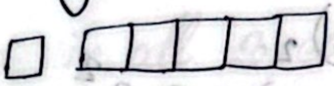
CSE330 [Playlist RRH]

CH/Module-1 [1·1 & 1·2]

Floating Point Arithmetic

64-Bit $\rightarrow 2^{64}$

signed



$+\ 2^{63}-1$

[1.1] Fixed Point Numbers

$$x = \pm(d_1 d_2 \cdots d_{k-1} \cdot d_k d_{k+1} \cdots d_n)_\beta \quad \beta \rightarrow \text{Base}$$

$\cancel{\rightarrow (d_1 d_2 \cdots \beta-1)}$

$\rightarrow d_i \in \{0, 1, \cdots, \beta-1\}$

ex 1. $(\overset{d_1}{1}\overset{d_2}{0}\cdot\overset{d_3}{1})_2$ [convert to decimal]

$\rightarrow 1\times 2^1 + 0\times 2^0 + 1\times 2^{-1}$ $\quad \mathbb{R} \quad \mathbb{Z} \rightarrow$ set of integers

$= 2 + \frac{1}{2}$

$= (2\cdot 5)_{10}$

Floating Point Representation

(i) $F \subset \mathbb{R}$

(ii) $F = \pm(\underbrace{0.d_1 d_2 d_3 d_4 \cdots d_m}_{\text{Mantissa/fraction}})_\beta\, \beta^e$ where $\beta, m, e \in \mathbb{Z}$ ($\beta \rightarrow$ Base) & $e_{min} \le e \le e_{max}$

Ex

$(123.45)_{10}$ [bring . at the left]

$= 0.12345 \times 10^{3}$ (m = 0.12345, β = 10, e = 3)

$\beta = 10$

$m = 5$

$e = 3$

$\beta^{(+)} \leftarrow .$

$\beta^{(-)} \; . \longrightarrow$

Ex. $(11.101)_2$

$(0.11101)_2 \times \cancel{10}\, 2^2$

$\cancel{1\times 2^1 + 1\times 2}$

$\beta = 2$

$m = 5$ (just count digits after the point)

$e = 2$

\# $(0123)_{10}$

$= (0.\underline{0}123)_{10} \times 10^4$ (1st problem)

Conventions

(i) $F = \pm (0.d_1 d_2 d_3 \dots d_m)_\beta \cdot \beta^e$

$d_1 = 1$ always

[lecture note form]

constraints

$d_1 = 1$

starts with 0.

(ii) $F = \pm (1.d_1 d_2 d_3 \dots d_m)_\beta \cdot \beta^e$

starts with 1

[Normalized form]

(iii) $F = \pm(0.1d_1d_2d_3 \dots d_m)_\beta \cdot \beta^e$ (Best form)

[denormalized form]

Ex

$\beta = 2, m = 3$, $e_{min} \le e \le e_{max}$

Q. (a) Highest / max no in all 3 convention

(1) $F = \cancel{\pm(0.1993)_2 \times 2}$

$= \pm(0.111)_2 \times 2^{e_{max}}$

(2) $F = \pm(1.111)_2 \times 2^{e_{max}}$

(3) $F = +(0.1111)_2 \times 2^{e_{max}}$

lowest.

(1) $F = -(0.111)_2 \times 2^{e_{min}}$

(2) $F = -(1.111)_2 \times 2^{e_{min}}$

(3) $F = -(0.1111)_2 \times 2^{e_{min}}$

Task

Q. Given, $\beta = 2, m = 3$, $e_{min} = -1$, $e_{max} = 2$

(a) Find how many no. can be represented using this system if we follow the lecture note form?

$2^3\ 2^2\ 2^1\ 2^0$
$1\ 0\ 0$

Floating Point Arithmetic

Ex $\beta = 2,\ m = 3,\ e_{min} = -1,\ e_{max} = 2$

using ~~con~~ the lecture note form, how many different no. s can we represent?

$$\pm (0.d_1d_2d_3 \ldots d_m)_\beta \beta^e$$

$$= \pm (0.d_1\square\square)_\beta \beta^e \qquad e \in [-1, 2]$$

$d_1 \to 1$; $\square\square$: 00, 10, 01, 11 $\times 2^{-1,0,1,2}$ → $-1, 0, 1, 2$

$\underbrace{\quad}_{4}$ $\quad 4 \times 4 \quad$ $\underbrace{\quad}_{4}$

$= 16$ numbers

(b) Draw a number line for all the combination

how many e → that many sets

(1) $e = -1$

$\to (0.100)_2 \times 2^{-1} = 2^{-1} * 2^{-1} = \frac{1}{4}$

$\to (0.101)_2 \times 2^{-1} = 2^{-1} * 2^{-3} * 2^{-1} = \frac{1}{2} + \frac{1}{8} \times \frac{1}{2} \times 2^{-1}$

$= \cancel{\frac{7}{4}}\ \frac{9}{8} \times 2^{-1}$

$= \frac{9}{8} \cdot \frac{1}{2} = \frac{9}{16}$

lowest no → $-\frac{7}{2}$
highest " → $7/2$
lowest non neg no
highest " " " →

Bin to decimal

$$\begin{array}{cccccccc} 2^3 & 2^2 & 2^1 & 2^0 & & 2^{-1} & 2^{-2} & 2^{-3} \\ 1 & 1 & 0 & 0 & . & 1 & 0 & 1 \end{array}$$

$2^3 + 2^2 + 2^{-1} + 2^{-3}$

$8 + 4 + \frac{1}{2} + \frac{1}{8} = 12.625$

$(43.167)_{10}$

$$\begin{array}{r|l} 2 & 43 \\ 2 & 21 \;-1 \\ 2 & 10 \;-1 \\ 2 & 5 \;-0 \\ 2 & 2 \;-1 \\ & 1 \;-0 \end{array}$$

$0.167 \times 2 \to 0.334 \to 0$

$0.334 \times 2 \to 0.668 \to 0$

$0.668 \times 2 \to 1.336 \to 1$

$0.336 \times 2 \to 0.672 \to 0$

$0.672 \times 2 \to 1.344 \to 1$

101011.00101

highest non-negative: $\beta = 2, m = 3, e \in [-1, 2]$

$(0.111)_2 \, 2^2$

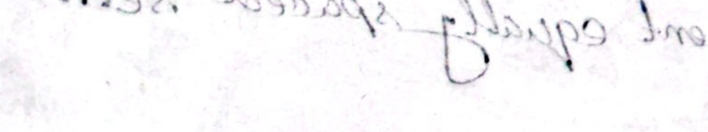
$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdot 4$

$= \frac{7}{8} \cdot 4$

$= \frac{7}{2}$

lowest non-neg no.

$(0.100)_2 \cdot 2^{-1} = 2^{-1} \cdot 2^{-1} = \frac{1}{4}$



Pattern

$e = -1, 0, 1, 2$

(1) $+(0.100)_2 . 2^{-1} = \frac{1}{4}$

$+(0.101)_2\ 2^{-1} = \frac{5}{16}$

$+(0.110)_2\ 2^{-1} = \frac{3}{8}$

$+(0.111)_2 \times 2^{-1} = \frac{7}{16}$

distance:

(3) $+(0.100)_2\ 2^1 = 1$

$+(0.101)_2\ 2^1 =$

$+$

$+$

(2) $+(0.100)_2\ 2^0 = \frac{1}{2}$

$e=0$ $\ +(0.101)_2\ 2^0 = 5/8$

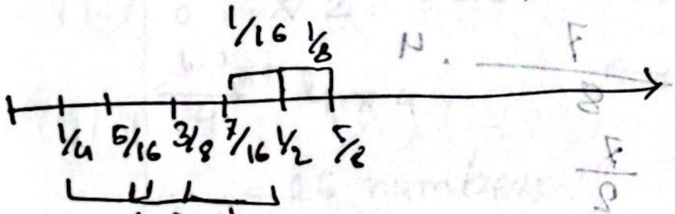
$+(0.110)_2\ 2^0 =$

(4) $+$

$+$

$+$

$+(0.111) \times 2^3 = \frac{7}{4}$



distances: $\frac{5}{16} - \frac{1}{4} = \frac{1}{16}$ $\quad$ $1/16$ $\quad$ $1/16$

$1/2 - \frac{7}{16} = \frac{1}{16}$

$5/8 - \frac{1}{2} = \frac{1}{8}$

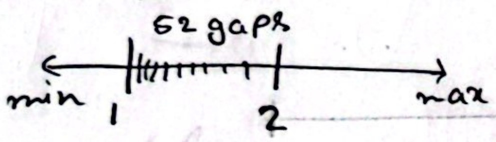
→ The individual sets are equally spaced

→ Each of the equally spaced sets have different spaces.

⟶ different equally spaced sets

IEEE Standard (1985) for double precision (64-bit) arithmetic

$\beta = 2$, 52-bits for the fraction/mantissa → precision
11-bits for the exponent [range]
1-bit " " sign



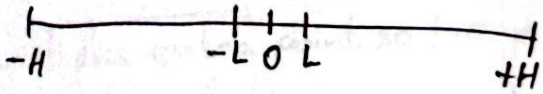
__normalized form:__ $\pm(1.d_1 d_2 \ldots d_{52}) \times 2^e$ (2^{11})

$2^{11} = 2048$

$e \in [0, 2047]$ → emin, emax

lowest non negative no.
$\leq (1.000\ldots 0) \times 2^0 = 1$

0.5? 0.4?



highest non negative:
$H\ (1.1111 \ldots 11) \times 2^{2047}$

__we do exponent bias:__

$\boxed{(1.d_1 d_2 \ldots d_{52}) \times 2^{e-1023}}$ bias

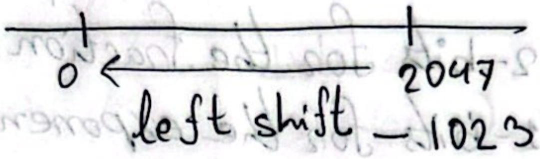
Bias $= 2^{(\text{ebits}-1)} - 1$
$= 2^{10} - 1$
$= 1023$

if ~~8 bits~~ e = 4 bits
Bias $= 2^{(4-1)} - 1$
$= 2^3 - 1$
$= 7$

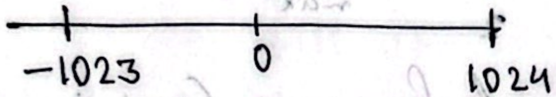
$2^{9.97} = 1002.93$
$\log 501 = (e-1)\log 2$
$e - 1 = \frac{\log 501}{\log 2}$
bias $= 500$
$500 = 2^{e-1} - 1$
$501 = 2^{e-1}$

[44:26]

$e \in [0, 2047]$



now, $e \in [-1023, 1024]$



Normalized: $(1.d_1 d_2 \cdots d_{52}) . 2^{e-1023}$

$e \in [0, 2047]$

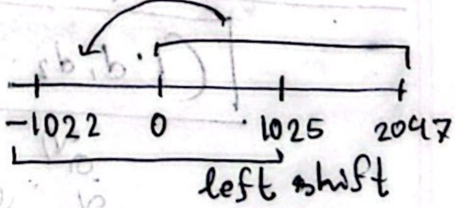
Denormalized:

$(0.1 d_1 d_2 \cdots d_{52}) . 2^{e-1023+1}$ → it is done because the decimal is being left shifted one space.

$= (0.1 d_1 d_2 \cdots d_{52}) . 2^{e-1022}$

$e \in [-1022, 1025]$ → [Range increased]

lowest non-neg → $(0.1) \times 2^{-1022} \approx 2.225 \times 10^{-308}$



Highest non-neg → $(0.1111 \cdots 1) \times 2^{1025}$
→ $(0.1111 \cdots 1)_2 \times 2^{1024} \simeq 1.798 \times 10^{308}$

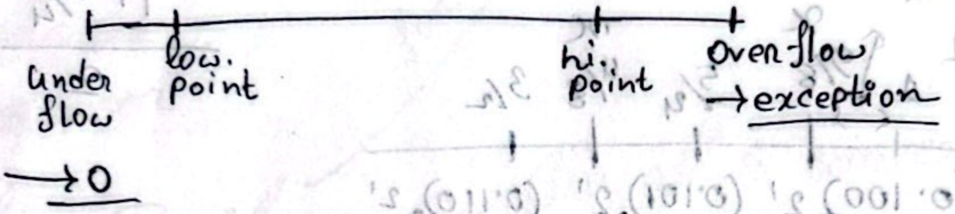
as 1025 goes to infinity

$$\boxed{\begin{aligned} 2^{1025} &\to \pm\infty \\ 2^{-1022} &\to 0 \end{aligned}}$$

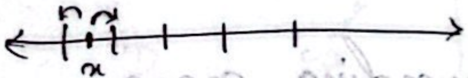
Q: bits will be changed then highest val, lowest value, how many numbers; expo. bias

lowest non neg → $(0.100\ldots0)_2 \times 2^{-1022}$ → can be taken as there is a 1 after the decimal &.

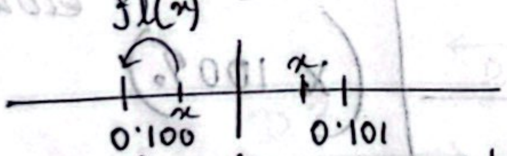
$\simeq 2.225 \times 10^{-308}$



Module 1.4 Rounding Error



$x \to fl(x)$



exact middle → check even/odd

(round up) ← x (middle point)

0.1001 (middle point has extra point so 1 is added at the last)

constraint

→ it will get rounded to the nearest even float no.

Binary odd = ends w 1
" even = " w 0

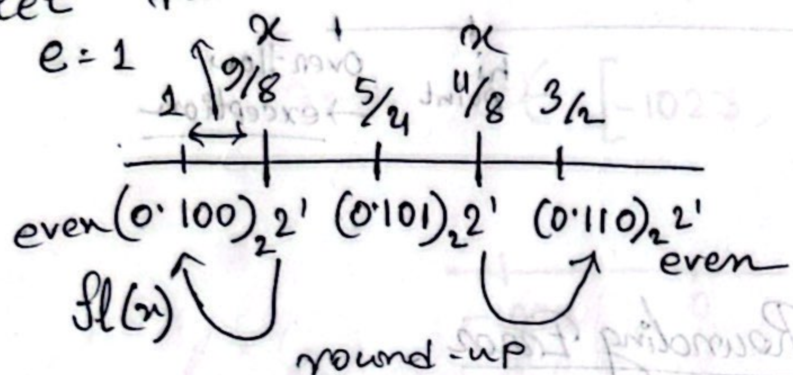
alice@aj

Ex $\beta = 2$, $m = 3$, $e_{min} = -1$, $e_{max} = 2$

convention 1: $(0.d_1 d_2 d_3)_2 \; 2^e$

let rounding error

$e = 1$



$$\frac{1 + 5/4}{2} = 9/8$$

Rounding error

$$\delta = \boxed{\frac{|fl(x) - x|}{|x|}} \; (\times 100\%)$$

this gives percentage error

Scale invariant error → because it is a ratio.

$$fl(x) = x(1 + \delta)$$

$$= x + \underbrace{x\delta}_{\text{error}}$$

$\boxed{1.4 \text{ part } 1}$

class 3

1·4 Rounding Error

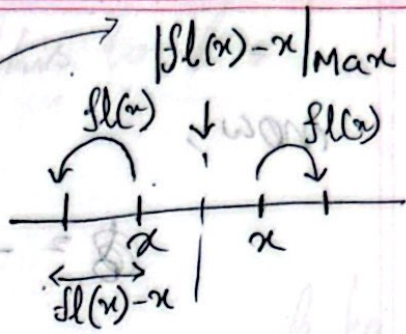
$fl(x)$ rounded value

$fl(x) \neq x$

$$\delta = \frac{|fl(x)-x|_{max}}{|x|_{min}}$$

δ → scale invariant Rounding error

for $|fl(x)-x|_{Max}$

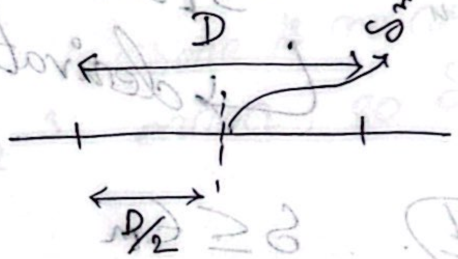


Machine Epsilon, ϵ_m (for all 3 conventions)

→ Maximum possible scale invariant error

con-1

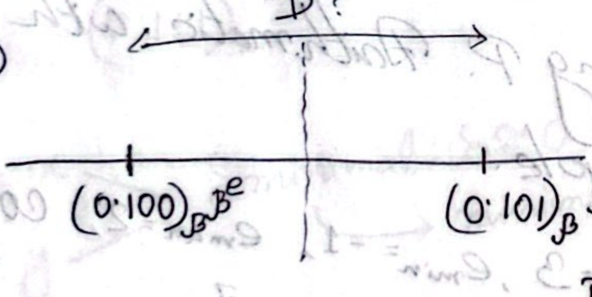
lecture Note [m=3]



$(0.001)_2$
$= 1 \times 2^{-3}$
$= 1 \times \beta^{-3}$
$= \beta^{-m}$ [m = 3]

step-1

finding the middle part



$$D = \{(0.101)-(0.100)\}_\beta \beta^e$$
$$= (0.001)_\beta \beta^e$$

$$D = \beta^{-m} \cdot \beta^e$$ → $|fl(x)-x|_{max}$

$$\frac{D}{2} = \boxed{\frac{1}{2}(\beta^{-m} \cdot \beta^e)}$$

step-2 $|x|_{min}$

lecture note [m=3] $(0.100)_\beta \cdot \beta^e$
$= \beta^{-1} \cdot \beta^e$
$= |x|_{min}$

$(0.100)_2$
$= 2^{-1}$
$= \beta^{-1}$

now,

$$\delta = \frac{\frac{1}{2}\beta^{-m} \cdot \cancel{\beta^{e}}}{\beta^{-1} \cdot \cancel{\beta^{e}}}$$

e will not affect the value of machine epsilon.

$$\epsilon_{m_L} = \frac{1}{2}\beta^{1-m} \quad \text{(lecture note form.)}$$

convention-2 [Normalize] $\epsilon_{M_N} = \frac{1}{2}\beta^{-m}$

" -3 [denormalized] $\epsilon_{M_D} = \frac{1}{2}\beta^{-m}$

↳ derivation

Point : $\delta \le \epsilon_m$

Ex : Floating P. Arithmatic with rounding error example.

$\beta = 2$, $m = 3$, $e_{min} = -1$, $e_{max} = 2$, conv = 1

(i) Given, $x = \frac{5}{8}$, $y = \frac{7}{8}$

find $fl(x)$ & $fl(y)$

(ii) Find $fl(x^* y)$

ⓐ $x = \frac{5}{8}$ ——— convert this to bin

$fl(x) = \frac{4}{8} + \frac{1}{8}$

$= \frac{1}{2} + \frac{1}{8}$

$= 2^{-1} + 2^{-3}$

$= (0.101)_2\, 2^0$ → if nothing needs to be changed, exponent shall be 0.

$y = \frac{7}{8}$

$fl(y) = \frac{4}{8} + \frac{2}{8} + \frac{1}{8}$ There is no rounding error, so $\delta = 0$.

$= 2^{-1} + 2^{-2} + 2^{-3}$

$= (0.111)_2\, 2^0$

$\therefore fl(x) = x$, $fl(y) = y$ → δ minimal $\delta \simeq 0$ → minimal scale invariant.

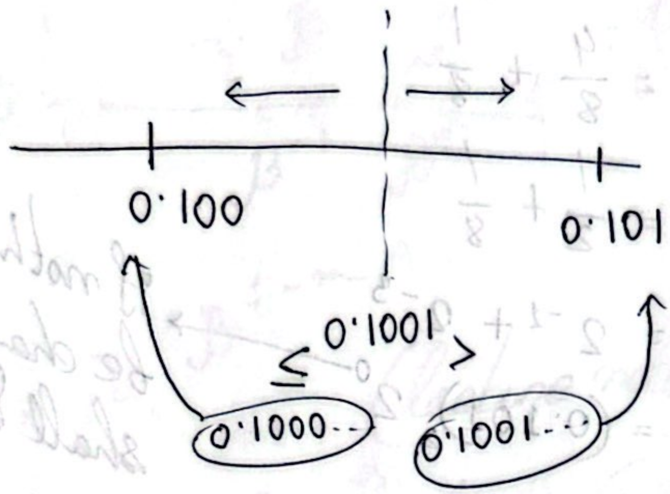
ⓑ $fl(x) * fl(y)$

$= \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} = \frac{32}{64} + \frac{2}{64} + \frac{1}{64}$

$= 2^{-1} + 2^{-5} + 2^{-6}$

$= (0.1000011)$

0.100011 check



for m^{th} bit,
if ${m+1}^{th}$ bit = 0
Round to left
else if,
$m+1^{th}$ bit = 1, Round to Right

$$fl(x^*y) = 0.100$$
$$= \frac{1}{2}$$
$$= \frac{32}{64}$$

$\therefore$ Rounding error $= \frac{35}{64} - \frac{32}{64}$

$$= \frac{3}{64} \text{ (Ans)}$$

$$\delta = \frac{|fl(x) - x|}{|x|}$$

1.5 Loss of Significance

$\Rightarrow \delta x = fl(x) - x$ Absolute Rounding error

$\Rightarrow fl(x) = x + \delta x$

$\Rightarrow fl(x) = x(1+\delta)$; $fl(y) = y(1+\delta)$

Scale invariant R.E

$fl(x) \neq x$, $fl(y) \neq y$

$fl(x) = x(1+\delta_1)$; $fl(y) = y(1+\delta_2)$

in case of $\frac{x-y}{~~neg~~}$

if $x \simeq y$ the denominator becomes zero and the error will be infinite and thats called loss of significance

$x \pm y$

$fl(x \pm y) = fl(x) \pm fl(y)$

$= x(1+\delta_1) \pm y(1+\delta_2)$

$$fl(x \pm y) = (x \pm y)\left(1 + \frac{x\delta_1 \pm y\delta_2}{x \pm y}\right)$$

$\frac{x\delta_1 \pm y\delta_2}{x \pm y}$ → scale invariant error of $(x \pm y)$

→ Where does it occur

$\underline{x+y}$

$x, y \simeq 0$ then it will become loss of significance.

$\underline{LOS}$ 4sf → significant figure

$x^2 - 56x + 1 = 0$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

human: $x_1 = 28 + \sqrt{783} = 55.98$

$x_2 = 28 - \sqrt{783} = 0.0\boxed{1786}$

computer: $x_1 = 28 + 27.98 = 55.98$ [4 s.f.]

$x_2 = 28 - 27.98 = \cancel{0.2}\ 0.02$ [1sf]

↳ $0.0\boxed{2000}$ [4 s.f]

↳ L.O.S.

2000 & 1786 has huge difference

significantly close enough

the answer

When I subtract two numbers, if both the number were much closer, then in case of rounding/ scale invariant error, the denominator will be

$6 - 5.56$
$= 0.$

very ~~high~~ low. In thate case, the scale invariant error will be very high or it tends to be infinity.

How to avoid loss of significance

$x^2 - 56x + 1 = 0$

$x_1 = 55.98$

$x_2 = 0.02000$

© α, β

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\alpha\beta = 1$

let $x_1 = \alpha$, $x_2 = \beta$

$x_1 . x_2 = 1$ —— ①

We know, $x_1 = 55.98$

using eqn 1,

$$x_2 = \frac{1}{x_1}$$

$$= \frac{1}{55.98}$$

$$= 0.01786$$

Module 2

"Polynomial Interpolation :-

degree = 3

$f(x) = 2 + 3x + 4x^2 + x^3$

lowest power ← highest power →

Finite ← $P_3(x) = 2 + 3x + 4x^2 + x^3$

2.1 Basic Terms

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \rightarrow$ degree = n

$[a_0, a_1, a_2, a_3, \ldots, a_n]$, $[1, x, x^2, \ldots, x^n]$

↳ set of coefficients ↳ Basis set

length of coeff = $(n+1)$

if deg = n,

length of coeff = $(n+1)$

$P_{20}(x) \rightarrow$ degree 20

↪ coeff = 21.

Vector Space

V → Region/space of vectors where we can Add/subtract vectors multiply with scalars

$P_2(x) = 1 + x + 2x^2$

$+$

$G_2(x) = 1 + 2x - 2x^2$

$2 + 3x$

$P_2(x) \times 5$

$= (1 + x + 2x^2) \times 5$

$= 5 + 5x + 10x^2$

polynomial is a part of vector space.

$$\boxed{P_n(x) \in V}$$

Basis : is a set of vectors that spans the vector space

$[1, x, x^2, \dots, x^n]$

set of basis

$P_2(x) = 1 + 2x^2$

↳ coeff set = [1, 0, 2] [allows 0]

↳ Deg = 2

↳ Basis = $[1, x, x^2]$

↳ len of coeff = 3

↳ len of basis = 3

basis will always stay like this [Generic]

→ Degree = n

→ $\text{coeff}_{len} = n+1$

→ $\text{basis}_{len} = n+1$

↳ Dimension / Dimensional space

Ex:

$P_{58}(x)$

Degree = ? 58

coeff = 59

dim of space = 59

| state the dimension |

$[1, x, x^2, \ldots\ x^{58}]$

state of the coeff $[a_0, a_1, \ldots, a_{58}]$

Quiz 2 → 2.4 (vandermonde matrix)

2.2

Weierstrass Approximation Theorem

Prereq:

Natural basis = $[1, x, x^2, \ldots, x^n]$

(Basis length) Dimension = $n+1$

$\boxed{P_n(x) \in V^{n+1}}$ → Polynomial space

function: $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Basis = $[1, x, x^2, x^3, \ldots]$ → infinite Basis

dim = ∞

Functional space: $\boxed{f(x) \in V^{\infty}}$

$f(x) = 2 + 3x + 4x^2 + \underbrace{9x^3 + x^4 + 6x^5 + \cdots}_{\text{Truncation Error}}$ → polynomial version

$P_2(x) = a_0 + a_1 x + a_2 x^2$ (find the coefficients)

$P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

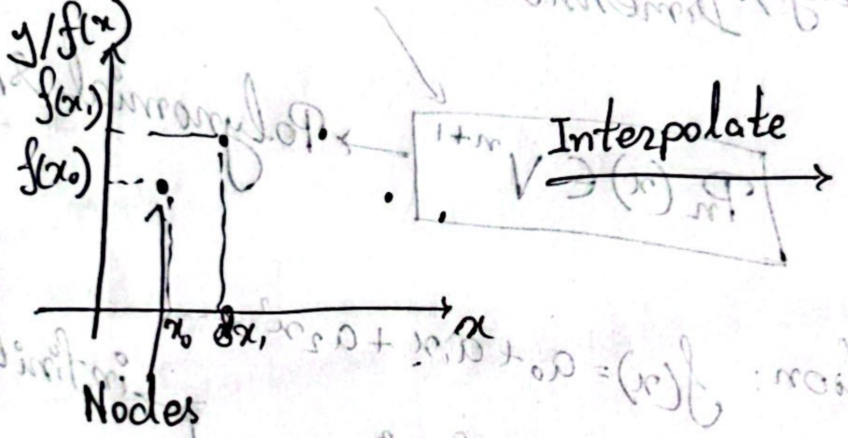
Quiz 2 → 2.4 (vandermonde matrix)

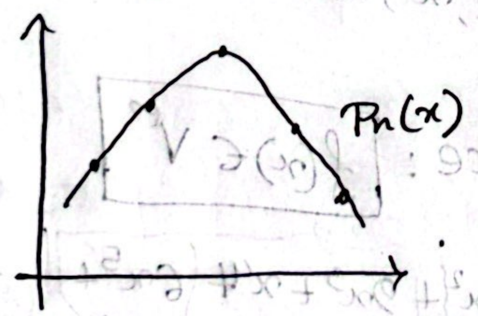
Truncation Error,

$$T.E. = |f(x) - P_n(x)|$$

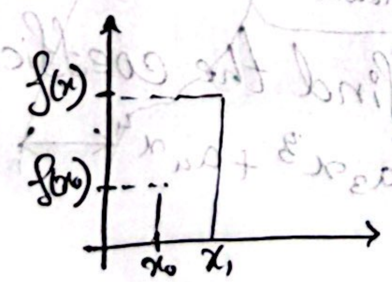
if $n\uparrow$ (degree), value of the T.E.$\downarrow$

Module 2 Part-3





Taylor Series



$f'(x_0)$

$f^2(x_0)$

$f^3(x_0)$

$\vdots$

$f^n(x_1)$

Can I predict the value of the function at other points (x)? → yes.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + \dots$$

Ex

Expand the function $\sin(x)$ using taylor series, centered at $x_0 = 0$. [6th term]
↳ 5th derivative

Sol^n:

$$\left.\begin{aligned} f(x) = \sin(x) &\longrightarrow f(0) = \sin(0) = 0 \\ f'(x) = \cos(x) &\rightarrow f'(x_0) = \cos(0) = 1 \\ f^2(x) = -\sin(x) &\rightarrow f^2(0) = -\sin(0) = 0 \\ f^3(x) = -\cos(x) &\rightarrow f^3(0) = -\cos(0) = -1 \end{aligned}\right] \downarrow \text{repetative}$$

$$f^4(x) = \sin(x) \rightarrow f^4(0) = \sin(0) = 0$$

$$f^5(x) = \cos(x) \rightarrow f^5(0) = \cos(0) = 1$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0)^2 + \frac{f^3(x_0)}{3!}(x-x_0)^3 + \frac{f^4(x_0)}{4}(x-x_0)^4 + \frac{f^5(x_0)}{5}(x-x_0)^5$$

$$= 0 + 1(x-0) + 0 + \frac{(-1)}{3!}(x-0)^3 + 0 + \frac{1}{5!}(x-0)^5$$

$$f(x) = \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5} - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots$$

$x = 0.1$

$f(0.1)$

1st term $= 0.1$

2nd term $= x - \frac{1}{3!}x^3$

$= 0.1 - \frac{1}{3!}(0.1)^3$

e^x, e^{-x}

4th term

Taylor's theorem

Tay. S. $\Rightarrow \underbrace{f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^2(x_0)(x - x_0)^2}{2!}}_{P_2(x)} + \underbrace{\frac{f^3(x_0)(x - x_0)^3}{3!}}$ → Part of the error

3rd term

error bound [largest error]

Taylor's theorem:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k} + \boxed{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}}$$

$\xi \rightarrow$ Zhi

Taylor's polynomial of degree n.

Lagrange form of the remainder [error bound]

[Learn machine epsilon]

$$\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{(n+1)}$$

we don't know actual value of ξ but it will have a range. [we only know, ξ is a value between (a,b)

we can find maximum bound of the error

→ If $\xi \in (a,b)$, what is the maximum value of error?

$$f(x) = \sin(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{P_5(x)} \underbrace{- \frac{x^7}{7!} + \dots}_{\text{Error}}$$

can also be said, $= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0 \cdot x^6}_{P_6(x)}$ → $|f(x) - P_6(x)| =$

↳ Truncation error

$$\underbrace{\left|\frac{f^7(\xi)}{7!}(x-x_0)^7\right|}_{\text{error}}$$

using the taylors theorem,

$f^7(x) = -\cos(x)$

$\boxed{x = 0.1}$

$$|f(x) - P_6(x)| = \left|\frac{-\cos(\xi)}{7!}(x-x_0)^7\right|$$

$$= |f(0.1) - P_6(0.1)| = \underbrace{\left|\frac{-\cos(\xi)}{7!}(0.1)^7\right|}$$

$$\frac{\text{max value of } \cos(\xi)}{1}$$

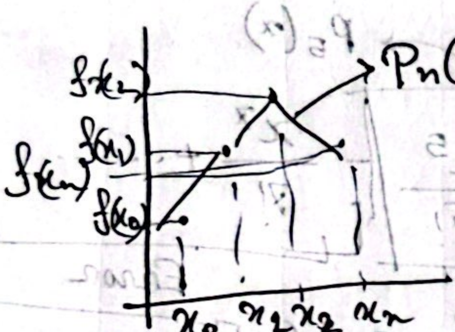
[learn machine epsilon]

$$\underbrace{\left|f(0.1) - P_6(0.1)\right|}_{\text{Truncation Error}} \le \underbrace{\left|\frac{1}{7!}(0.1)^7\right|}_{\text{Max bound / Error}}$$

$$\le \boxed{1.984 \times 10^{-11}}$$

~~2.3~~ 2.5

Polinomial Interpolition (Vandermonde Matrix)



$P_n(x) \longrightarrow (n+1)$ nodes

$(n+1)$ nodes like coefficient $(n+1)$ coeffs

$$P_n(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

$$= \sum_{k=0}^{n} a_k x^k$$

$a_k \rightarrow$ coefficients

$x^k \rightarrow$ natural basis

coefficients

$\hookrightarrow$ Find out the co-efficients

Impoatant

I have given $(n+1)$ nodes,

it will satisfy $(n+1)$ conditions.

to find $(n+1)$ coeffs.

Ex: Given

x_0 ; $f(x_0)$

x_1 ; $f(x_1)$

x_2 ; $f(x_2)$

$\vdots$

x_n , $f(x_n)$

conditions/equations

$$\underbrace{\begin{aligned} P_n(x_0) &= f(x_0) \\ P_n(x_1) &= f(x_1) \\ P_n(x_2) &= f(x_2) \\ &\vdots \\ P_n(x_i) &= f(x_i) \end{aligned}}_{n+1 \text{ conditions}} \quad [\because i = 0, 1, 2, \dots, n]$$

$$P_n(x_0) = f(x_0)$$

$$a_0 + a_1 x_0 + a_2 x_0^2 + \dots a_n x_0^n = f(x_0) \dots \text{(i)}$$

$$P_n(x_1) = f(x_1)$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \dots a_n x_1^n = f(x_1) \dots \text{(ii)}$$

$$a_0 + a_1 x_2 + a_2 x_2^2 + a_3 \cdot x_2^3 + \dots a_n x_2^n = f(x_2) \dots \text{(iii)}$$

$\vdots$

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \dots a_n x_n^n = f(x_n) \dots (n+1)$$

Generalized: [matrix method]

vandermond matrix, V

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix}}_{\cancel{[m \times n]}\ [n+1,\, n+1]} \times \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\substack{A \\ (n+1,\,1)}}$$

$$= \underbrace{\begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}}_{F} \quad (n+1,\,1)$$

(i) $\boxed{V \cdot A = F}$

(ii) $\boxed{A = V^{-1} \cdot F}$ [Matrix V must be invertible] determinant $(V) \neq 0$

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \text{---}$

Example

$x_0 = 0 \qquad x_1 = \frac{\pi}{2} \qquad x_2 = \pi$

$f(x_0) = 1 \qquad f(x_1) = 0 \qquad f(x_2) = -1$

ⓐ Determine the degree of the polynomial hence the interpolating polynomial equation.

$P_n(x) = ?$

$n=2$

3 nodes $\longrightarrow$ degree $= 2 \rightarrow P_2(x) = a_0 + a_1 x + a_2 x^2$

$3 = n+1$

$\therefore n = 2$

$V \rightarrow (3\times 3)$

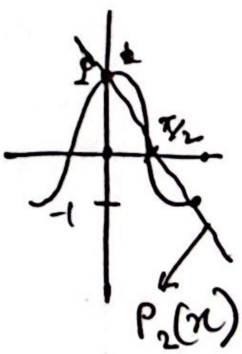
$V \cdot A = f$

$$\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} 1 & 0^1 & 0^2 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & \pi^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & \pi^2/4 \\ 1 & \pi & \pi^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/\pi \\ 0 \end{bmatrix}$$

$P_2(x) = 1 - \frac{2}{\pi} x$



ⓑ Hence find $P_n\left(\frac{\pi}{4}\right)$

CH: "Lagrange Interpolation/ Polynomial

$\{1, x, x^2, \ldots, x^n\}$
Natural Basis

$P_n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

$P_n = \sum_{k=0}^{n} a_n x^k$ → Variables/Natural basis

↳ coefficients [Find]

Problems of vandermonde Matrix Method:

$VA = F$
$A = V^{-1} F$

* V has to be invertible
[$\det(V) \neq 0$]

* Calc V^{-1} when new node is added needs high computational power

$P_n = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x) \ldots + f(x_n)\, l_n(x)$

↪ $\{l_0(x), l_1(x), l_2(x), \ldots l_n(x)\}$

$\Rightarrow P_n = \sum_{k=0}^{n} f(x_k)\, l_k(x)$

↳ lagrange basis/ variables [Find]

↳ coeff / Lagrange coeffs (Given)

[Eg]

$x_0 = -\frac{\pi}{4}$ | $x_1 = 0$ | $x_2 = \frac{\pi}{4}$

$f(x_0) = \frac{1}{\sqrt{2}}$ | $f(x_1) = 1$ | $f(x_2) = \frac{1}{\sqrt{2}}$

[using lagrange interpolation

(a) $P_n(x)$

(b) $P_n(1.6)$

(a) Nodes: 3 $\downarrow$ 3 coefficients $\rangle$ 2 degree $= P_2(x)$

$$\boxed{P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x)}$$

num of degree = That many fraction multiplys.

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2} = \frac{x-0}{-\pi/4 - 0} \cdot \frac{x - \pi/4}{-\pi/4 - \pi/4}$$

degree = 2

main $x = x_0$

$\frac{x -}{x_0 -}$ | if $x = x_1$ $\frac{x -}{x_1 -}$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_2}{x_1 - x_2}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1}$$

lagrange → easy

$$l_0(x) = \frac{8}{\pi^2} x\left(x - \frac{\pi}{4}\right)$$

$$l_1(x) = \frac{x + \frac{\pi}{4}}{0 + \frac{\pi}{4}} \times \frac{x - \frac{\pi}{4}}{0 - \frac{\pi}{4}}$$

$$= -\frac{16}{\pi^2}\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$l_2(x) = \frac{8}{\pi^2} x\left(x + \frac{\pi}{4}\right)$$

$$P_2(x) = \frac{1}{\sqrt{2}}\left(\frac{8}{\pi^2}x\right)\left(x - \frac{\pi}{4}\right) + 1 \cdot \left(\frac{-16}{\pi^2}\right)\left(x + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$+ \frac{1}{\sqrt{2}}\left(\frac{8}{\pi^2}x\right)\left(x + \frac{\pi}{4}\right)$$

Advantages of lagrange interpolation

↳ No need to inverse a matrix

Problems:

→ New nodes cannot be added, if added, need to calculate the $l_k(x)$ all over again as each node increases multiplying.

CH = 2.4

Newton's Interpolation

$\rightarrow$ "Newton's Divided Difference Interpolation"

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

Basis: $\{1, \{(x-x_0)\}, \{(x-x_0)(x-x_1)\}, \{(x-x_0)(x-x_1)(x-x_3)\} \dots\} \longrightarrow$ Newtons basis

$$= \{n_0(x), n_1(x), n_2(x), n_3(x), \dots n_n(x)\}$$

$$\therefore P_n(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x) + a_3 n_3(x) + \dots a_n n_n(x)$$

$$\Rightarrow P_n(x) = \sum_{k=0}^{n} a_k n_k(x)$$

$n_k(x) \longrightarrow$ Newton's Basis [Given]

$a_k \longrightarrow$ Newtons coeff. [Find]

Newton's Coeff

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$a_3 = f[x_0, x_1, x_2, x_3]$

$\vdots$

$a_n = f[x_0, x_1, x_2, x_3 \ldots x_n]$

$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2]$
$(x-x_0)(x-x_1) + \ldots f[x_0, x_1, x_2, \ldots x_n]$
$(x-x_0)(x-x_1) \ldots (x - x_{n-1})$

Eg:

$x_0 = -1$	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$
$f(x_0) = 5$	$f(x_1) = 1$	$f(x_2) = 1$	$f(x_3) = 11$

(a) $P_n(x)$

(b) $P_n(1.6)$

4 nodes
4 coeffs $\Big>$ degree 3 $= P_3(x)$

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$\Rightarrow P_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

Tabular method

Newton's coeff

x_0, x_1, x_2, x_3

$x_0 = -1$ $\quad f[x_0] = 5$

$$f[x_0,x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0} = \frac{1-5}{0+1} = -4$$

$x_1 = 0$ $\quad f[x_1] = 1$

$$f[x_1,x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1} = \frac{0}{1-0} = 0$$

$x_2 = 1$ $\quad f[x_2] = 1$

$$f[x_2,x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2} = \frac{10}{1} = 10$$

$x_3 = 2$ $\quad f[x_3] = 11$

$$f[\underbrace{x_0,x_1}, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{0+4}{1+1} = 2$$

$$f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{10-0}{2-0} = 5$$

$$\rightarrow f[x_0, x_1, x_2, x_3]$$

$$= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

[Landscape]

$$= \frac{5-2}{2+1} = 1$$

$$P_3(x) = 5 - 4(x+1) + 2(x+1)(x) + 1(x+1)(x)(x-1)$$

new row can be added in terms of adding a new node.

Advantage of newton's divided Difference

→ can keep on adding nodes but no need to do calculations from the very beginning

Quiz → 2.2, → 2.5

Quizz → 2.2, 2.3, 2.4

2.2 → Vandermonde

2.3 → Lagrange

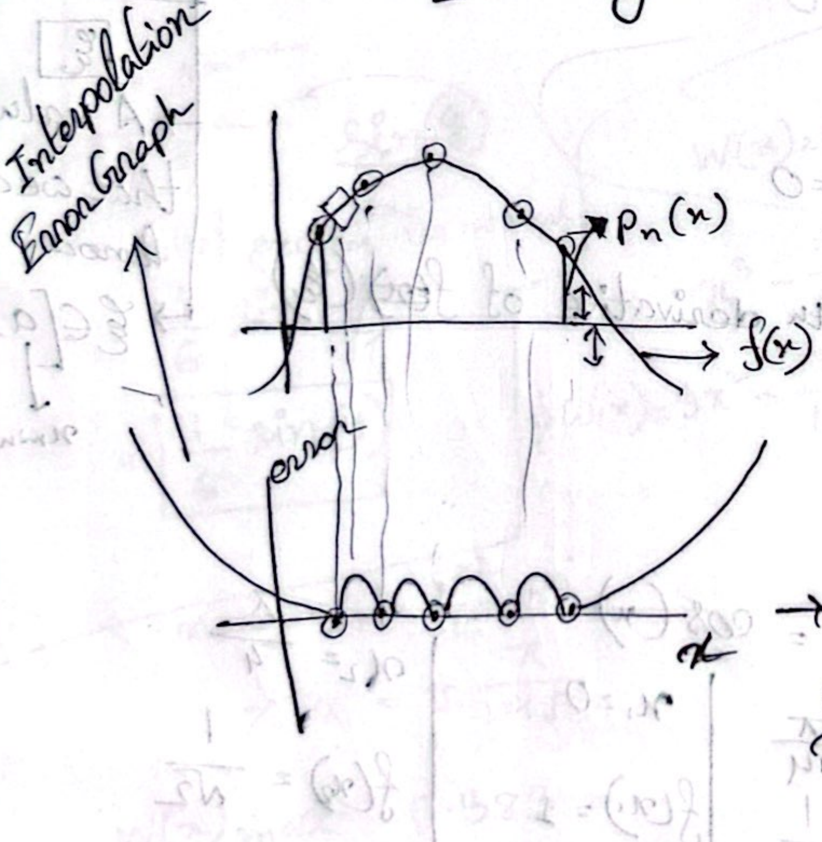
2.4 → Newtons devide

CH - 2.6

~~Chebyshev~~

2.5 Interpolation Error

[Cauchy's Theorem]



$$|f(x) - P_n(x)| \rightarrow \text{error}$$

→ Error = 0 at the nodes

$f(x_i) = P_n(x_i)$

$f(x_0) = P_n(x_0)$

$f(x_1) = P_n(x_1)$

→ We can never find the exact error at any given point.

→ We're worried about maximum error

Find out the maximum error = Cauchy's Theorem

Cauchy's Theorem:

nodes
x_0
x_1
x_2

$$\left|f(x) - P_n(x)\right| = \left|\frac{f^{n+1}(\xi)(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!}\right|$$

ξ

→ A value of x that we don't know.

→ $\xi \in [a, b]$

$a \to x_{min}$, $b \to x_{max}$

$\boxed{I}$ → At, $x = x_0$ R.H.S = 0

$\boxed{II}$ → $f^{n+1}(\xi)$ = $(n+1)^{th}$ derivative of $f(\xi)$

$\boxed{III}$ →

Example:

$f(x) = \cos(x)$

$x_0 = -\frac{\pi}{4}$	$x_1 = 0$	$x_2 = \frac{\pi}{4}$
$f(x_0) = \frac{1}{\sqrt{2}}$	$f(x_1) = 1$	$f(x_2) = \frac{1}{\sqrt{2}}$

$[-1, 1]$ interval/values of x

(a) $P_n(x)$

(b) Max^m error of the interp. ($P_n(x)$) [using cauchy] OR $[-1, 1]$

3 Nodes $\longrightarrow P_2(x)$

$$|f(x) - P_2(x)| = \left|\frac{f^3(\xi)}{3!}\left(x+\frac{\pi}{4}\right)(x-0)\left(x-\frac{\pi}{4}\right)\right|$$

maximum value

$$= \left|\underbrace{\frac{\sin(\xi)}{6}}\ \underbrace{\left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)}_{W(x)}\right|$$

separately max,

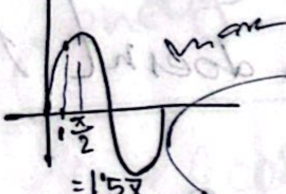
$f(x) = \cos(x)$

$f(\xi) = \cos(\xi)$

$f'(\xi) = -\sin(\xi)$

$f''(\xi) = -\cos(\xi)$

$f'''(\xi) = \sin(\xi)$



$\frac{\sin(\xi)}{6}$

but it exceeds the interval.

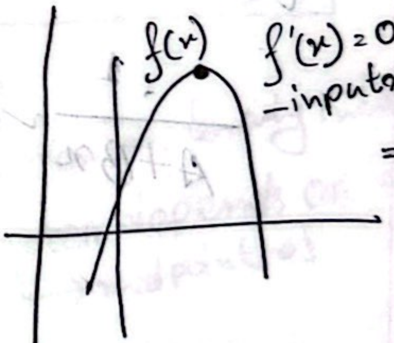
$= \frac{1}{6}\sin(\xi)$

$= \boxed{\frac{1}{6}\sin(1)}$

$W(x) = \left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)$ max ξ

$= x^3 - \frac{\pi^2}{16}x$

$\Rightarrow W'(x) = 3x^2 - \frac{\pi^2}{16}$



$f'(x) = 0$, $x = --$

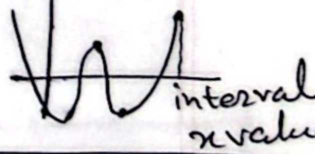
input $x \to f(x)$

$= \square$

max/min

now, $3x^2 - \frac{\pi^2}{16} = 0$

$\Rightarrow x = \pm\frac{\pi}{4\sqrt{3}}$



interval x value

$W(x)$ max $= 0.383$

final error max:

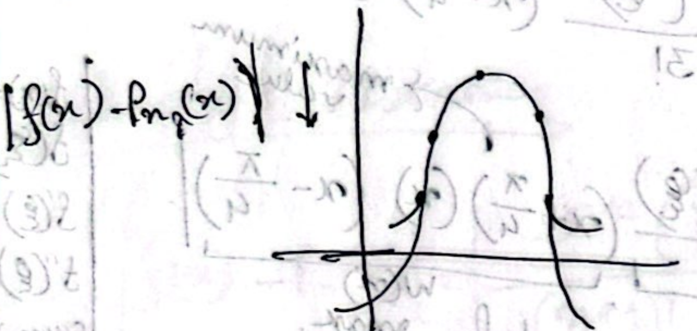
$= \frac{\sin(1)}{6} \cdot 0.383$

x	$W(x)$
$-\pi/4\sqrt{3}$	-0.186
$+\pi/4\sqrt{3}$	$+0.186$
-1	-0.383
	0.383

Trigonometric values → calc → r mode

CH - 2.6

CHEBYSHEV Nodes

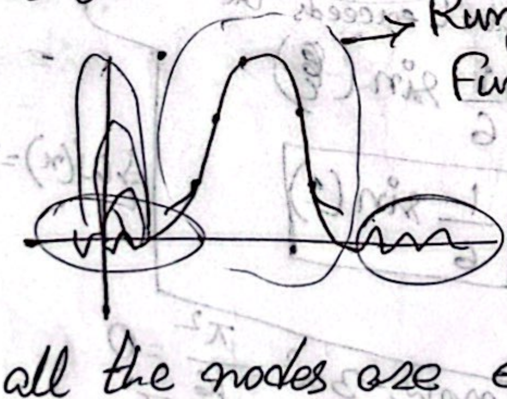


$n \to \infty$, Error $\to 0$

Runge

There are some function who doesn't follow the above scenario.

$$\frac{1}{A + Bx^2}$$



→ all the nodes are equally spaced.

→ nodes at the corner are less

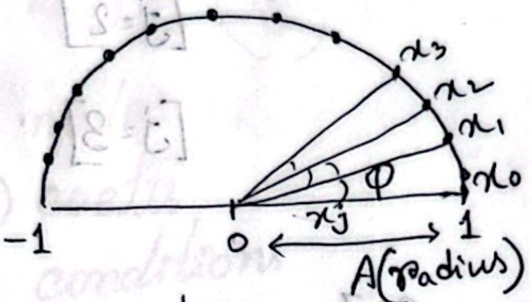
→ $\boxed{n \to \infty, \text{ error} \to \infty}$

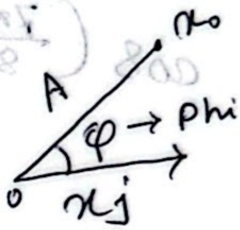
↳ Runge Phenomenon

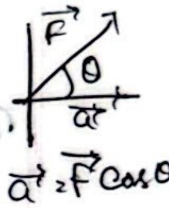
Chebyshev

→ Do not take equally spaced nodes. Take equally spaced angles

angles algo → node's value







$x_j = A\cos\varphi + \text{center}$ [$j = 0, 1, 2, \cdots n$]

chebyshev nodes

$$\boxed{x_j = A\cos\frac{(2j+1)\pi}{2(n+1)} + \text{center}}$$ [$j = 0, 1, 2, \ldots n$]

$$\boxed{\varphi = \frac{(2j+1)\pi}{2(n+1)}}$$

Ex $f(x) = \dfrac{1}{1+25x^2}$ → Runge on $I = [-1, 1]$ → center to any interval

A & center depends on interval
↳ mid point of "

[$P_3(x)$]

Find out all the nodes.

Soln

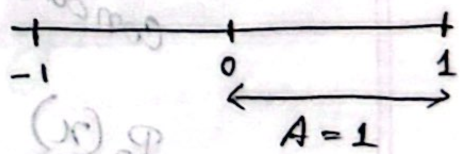
4 nodes.

$x_0 =$

$x_1 =$

$x_2 =$

$x_3 =$



$x_0 = 1.\cos\left(\frac{1}{8}\pi\right) + 0 \quad [j=0]$

$x_1 = 1.\cos\left(\frac{3\pi}{8}\right) \quad [j=1]$

$x_2 = \cos\left(\frac{5\pi}{8}\right) \quad [j=2]$

$x_3 = \cos\left(\frac{7\pi}{8}\right) \quad [j=3]$

CH - 2.7

Hermite Interpolation

x_0	$f(x)$
0	10
1	-20
2	40

$$P_n(x) = \sum_{k=0}^{n} a_k \underbrace{x^k}_{\text{natural Basis}}$$] V. Matrix

$$P_n(x_i) = f(x_i) \quad \ldots \quad [i = 0, 1, 2, \ldots n]$$

$(n+1)$ conditions

$P_2(x)$

$n = 2$

$$3\left\{\begin{aligned} P_2(x_0) &= f(x_0) \\ P_2(x_1) &= f(x_1) \\ P_2(x_2) &= f(x_2) \end{aligned}\right.$$

3

in hermite,

x_0	$f(x)$	$f'(x)$
0	10	2
1	-20	-4
2	40	6

$(n+1)$ nodes

n – $(n+1)$ coeffs

$(n+1)$ conditions

new condition:

$$P_n'(x_i) = f'(x_i)$$

$$P_2(x_0) = f'(x_0)$$

$$P_2(x_1) = f'(x_1)$$

$$P_2(x_2) = f'(x_2)$$

3

$$(3+3) = 6 - 1 = 5$$

$\therefore$ Hermite int $= P_5(x)$

Before $n+1$ conditions $\xrightarrow{(-)}$ n degree

After $(n+1)(n+1) = 2n+2 \xrightarrow{(-1)} 2n+1$

[Hybrid interpolation]

$\therefore$ degree of hermite $= \boxed{2n+1}$

$= 2(2)+1$

$= 5 \Rightarrow P_5(x)$

$$P_n(x) = \sum_{k=1}^{n} f(x_k)\, l_k(x)$$] Lagrange

Hat

$$P_{2n+1}(x) = \sum_{k=1}^{n} \left(f(x_k)\, h_k(x) + f'(x_k)\, \hat{h}_k(x) \right)$$

derivative

$$\boxed{\begin{aligned} h_k(x) &= \left[1 - 2(x - x_k)\, l_k'(x_k)\right] l_k^2(x) \\ \hat{h}_k(x) &= (x - x_k)\, l_k^2(x) \end{aligned}}$$

<u>advantages</u>

→ Error gets lessen, by the use of same number of nodes than other interpolation technique.

Hermite increases the degree of the polynomial

Example

$f(x) = \sin(x)$

x_0	$f(x)$	$f'(x)$
0	0	1
$\frac{\pi}{2}$	1	0

\# n's value & degree will be different.

ⓐ $P_n(x) = ?$

ⓑ $P_n(1/6) = ?$

Ans

ⓐ $\boxed{n = 1}$

degree of hermite $= 2.(1)+1$

$= 3$ [degree of the polynomial]

$$P_3(x) = \cancel{f(x_0)}^{0} h_0(x) + f(x_1)^{1} h_1(x) + \cancel{f'(x_0)}_{1} \hat{h}_0(x) + \cancel{f'(x_1)}_{0} \hat{h}_1(x)$$

mid's syllabus: 3.2

$$P_3(x) = h_1(x) + \hat{h}_0(x)$$

expand $h_1(x)$ & $\hat{h}_0(x)$

CH - 2.7
Hermite Interpolation

Example

$f(x) = \sin(x)$

		$f(x)$	$f'(x)$
x_0	0	0	1
x_1	$\frac{\pi}{4}$	1	0

$n = 1$

$2n+1$

$2.(1)+1 = 3$

$P_3(x) = h_1(x) + \hat{h}_0(x)$

$\boxed{k=1}$

$h_1(x) = [1 - 2(x - x_1)\, l_1'(x_1)]\, l_1^2(x)$

$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{\pi/2} = \frac{2x}{\pi}$

$l_1'(x) = \frac{2}{\pi}$

$l_1'\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$

$$h_1(x) = \left[1 - 2\left(x - \frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left(\frac{2x}{\pi}\right)^2$$

$$= \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

$$\hat{h}_0(x) = (x - x_0)\, l_0^{\,2}(x)$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \frac{\pi}{2}}{-\frac{\pi}{2}} = 1 - \frac{2}{\pi}x$$

$$\therefore \hat{h}_0(x) = (x - 0)\left(1 - \frac{2}{\pi}x\right)^2$$

$$\hat{h}_0(x) = x\left(1 - \frac{2}{\pi}x\right)^2$$

$$\therefore P_3(x) = \frac{4}{\pi^2}x^2\left(3 - \frac{4}{\pi}x\right) + x\left(1 - \frac{2}{\pi}x\right)^2$$

[Highest power of both sides will be equal] (Degree)

Mid
$\rightarrow$ 3.2 "Richardson Extrapolation & Rounding Error

CH - 3.1

"Differentiation"

Slope/Gradient
Tangent

$y = ax^n$

$\frac{dy}{dx} = n.ax^{n-1}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$

$f(x) = ax^n$

$f'(x) = n.ax^{n-1}$

Rules

$\rightarrow$ Chain Rule

$\rightarrow$ Product Rule

$\rightarrow$ Quadrent Rule [u/v]

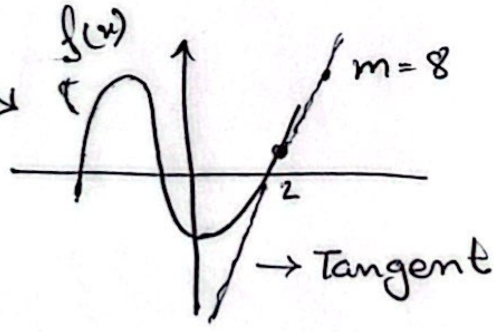
Ex

$f(x) = x^3 - 4x + 1$

$f'(x) = 3x^2 - 4$

$f'(2) = 3(2)^2 - 4$

$= 8$



Numerical Approach

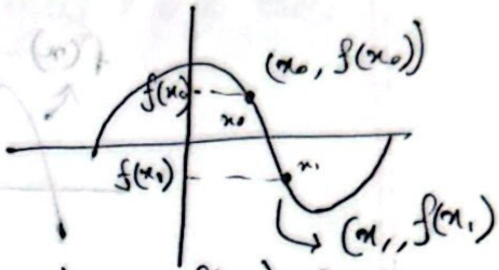
① Forward ~~Differentiation~~ Difference [FD]

② Backward Differentiation [BD]

③ Central Differentiation [CD]

Forward Differentiation

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$



$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0}$$

$$f' = \frac{f'(x_0 + h) - f(x_0)}{h}$$

h = step size
↳ it has a small value (< 1)
↳ The smaller the value of h, ~~th~~ more accuracy & less error it gets.

Ex

$f(x) = x^3 - 4x + 1$

find $f'(x)$ using FD at $x = 2$ & $h = 0.1$

or $x_0 = 2, x_1 = 2.1$

$\therefore h = x_1 - x_0$

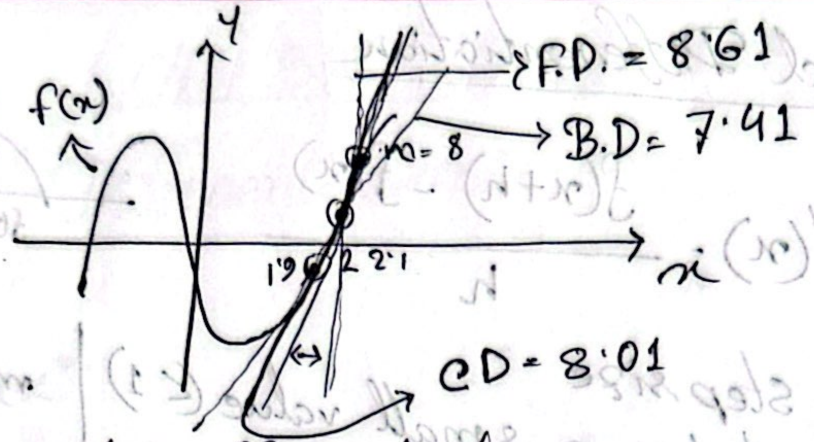
Ans:

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{f(2 + 0.1) - f(2)}{0.1}$$

$$= \frac{\left((2.1)^3 - 4.(2.1) + 1\right) - \left((2)^3 - 4(2) + 1\right)}{0.1}$$

$$= 8.61$$



Backward Differentiation

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

Ex $f(x) = x^3 - 4x + 1$

find $f'(x)$ using BD at $x = 2$ & $h = 0.1$

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(2) - f(1.9)}{0.1}$$

$$= 7.41 \text{ (Ans)}$$

least error

Central Differentiation (avg of FD & BD)

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

CD gives us the least error compare to FD & BD.

Ex $f(x) = x^3 - 4x + 1$

find $f'(x)$ using CD $h = 0.1$, $x = 2$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$= \frac{f(2.1) - f(1.9)}{0.2}$$

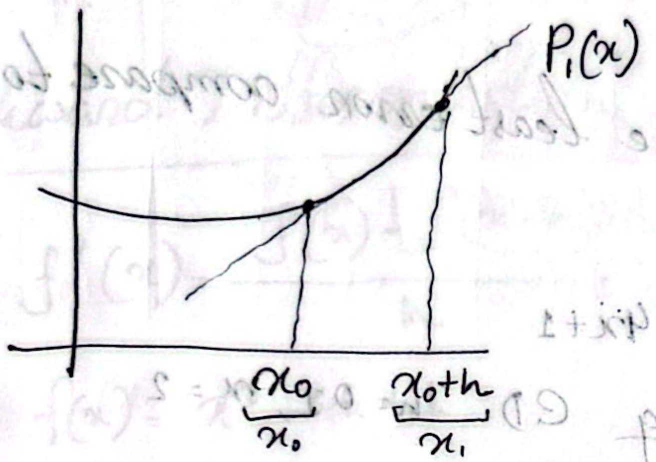
$$= 8.01$$

Quiz 3 → 2.6, 2.7, 3.1

Chebyshev, hermite, Differentiation

Numerical Differenciation

Forward Difference Error:



using lagrange,

$$P_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

$$= \frac{x-x_1}{x_0-x_1} \cdot f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

Cauchy's Theorem:

$$f(x) = P_1(x) + \boxed{\text{Error}} \rightarrow \text{Cauchy's } T$$

$$= \underbrace{\frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)}_{P_1(x)} + \overbrace{\frac{f''(\xi)}{2}}^{u} \overbrace{(x-x_0)(x-x_1)}^{v}$$

$$\frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1) \ldots (x-x_n)$$

$$f'(x) = \frac{1}{x_0-x_1}f(x_0) + \frac{1}{x_1-x_0}f(x_1) + \frac{f''(\xi)}{2}(2x-x_0-x_1) + \frac{f'''(\xi)}{2}\cdot\frac{d(\xi)}{dx}(x-x_0)(x-x_1)$$

$\hookrightarrow -\frac{1}{x_1-x_0}f(x_0)$

Plug $x = x_0$

$$\boxed{f'(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0} + \frac{f''(\xi)}{2}(x_0-x_1)}$$

x_0+h

x_0+h-x_0

$\boxed{x_1 = x_0+h}$ [FD]

$$\frac{f(x_0)x - f(x_0)x_1}{x_0-x_1}$$

$$\frac{f(x_0)\cdot x}{x_0-x_1} - \cancel{\frac{f(x_0)x_1}{x_0-x_1}}$$

$$uv = u\frac{d}{dx}v + v\frac{d}{dx}u$$

$$f'(x_0) = \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\text{FD}} + \underbrace{\frac{f''(\xi)}{2}(-h)}_{\text{Truncation Error}}$$

Same for Backward Diff. Error

T. Error $\propto h$

Example

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$x=2$, FD, $h = 1, 0.1, 0.01, 0.001$

Find error using FD.

Using Forward Differenciation

	h	$f'(x_0)$	Truncation Error	
$\div 10$	1	0.405465	0.0945349	$\div 10$
$\div 10$	0.1	0.487902	0.0120984	$\div 10$
$\div 10$	0.01	0.498754	0.00124585	$\div 10$
	0.001	0.499875	0.00012000	

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \frac{\ln(2+h) - \ln(2)}{h}$$

$h = 1$,

$$\frac{\ln(3) - \ln(2)}{1}$$

$h = 0.1$,

$$\frac{\ln(2.1) - \ln(2)}{0.1}$$

T.E = |Actual Diff val − FD value|

$f'(2) = \frac{1}{2} = 0.5$ (Actual val)

→ |0.5 − FD value|

TE ($h=1$) = $0.5 - 0.405465$

Moral: If the h is divided by 10, error also gets divided by 10.

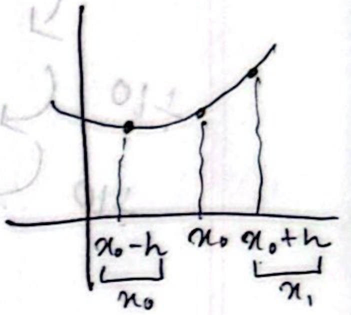
$\therefore$ error $\propto h$.

Backward Differenciation error:

Same as the Forward Differenciation error

$[f(x_0+h) \longrightarrow f(x_0-h)]$; Error $\propto h$

Central Differenciation Error



$x_0 \rightarrow x_0 - h$

$x_1 \rightarrow x_0 + h$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + \underbrace{\left| \frac{f'''(\xi)}{3!} (-h^2) \right|}_{\text{Trun. Error}}$$

$\rightarrow$ Error $\propto h^2$

Eg $f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$; $f'(2) = \frac{1}{2} = 0.5$

$x_0 = 2$, $h = 1, 0.1, 0.01, 0.001$

maintain significant error

h	$f'(x_0)$	Trunc. Error
	$\dfrac{f(x_0+h)-f(x_0-h)}{2h}$	|Actual val - cent. Diff val|
1	0.479306	0.0493061
0.1	0.500417	0.000417293
0.01	0.5000004	4.16673×10^{-6}
0.001	0.5000000	4.16666×10^{-8}

$\div 10$ $\div 10$ $\div 10$

$\div 10^2$ $\div 10^2$ $\div 10^2$

[if h is divided by 10, error gets divided by 100.
error $\propto h^2$]

—— end of Ch-3.1 ——

Quiz-3 up until this (2.6, 2.7, 3.1)

CH-3.2

Rounding Error

Richardson Extrapolition

Rounding Error

C.Diff: $f'(x) = \frac{f(x+h) - f(x-h)}{2h}$

→ smaller the step size (h), better result

Scale Invariant Error ⟶ Rounding error

$\delta = \frac{|fl(x) - x|}{|x|}$

→ However if h is very small, f(x+h) & f(x-h) will have similar value.

Eg $f(2+0.0001) - f(2-0.0001)$

$fl(x) = (1+\delta)x$

→ In CH-1: Loss of Significance

→ very small h creates a problem → Loss of Significance

$\cancel{fl}\ fl(x) = (1+\delta_1)x$

$fl[f(x_1+h)] = (1+\delta_1) f(x_1+h)$

$fl[f(x_1-h)] = (1+\delta_2) f(x_1-h)$

$\boxed{|\delta_1|, |\delta_2| \le \epsilon_M}$ ← CH-1
Machine Epsilon

Round Error:

$$\left| \begin{matrix}\text{Actual} \\ \text{value}\end{matrix} \text{ of Diff} - \begin{matrix}\text{value of Diff} \\ \text{by Numerical Approach}\end{matrix} \right|$$

$P_2(x) + \frac{f'''(\xi)}{5!}(-h^2)$

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \underbrace{\frac{f'''(\xi)}{3!}}_{\text{Cauchy's T}} h^2 - \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h} \right|$$

cancel [57:10]

$$= \left| \underbrace{\frac{f(x_1+h) - f(x_1-h)}{2h}}_{\text{cancel out}} - \frac{f'''(\xi)h^2}{3!} - \frac{(1+\delta_1)f(x_1+h) - (1+\delta_2)f(x_1-h)}{2h} \right|$$

$$= \left| \underbrace{\frac{-f'''(\xi)h^2}{3!}}_{a} - \underbrace{\frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h}}_{b} \right|$$

$|a+b| \le |a| + |b| \longrightarrow |5+(-1)| \qquad |5| + |-1|$

$|4| \qquad 5+1$

$4 \le 6$

$$\le \left| \underbrace{\frac{-f'''(\xi)h^2}{3!}}_{a} \right| + \left| \underbrace{-\frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h}}_{b} \right|$$

$$\leq \left| \frac{f'''(\xi) h^2}{3!} \right| + \left| \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

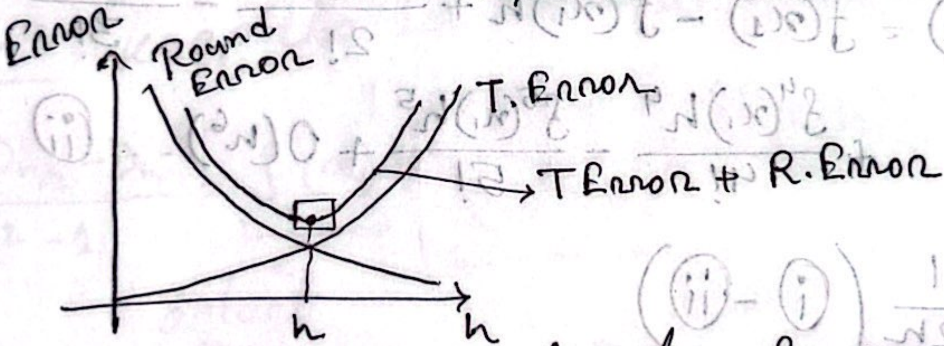
$|\delta_1|, |\delta_2| \leq \epsilon_M$

$$\leq \underbrace{\left\{ \frac{|f'''(\xi) h^2|}{3!} \right.}_{\text{Truncation Error}} + \underbrace{\epsilon_M \frac{\left(|f(x_1+h)| + |f(x_1-h)| \right)}{2h}}$$

↳ (upper bound of the rounding error)

→ smaller h, less Error

⇒ Smaller h, larger rounding error
→ h is in denominator



↘ optimal value for h.

Q. State a concequence of decreasing the value of h, when it comes to Numerical Differenciation

↳ h↓ T. Error↓, R-Error↑

CH-3.2

Richardson Extrapolation

$$D_h \leftarrow f'(x_1) = \frac{f(x_1+h)-f(x_1-h)}{2h}$$

$x_1+h \rightarrow x_2$ [Centered val]
x_1+h

Step-1

Taylor series breakdown (centered at x_1)

$$f(x_1+h) = f(x_1) + f'(x_1)h + \frac{f^2(x_1)h^2}{2!} + \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} + \frac{f^5(x_1)h^5}{5!} + O(h^6) \quad \text{--- (i)}$$

$$f(x_1-h) = f(x_1) - f'(x_1)h + \frac{f^2(x_1)h^2}{2!} - \frac{f^3(x_1)h^3}{3!} + \frac{f^4(x_1)h^4}{4!} - \frac{f^5(x_1)h^5}{5!} + O(h^6) \quad \text{--- (ii)}$$

$$D_h = \frac{1}{2h}\left(\text{(i)} - \text{(ii)}\right)$$

$$= \frac{1}{\cancel{2}\cancel{h}}\left[\cancel{2}f'(x_1)\cancel{h} + \frac{2f^3(x_1)h^3}{3!} + \frac{2f^5(x_1)h^5}{5!} + O(h^7)\right]$$

$$D_h = \underbrace{\boxed{f'(x_1)}}_{\text{actual Diff}} + \underbrace{\frac{f^3(x_1)h^2}{3!} + \frac{f^5(x_1)h^4}{5!} + O(h^6)}_{\text{Error}}$$

actual Diff

Error

Dominating error term

Step-2 Vanish the dominating error term

$$D_{h/2} = f'(x_1) + \frac{f^3(x_1)(h/2)^2}{3!} + \frac{f^5(x_1)(h/2)^4}{5!} + O((h/2)^6)$$

$$2^2 D_{h/2} = 2^2 f'(x_1) + \frac{f^3(x_1)h^2}{3!} + \frac{1}{2^2}\frac{f^5(x_1)h^4}{5!} + O\left(\frac{h^6}{2^4}\right)$$

$$\boxed{2^2 D_{h/2} - D_h}$$

$$= (2^2-1)\underbrace{f'(x_1)} + \left(\frac{1}{2^2}-1\right)\frac{f^5(x_1)h^4}{5!} + O(h^6)$$

Step-3 keep the 1st term of $2^2 D_{h/2} - D_h$ as $f'(x_1)$

$$\frac{2^2 D_{h/2} - D_h}{2^2-1} = \underbrace{f'(x_1)}_{\text{actual Diff}} + \underbrace{\left(\frac{\frac{1}{2^2}-1}{2^2-1}\right)\cdot\frac{f^5(x_1)h^4}{5!} + O(h^6)}_{\text{Error}}$$

$\to$ Dominating term $= h^4$

$$\therefore D_h^{(1)} = \frac{2^2 D_{h/2} - D_h}{2^2-1}$$

$$D_h^{(2)} = ?$$

Question Format

$$D_h^{(1)} \to D_{h/2}^{(1)}$$
$$\to 2^2 D_{h/2}^{(1)} - D_h^{(1)}$$

$x_0 = 2$

~~$x_1 = 1$~~

$h = 1, 0.5$, use R.E. to find out the value for the given h.

$$D_h^{(1)} = \frac{2^2 D_{0.5} - D_1}{2^2 - 1}$$

$D_h = CD$

in CD, $h = 0.5$; value $\times 2^2 - CD, h = 1$,

devided by $2^2 - 1$

CH - 3.2

RICHARDSON EXTRAPOLATION

$$D_h \to f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h}$$

step-1

Expand D_h in Taylor Series along with the Error terms.

$D_h \to$ Taylor Series

$$= \frac{1}{2h}\left(2f'(x)h + 2(h^3) + 2(h^5) + O(h^7)\right)$$

$\to$ Dominating term

$$D_h = \underbrace{f'(x)}_{\text{Diff}} + \underbrace{c_1 h^2 + c_2 h^4 + O(h^6)}_{\text{Error}}$$

step-2 Vanish the dominating term.

$$D_{h/2} \to K_1 \left(\tfrac{h}{2}\right)^2$$

$$2^2 D_{h/2}$$

step 3 $2^2 D_{h/2} - D_h$

step 4

$$\boxed{\frac{2^2 D_{h/2} - D_h}{2^2 - 1}} \to D_h^{(1)}$$

as $f'(x)$ gets multiplied by (2^2-1) in step-3.

$D_h^{(2)}$

↳ step 2 Repeat but with $D'_{h/2}$ & D'_h

$$D_h^{(1)} = \frac{2^2 D_{h/2} - D_h}{2^2 - 1}$$

$$D_h^{(2)} = \frac{2^4 D_{h/2}^{(1)} - D_h^{(1)}}{2^4 - 1}$$

$$\boxed{D_h^{(k)} = \frac{2^n D_{h/2}^{(k-1)} - D_h^{(k-1)}}{2^n - 1}}$$

$$\boxed{n = 2k}$$

Eg - 1

$f(x) = e^x \sin(x)$

$h = 0.5$

$h = 0.25$

Q. find $f'(1)$ using R.E

Ans: Tabular Approach

$D_h \to CD \to \frac{f(x+h)-f(x-h)}{2h}$

$h_1 = 0.5 \quad D_{h_1} = \frac{f(x+h)-f(x-h)}{2h}$

~~h_2~~ $= \frac{e^{1.5}.\sin(1.5) - e^{0.5}.\sin(0.5)}{2\times 0.5}$

$[D_h] = 3.68$

$D'_h = \frac{2^2 D_{h/2} - D_h}{2^2-1}$

$h_2 = 0.25 \quad D_{h_2} = \frac{e^{1.25}\sin(1.25) - e^{0.75}\sin(0.75)}{2\times 0.25}$

$= \frac{2^2 \cdot 3.7385 - 3.68}{3}$

$[D_{h/2}] = 3.7385$

$= 3.757$

$\therefore D^1_h = 3.757$ ✓ first order richardson

$f'(x) = e^x \cos x + e^x \sin(x)$

$f'(1) = e^1 \cos 1 + e^1 \sin(1)$

$= 3.756$

E.g-2

$h = 0.1$

$h = 0.2$

$f'(1) = 0.7 \quad f'(1) = 0.5$

$[D_{h_1}] \{D_{h/2}\} \quad [D_{h_2}] \{D_h\}$

Q. find $f'(1)$ using R.E.

Ans:

$$D^1_h = \frac{2^2 D_{h/2} - D_h}{2^2 - 1}$$

$$\therefore D^1_h = \frac{2^2 (0.7 - 0.5)}{3}$$

Theory:

How/ Why Richardson Extrapolition is the better technique than all the other 3 methods

↳ It vanishes the dominating error term.

Review [Missed Part]

Cauchy's Theorem

$$P_n(x) + \underbrace{\left|\frac{f^{n+1}(\xi)}{(n+1)!}\right|}_{\text{Graph}} \underbrace{\left|(x-x_0)(x-x_1)\ldots(x-x_n)\right|}_{w(x)}$$

Lagrange

$$\frac{f^{n+1}(\xi)}{(n+1)!} (x - x_0)^n$$

Q. What if the interval is not given

→ For finding $W(x)_{max}$: Leftmost Interval = x_0
Rightmost Interval = x_n

→ For finding $\left|\frac{f^{n+1}(\xi)}{(n+1)!}\right|_{max}$ → Draw the graph & find out highest value in graph

$\sin x - \cos x$
$a - b$
$a_{max} - b_{min}$
$\Rightarrow \sin x_{max} - \cos_{min}$
$= 1 - 0$

Question: How to find the Truncation Error.

Mid Term Review

Error

CH-① Rounding Error → $|fl(x)-x|$

Scale Invariant → $\delta = \frac{|fl(x)-x|}{|x|}$

CH-② Truncation Error → Actual T.E. $|f(x)-P_n(x)|$

(Interpolation Error). Cauchy's Theorem (upper bound).

Lagrange form of upper bound + taylor polynomial

CH-③ Truncation Error → Rounding Error ϵ

Exact / Actual

Upper band of the Truncation Error → Cauchy's Theorem

$$\rightarrow F.D / B.D = P_1(x) + \frac{f''(\xi)}{2!}(-h)$$

$$C.D = P_2(x) + \frac{f'''(\xi)}{3!}\underbrace{(-h^2)}_{max}$$

h^2, $\frac{d}{dh}(h^2) = 2h = 0$

$\Rightarrow h = 0$

if there is no interval given,
highest value $= (x_0 + h)$
lowest val $= (x_0 - h)$

Relative Error

$$= \frac{|f'(x) - \text{Numerical diff value}|}{|\text{Numerical diff value}|}$$