $g(x)=x$.

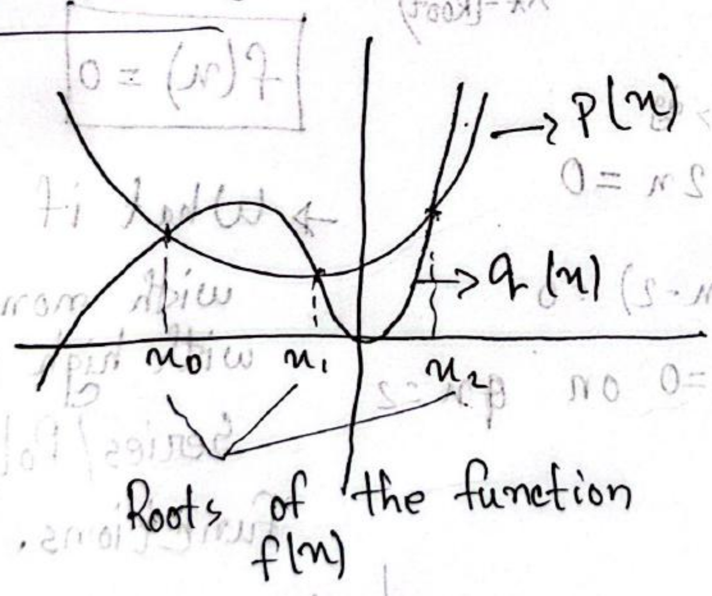
*

Algebra Recap

$$\underbrace{x^2+5x+7}_{p(x)} = \underbrace{x^3+3x^2}_{q(x)} \longrightarrow \underbrace{x^3+2x^2-5x-7=0}_{f(x)}$$

$\boxed{f(x)=0}$ → roots.

Solving Graphically:



→ Intersection of $g(x)=x$ should give the roots of $f(x)=0$

Eg. $\overbrace{\qquad\qquad f(x)}$

$f(x) = -\frac{1}{2}x + 1 = 0$ [$x = 2 \to$ root]

$$\underbrace{\frac{x+2}{2}}_{g(x)} = \underbrace{x}_{x}$$

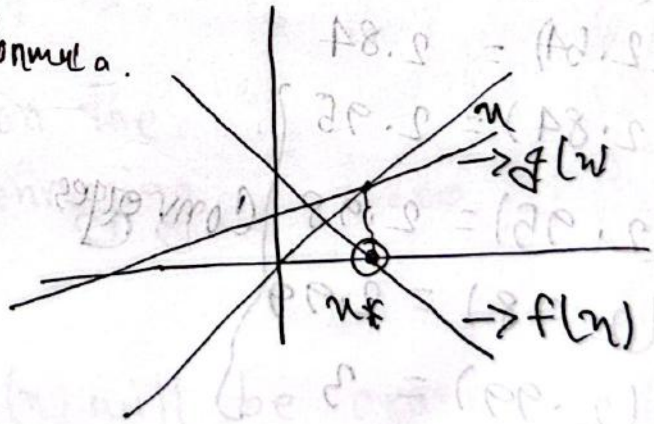
Solving Numerically

$g(x)$ $\left[\frac{x+2}{2} = x\right.$

$x + 2 = 2x$

$\boxed{x = 2}$

Solving Graphically



Convert $f(x)=$ into $g(x)$.

$f(x) = x^2 - 2x - 3 = 0$ [Roots]

are at $x = -1, 3$

[Midterm]

→ Quadratic formula.

$g(x) = x$

Make x the subject

(i) $\underbrace{x^2 - 2x - 3}_{f(x)} = 0$

$x^2 = 2x + 3$

$$\underbrace{x}_{x} = \underbrace{\sqrt{2x+3}}_{g(x)}.$$

② $x^2 - 2x - 3 = 0$

$x^2 - x - x - 3 = 0$

$\Rightarrow \underset{x}{\underbrace{x}} = \underbrace{x^2 - x - 3}_{g(x)}$

$x_* = -1, 3$

Root Finding formula

$k = 0, 1, 2 \ldots \quad g(x_k) = x_{k+1}$

① $g(x) = \sqrt{2x+3}, \quad x_0 = 0$

$g(0) = 1.73$

$g(1.73) = 2.54$

$g(2.54) = 2.84$

$g(2.84) = 2.95$

$g(2.95) = 2.98$ Converges

$g(2.98) = 2.99$

$g(2.99) = 3$

$\boxed{g(3) = 3}$ → fixed Point reached

③ $x^2 - 2x - 3 = 0$

$\Rightarrow -2x = -x^2 + 3$

$(+2x^2) \quad (+2x^2)$

$\Rightarrow (2x^2 - 2x) = 2x^2 - x^2 + 3$

$\Rightarrow 2x^2 - 2x = x^2 + 3$

$\Rightarrow x(2x-2) = x^2 + 3$

$\Rightarrow \underset{g}{\underbrace{x}} = \underbrace{\frac{x^2+3}{2x-2}}_{g(x)}$

② $g(x) = x^2 - x - 3, \quad x_0 = 6$

$g(6) = -3$

$g(-3) = 9$

$g(9) = 69$

$g(x)$ Diverges

(3) $g(x) = \dfrac{x^2+3}{2x-2}$, $x_0 = 0$

$g(0) = -1.5$

$g(-1.5) = -1.05$

$g(-1.05) = -1$

$g(-1) = -1$

converges.

fixed point reached.

2 Qs arises

1. At which root will it converge to?

It will depend on the initial choice of x_0 → the root closer to x_0, will be the fixed point.

It will also depend on the value of the converging rate (λ)

2. Which form of $g(x)$ will be convergent?

it will also depend on the value of the converging Rate. λ

$x = -1, 3$

① $g(x) = \sqrt{2x+3}$, $x_0 = 0$

$\vdots$

$g(3) = 3$

$g(x) = \sqrt{2x+3}$, $x_0 = 42$

$g(42) = 9.33$

$\vdots$

$g(3) = 3$

② $g(x) = x^2 - x - 3$

$g(0) = -3$

$= -9$

69

$\vdots$ Diverges

$g(x) = x^2 - x - 3$, $x_0 = 42$

$g(42) = 1.72 \times 10^3$

$g(1.72 \times 10^3) = 2.95 \times 10^6$

} Diverges.

$x_0 = -1, 3$

③ $g(x) = \dfrac{x^2+3}{2x-2}$ $x_0 = 0$

$g(0) = -1.5$

$\vdots$

$g(-1) = -1$

$g(x) = \dfrac{x^2+3}{2x-2}$, $x_0 = 42$

$g(42) = 21.6$

$g(21.6) = 11.4$

$g(11.4) = 6.39$

$g(6.39) = 4.07$

$g(3) = 3$

→ $x_0 = 0$ converges to the nearest root, which is -1

→ $x_0 = 4$ " " " " , which is 3

Contraction Mapping Theorem

→ find $g'(x)$

→ Find $g'(root_k)$ for each root

if $|g'(root_k)| < 1$, $g(x)$ will be convergent

$$\lambda = |g'(root_k)|$$

↑ converging rate.

$x_* = -1, 3$

① $g(x) = \sqrt{2x+3} = (2x+3)^{\frac{1}{2}}$

$g'(x) = \frac{1}{\cancel{2}}(2x+3)^{-\frac{1}{2}} \cdot \cancel{2}$

$= (2x+3)^{-1/2}$

$g'(x) = \frac{1}{\sqrt{2x+3}}$

$\lambda = |g'(-1)| = 1$ [Not <1] → Diverges.

$\lambda = |g'(3)| = \frac{1}{3}$ [<1] → Converges

∴ Both $x_0 = 0$, $x_0 = 42$ converges to the root $x_* = 3$. ①

2. $g(x) = x^2 - x - 3$

$g'(x) = 2x - 1$

$\lambda = |g'(-1)| = |-3| = 3$. [Not <1] $\rightarrow$ Diverges

$\lambda = |g'(3)| = 5$ [Not <1] $\rightarrow$ Diverges.

$\therefore$ Both $x_0 = 0$, $x_0 = 42$ Diverges.

3. $g(x) = \dfrac{x^2+3}{2x-2}$.

$g'(x) = \dfrac{u}{v}$ ~

$\lambda = |g'(-1)| = 0$ [<1] $\rightarrow$ converges.

$\lambda = |g'(3)| = 0$ [<1] $\rightarrow$ converges.

$x_0 = 0$ converge to Root $x_* = -1$ [Closer].

$x_0 = 42$ " " " $x_* = 3$ [Closer]

Order of Convergence

1. $\lambda = 0$ → Super Linear Convergence
   → fastest convergence
   ↳ Less iterations required to converge.

2. $0 < \lambda < 1$ → Linear Convergence → will converge
   ↳ But not as fast as $\lambda = 0$.

3. $\lambda = 1$ [Fixed Point] → Ignore ✗

4. $\lambda > 1$ ⇒ Divergence.

CH - 4.4

Newton's Method / Newton Raphson Method

$f(x)=0$

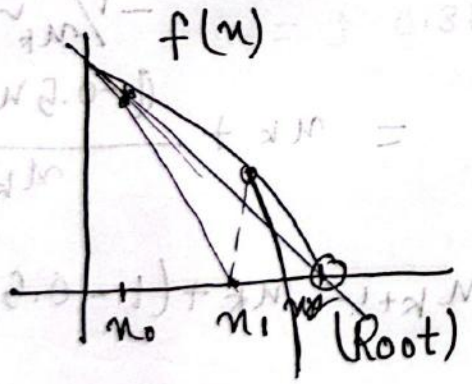
$x=g(x)$

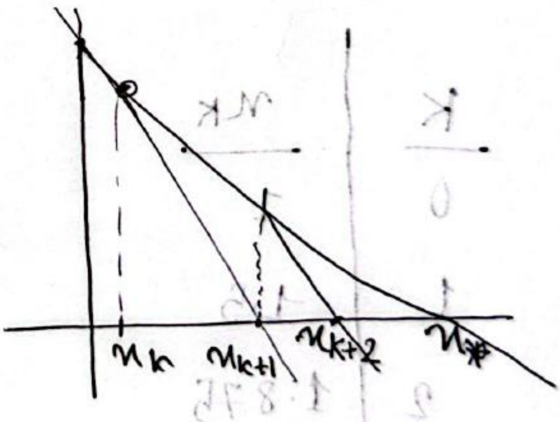
Converging Rate

$\lambda = |g'(x)|$ → Roots

- → $\lambda = 0$ [Super Linear Convergence] → Fast conv.
- → $0 < \lambda < 1$ [Linear Conv.]
- → $\lambda \geq 1$ [Divergence]

Newton's Method → [Super Lin. Con.]





Slope of the tangent line:

$$\text{Slope} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

$$= \frac{-f(x_k)}{x_{k+1} - x_k} \quad \text{— (i)}$$

$$\text{Slope} = f'(x_k) \quad \text{— (ii)}$$

Eq^n (i) & (ii)

$$f'(x_k) = \frac{-f(x_k)}{x_{k+1} - x}$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

$$\underbrace{}_{g(x)}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

E.g.

$$f(x) = \frac{1}{x} - 0.5$$

$x_0 = 1$

K	x_K
0	1
1	1.5
2	1.875
3	1.9921875
4	
⋮	
5	2

$$x_{k+1} = x_k - \frac{\frac{1}{x_k} - 0.5}{\frac{-1}{x_k^2}}$$

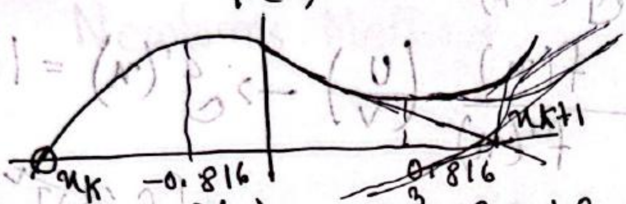
$$= x_k - \frac{\frac{1 - 0.5x_k}{x_k}}{-1/x_k^2}$$

$$= x_k + \frac{(1 - 0.5x_k)x_k^2}{x_k}$$

$$x_{k+1} = x_k + (1 - 0.5x_k)x_k$$

Problem with Newton's Method

Newton's Method will not work if there is a turning point between x_k & x_{k+1}



$f(x) = x^3 - 2x + 2$

Turn. pt.

$x_0 < -0.816$

$f'(x) = 0$

$3x^2 - 2 = 0$

$x = \pm 0.816$

Deriving Sup.Lin.Conv. of Newton's Method

$$\longrightarrow \underbrace{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$$

$$g(x) = x - \frac{f(x)}{f'(x)} \left(\frac{u}{v}\right) \rightarrow g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2}$$

$$= \frac{\cancel{[f'(x)]^2} - \cancel{[f'(x)]^2} + f(x)f''(x)}{[f'(x)]^2}$$

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$\lambda = |g'(x)|$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right) = \frac{f'(x).f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

Root $= x_*$

$$\lambda = |g'(x_*)| = \frac{\cancelto{0}{f(x_*)} f''(x_*)}{[f'(x_*)]^2}$$

$\therefore \lambda = 0$ [Proven]

$\rightarrow g'(x)$

Newton's Method Example

$f(x) = x^2 - 2xe^{-x} + e^{-2x}$, $x_0 = 1$

Q. Find the solution of this function within 10^{-5} → error bound
using Newton's Method.

Note: → within 10^{-5} means the root must be accurate upto 5 decimal points.
[calculate 6 dp, round up to the 5^{th} d.p]

→ What should we compare our answer to?
↳ $f(x)=0$
→ Actual Root, (x_k).

$\boxed{f(x)=0}$ → Reach our answer when $|f(x_k)| < 0.0001$

→ Actual Root, x_* → Reach our final Ans.
$|x_* - x_k| < 0.00001$

→ we compare to $f(x) = 0$, when x_* (Root) is not given or cannot be found numerically.

NM

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Soln

$$f(x) = x^2 - 2xe^{-x} + e^{-2x}$$

$$f'(x) = 2x - [2x \cdot (-1) \cdot e^{-x} + e^{-x}(2)] + (-2) \cdot e^{-2x}$$

$$= 2x - 2x(-e^{-x}) - 2e^{-2x}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{x_k^2 - 2x_k e^{-x_k} + e^{-2x_k}}{2x_k - 2x_k(-e^{-x_k}) - 2e^{-2x_k}}$$

Table

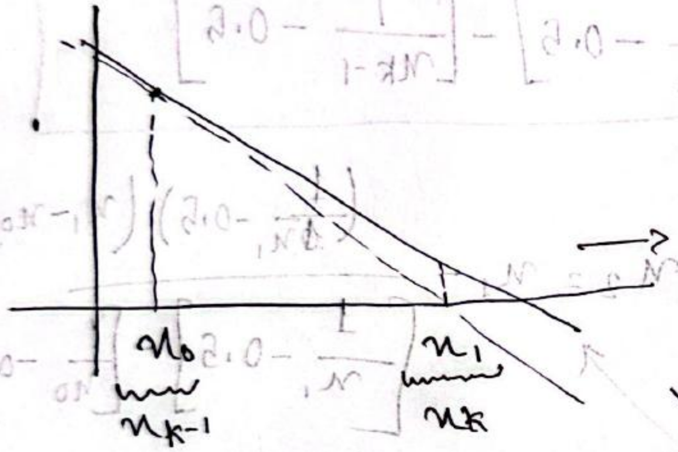
k	x_k	$f(x_k)$	if $\lvert f(x_k)\rvert < 0.00001$? 1×10^{-5}
0	1 f(1) Put(1) init(1)	0.399576	No
1	0.768941	0.093292	No
2	0.664590	0.022532	No
⋮			
86	0.573041	0.000085	No
7	0.570089	0.000020	No
8	0.568615	0.5×10^{-5}	Yes

Answer: $x_p = 0.56861$

CH- 4.7 [Quasi-Newton Method / Second Method]

Newton's Method $\rightarrow x_{k+1} = x_k - \frac{f(x_k)}{\underbrace{f'(x_k)}}$

$\rightarrow$ Replace $f'(x_k)$ with Backward differentiation



$\rightarrow$ Backward diff

$$\frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f'(x_k) = \left[\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right] \text{N.M.}$$

Iteration formula for Second Method

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}}$$

Note: We need 2 starting points for the second method (x_0 & x_1)

Advantage: $f'(x)=0$ হলেও কাজ করবে } No worries about turning point & No diff techniques.

Example:

$$f(x) = \frac{1}{x} - 0.5 \qquad x_0 = 0.25,\ x_1 = 0.5$$

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$\boxed{x_{k+1} = x_k - \frac{\left(\frac{1}{x_k} - 0.5\right)(x_k - x_{k-1})}{\left[\frac{1}{x_k} - 0.5\right] - \left[\frac{1}{x_{k-1}} - 0.5\right]}}$$

K	x_k
0	0.25
1	0.5
start → 2	0.687500
3	1.015620
4	1.35400
5	
⋮	
12	$\boxed{2.000000}$ Ans

$$x_2 = x_1 - \frac{\left(\frac{1}{x_1} - 0.5\right)(x_1 - x_0)}{\left[\frac{1}{x_1} - 0.5\right] - \left[\frac{1}{x_0} - 0.5\right]}$$

$$x_3 = x_2 - \frac{\left(\frac{1}{x_2} - 0.5\right)(x_2 - x_1)}{\left[\frac{1}{x_2} - 0.5\right] - \left[\frac{1}{x_1} - 0.5\right]}$$

$x_4 =$

Second Meth. Adv. over Newton's M:

→ We don't need to find the $f'(x)$

→ Calculations are simple.

Disadv.

→ 2 Initial Points.

CH-5 "Linear Equations"

→ Powers of variables → $\boxed{0, 1}$

5.1

e.g. $x + y = 5$ —— (i)

$x - 2y = 10$ —— (ii)

→ Linear System / System of Linear equations

Rule

No. of variables = No. of equations.

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Matrix Inverse Method

System:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots a_{mn}x_n = b_m$$

det(A) ≠ 0

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}}_{[n \times n]} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

$$A \times x = b$$

$$\boxed{Ax = b.}$$

$$x = A^{-1}b$$

Matrix Inverse Method

det(A) ≠ 0

→ A → Non-Singular Matrix

Scanned with CamScanner

A → Triangular Form / Matrix.

5.1 → "Triangular Forms"

Triangular Matrix

Lower Triangular (L) ← → Upper Triangular (U)

Lower Triangular → $L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$

a_{ij}

$i \to$ Row #

$j \to$ Col #

$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$

$$\begin{array}{ccc} n \times n & n \times 1 & \\ L = \begin{bmatrix} l_{11} & 0 & 0 \ldots 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} \ldots 0 \\ l_{n1} & l_{n2} & l_{n3} \ldots l_{nn} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = & \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} \end{array}$$

*det(L)/det(U) = Product of diagonal elements

Coeff Matrix

$L \cdot x = b$ [1 div]

$l_{11} x_1 = b_1 \longrightarrow x_1 = \frac{b_1}{l_{11}}$

[1 div + (1s + 1m)] $l_{21} x_1 + l_{22} x_2 = b_2 \longrightarrow x_2 = \frac{b_2 - l_{21} x_1}{l_{22}}$

[1 div + 2(1s + 1m)] $l_{31} x_1 + l_{32} x_2 + l_{33} x_3 = b_3 \to x_3 = \frac{b_3 - l_{31} x_1 - l_{32} x_2}{l_{33}}$

x_n - [1 div + (n - 1) {s + m}] $l_{n1} x_1 + l_{n2} x_2 + l_{n3} x_3 + l_{nn} x_n = b_n$

$\longrightarrow x_n = \frac{b_n - l_{n1} x_1 - l_{n2} x_2 - \ldots l_{n(n-1)} x_{n-1}}{l_{nn}}$

Forward Substitution

$(j = 1, 2, \ldots n)$

$\therefore x_n = \frac{b_n}{l_{nn}}$

$$\boxed{x_j = \frac{b_j - \sum_{k=1}^{j-1} l_{jk} \cdot x_k}{l_{jj}}}$$

$U \to$ Derive

x_j

Backward Substitution

x_n

$$\rightarrow \sum_{j=1}^{n} (1 + 2(j-1))$$

$s+m$

$$\sum_{j=1}^{n} (1+2j-2) = (2j-1)$$

$$\Sigma j = 1+2+3 \ldots + n. = n(n+1)$$

$$= 2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1$$

$$n(n+1) - n \quad \boxed{O(n^2)}$$

CH - 5.2

"Gaussian Elimination"

$\boxed{Ax=b}$

→ A technique which transforms matrix A into a triangular form (Upper or Lower)

→ Solves $Ax=b$ without finding the inverse

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & 0 \\ l_{31} & l_{32} & \cdots & l_{nn} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

~~→ Convert~~ $\boxed{Lx=b}$ Forward substitution $\boxed{Ux=b}$ Backward substitution

Steps:

① Convert A to its upper Triangular form (U). [Elementary Row Operations]

② Apply Backward Substitution to find the variables.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xRightarrow{1^{st}\ row} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

$R_2 \to a_{21} \to 0$

$R_2' = R_2 - \boxed{} R_1$

1st row operation. Diagonal elements → Denominator $\neq 0$

m_{21}

$* \; R_2' = R_2 - \boxed{\dfrac{a_{21}}{a_{11}}} R_1$

$* \; R_3' = R_3 - \boxed{\dfrac{a_{31}}{a_{11}}} R_1$

M_{31} → Row Multiplier

(M_{**}) → Denotation.

$R_2' = R_2 - \boxed{} R_1$ → $\dfrac{a_{21}}{a_{11}}$

$R_3' = R_3 - \boxed{} R_1$ → $\dfrac{a_{31}}{a_{11}}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

$a_{32}' \to 0$

2nd row operation ⟹

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix}$$

$R_3' = R_3 - \boxed{} R_2$ → $\dfrac{a_{32}}{a_{22}'}$ → M_{32}

if any diagonal $= 0$ at any stage then we use "Pivoting"

Final $A \to U$

No. of variables $= 3 \; (n)$

of Row operations $= 2 \; (n-1)$

$x_1 + 2x_2 = 2$

$2x_1 - 3x_2 = 4$

$Ax = b$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{b}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$A = [\quad]$ $\quad b = [\quad]$

change

→ A & b merge

Augmented matrix → $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 4 \end{array}\right]$

A → যদি dim (n) হয় তবে,

Aug. Mat. $(n \times n+1)$ Dim → (2×3)

E.g.-1

$$x_1 + 2x_2 + x_3 = 0$$

$$x_1 - 2x_2 + 2x_3 = 4$$

$n=3$ $\quad 2x_1 + 12x_2 - 2x_3 = 4$

$a_{22} = -2 - \left(\frac{1}{1}\right)2$

$a_{22} = -2-2$
$= -4$

Augmented Matrix →

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 2 & 12 & -2 & 4 \end{array}\right] \xrightarrow[R_3' = R_3 - \left(\frac{2}{1}\right)R_1]{R_2' = R_2 - \left(\frac{1}{1}\right)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 8 & -4 & 4 \end{array}\right]$$

2nd row operation.

$\Longrightarrow$

$R_3' = R_3 - \left(\frac{8}{-4}\right)R_2$

$a_{23} = 2 - \left(\frac{1}{1}\right) \cdot 1$
$= 1$

$a_{24} = 4 - 1 \cdot 0$

$a_{32} = 12 - \left(\frac{2}{1}\right)2 = 8$

$a_{33} = -2 - 2 \cdot 1$

$a_{34} = 4 - 2 \cdot 0$
$= 4$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -4 & 1 & 4 \\ 0 & 0 & -2 & 12 \end{array}\right]$$

$a_{33}'' = -4 - \left(\frac{8}{-4}\right) \cdot 1$

$= \cancel{4} - 2$

$a_{34} = 4 - \left(\frac{8}{-4}\right)4$

Step-2 : Backward Substit.

$$Ax = b$$

$$\cancel{Ux = b}$$

$$Ux = y$$

$$\begin{array}{c} \text{iii} \\ \text{ii} \\ \text{i} \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

(i)	(ii)	(iii)
$-2 \cdot x_3 = 12$	$-4x_2 + x_3 = 4$	$x_1 + 2x_2 + x_3 = 0$
$\boxed{x_3 = -6}$	$\boxed{\Rightarrow x_2 = -2.5}$	$\boxed{\Rightarrow x_1 = 11}$

~~370, 422, 423, 495~~

24.04.25

PIVOTING

$\boxed{a_{11}; a_{22} \neq, a_{33} = 0}$ → pivot points

$a_0 x_0 + a_1 x_1 + a_2 x_2$

$a_1 x_1 + a_0 x_0$

Sol^n Row/col swap

$m_{21} = \frac{2}{0}$ undefined

$$\underbrace{\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

Swap $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

CH - 5.3

24.04.25

LU Decomposition

$Ax = b$

$[A | b]$

Problem with Gaussian Elimination

→ Dependent on the constant Matrix [b]

→ If any value(s) of b-matrix is changed, need to start from first iteration again.

$A \xrightarrow{\text{breakdown}} LU$

$Ax = b$

(i) $LUx = b$

(ii) let, $Ux = y$ —— (1)

(iii) $Ly = b$ —— (II)

find y

(iv) → Put y into eqn (1) & find x

Step-1

$A \longrightarrow LU$

26.04.25

LU Decomposition

→ Q. Determine if matrix-A is UNIQUE or not.

Ans. if determinant, $det(A) == 0$, then there will be No UNIQUE Sol^n.

→ PIVOTING of mat(A).

→ Diagonal element(s) == 0, then need to do PIVOTING.

Sol^n Row / Col → Swap

Mat(A) & Mat(b) changes.

Col Swap

Mat(A) & mat(x) changes.

Col swap

$$
\begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \quad A \qquad x \qquad b
$$

$$
\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

$$
\begin{array}{ccc} & x_2 & x_3 \end{array}
$$

$$
= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
$$

$$Ax = b \quad \text{--- (i)}$$

Step-1

$$A \longrightarrow LU$$

Step-2

let, $LUx = b$ --- (ii)

let, $Ux = y$ --- (iii)

$\Rightarrow Ly = b$

$\rightarrow$ Find y

Step-3

Input y to eqn (iii)

find x

E.g.

$n \to$ no. of variables

$x_1 + 2x_2 + x_3 = 0$

$x_1 - 2x_2 + 2x_3 = 4$

$2x_1 + 12x_2 - 2x_3 = 4$

Constructing L & U from A

$F^{(n)}$ matrix
-> Frobenius matrix
-> Matrix that contains all.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

Negative Multipliers on the position of the multiplier

step start

(i) $A^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ \boxed{1} & -2 & 2 \\ \boxed{2} & 12 & -2 \end{bmatrix}$ $M_{21} = \frac{1}{1}$ $M_{31} = \frac{2}{1}$ 1st row operation

-> Diagonal elements are all 1

-> has the same shape / dim of A

-> Rest elements are all 0.

(ii) $F^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(iii) $A^{(2)} = F^{(1)} \times A^{(1)}$ -> Matrix Cross Product.

[Calculator]

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & 12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix}$$ $M_{21} = \frac{0}{1} = 0$, $M_{31} = 0$

Check

$A \to LU$

LXU

$$F^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad M_{32} = \frac{8}{\cancel{7}4}$$

$$A^3 = F^{(2)} \times A^{(2)} \qquad \frac{n}{U} = F^{(n-1)} A^{(n-1)} = A^n$$

$$U = A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 8 & -4 \end{bmatrix} \qquad n = 3$$

$$U = F^n A^n$$

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad \begin{aligned} l_{21} &= m_{21} \\ l_{31} &= m_{31} \\ l_{32} &= m_{32} \\ l_{nn} &= 1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Step-1 $Ax = b$

$LUx = b$ — (I)

Step-2

Let, $Ux = y$ — (II)

Solve, $Ly = b$ — (III) → Forward Substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$y_1 = 0$

$y_1 + y_2 = 4$

$y_2 = 4$

$$y = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$\rightarrow 2y_1 - 2y_2 + y_3 = 4$

$\boxed{y_3 = 12}$

Step-3

Solve, $Ux = y$ → Backward

$x_1 + 2x_2 + x_3 = 0$

$x_1 = 11$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 12 \end{bmatrix}$$

$-2x_3 = 12$

$$\boxed{x_3 = -6}$$

$-4x_2 + x_3 = 4$

$$\boxed{x_2 = -2.5}$$

Ch - 6.1

"Least Squares Approximation"

/ Discrete Least Sq "

e.g.

#eqⁿ = #unknowns, perfectly determined system

$x_1 + 2x_2 + x_3 = 0$

$x_1 - 9x_2 + 7x_3 = 2$

$2x_1 + 3x_2 + 5x_3 = 5$

$$A \quad x = b$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 2 & 3 & 5 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}}_{n\times 1}$$

3×3 square matrix $n \times n$

n = # No of unknowns → $n = 3$

Final matrix

$n \times n \quad n \times 1 \rightarrow n \times 1$

→ If we have system where no. of eqⁿs > no. of variables.

then: it is called

→ "Over-determined system"

→ How do we solve these OD systems.

→ Least Sq. Appr.

→ QR-Decomposition

e.g.

$n = 3$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 9x_2 + 7x_3 = 2$$
$$x_1 + 3x_2 + 5x_3 = 2$$
$$2x_1 + 11x_2 - 9x_3 = 5$$
$$9x_1 + x_2 - x_3 = 7$$

$$\longrightarrow \underset{\substack{5\times 3 \\ m\times n}}{\overset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 1 & -9 & 7 \\ 1 & 3 & 5 \\ 2 & 11 & -9 \\ 9 & 1 & -1 \end{bmatrix}}} \underset{\substack{3\times 1 \\ n\times 1}}{\overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \underset{\substack{5\times 1 \\ m\times 1}}{\overset{b}{\begin{bmatrix} 0 \\ 2 \\ 2 \\ 5 \\ 7 \end{bmatrix}}}$$

$m\times n \longleftarrow 5\times 3$

$n = 3$, $m = 5$ $\rightarrow$ $\boxed{m > n}$ $\rightarrow$ O-D system

$\rightarrow$ Least Sq. Appn. $\rightarrow$ a way to find an approximate solution of an O-D system.

O.D

$$\underset{(m\times n)}{A} \cdot \underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

Row $\rightarrow$ col
col $\rightarrow$ row

$A = (m \times n)$

A-transpose $\longleftarrow$ $A^T = (n \times m)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

$OD \rightarrow A\ x = b$ ——— ①

Soln [LSA method]

→ multiply by A^T on both sides of eqn - ①

$$A x = b$$

$$A^T.(A.x) = A^T(b)$$

$$\rightarrow \underbrace{\underbrace{A^T}_{n\times m}.\underbrace{A}_{m\times n}}_{=(n\times n)}.\underbrace{x}_{(n\times 1)} = \underbrace{\underbrace{A^T}_{n\times m}.b}_{m\times 1} \rightarrow n\times 1$$

Apply GE/ LUD/ Matrix Inv. to solve this system

Example

From the Polynomial Chapter

If we had $(n+1)$ nodes, we calculated the values of $(n+1)$ coefficients.

$x_0, x_1 \dots x_n$ $\quad [x_0, f(x_0)], (x_1, f(x_1))$ $\quad a_0, a_1 \dots a_n$

Using Vandermonde matrix

Well defined system $\rightarrow$

$$\underbrace{\begin{bmatrix} 1 & x_0^1 & x_0^2 & \dots & x_0^n \\ 1 & x_1^1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \\ 1 & x_n^1 & x_n^2 & \dots & x_n^n \end{bmatrix}}_{V\,(n \times n)} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\substack{a \\ (n \times 1)}} = \underbrace{\begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}}_{\substack{b \\ (n \times 1)}}$$

But now, let's say,

we have $(m+1)$ nodes, & we need to calculate $(n+1)$ coeff. $\quad m > n$

$x_0, x_1, \dots x_n \dots x_m$ $\quad a_0, a_1 \dots a_n$

$$\begin{bmatrix} 1 & x_0^1 & x_0^2 & \cdots & x_0^n \\ 1 & x_1^1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_m^1 & x_m^2 & \cdots & x_m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{bmatrix}$$

O-D System

$(m \times n)$ $\quad (n \times 1) \longrightarrow (m \times 1)$

$P_1(x) = \underbrace{a_0 + a_1 x}_{2 \text{ coeffs}}$ $\quad a_0 \to a_n$ 2 coeffs

degree = 1

$\boxed{\text{E.g}}$

We want to fit a straight line through the following nodes.

$x_0 = 3$ $\quad x_1 = 0$ $\quad x_2 = 6$

$f(x_0) = 0$ $\quad f(x_1) = 0$ $\quad f(x_2) = 2$

no. of nodes = 3

$\longrightarrow P_1(x_0) = f(x_0) \longrightarrow a_0 + a_1(x_0) = f(x_0) \rightarrow a_0 + a_1(-3) = 0$

$\longrightarrow P_1(x_1) = f(x_1) \longrightarrow a_0 + a_1(x_1) = f(x_1) \rightarrow a_0 + a_1(0) = 0$

$\longrightarrow P_1(x_2) = f(x_2) \longrightarrow a_0 + a_1(x_2) = f(x_2) \rightarrow a_0 + a_1(6) = 2$

a. Matrix Form $\longrightarrow$

$$\underbrace{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{b}$$

$m > n$; $3 > 2$ O-D system

3×2

6. Is it an O-D system or not?

→ Yes, ~~Rows >~~ the number of rows of A is greater than the number of column of b.

$m > n$.

Or, not a sq. matrix $(m \times n)$

6. Apply LSA to find the unknowns.

St-1 Multiply by A^T on both sides.

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix}_{2\times 3}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} 10 & 1 & -17 \\ 1 & 1 & 1 \\ -17 & 1 & 37 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} =}$$

well def. ← $$= \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 2 \\ 12 \end{bmatrix}}$$

Now apply GE / LU / Mat Inv. to find out the values of a_0 & a_1

Applying Inverse method

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 5/21 \end{bmatrix}$$

$$\therefore P_1(x) = a_0 + a_1(x)$$

$$\boxed{P_1(x) = \frac{3}{7} + \frac{5}{21}x}$$

Ch- 6.3

"Q R Decomposition"

Orthogonality

→ We need to understand vector Dot Product / Inner Product.

$\vec{a}.\vec{b}$ = scalar vector

→ Vector Dot Product returns a Scalar value. (an.)

2 types of notations

$\vec{x_0} = 1\hat{i} + 2\hat{j} + 3\hat{k}$ → Matrix Notation → $x^T.y$

→ Vector Notation → $\vec{x}.\vec{y}$

Matrix notation.

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

① Matrix Notation : $x^T.y$.

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Dim (1×3) (3×1)

1×1

= a single value.

$= [4+10+18]$

$= [32]$

Matrix $= x^T y$
Vector $= \vec{x}.\vec{y}$

2. Vector Notation : $\vec{x}.\vec{y}$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$= (1\hat{i}+2\hat{j}+3\hat{k}).(4\hat{i}+5\hat{j}+6\hat{k})$$

$$= (1\times4)+(2\times5)+(3\times6)$$

$$= 32$$

8. Length/Magnitude/Norm of a vector

$$a = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$

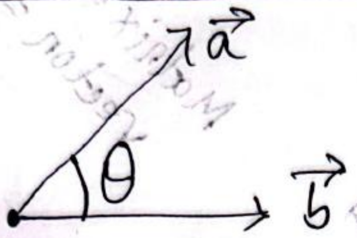
$$|a| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$$

Dot Prod

$$a = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \quad b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$ab = a_1b_1+a_2b_2+a_3b_3$$

Dot Product (Second Approach) :



vector meet করলে

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,.\,|\vec{b}|\,.\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|}$$

Ø Orthogonal Vectors

if the angle ($\angle$) between 2 vectors $= 90° / \pi/2$, then orthogonal vectors.

→ In other word, if $v_1 \perp v_2$, then v_1 & v_2 are orthogonal.

For orthogonal vectors, dot product is, dot product is 0

$$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\,\overset{0}{\cancel{\cos(\pi/2)}}$$

$$\vec{a}.\vec{b} = 0$$

orthogonality check করতে বললে, dot product 0 আসলে, they are orthogonal.

OR:

$a^T . b = 0$ [Matrix Notation]

Set of vectors

$$s = \{\vec{a}, \vec{b}, \vec{c}\}$$

when do I call the set - S an orthogonal set

→ when $\vec{a} . \vec{b} = 0$

$\vec{b} . \vec{c} = 0$

$\vec{c} . \vec{a} = 0$.

each vector is ⊥ to each other.

Orthonormality

Conditions

orthogonal + Normal

① the vectors are orthogonal (dot product = 0)

② the length of each vector = 1 (Unit vector) / norm

Then the vectors are orthonormal.

Eg. $\vec{a} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

Convert this to orthonormal:

① Check for orthogonality:

$$\vec{a} . \vec{b} = (4 \times 1) + (2 \times -3) + (1 \times 2)$$

$$= 4 - 6 + 2$$

$$= 0 \quad [\therefore \text{orthogonal}$$

Convert to orthonormal $\sqrt{4^2+2^2+1^2}$ Unit Vect

Normalize the vector

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix} = \frac{\text{Vector}}{\text{Magnitude of vect}}$

$|\hat{a}| = \sqrt{(4/21)^2 + (2\sqrt{21})^2 + (1/\sqrt{21})^2}$

$|\hat{a}| = 1$

$\hat{b} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2\sqrt{14} \end{bmatrix}$

$|\hat{b}| =$

Example Consider the set of vector, s:

$s = \left\{ \frac{1}{\sqrt{5}}(2,1)^T, \frac{1}{\sqrt{5}}(1,-2)^T \right\}$

Q. Show if the set S is Orthonormal or not

Sol^n $S = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

$S = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \right\}$

$\quad u_1 \qquad\qquad u_2$

(1) Orthogonality:

$\vec{u_1} \cdot \vec{u_2} = \left(2/\sqrt{5} \times \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} \times \frac{-2}{\sqrt{5}}\right) = 0$

$\Rightarrow \vec{u_1} \cdot \vec{u_2} = 0$ [Orthogonal Proved]

(ii) Normalized ?

$$|\vec{u_1}| = \sqrt{(2\sqrt{5})^2 + (1/\sqrt{5})^2} = 1$$

$$|\vec{u_2}| = \sqrt{(1/\sqrt{5})^2 + (-2/\sqrt{5})^2} = 1$$

[Normalized]

$\therefore \vec{u_1}$ & $\vec{u_2}$ are orthonormal

Therefore Set S is orthonormal.

Orthonormal Matrix :

$$S = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Every col orthonormal

→ Whole Matrix - S is orthonormal

→ for a square orthonormal Matrix

Q ↳ orthonormal Matrix. → (1) The inverse is simply its Transpose.

Properties :

#(i) $Q^{-1} = Q^T$

$Q.Q^T = I$

$Q^T.Q = I$

QR Decomposition:

O-D system $\quad Ax = b$

$A \rightarrow (m \times n)$, $x \rightarrow (n \times 1)$, $b \rightarrow (m \times 1)$

$$A = QR$$

where, Q is a $(m \times n)$ matrix with orthonormal columns.

R is an upper Tri-angular Matrix (U) of shape $(n \times n)$

$\boxed{Q \cdot Q^T = I}$, $\boxed{Q^T \cdot Q = I}$

$\rightarrow \boxed{A = Q \cdot R} \rightarrow$ ① eq^n

A: $(m \times n)$; Q: orthonormal matrix $(m \times n)$; R: $(n \times n)$

→ Multiply by Q^T on both side of eq-①

$$Q^T \cdot A = \boxed{Q^T \cdot Q \cdot R}$$

$IA = A$

$\underbrace{Q^T \cdot Q \cdot R}_{IR}$

$\Rightarrow Q^T A = R$

$\boxed{R = Q^T \cdot A} \rightarrow U$

Q → orthonormal

03.05.25

CH-6.3

"QR Decomposition"

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

u_1 u_2

$Q^T = Q^{-1}$

G-S Process

$$P_K = u_K - \sum_{i=0}^{K-1} (u_K^T q_i) q_i$$

$A = QR$.

How to find out Q.

Steps — Gram-Schmidth Process

(i) Matrix A -> Orthogonal, P

(ii) Matrix P - Nomalized Form. Q.

$A^{-1}A = I \quad m > n$

$Q^T \cdot Q$

$\cancel{I} \to I$

Orthogonal vectors $\to \vec{P_1}$, $\to \vec{P_2}$

$\vec{u_1} \to \vec{p_1} \to \vec{q_1}$

$\vec{u_2} \to \vec{p_2} \to \vec{q_2}$

$$p_1 = u_1 \cancel{\sum_{i=0}^{0} (u_1^T q_i) q_i} \to 0$$

$p_1 = u_1 \longrightarrow p_1$ will always be u_1

$$q_k = \frac{p_k}{|p_k|}$$

$$p_2 = u_2 - \sum_{i=1} (u_2^T q_i) q_i$$

$$= u_2 - (u_2^T q_1) q_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 45/\sqrt{45} \\ 90/\sqrt{45} \\ 0 \end{bmatrix}$$

$u_2^T q_1$

$$\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3/\sqrt{45} \\ 6/\sqrt{45} \\ 0 \end{bmatrix}$$

$$= 3/\sqrt{45} + 12/\sqrt{45}$$

$$u_2^T q_1 = 15/\sqrt{45}$$

$$q_1 = \frac{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}{\sqrt{3^2+6^2+0^2}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 3/\sqrt{45} \\ 6/\sqrt{45} \\ 0 \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|}$$

$$= \frac{\begin{bmatrix}0\\0\\2\end{bmatrix}}{\sqrt{2^2}}$$

$$q_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

~~370~~
422
~~260~~
423

$$Q = [q_1, q_2]$$

$$Q = \begin{array}{c}\begin{bmatrix} 3/\sqrt{45} & 0 \\ 6/\sqrt{45} & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{matrix} q_1 & \quad q_2 \end{matrix}\end{array}$$

*Q. No. of func^n.

$$R = \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ 0 & a_2^T q_2 \end{bmatrix}$$
$[2 \times 2]$

$$R = Q^T A$$

$$= \underset{\boxed{2} \times 3}{\begin{bmatrix} 3/\sqrt{45} & 6/\sqrt{45} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \times \underset{3 \times \boxed{2}}{\begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}$$

Alternate

Step-1 $A = Q \cdot R$

$= (3 \times 2) \times (2 \times 2)$

$= (3 \times 2)$

Step-2 $Ax = b$

Least Square Approximation. → Multiply both side with A^T

$\Rightarrow A^T . Ax = A^T b$

$\Rightarrow (QR)^T . (QR)x = (QR)^T b$

$\Rightarrow \cancel{Q^T Q} . \cancel{R^T} . R x = Q^T . \cancel{R^T} . b$

$Rx = Q^T . b$ [Backward Substitution]

$\Rightarrow x = (2 \times 3)(3 \times 1)$

$= 2 \times 1$

Ch. 7.1

"Numerical Integration"

Actual

$I = \int_a^b f(x)\,dx \quad P_n(x)$ [Lagrange form]

$[a,b] \to$ Interval

$I_n = \int_a^b P_n(x)\,dx \quad P_n(x) = \sum_{k=0}^{n} l_k(x) f(x_k)$

Numerical

weight function/factor

$I_n = \int_a^b \sum_{k=0}^{n} l_k(x) f(x_k)\,dx \qquad \sigma_k$

sigmaid

$$\boxed{I_n(x) = \sum_{k=0}^{n} \sigma_k f(x_k)}$$

$n \Rightarrow$ degree.

$(n+1)$ nodes

* If the nodes are equidistant, then we call $I_n(x) \to$ Newton's Cotes Formula.

Newton's Cotes formula

Closed → Open

Closed Newton's Cotes Formula $[a,b]$

$x_0, x_1, x_2 \dots x_n$

$x_0 = a \qquad x_n = b$

$\text{depth},\ h = \frac{b-a}{n}$

Open Newton's Cotes Formula

$[a,b]$

$\underbrace{a, x_0}_{h}, x_1, x_2 \; \underbrace{x_n, b}_{h}$

$x_0 = a + h$

$$\boxed{h = \frac{b-a}{n+2}}$$

$x_1 = b - h$

7.2 Trapezium Rule

Closed Newton's Cotes Formula

Derive → with $n = 1$

* nodes = 2

$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$

$x_0 \to a$, $x_1 \to b$

$l_0(x) = \frac{x - x_1}{x_0 - x_1}$ $\quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$

$h = \frac{b-a}{n} = \frac{b-a}{1(n)}$

$\boxed{\therefore h = b - a}$

sigmoid,

$\sigma_k = \int_a^b l_k(x)\, dx$

$$\sigma_0 = \int_a^b \frac{x - x_1}{x_0 - x_1}\, dx = \frac{1}{a-b}$$

$$= \frac{1}{a-b} \int_a^b (x - b)\, dx$$

$$= \frac{1}{a-b} \left[\frac{1}{2}x^2 - bx\right]_a^b$$

$$\sigma_0 = \frac{b-a}{2}$$

$$\sigma_1 = \int_a^b \frac{x - x_0}{x_1 - x_0}\, dx$$

$$= \frac{1}{b-a} \int_a^b (x-a)\, dx$$

$$= \frac{1}{b-a}\left[\frac{1}{2}x^2 - ax\right]_a^b$$

$$\sigma_1 = \frac{b-a}{2}$$

$$I_n = \sum_{k=0}^{1} \sigma_k f(x_k)$$

$$I_1(x) = \sigma_0 f(x_0) + \sigma_1 f(x_1)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

$$\therefore I_1(x) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

E.g. $\overset{a\ b}{[0, 2]}$ & $f(x) = e^x$

Q. a. Actual Integration of $f(x)$

f/w

b. Numerical Integration using Trapezium Rule

c. Percentage Error

Soln

a. $\int_0^2 e^x \, dx$

$= [e^x]_0^2 = 6.3891$

b. $I_1(x) = \frac{2-0}{2}(e^0 + e^2)$

$= (e^0 + e^2)$

$= 8.3891$

c. $\% \text{ Err.} = \left| \frac{\text{Diff.}}{\text{Actual}} \right| \times 100\%$

$= \left| \frac{8.3891 - 6.3891}{6.3891} \right| \times 100\%$

$= 31.65\%$

Cauchy's Theorem

Normal $p_n(x)$

$$\text{Upper Bound of error} = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|_{\max \to \xi \in (a,b)} \overbrace{(x-x_0)(x-x_1)\ldots(x-x_n)}^{W(x)}$$

Integration

$$\text{Integral } I_n(x) \; u = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|_{\max \to \xi \in (a,b)} \int_a^b (x-x_0)(x-x_1)\ldots(x-x_n)$$

E.g. $f(x) = e^x \quad [\overset{a}{0}, \overset{b}{2}]$

(a) $\text{Upp. Bound} = \left| \dfrac{f^{(2)}(\xi)}{2!} \right|$

$$= \left| \frac{e^2}{2!} \right|$$

$$W(x) = (x-0)(x-2) \longrightarrow \int_0^2 w(x)\,dx$$

$$w(x) = x^2 - 2x$$

$$= \int_0^2 (x^2 - 2x)\,dx$$

$$= \left[\frac{1}{3} x^3 - x^2 \right]_0^2 = \frac{4}{3}$$

$$\text{Upper Bound} = \frac{e^x}{2} \times \frac{4}{3}$$

7.3

→ Composite Newton's Cotes Formula

→ This method ~~in~~ improves the result without increasing the actual node numbers.

$$[a, b] \xrightarrow{\text{sub-interval}} [a, c_1] \cup [c_1, c_2] \cup [c_2, b]$$

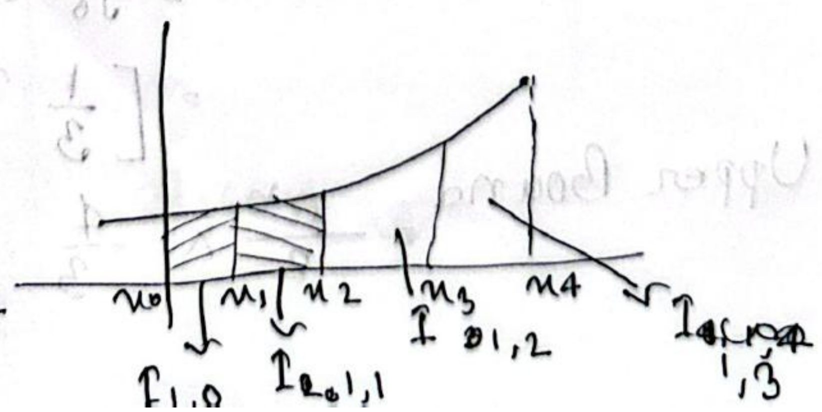
$[a, c_1]$		$[c_1, c_2]$		$[c_2, b]$
Apply Trap. rule	+	Apply Trap. rule	+	Apply Trap. rule

→ Divide the whole interval into m equal subintervals

→ Integration is the sum of all sub-integrations.

$$h = \frac{b-a}{m} \qquad I_1(h) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$m = 4$



$$I_{1,0} = \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$

$$I_{2,1} = \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$I_{2,1} = \frac{h}{2}\left[f(x_2) + f(x_3)\right]$$

$$I_{3,1} = \frac{h}{2}\left[f(x_3) + f(x_4)\right]$$

$$C_{1,4} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$C_{1,n} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots 2f(x_{m-1}) + f(x_m)\right]$$

e.g. $[0,2]$ $f(x) = e^x$ $\boxed{m=2}$

St-1 $h = \frac{2-0}{2} = 1$

St-2 $x_0 \quad x_1 \quad x_2$

$= 0 \quad = x_0 + h = 1 \quad = 2$

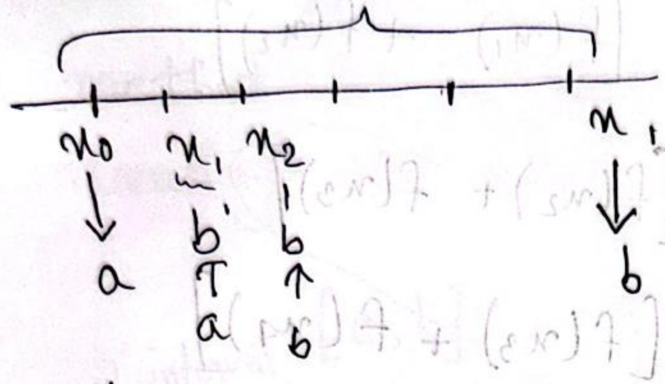
$$C_{1,2} = \frac{1}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right)$$

$$= \frac{1}{2}\left[e^0 + 2 \cdot e^1 + e^2\right]$$

The error term increases by the factor of 4 as m increases by 1.

Composite Newton - Cotes Formula

$$h = \frac{b-a}{m}$$ m-subintervals.



Eg-1

$f(x) = e^x$ Int $\rightarrow [0,2]$

$a=0,\ b=2$

Actual/Exact $\rightarrow \int_0^2 e^x\,dx = 6.3891$

(a)
$\rightarrow$ Find Newton's Cotes with no. of sub-intervals $= 2\ (m=2)$

Step-1
find h

$$h = \frac{b-a}{m} = \frac{2-0}{2} = 1$$

Step-2 find $x_0, x_1, x_2 \text{---} x_n$

$m=2 \quad x_0, x_1, x_2$

$m=3 \quad x_0, x_1, x_2, x_3$

$\boxed{h=1}$ $x_1 = x_0 + h$

$x_2 = x_1 + h$

$\boxed{m=2}$ $x_0 = a = 0$

$x_1 = a + h = 1$

$x_2 = x_1 + h = 2$

$C_{n,m}$

Step-3 : Find Composite New. cote. $m=1$

$C_{1,2}(f)$

$$C_{1,2}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$= \frac{1}{2}\left[e^0 + 2e^1 + e^2\right]$$

$$= 6.91281$$

b. $\boxed{m=3}$ x_0, x_1, x_2, x_3

St-1 $h = \frac{b-a}{m}$, $h = \frac{2-0}{3} = \frac{2}{3}$

Step-2 $x_0, - - x_3$

$x_0 = a = 0$

$x_1 = 0 + \frac{2}{3} = \frac{2}{3}$

$x_2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

$x_3 = \frac{4}{3} + \frac{2}{3} = 2$

Step-3

$$C_{1,3}(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{2/3}{2}\left[e^0 + 2e^{2/3} + 2e^{4/3} + e^2\right]$$

$$\boxed{= 6.62395}$$

$\boxed{m=4}$

x_0, x_1, x_2, x_3, x_4 $\quad h = \frac{2-0}{4} = 0.5$

$C_{1,4}(f) = 6.52161$

Error decreases, as M increases. ↳ (by a factor of 1)

by a

factor of 4

Simpson's Rule

Lagrange form

— Trapezium Rule: $\int_a^b P_1(x)\,dx$

Simpson's Rule: $\int_a^b P_2(x)\,dx$

$I_2(f) = \int_a^b P_2(x)\,dx$, $P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$

$I_2(f) = \int_a^b \left(l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) \right) dx$

$I_2(f) = \underbrace{\int_a^b l_0(x)}_{\sigma_0} f(x_0) + \underbrace{\int_a^b l_1(x_1)}_{\sigma_1} f(x_1) + \underbrace{\int_a^b l_2(x)}_{\sigma_2} f(x_2)\, dx$

$\therefore I_2(f) = \sigma_0 f(x_0) + \sigma_2 f(x_{1/2}) + \sigma_2 f(x_2).$