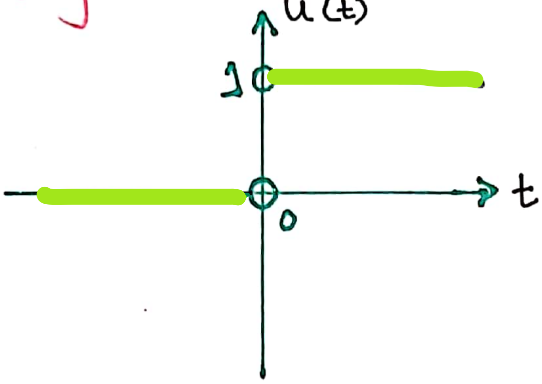
Laplace Transformation (Part B)

→ UNIT STEP FUNCTION or HEAVISIDE FUNCTION

Say we have the following signal:

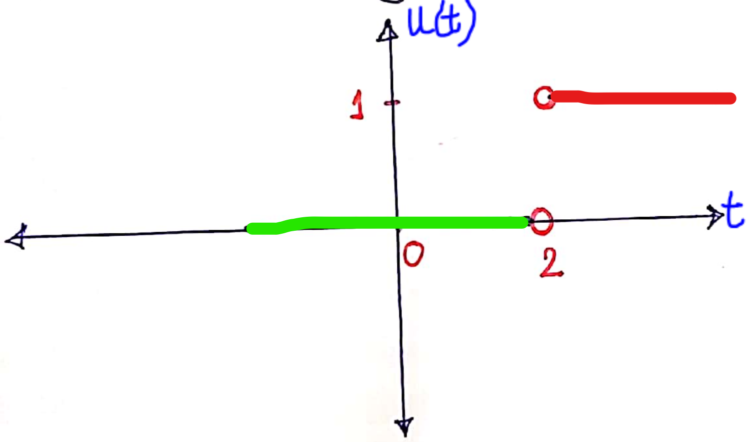


$$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t > 0 \end{cases}$$

This is unit step function

$$u(t) \Rightarrow u(t-0)$$

Write the following graph in terms of function:



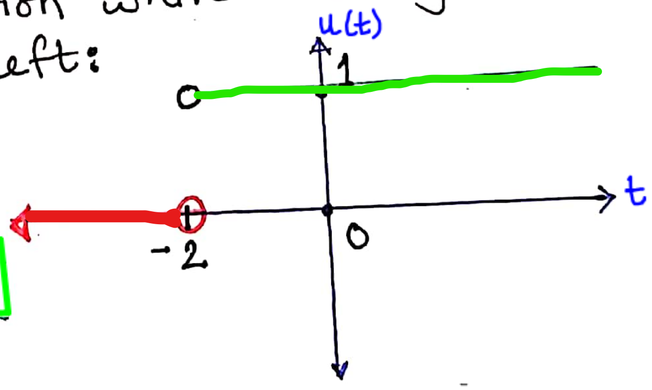
We are shifting towards right side

$$u(t-2) = \begin{cases} 0 & , \boxed{t < 2} \\ 1 & , \boxed{t > 2} \end{cases}$$

$$\begin{cases} t < 2 \Rightarrow t-2 < 0 \\ t > 2 \Rightarrow t-2 > 0 \end{cases}$$

Graph the following function while the signal is shifting two units towards left:

$$u(t+2) = \begin{cases} 0 & , \boxed{t < -2} \\ 1 & , \boxed{t > -2} \end{cases}$$



$$\begin{cases} t < -2 \Rightarrow t+2 < 0 \\ t > -2 \Rightarrow t+2 > 0 \end{cases}$$

Consider the following step function:

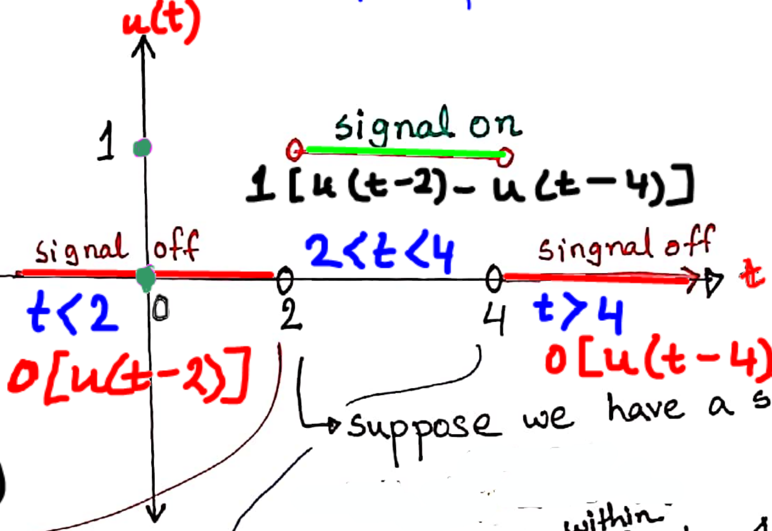
"a" is an arbitrary +ve number

General structure of Unit Step Function:

$$u(t-a) = \begin{cases} 0, & \underbrace{t<a}_{} \text{ -ve input} \\ 1 & \underbrace{t>a}_{\text{+ve input}} \end{cases} \quad a \geq 0$$

$t - a < 0 \Rightarrow t < a$

$t - a > 0 \Rightarrow t > a$



$f(t) = 0[u(t-2)] + 1[u(t-2) - u(t-4)] + 0[u(t-4)]$

$\therefore f(t) = u(t-2) - u(t-4)$

Suppose we have a signal while t within 2 to 4 signal is on & when $t > 4$, the signal is off.

Lets express this in terms of step function.

$$\boxed{f(t) = u(t-2) - u(t-4)}$$

The signal of the above graph can be presented mathematically in this way.

let $t = 1$

$t<2 \Rightarrow f(t) = \underset{\text{-ve input}}{u(t-2)} - \underset{\text{-ve input}}{u(t-4)} = 0 - 0 = 0$

graphically output should be "0"

let $t = 3$

$2<t<4 \Rightarrow f(t) = \underset{\text{+ve input}}{u(t-2)} - \underset{\text{-ve input}}{u(t-4)}$ $= 1 - 0 = 1$

graphically output should be "1"

$t > 2$ & $t < 4 \Rightarrow u(t-2)$ is +ve

$t > 2$ & $t < 4 \Rightarrow u(t-4)$ is -ve

$\begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

let $t = 5$

$t > 4 \Rightarrow f(t) = \underset{\text{+ve input}}{u(t-2)} - \underset{\text{+ve input}}{u(t-4)}$ $= 1 - 1 = 0$

graphically output should be "0"

it has to be "−" so the final answer is "0"

$t>4 \Rightarrow u(t-2)$ is +ve output $= 1$

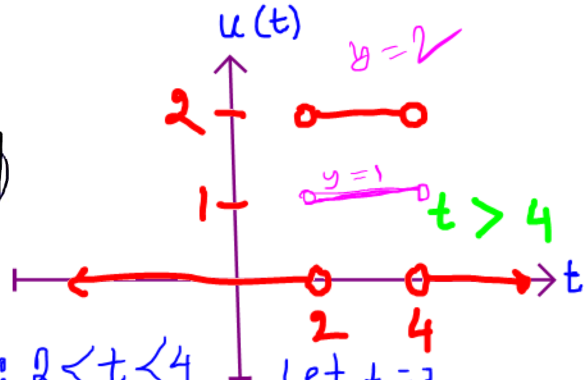
$t>4 \Rightarrow u(t-4)$ is +ve $\therefore$ output $= 1$

If we change the function

$f(t) = 2\,[u(t-2) - u(t-4)]$
+ve input   +ve input

$t > 4$
let $t = 5$

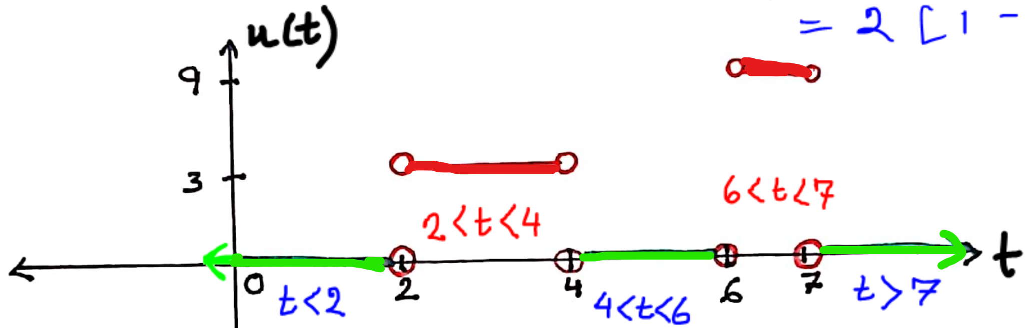
$= 2(1-1) = 0$



for: $2 < t < 4$ Let $t = 3$

$f(t) = 2[u(t-2) - u(t-4)]$
$= 2[1 - 0] = 2$ etc.

Consider another signal:



$f(t) = 0[u(t-2)] + 3[u(t-2) - u(t-4)] + 0[u(t-4) - u(t-6)] + 9[u(t-6) - u(t-7)]$
$+ 0[u(t-7)]$

$\therefore f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$

Represent the above function:

Step Function

$$f(t) = \begin{cases} 3, & 2 < t < 4 \\ 9, & 6 < t < 7 \\ 0, & o/w \end{cases}$$

$t < 2$ *, $t > 7$
$4 < t < 6$ **
Not given
$\therefore$ assuming over those intervals the signal is off.

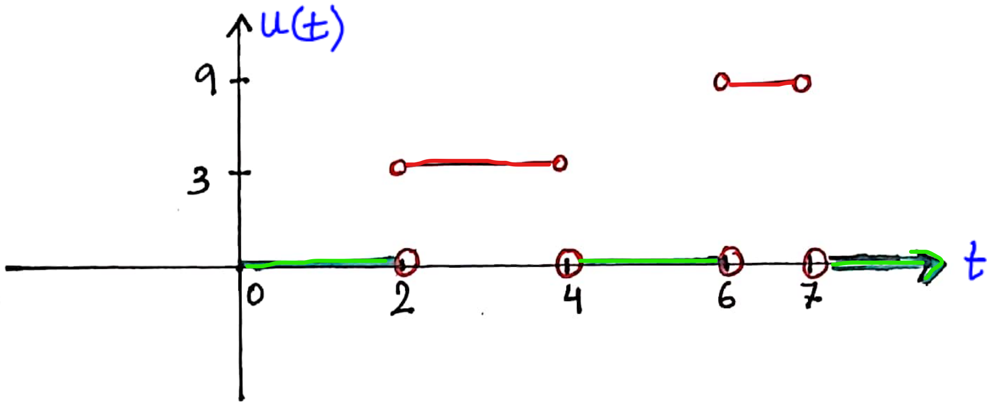
Express this function in one line: (unit step function)

$f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$

* $t < 2 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{-ve \\ "0"}} - \underbrace{u(t-4)}_{\substack{-ve \\ "0"}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ "0"}} - \underbrace{u(t-7)}_{\substack{-ve \\ "0"}}] = 0$
*let "t = 1" $\therefore$

$2 < t < 4 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{+v \\ "1"}} - \underbrace{u(t-4)}_{\substack{-ve \\ "0"}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ "0"}} - \underbrace{u(t-7)}_{\substack{-ve \\ "0"}}] = 3$
let "t = 3"

** $4 < t < 6 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{+ve \\ "1"}} - \underbrace{u(t-4)}_{\substack{+ve \\ "1"}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ "0"}} - \underbrace{u(t-7)}_{\substack{-ve \\ "0"}}] = 0$
let "t = 5"



$$6<t<7 \Rightarrow f(t) = 3[u(\underbrace{t-2}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-4}_{\substack{+ve \\ "1"}})] + 9[u(\underbrace{t-6}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] = 9$$

let "$t = 6.5$"

Unit step function helps us to write the **step** function in one line.

Example: **Reading**

$$f(t) = \begin{cases} t & ; \ 1<t<3 \\ \sin t & ; \ 6<t<7 \\ e^{2t} & ; \ t>7 \end{cases}$$

$\rightarrow \therefore t<1 \rightarrow$ output $= 0$

$3<t<6 \rightarrow$ " $= 0$

$\because$ this intervals are missing in the information

Unit step function will be as follows:

$$f(t) = t[u(t-1) - u(t-3)] + \sin t[u(t-6) - u(t-7)] + e^{2t}[u(t-7)]$$

$$t<1 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-3}_{\substack{-ve \\ "0"}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] + e^{2t}(u(\underbrace{t-7}_{\substack{-ve \\ "0"}}))$$

$$= 0$$

$$1<t<3 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-3}_{\substack{-ve \\ "0"}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] + e^{2t}[u(\underbrace{t-7}_{\substack{-ve \\ "0"}})]$$

$$= t$$

$$6<t<7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{+ve\ "1"}) - u(\underbrace{t-3}_{+ve\ "1"})] + \sin t\,[u(\underbrace{t-6}_{+ve\ "1"}) - u(\underbrace{t-7}_{-ve\ "0"})] + e^{2t}\,[u(\underbrace{t-7}_{-ve\ "0"})]$$

$$= t\,[1-1] + \sin t\,[1-0] + e^{2t}\,[0]$$

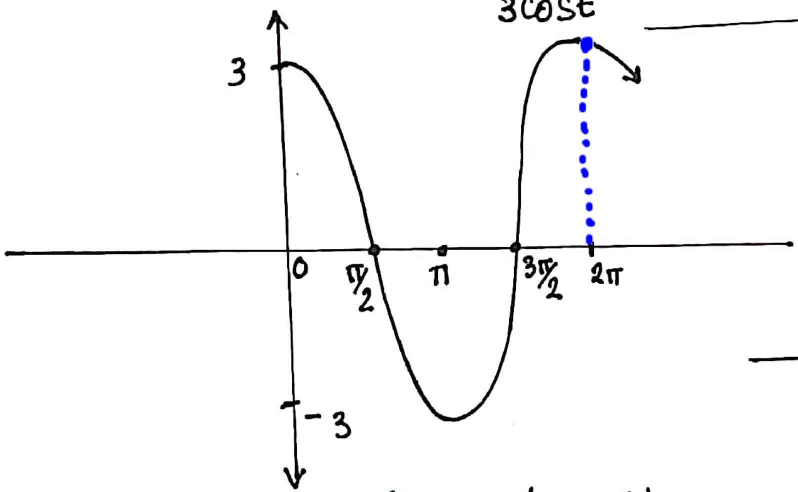
$$= \sin t$$

$$t>7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{+ve}) - u(\underbrace{t-3}_{+ve})] + \sin t\,[u(\underbrace{t-6}_{+ve}) - u(\underbrace{t-7}_{+ve})] + e^{2t}[u(\underbrace{t-7}_{+ve})]$$

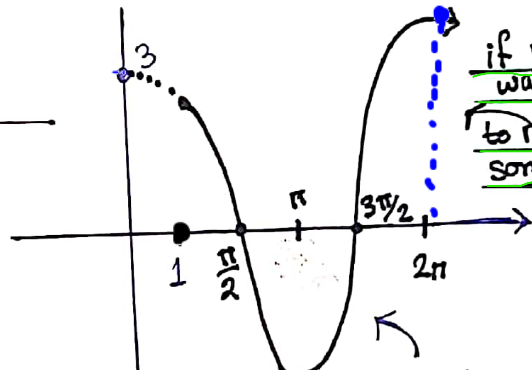
$$= t\,[1-1] + \sin t\,[1-1] + e^{2t}\,(1)$$

$$= e^{2t}$$

Omit these Graphs ⟶ TRANSLATIONS



$\longrightarrow f(t) = 3\cos t$

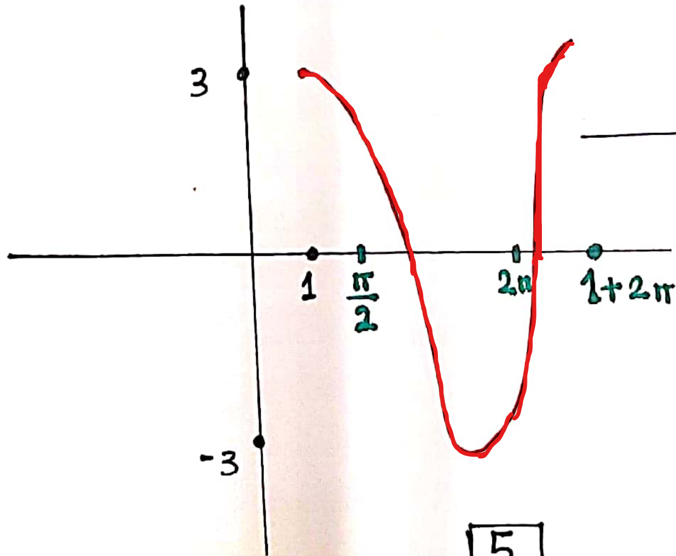


if you want to remove some part

-3 $\quad f(t) = 3\cos t\,[u(t-1)]$

$= f(t)\,u(t-1)$

shift it forward by 1 unit



$\longrightarrow f(t) = 3\cos(t-1)[u(t-1)]$

$= f(t-1)\,u(t-1)$

Translation

Example $\boxed{1}$

$$f(t) = \begin{cases} 0 & ; \; 0<t<1 \\ t-1 & ; \; 1<t<2 \\ t+1 & ; \; t>2 \end{cases}$$

Find the Laplace transformation of the above function

Unit step function:

$$f(t) = 0[u(t-0)-u(t-1)] + (t-1)[u(t-1)-u(t-2)] + (t+1)[u(t-2)]$$

$$= tu(t-1) - tu(t-2) - u(t-1) + u(t-2) + tu(t-2) + u(t-2)$$

$$= tu(t-1) - u(t-1) + 2u(t-2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{tu(t-1)\} - \mathcal{L}\{u(t-1)\} + 2\mathcal{L}\{u(t-2)\}$$

take laplace on both sides

2nd Translation Theorem:

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\,u(t-1)\} - \mathcal{L}\{1u(t-1)\} + 2\mathcal{L}\{1u(t-2)\}$$

$a=1$ $f(t)=1$ $a=1$ $f(t)=1$ $a=2$

$f(t)=t$ $\therefore f(t+a) = f(t+1) = 1$ $\therefore f(t+a) = f(t+2) = 1$

$f(t+a) = f(t+1) = t+1$

$$\mathcal{L}\{f(t)\} = e^{-s(1)}\mathcal{L}\{f(t+1)\} - e^{-s(1)}\mathcal{L}\{f(t+1)\} - 2e^{-s(2)}\mathcal{L}\{f(t+2)\}$$

$$= e^{-s}\mathcal{L}\{t+1\} - e^{-s}\mathcal{L}\{1\} - 2e^{-2s}\mathcal{L}\{1\}$$

$$= e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) - e^{-s}\cdot\frac{1}{s} + 2e^{-2s}\frac{1}{s}$$

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$$= e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s} - e^{-s}\frac{1}{s} + 2e^{-2s}\frac{1}{s}$$

$$= \frac{1}{s^2}e^{-s} + \frac{2}{s}e^{-2s}$$

Example $\boxed{2}$ Find the Laplace Transformation of

$$f(t) = \begin{cases} 2 ; & 0<t<3 \\ t^2 ; & 3<t<5 \\ t+1 ; & t>5 \end{cases} \qquad \mathcal{L}\{u(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

Unit step function:

$$f(t) = 2[u(t-0) - u(t-3)] + t^2[u(t-3) - u(t-5)] + (t+1)[u(t-5)]$$

$$= 2u(t-0) - 2u(t-3) + t^2 u(t-3) - t^2 u(t-5)$$
$$+ t(u(t-5)) + 1u(t-5)$$

take Laplace on both sides:

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1u(t-\overset{a}{0})\} - 2\mathcal{L}\{1u(t-\overset{a}{3})\} + \mathcal{L}\{t^2u(t-\overset{a}{3})\}$$
$$\begin{matrix} f(t)=1 \\ f(t+0)=1 \end{matrix} \qquad \begin{matrix} f(t)=1 \\ f(t+3)=1 \end{matrix} \qquad \begin{matrix} f(t)=t^2 \\ f(t+3)=(t+3)^2 \end{matrix}$$

$$- \mathcal{L}\{t^2u(t-\overset{a}{5})\} + \mathcal{L}\{t\,u(t-\overset{a}{5})\} + \mathcal{L}\{1u(t-\overset{a}{5})\}$$
$$\begin{matrix} f(t)=t^2 \\ f(t+5)=(t+5)^2 \end{matrix} \qquad \begin{matrix} f(t)=t \\ f(t+5)=t+5 \end{matrix} \qquad \begin{matrix} f(t)=1 \\ f(t+5)=1 \end{matrix}$$

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$= 2e^{-s(0)}\mathcal{L}\{f(t+0)\} - 2e^{-s(3)}\mathcal{L}\{f(t+3)\} + e^{-s(3)}\mathcal{L}\{f(t+3)\}$$
$$- e^{-s(5)}\mathcal{L}\{f(t+5)\} + e^{-s(5)}\mathcal{L}\{f(t+5)\} + e^{-s(5)}\mathcal{L}\{f(t+5)\}$$

$$= 2\cdot1\cdot\mathcal{L}\{1\} - 2e^{-3s}\mathcal{L}\{1\} + e^{-3s}\mathcal{L}\{t+3\}^2$$
$$- e^{-5s}\mathcal{L}\{(t+5)^2\} + e^{-5s}\mathcal{L}\{t+5\} + e^{-5s}\mathcal{L}\{t+5\}$$

$$= 2\cdot\frac{1}{s} - 2e^{-3s}\frac{1}{s} + e^{-3s}\mathcal{L}\{t^2+6t+9\}$$
$$+ e^{-5s}\mathcal{L}\{t^2+10t+25\} + e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}\right) + e^{-5s}\cdot\frac{1}{s}$$

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$$= \frac{2}{s} - \frac{2}{s}e^{-3s} + e^{-3s}\left(\frac{2!}{s^{2+1}} + 6 \cdot \frac{1}{s^2} + \frac{9}{s}\right)$$

$$+ e^{-5s}\left(\frac{2!}{s^{2+1}} + 10 \cdot \frac{1}{s^2} + 25\frac{1}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right) + \frac{e^{-5s}}{s}$$

$$= \frac{2}{s} - \frac{2e^{-3s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{6e^{-3s}}{s^2} + \frac{9e^{-3s}}{s}$$

$$+ \frac{2e^{-5s}}{s^3} + \frac{10e^{-5s}}{s^2} + \frac{25e^{-5s}}{s} + \frac{e^{-5s}}{s^2} + \frac{5e^{-5s}}{s} + \frac{e^{-5s}}{s}$$

$$= \frac{2}{s} + \frac{7e^{-3s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{6e^{-3s}}{s^2}$$

$$+ \frac{2e^{-5s}}{s^3} + \frac{11e^{-5s}}{s^2} + \frac{31e^{-5s}}{s}$$

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