Transform of Piecewise Continuous Function

Evaluate $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^3 e^{-st} (0)\, dt + \int_3^\infty e^{-st} (2)\, dt$$

$$= 0 + \lim_{l \to \infty} \int_3^l 2e^{-st}\, dt$$

$$= 2 \lim_{l \to \infty} \left[\frac{e^{-st}}{-s} \right]_3^l$$

$$= -\frac{2}{s} \lim_{l \to \infty} \left[e^{-sl} - e^{-s3} \right]$$

$$= -\frac{2}{s} \lim_{l \to \infty} \left[\frac{1}{e^{sl}} - \frac{1}{e^{3s}} \right]$$

$$= -\frac{2}{s} \left[\frac{1}{e^{\infty}} - \frac{1}{e^{3s}} \right]$$

$$= \frac{2}{se^{3s}}, \quad s > 0$$

The following piecewise-defined function

$$f(t) = \begin{cases} g(t) ; & 0 \le t < a \\ h(t) ; & t \geq a \end{cases}$$

Can be written as a Unit Step function:

$$f(t) = g(t)\left[u(t-0) - u(t-a)\right] + h(t)\left[u(t-a)\right]$$

$$= g(t)\,u(t) - g(t)\,u(t-a) + h(t)\,u(t-a)$$

The following piecewise-defined function

$$f(t) = \begin{cases} 0 , & 0 \le t < a \\ g(t) , & a \le t < b \\ 0 , & t \geq b \end{cases}$$

can be written as a Unit Step function:

$$f(t) = 0\left[u(t-0) - u(t-a)\right] + g(t)\left[u(t-a) - u(t-b)\right] + 0\left[u(t-b)\right]$$

$$\therefore f(t) = g(t)\left[u(t-a) - u(t-b)\right]$$

Express $f(t) = \begin{cases} 20t, & 0 \leq t < 5 \\ 0, & t \geq 5 \end{cases}$ in terms

of unit step function.

$$f(t) = 20t\left[u(t-0) - u(t-5)\right] + \cancel{0[u(t-5)]}$$

$$= 20t\,(u(t) - u(t-5))$$

Introducing Laplate into unit step function:

Refer page 6 Part B Lec Note:

Known as 2nd Translation Theorem

$$\mathcal{L}(u(t-a)f(t)) = e^{-sa}\,\mathcal{L}\{F(t+a)\}$$

$$= e^{-sa} f(s+a)$$

OR

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\,\mathcal{L}\{f(t)\}$$

$$= e^{-as} F(s)$$

Find the laplace transformation of

$$f(t) = \underbrace{2}_{f(t)=1} - 3u(t-\overset{a}{2}) + u(t-\overset{a=3}{3})$$

$f(t)=1$ $f(t)=1$

$f(t+a) = f(t+2) = 1$ $\cancel{f(t+a)}$ $f(t+3) = 1$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{u(t-2)\cdot 1\} + \mathcal{L}\{u(t-3)\cdot 1\}$$

$$= 2\mathcal{L}\{1\} - 3e^{-2s}\mathcal{L}\{1\} + e^{-3s}\mathcal{L}\{1\}$$

$$= 2\frac{1}{s} - 3e^{-2s}$$

$$= 2/s - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

$\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$
$$= e^{-as}f(s+a)$$
$$u(t-a)f(t) = \mathcal{L}^{-1}\{e^{-as}f(s+a)\}$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$$
$$= e^{-as}f(s)$$
$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-as}f(s)\}$$

$$\left\{\begin{array}{l} \mathcal{L}\{u(t-a)f(t)\} = e^{-sa}\mathcal{L}\{F(t+a)\} \\ \therefore \mathcal{L}\{u(t-a)f(t)\} = e^{-sa}f(s+a) \\ \therefore u(t-a)f(t) = \mathcal{L}^{-1}\{e^{-sa}f(s+a)\} \end{array}\right.$$
$$\Rightarrow u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}f(s)\}$$

<u>Example:</u> $\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s-4}}_{f(s)}\underbrace{e^{-2s}}_{e^{-as}}\right\}$ $\therefore a=2$

$$f(s) = \frac{1}{s-4}$$

$$= u(t-2)f(t-2)$$ $\therefore f(t) = e^{4t}$

$$= u(t-2)e^{4(t-2)}$$ $f(t-2) = e^{4(t-2)}$

Example: $\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi s}{2}}\right\}$

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}f(s)\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi}{2}s}\right\} \quad \rightarrow a=\frac{\pi}{2}$$

$$f(s)=\frac{s}{s^2+9}$$

$$\therefore f(t) = \cos 3t$$

$$\therefore f\left(t-\frac{\pi}{2}\right) = \cos 3\left(t-\frac{\pi}{2}\right)$$

$$= u\left(t-\frac{\pi}{2}\right) f\left(t-\frac{\pi}{2}\right)$$

$$= u\left(t-\frac{\pi}{2}\right)\cos 3\left(t-\frac{\pi}{2}\right)$$

Example Solve $Y' + Y = f(t)$, $Y(0)=5$

where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$ $\quad \begin{matrix} \rightarrow 0[u(t-0)-u(t-\pi)] \\ \rightarrow 3\cos t\, u(t-\pi) \end{matrix}$

$$\mathcal{L}\{Y'\} + \mathcal{L}\{Y\} = \mathcal{L}\{0[u(t-0)-u(t-\pi)] + 3\cos t\,[u(t-\pi)]\}$$

$$\Rightarrow [S^1\mathcal{L}\{Y\} - S^0 Y(0)] + \mathcal{L}\{Y\} = 3\mathcal{L}\{\cos t\,(u(t-\pi))\}$$

$$\mathcal{L}\{u(t-a)F(t)\} = e^{-sa}\mathcal{L}\{F(t+a)\} = e^{-s\pi}\mathcal{L}\{F(t+\pi)\}$$

$$sy - 5 + y = 3\mathcal{L}\{\cos t\, u(t-\pi)\}$$
$$= -3\frac{s}{s^2+1}\cdot e^{-\pi s}$$

$$F(t) = \cos t$$
$$F(t+\pi) = \cos(t+\pi)$$
$$F(t+\pi) = -\cos t$$
$$\therefore \mathcal{L}\{F(t+\pi)\} = \mathcal{L}\{-\cos t\} = \frac{-s}{s^2+1}$$

$$y(s+1) = 5 - \frac{3s}{s^2+1}e^{-\pi s}$$

$$y = \frac{1}{s+1}\left\{5 - \frac{3s}{s^2+1}e^{-\pi s}\right\}$$

$$y = \frac{5}{s+1} - \frac{3se^{-\pi s}}{(s+1)(s^2+1)}$$

$$\mathcal{L}\{Y\} = \frac{5}{s+1} - \frac{3se^{-\pi s}}{(s+1)(s^2+1)}$$

$$Y = \mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{(s+1)(s^2+1)}\right\}$$

$$\frac{se^{-\pi s}}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$se^{-\pi s} = A(s^2+1) + (Bs+C)(s+1)$$

$$se^{-\pi s} = As^2 + A + Bs^2 + Cs + Bs + C$$

"s^2"

$0 = A + B$

$B = -A$

"s"

$e^{-\pi s} = C + B$

$e^{-\pi s} = C - A$ — (ii)

"constant"

$0 = A + C$ — (iii)

$\therefore C = -A$

(ii) + (iii) $2C = e^{-\pi s}$

$C = \frac{1}{2}e^{-\pi s}$

$A = -\frac{1}{2}e^{-\pi s}$

Also $B = C$ $\quad \because B = -A$, $C = -A$

$\therefore B = \frac{1}{2}e^{-\pi s}$

$$Y = 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{(s+1)(s^2+1)}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\left[\mathcal{L}^{-1}\left\{\frac{A}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{Bs}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s^2+1}\right\}\right]$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\left[\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}e^{-\pi s}}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\frac{1}{2}e^{-\pi s}\right\}\right.$$

$$\left. + \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}e^{-\pi s}}{s^2+1}\right\}\right]$$

$$= 5e^{-t} + \frac{3}{2}\left[\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s+1}}_{e^{-t}} e^{-\pi s}\right\} - \mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+1}}_{\cos t} e^{-\pi s}\right\}\right.$$

$\cos t\ ;\ f(t-\pi) = \cos(t-\pi)$

$$\left. - \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1}}_{\sin t} e^{-\pi s}\right\}\right]$$

$f(t-\pi) = e^{-(t-\pi)}$

$\sin t\ ;\ f(t-\pi) = \sin(t-\pi)$

$\boxed{a = \pi}$

find $f(t-a) = f(t-\pi)$

$$\boxed{u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}f(s)\}}$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)}u(t-\pi) - \cos(t-\pi)u(t-\pi) - \sin(t-\pi)u(t-\pi)\right]$$

$$= 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} - \cos(t-\pi) - \sin(t-\pi)\right)u(t-\pi).$$

$$= 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} + \cos t + \sin t\right)u(t-\pi).$$

$\cos x\cos(-\pi) - \sin x\sin(-\pi)$

$\cos(x-\pi) = -\cos x$

$\sin(x-\pi) = -\sin x$

$\sin x\cos(-\pi) + \cos x\sin(-\pi)$