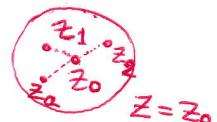


SINGULARITIES

A singularity of an analytic function $f(z)$ is the point at which the function ceases to be analytic ex the function $f(z) = \frac{1}{z-a}$ is analytic except at the point $z=a$. Thus the point $z=a$ is singularity of $f(z)$.

Types of Singularities1] Isolated singularities:

Let $z=z_0$ be a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z=z_0$, then this point is said to be an isolated singularity and otherwise it is termed as non-isolated singularity.

example: ① $f(z) = \frac{z+1}{(z-1)(z-2)}$ is analytic everywhere

except at $z=1, 2$ which are its singularities. Also there are no other singularities of $f(z)$ in the neighbourhood of these points and as such these are isolated singularities.

example ii $f(z) = \frac{z+3}{z^2(z^2+2)}$ possesses three isolated
(belongs to)

singularities at $z=0$, $z=\sqrt{2}i$ and $z=-\sqrt{2}i$

2] Poles:

If we can find a positive integer n such that
 $\lim_{z \rightarrow z_0} (z-z_0)^n f(z) = A \neq 0$, then $z=z_0$ is called
a pole of order ' n '. **f(z) is a rational function.**

If $n=1$, z_0 is called a simple pole.

example $f(z) = \frac{3z-2}{(z-1)^2(z+1)(z-4)}$ has a pole of

order 2 at $z=1$ & simple poles at $z=-1, 4$.

3] Branch Points:

Branch points of multiple valued functions are singular points.

example: $f(z) = \ln(z^2+z-2)$

$$f'(z) = \frac{2z+1}{z^2+z-2}; \quad \begin{aligned} z^2+z-2 &\neq 0 \\ z^2+2z-z-2 &\neq 0 \\ (z+2)(z-1) &\neq 0 \\ z &\neq -2, 1 \end{aligned}$$

$f'(z)$ is not analytic at $z=1, -2$

Hence $f(z)$ has branch points at $z=1, -2$

4] Removable Singularities:

The singular point z_0 is called a Removable singularity of $f(z)$ if $\lim_{z \rightarrow z_0} f(z)$ exists.

example The singular point $z=0$ is a removable singularity of $f(z) = \frac{\sin z}{z}$ since $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$

$$\text{i.e. } \lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{0}{0}$$

$$\therefore \lim_{z \rightarrow 0} \frac{\frac{d}{dz} \sin z}{\frac{d}{dz} z} = \lim_{z \rightarrow 0} \frac{\cos z}{1} = \cos 0^\circ = 1$$

(L'Hôpital's rule)

5] Essential Singularities:

A singularity which is not a pole, or removable singularity is called an essential singularity.

example: (i) $z=0$ for $\log(z) = f(z)$

(ii) $z=0$ for $f(z) = e^{\frac{1}{z}}$

The easiest way to define essential singularity of a function involves a "Laurent Series" which we will discuss at the end of Topic 5.

Examples

Locate and name the singularities

1. $f(z) = \frac{z}{(z+1)(z+2)}$

$$\lim_{z \rightarrow -1} (z+1) \frac{z}{(z+1)(z+2)} = \frac{-1}{-1+2} = -1$$

$$\lim_{z \rightarrow -2} (z+2) \frac{z}{(z+1)(z+2)} = \frac{-2}{-2+1} = 1$$

$\therefore z = -1$ and -2 are poles of order 1.

2. $f(z) = \frac{z}{(z^2+4)^2}$

$$\frac{z}{(z^2+4)^2} = \frac{z}{(z^2-2^2i^2)^2} = \frac{z}{(z+2i)^2(z-2i)^2}$$

$$\therefore \lim_{z \rightarrow 2i} (z-2i)^2 \cdot \frac{z}{(z+2i)^2(z-2i)^2} = \lim_{z \rightarrow 2i} \frac{z}{(z+2i)^2}$$

$$= \frac{2i}{16i^2} = -\frac{i}{8} \neq 0$$

$\therefore z = 2i$ is a pole of order 2.

$\therefore z = -2i$ is a pole of order 2.

Similarly $z = -2i$

3. $f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}}$

$$\therefore \lim_{z \rightarrow 0} \frac{\sin \sqrt{z}}{\sqrt{z}} = 1, \quad z = 0 \text{ is a removable singularity}$$

$$\lim_{z \rightarrow 0} \frac{\sin \sqrt{z}}{\sqrt{z}} = \frac{0}{0} \Rightarrow \lim_{z \rightarrow 0} \frac{(\sin \sqrt{z})'}{(\sqrt{z})'} = \lim_{z \rightarrow 0} \frac{\cos \sqrt{z} \cdot \frac{1}{2\sqrt{z}}}{\frac{1}{2\sqrt{z}}}$$

$$= \lim_{z \rightarrow 0} \cos \sqrt{z} = \cos \sqrt{0} = 1$$

The Residue Theorem

Residues:

Let $f(z)$ be single-valued and analytic inside and on a circle ' C ' except at the point $z = z_0$ chosen as the centre of ' C '. Then, $f(z)$ has a Laurent series about $z = z_0$ given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{--- (i)}$$

where $b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

see Cauchy's Integral
Formula "Topic 4"
Page 31

when $n = 1$, this expression for b_n can be written as

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz \Rightarrow R = \frac{1}{2\pi i} \int_C f(z) dz$$

$$\Rightarrow 2\pi i R = \int_C f(z) dz$$

The complex number b_1 , which is the coefficient of $\frac{1}{z - z_0}$ in expansion (i), is called the residue of f at the isolated singular point $z = z_0$.

We often use the notation,
residue at $z_0 = \operatorname{Res}_{z=z_0} f(z)$

Calculation of Residues

To obtain the residue of a function $f(z)$ in the case of $z = z_0$ is a pole of order k , there is a simple formula for b_k given by

$$b_k = \lim_{z \rightarrow z_0} \frac{1}{(k-1)!} \cdot \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-z_0)^k f(z) \right\}$$

Note
 $0_0 = 1$
 $\rightarrow f(z) = \frac{g(z)}{(z-z_0)^k}$

If $K=1$

$$\text{then } b_1 = \lim_{z \rightarrow z_0} (z-z_0) f(z)$$

$\rightarrow f(z) = \frac{g(z)}{(z-z_0)}$

Examples

For the following functions, determine the poles and the residues at the poles:

$$\textcircled{1} \quad f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z-2)(z+1)}$$

$f(z)$ has simple poles at $z=2$ and $z=-1$

Residue at $z=2$ is

$$\lim_{z \rightarrow 2} (z-2) \cdot \frac{2z+1}{(z-2)(z+1)} = \frac{5}{3}$$

Residue at $z=-1$ is

$$\lim_{z \rightarrow -1} (z+1) \frac{2z+1}{(z-2)(z+1)} = \frac{-2+1}{-1-2} = \frac{-1}{-3} = \frac{1}{3}$$

$$\textcircled{II} \quad f(z) = \left(\frac{z+1}{z-1} \right)^2$$

$$f(z) = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$ has a double pole at $z=1$

Residue at $z=1$ is:

$$\lim_{z \rightarrow 1} \frac{1}{1!} \frac{d}{dz} \left\{ (z-1)^2 \frac{(z+1)^2}{(z-1)^2} \right\}$$

$$= \lim_{z \rightarrow 1} 2(z+1) = 4$$

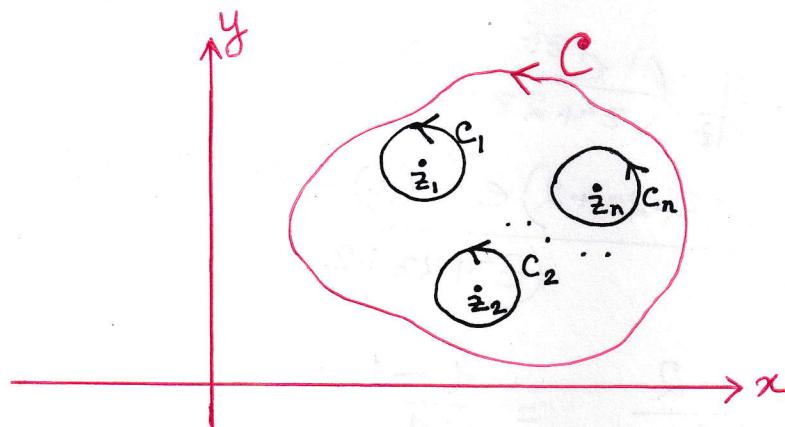
Residue Theorem

If $f(z)$ is analytic inside and on a simple closed curve ' C ' except at a finite number of singular points

z_k ($k=1, 2, \dots, n$) inside ' C ', then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n R_k$$

$$= 2\pi i (R_1 + R_2 + \dots + R_n)$$



POLES

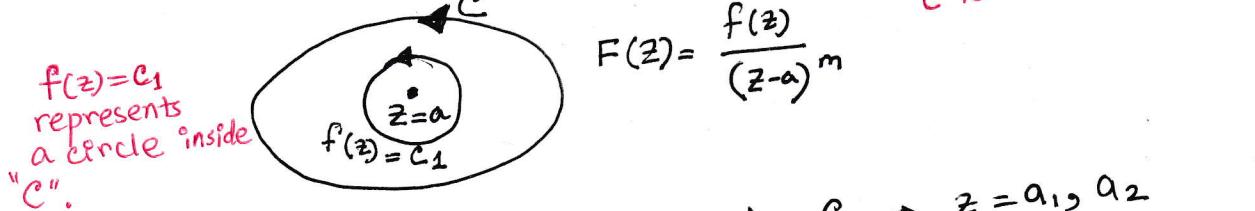
If $\lim_{z \rightarrow z_0} (z - z_0)^n F(z) = A \neq 0$ while $F(z) = \frac{f(z)}{(z - z_0)^n}$

then $F(z)$ has a pole of order n at $z = z_0$ or
 $z = z_0$ is called a pole of order ' n '.

Theorem of Residue (remains / rest / remainder)

If $f(z)$ is analytic on and within C (closed curve)
except for a pole of order m at $z = a$ inside C
where $f(z)$ is undefined

Then $\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$



$$F(z) = \frac{f(z)}{(z-a)^m}$$

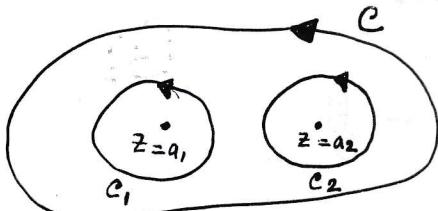
"C" is a closed curve

If more than one pole inside $C \Rightarrow z = a_1, a_2$

$$\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \rightarrow a_1} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \{(z-a_1)^{m_1} F(z)\} + \lim_{z \rightarrow a_2} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \{(z-a_2)^{m_2} F(z)\}$$

$$\Rightarrow \oint_C F(z) dz = 2\pi i [R_1 + R_2]$$

limpts are denoted by R_1 & R_2 which represents residues of $F(z)$.



C_1, C_2 are non overlapping simple closed curve inside C

Example Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz$ around the circle 'C' with equation $|z|=3$.

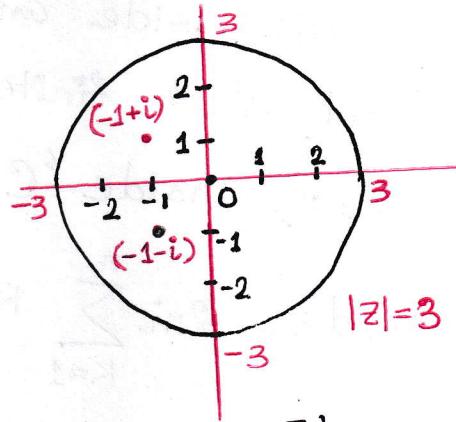
$$f(z) = \frac{e^{zt}}{z^2(z^2+2z+2)}$$

$$= \frac{e^{zt}}{z^2(z+1-i)(z+1+i)}$$

$$\begin{aligned} z^2 + 2z + 2 &= 0 \\ z^2 + 2 \cdot z \cdot 1 + 1^2 + 1 &= 0 \\ (z+1)^2 - i^2 &= 0 \quad \begin{cases} i^2 = -1 \\ -i^2 = +1 \end{cases} \\ (z+1-i)(z+1+i) &= 0 \end{aligned}$$

$f(z)$ has a double pole at $z=0$
and two simple poles at $z=-1-i, z=-1+i$.

all these poles are
inside 'C'.



$$R_1 f(z) = \lim_{\substack{z \rightarrow 0 \\ z=0}} \frac{1}{2\pi i} \frac{d}{dz} \left\{ z^2 \cdot \frac{e^{zt}}{z^2(z^2+2z+2)} \right\}$$

$$= \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{e^{zt}}{z^2+2z+2} \right)$$

$$= \frac{(z^2+2z+2)e^{zt}(t) - e^{zt}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t-2}{2^2} = \frac{t-1}{2}$$

$$\begin{aligned}
 R_2 \quad f(z) &= \lim_{z \rightarrow -1+i} \left\{ (z+1-i) \frac{e^{zt}}{z^2(z^2+2z+2)} \right\} \\
 &= \lim_{z \rightarrow -1+i} \left\{ (z+1-i) \frac{e^{zt}}{z^2(z+1-i)(z+1+i)} \right\} \\
 &= \lim_{z \rightarrow -1+i} \frac{e^{zt}}{z^2(z+1+i)} \\
 &= \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)} \\
 &= \frac{e^{(-1+i)t}}{(1-2i+i^2)(2i)} = \frac{e^{(-1+i)t}}{(-2i)(2i)} = \frac{e^{(-1+i)t}}{4}
 \end{aligned}$$

$$\begin{aligned}
 R_3 \quad f(z) &= \lim_{z \rightarrow -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z^2+2z+2)} \right\} \\
 &= \lim_{z \rightarrow -1-i} \left\{ (z+1+i) \frac{e^{zt}}{z^2(z+1-i)(z+1+i)} \right\} \\
 &= \lim_{z \rightarrow -1-i} \frac{e^{zt}}{z^2(z+1-i)} \\
 &= \frac{e^{(-1-i)t}}{(-1-i)^2(-1-i+1-i)} \\
 &= \frac{e^{(-1-i)t}}{(1+2i+i^2)(-2i)} = \frac{e^{(-1-i)t}}{(2i)(-2i)} = \frac{e^{(-1-i)t}}{4}
 \end{aligned}$$

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz = 2\pi i [R_1 + R_2 + R_3]$$

$$= 2\pi i \left[\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4} \right]$$

$$\therefore \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz = \frac{t-1}{2} + \frac{1}{4} [e^{-t} e^{it} + e^{-t} e^{-it}]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4} [cost + isint + cos(t) + isint(t)]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4} [cost + isint + cost - isint]$$

$$= \frac{t-1}{2} + \frac{e^{-t}}{4} [2cost]$$

$$= \frac{t-1}{2} + \frac{1}{2} e^{-t} cost.$$

$$R_1(z) = -\cos\left(\frac{\pi}{4}\right)$$

$$= -\frac{1}{2}(1 - \frac{1}{2}i) \cdot \frac{1}{2} + \frac{1}{4}i \cdot \frac{1}{2} = -\frac{1}{4} + \frac{1}{8}i$$

$$= -\frac{1}{4} + \frac{1}{8}i \cdot \frac{1}{2} = -\frac{1}{8} + \frac{1}{16}i$$

$$R_2(z) = \text{coefficient of } \frac{1}{z}$$

$$z=0$$

$$= -\frac{1}{8}$$

III

Overview

Cauchy Riemann Equations

Refer to Topic 2

If $f(z) = u(x, y) + i v(x, y)$ then

$$\left. \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array} \right\} \text{known as necessary condition}$$

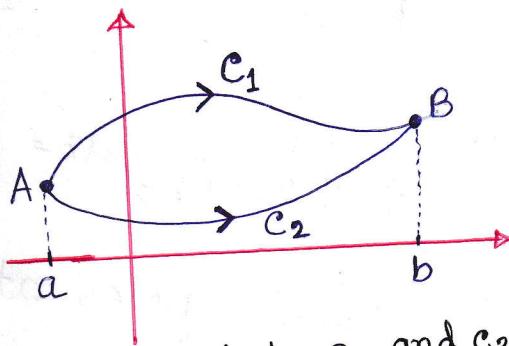
$$\left. \begin{array}{l} f'(z_0) = u_x + i v_x \\ \text{or } f'(z_0) = v_y - i u_y \end{array} \right\} \text{known as sufficient condition}$$

Refer to Topic 4
Page 9 and 10

Cauchy's Integral / Cauchy-Goursat Theorem

$$\int_C f(z) dz = 0$$

$$\Rightarrow \int_{C_1} f(z) dz + \left[- \int_{C_2} f(z) dz \right] = 0$$

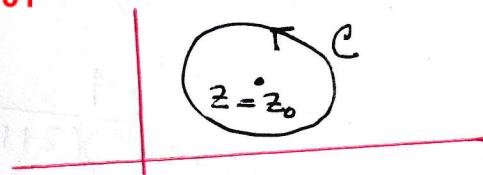


From a to b, C_1 and C_2 are independent of path

Cauchy's Integral Formula

Topic 4
Page 31

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$$



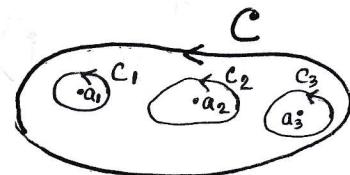
n^{th} derivative:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n=1, 2, 3, \dots$$

Theorem of Residue

$$\int_C F(z) dz = 2\pi i [R_1 + R_2 + \dots + R_n]$$

$$R = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{ (z-a)^m F(z) \}$$

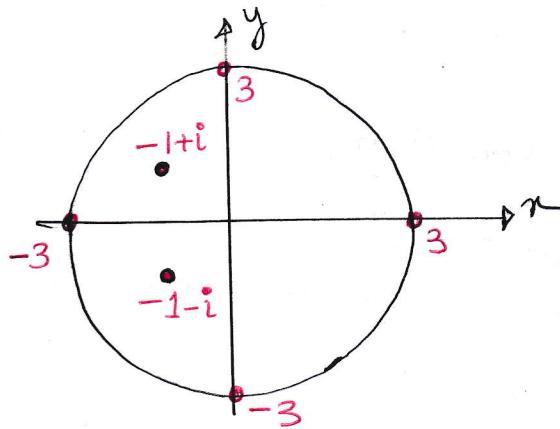


$$F(z) = \frac{f(z)}{(z-a)^m}$$

$f(z)$ has pole of order m at $z=a$.
 $f(z)$ is analytic everywhere except $z=a$.

(Exercise Sheet #5)

⑤ Evaluate $\oint_C \frac{z^2+4}{z^3+2z^2+2z} dz$ using the residue at the poles, around the circle $|z|=3$.



$$\begin{aligned}
 & z^3 + 2z^2 + 2z \\
 &= z(z^2 + 2z + 2) \\
 &= z(z^2 + 2z + 1 + 1) \\
 &= z((z+1)^2 + 1) \\
 &= z[(z+1)^2 - i^2] \\
 &= z[(z+1+i)(z+1-i)]
 \end{aligned}$$

$$\oint_C \frac{z^2+4}{z^3+2z^2+2z} dz = \oint_C \frac{z^2+4}{(z-0)\{(z-(-1-i))(z-(-1+i))\}} dz$$

$f(z)$ has simple poles at $\underbrace{z=0, -1-i, -1+i}_{\text{all inside the region}}$

$$\begin{aligned}
 R_1 &= \lim_{z \rightarrow 0} z \cdot f(z) \\
 &= \lim_{z \rightarrow 0} \frac{z^2+4}{(z+1+i)(z+1-i)} \\
 &= \frac{4}{(1+i)(1-i)} = \frac{4}{1-i^2} = \frac{4}{2} = 2
 \end{aligned}$$

$$R_2 = \lim_{z \rightarrow -1-i} (z+1+i) \frac{z^2+4}{z(z+1+i)(z+1-i)}$$

$$= \lim_{z \rightarrow -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2 + 4}{(-1-i)(-1-i+1-i)} = \frac{1+2(-1)(-i)+(-i)^2+4}{(-1-i)(-2i)}$$

$$R_3 = \lim_{z \rightarrow -1+i} (z+1-i)f(z) = \frac{1+2i-1+4}{2i+2i^2}$$

$$= \frac{2i+4}{2i-2} = \frac{i+2}{i-1} \cdot \frac{-i-1}{-i+1}$$

$$= \frac{-i^2-2i-i-2}{(-1)^2-i^2}$$

$$= \frac{-1-3i}{2}$$

$$= -\frac{1}{2} - \frac{3}{2}i$$

$$= \frac{-2i+4}{-2i+2i^2}$$

$$= \frac{-2i+4}{-2i-2}$$

$$= \frac{i-2}{i+1} \cdot \frac{-i+1}{-i+1}$$

$$= \frac{-i^2+2i+i-2}{1^2-i^2}$$

$$= -\frac{1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$$

$$\oint_C \frac{z^2+4}{z^3+2z^2+2z} dz = 2\pi i (R_1 + R_2 + R_3)$$

$$= 2\pi i \left(2 - \frac{1}{2} - \frac{3i}{2} - \frac{1}{2} + \frac{3i}{2}\right)$$

$$= 2\pi i (2-1)$$

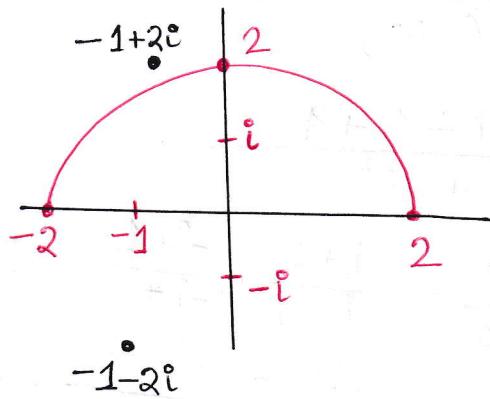
$$= 2\pi i$$

(Exercise Sheet # 5)

Reading

⑥ Evaluate $\int_C \frac{ze^{i\pi z}}{(z^2+2z+5)(z^2+1)^2} dz$ using the residue

at the poles, where C is the upper half circle of the equation $|z|=2$.



$$\begin{aligned}
 & (z^2+2z+5)(z^2+1)^2 \\
 &= (z^2+2z+1+4)(z^2-i^2)^2 \\
 &= [(z+1)^2-(2i)^2] (z+i)^2(z-i)^2 \\
 &= (z+1+2i)(z+1-2i)(z-i)^2(z+i)^2 \\
 &\quad \underbrace{z-(1-2i)}_{z-(-1+2i)} \quad \underbrace{z-(-1+2i)}_{z-(-1-2i)}
 \end{aligned}$$

$f(z)$ has simple poles at $z = \underbrace{-1-2i}_{\text{outside region}}, \underbrace{-1+2i}_{\text{outside region}}$

$f(z)$ has poles of order 2 at $z = i, -i$ $\underbrace{i}_{\text{outside region}}, \underbrace{-i}_{\text{outside region}}$

$$\begin{aligned}
 R_1 &= \lim_{z \rightarrow i} \frac{1}{(2-1)!} \cdot \frac{d}{dz} \left\{ (z-i)^2 f(z) \right\} \\
 &= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ (z-i)^2 \cdot \frac{ze^{i\pi z}}{(z^2+2z+5)(z-i)^2(z+i)^2} \right\} \\
 &= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{ze^{i\pi z}}{(z^2+2z+5)(z+i)^2} \right\} \\
 &= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{ze^{i\pi z}}{(z^2+2z+5)(z^2+2zi-1)} \right\}
 \end{aligned}$$

$$= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{ze^{iz}}{z^4 + 2z^3 + 5z^2 + 2z^3i + 4z^2i + 10zi - z^2 - 2z - 5} \right\}$$

$$= \lim_{z \rightarrow i} \frac{d}{dz} \left\{ \frac{ze^{iz}}{z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3i + 4z^2i + 10zi} \right\}$$

$$= \lim_{z \rightarrow i} \frac{(z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3i + 4z^2i + 10zi) [e^{iz} + zi e^{iz}]}{d(z^4 + 2z^3 + 4z^2 - 2z - 5 + 2z^3i + 4z^2i + 10zi)^2} \\ + ze^{iz} (4z^3 + 6z^2 + 8z - 2 + 6iz^2 + 8iz + 10i)$$

$$= \frac{(1 - 2i - 4 - 2i - 5 + 2 - 4i - 10)(e^{-\pi} - \pi e^{-\pi}) + ie^{-\pi}(-4i - 6 + 8i - 2 - 6i - 8 + 10i)}{(1 - 2i - 4 - 2i - 5 + 2 - 4i - 10)^2}$$

$$= \frac{(-20 - 8i)(e^{-\pi} - \pi e^{-\pi}) - ie^{-\pi}(-16 + 8i)}{(-20 - 8i)^2}$$

$$= \frac{20e^{-\pi} - 8ie^{-\pi} - 11\pi e^{-\pi} - 8\pi ie^{-\pi} + 16ie^{-\pi} - \cancel{8i^2 e^{-\pi}} + \cancel{8e^{-\pi}}}{(-20 - 8i)^2}$$

$$= \frac{-12e^{-\pi} + 8ie^{-\pi} - 11\pi e^{-\pi} - 8\pi ie^{-\pi}}{(-20 - 8i)^2}$$

$$= \frac{-12e^{-\pi} - 11\pi e^{-\pi} + 8ie^{-\pi} - 8\pi ie^{-\pi}}{400 + 2(-20)(-8i) + (-8i)^2} = \frac{e^{-\pi}(-12 - 11\pi) + i8e^{-\pi}(1 - \pi)}{400 + 320i + 64} \\ = \frac{-2.01 - i0.74}{464 + 320i}$$

$$= \frac{(-2.01 - 0.74i)}{(464 + 320i)} \cdot \frac{(464 - 320i)}{(464 - 320i)}$$

$$= \frac{-933.343i + 322}{215296 + 102400} + 237i^2$$

$$= \frac{1170 - 21i}{317696}$$

$$\oint_C F(z) dz = 2\pi i R_1$$

$$= 2\pi i \left(\frac{1170 - 21i}{317696} \right)$$

There might be some numerical error but the process is correct