HW_3

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1 Earth 164 Planetary Atmospheres HW 3

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1.1 Bake the atmosphere from bottom

Suppose an observation is being made on the planetary atmosphere of a gas giant whose main source of heat energy is from the internal radiation flux coming from the surface of the planet. The atmosphere is composed of Hydrogen gas which is a diatomic molecule with the adiabatic index being $\gamma = \frac{7}{5}$ and a collision-induced absorption opacity of $\tau = Ap^2$ where A is a constant and p is pressure. An intrinsic flux F seeps out of the surface and radiates towards the atmosphere. In order to calculate the temperature profile of the air parcel within the atmosphere we must first apply a two-flow model. This model states that for any incoming intensity I that interacts with the atmosphere, a reflecting I^- and transmitting ray I^+ is radiated from the interaction where $I = I^+ + I^-$. We shall also assume that the atmosphere is in monochromative equilibrium so that $\frac{\partial F}{\partial z} = 0$. A blackbody distribution was applied to the atmosphere in order to state that for any radiation that is absorbed from the atmosphere, it is radiated out again as blackbody radiation B_ν. Thus, focusing in a single frequency of light:

$$I_\nu^+ = B_\nu(T) + \frac{1}{2\pi}F_\nu$$

$$I_\nu^- = B_\nu(T) - \frac{1}{2\pi}F_\nu$$

Recall that for any radiative transfer within a system, the following equation must hold true:

$$\mu\frac{\partial I_\nu}{\partial \tau_\nu} = -I_\nu + S_\nu$$

where in this case our only source of radiation is the planet's surface which acts as a blackbody, so $S_\nu = B_\nu$. In this image we will focus in three areas of interests: S_1, S_2, S_3. In S_1 the surface emits energy with intensity I_g. The energy flux is scattered into two flux beams, I_g^+ emitting upward and I_g^- emitting downward. Both intensities are absorbed and remitted as part of a blackbody spectrum V_ν within the atmosphere resulting in a net flux F emitting upward. At the top of the atmosphere there is no reflection and only scattering and emission contributes to the net flux flowing outwards.

Thus we are able to define the total radiative flux as a function of opacity if we are able to solve the differential equation above. Before we try to solve the equaiton, we must first define the constraints of the system. In order for the greenhouse effect to be true within this gas planet, two conditions must apply: 1. The radiative transfer at the surface of the planet must result in an upward intensity of $I_{\nu g}^{+} = B_\nu(T_1) + \frac{1}{2\pi} F_\nu = B_\nu(T_g)$. 2. At the top of the atmosphere the net flux must result from upward going fluxes which means that $I_{\nu 0}^{-} = 0$ 3. $\frac{\partial F}{\partial \tau} = 0$ with no gradient on the magnitude of the flux along the height of the atmosphere

With these conditions we are now able to define a singular solution for the temperature profile of the atmosphere. First, we know that the net normal flux from radiation as a function of intensity and the infinitesimal scattering solid angle $d\Omega$ is:

$$F = \int I \cos(\theta) d\Omega = \int I \cos(\theta) \sin(\theta) d\theta d\psi$$

For simplicity a substitution is defined such that

$$\mu = \cos(\theta)$$

and \$d $\mu = -\sin(\theta) d\theta \$ so : F = \int_0^{2\pi} \int_0^{\pi} I \cos(\theta) \sin(\theta) d\theta d\psi = \int_0^{2\pi} \int_{-1}^{1} I \mu d\mu d\psi$.Now let us use Eddington's method of solving this equation and determine the solution. First we may apply a first order approximation to the scattered intensity so that:

$$I = I_o + \mu I_1$$

where I_o and I_1 are seperate expansion terms that are independent of μ. The radiative flux can now be defined as such:

$$F = \int_0^{2\pi} \int_{-1}^{1} \mu (I_o + \mu I_1) d\mu d\psi = 2\pi * (\frac{2I_1}{3})$$

The integral wasn't that hard and we can actually take advantage of the fact that μ is an orthonormal function being integrated within a period. Thus, let us define the solid angle integral as a weighting function such that:

$$< 1 >= \int_0^{2\pi} \int_{-1}^{1} d\mu d\psi = 4\pi$$

$$< I >= \int_0^{2\pi} \int_{-1}^{1} (I_o + \mu I_1) d\mu d\psi = 4\pi I_o$$

$$< \mu^1 I >= \int_0^{2\pi} \int_{-1}^{1} \mu (I_o + \mu I_1) d\mu d\psi = \frac{4\pi I_1}{3} = F$$

$$< \mu^2 I >= \int_0^{2\pi} \int_{-1}^{1} \mu^2 (I_o + \mu I_1) d\mu d\psi = \frac{4\pi I_o}{3}$$

Now we can apply calculus and build a system of equations that will result in a nice equation. Then: 1.

$$< \mu \frac{\partial I_\nu}{\partial \tau_\nu} >=< -I_\nu + B_\nu >$$

2.

$$< \mu^2 \frac{\partial I_\nu}{\partial \tau_\nu} >=< -\mu I_\nu + \mu B_\nu >$$

Integrating equaiton 1 results in the following:

$$< \mu\frac{\partial I_\nu}{\partial \tau_\nu} >= \frac{\partial < \mu I_\nu >}{\partial \tau_\nu} = \frac{\partial F}{\partial \tau_\nu} = -4\pi I_{\nu o} + 4\pi B_\nu$$

Now integrating equation 2 results in:

$$< \mu^2\frac{\partial I_\nu}{\partial \tau_\nu} >= \frac{\partial < \mu^2 I_\nu >}{\partial \tau_\nu} = \frac{4\pi}{3}\frac{\partial I_{\nu o}}{\partial \tau_\nu} = -F$$

The flux F is a result from integrating the odd orthonormal function applied to B_ν thus making it zero and only focusing on the intensity. Now that we have our two equations we can now apply substitution. For the first modified equation we shall look into the derivative relative to the opacity. Thus:

$$\frac{\partial^2 F}{\partial \tau_\nu^2} = -4\pi\frac{\partial I_{\nu o}}{\partial \tau_\nu} + 4\pi\frac{\partial B_\nu}{\partial \tau_\nu}$$

From equation 2 we can substitute $I_{\nu o}$ with F and have:

$$\frac{\partial^2 F}{\partial \tau_\nu^2} = 3F + 4\pi\frac{\partial B_\nu}{\partial \tau_\nu}$$

In this case we know that $\frac{\partial^2 F}{\partial \tau_\nu^2} = 0$ and from the second constraint of the atmosphere $F = 2\pi B_\nu(T_o)$. Thus:

$$\frac{3}{2}B_\nu(T_o) = \frac{\partial B_\nu}{\partial \tau_\nu}$$

$$\frac{3}{2}B_\nu(T_o)d\tau = dB_\nu$$

Integrate both sides within the interval of optical depth where at the top-edge of the atmosphere $\tau = 0$ and the temperature is set at T_o:

$$\int_0^\tau \frac{3}{2}B_\nu(T_o)d\tau = \int_{B_\nu(T_o)}^{B_\nu(\tau)} dB_\nu$$

$$\frac{3}{2}B_\nu(T_o)\tau = B_\nu(\tau) - B_\nu(T_o)$$

$$B_\nu(T_o)(1 + \frac{3}{2}\tau) = B_\nu(\tau)$$

Now integrate the entire equation over all of velocity space so that $\int_0^\infty B_\nu d\nu = \sigma T^4$ to get:

$$\sigma T_o^4(1 + \frac{3}{2}\tau) = \sigma T^4(\tau)$$

The final relation is found such that the equilibrium radiative temperature as a function of optical depth is:

$$T_o^4(1 + \frac{3}{2}\tau) = T^4(\tau)$$

Temperature as a function of pressure is:

$$T_o^4(1 + \frac{3}{2}Ap^2) = T^4(p)$$

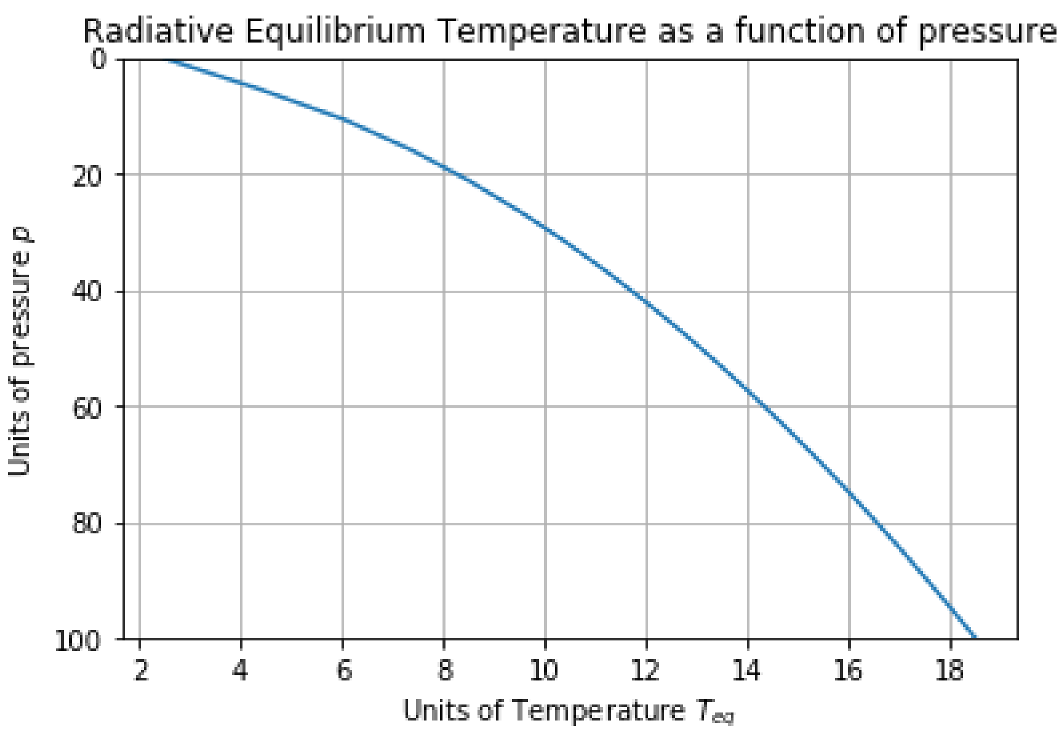
```python
In [1]: import numpy as np
        import matplotlib.pyplot as plt



        #Let us define the radiative equilibrium temperature as a function of p:
        def Temp(T_0, A, p):
            return T_0*((1+(1.5*A*(p**2)))**(0.25))

        interval_pressure = np.linspace(100, 0, 20)
        Temp_0 = 2.5 #mean temperature of the CMB
        A_guess = 0.2
        result_temp = Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess, len(int
        plt.plot(result_temp, interval_pressure)
        plt.ylabel(r'Units of pressure $p$')
        plt.xlabel(r'Units of Temperature $T_{eq}$')
        plt.ylim(100, 0)
        plt.title('Radiative Equilibrium Temperature as a function of pressure')
        plt.grid(True)
        plt.show()

        #
```



Let us analyze the potential temperature profile in order to get a cleaner picture of the adiabatic process of the air-parcel in this atmosphere. From the third law of thermodynamics we know that:

$TdS = C_p dT - VdP$. For an ideal gas we can modify the equation by stating the fact that $V = \frac{nRT}{P}$ at constant temperature. Then:

$$dS = C_p(\frac{dT}{T} - \frac{nRdP}{C_pP})$$

For the ideal case of one mole of Hydrogen gas in the atmosphere we further simplify the equation by noting the following: If $g(t) = \ln(x(t)y(t))$ then

$$d(g(t)) = \frac{x'(t)y(t) + x(t)y'(t)}{x(t)y(t)} = \frac{dx}{x} + \frac{dy}{y}$$

So:

$$\frac{dT}{T} - \frac{nRdP}{C_pP} = d(\ln(TP^{\frac{-R}{C_p}}))$$

Thus:

$$dS = C_p d(\ln(TP^{\frac{-R}{C_p}}))$$

$$\frac{\triangle S}{C_p} = \ln\left(\frac{TP^{\frac{-R}{C_p}}}{T_oP_o^{\frac{-R}{C_p}}}\right)$$

$$T_o e^{\frac{\triangle S}{C_p}} = T\left(\frac{P}{P_o}\right)^{\frac{-R}{C_p}} = \theta(T)$$

Notice that if entropy is held constant within the atmosphere then $\theta(T) = T_o$. In terms of pressure this equation is defined as:

$$\theta(P) = T_o(1 + \frac{3}{2}AP^2)^{1/4}\left(\frac{P}{P_o}\right)^{\frac{-R}{C_p}}$$

Since the atmosphere consists of mainly diatomic Hydrogen then $C_p = \frac{7}{2}R$ and:

$$\theta(P) = T_o(1 + \frac{3}{2}AP^2)^{1/4}\left(\frac{P}{P_o}\right)^{\frac{-2}{7}}$$

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt



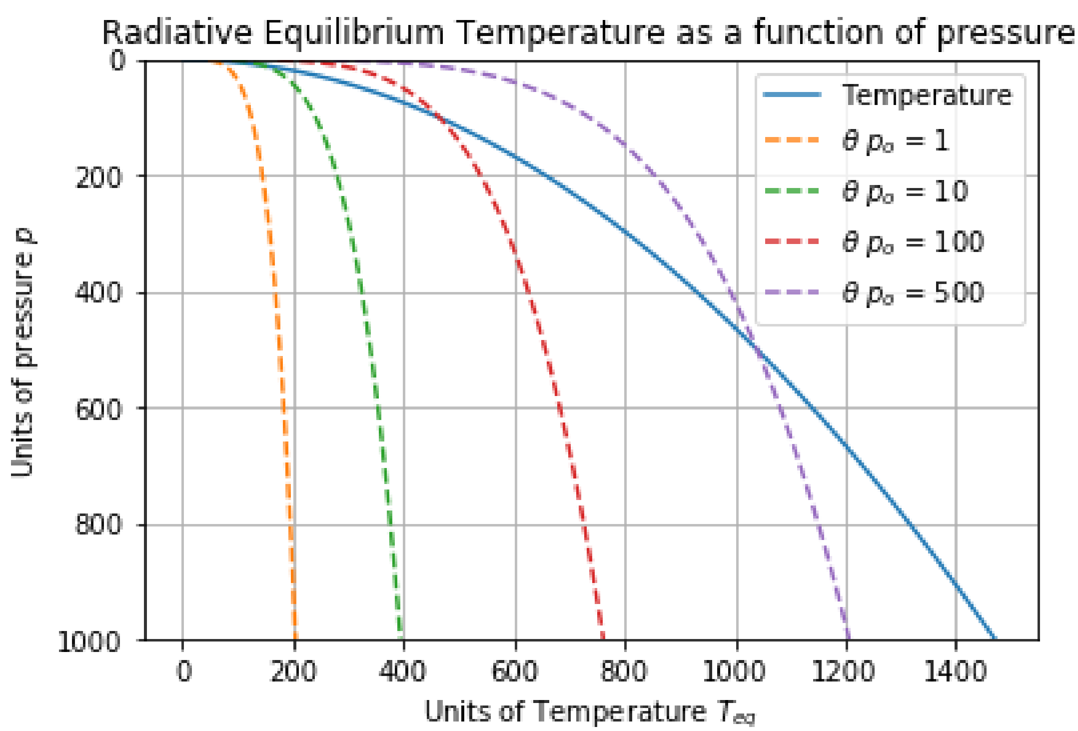
#Let us define the radiative equilibrium temperature as a function of p:
def Temp(T_0, A, p):
    return T_0*((1+(1.5*A*(p**2)))**(0.25))

def pot_Temp(T_0, A, p, p_o):
    return T_0*((1+(1.5*A*(p**2)))**(0.25))*((p/p_o)**(-2/7))

interval_pressure = np.linspace(0, 1000, 1000)
Temp_0 = 2.5 #mean temperature of the CMB
A_guess = 80000
result_temp = Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess, len(int
```

```
result_pot_temp1 = pot_Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess
result_pot_temp2 = pot_Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess
result_pot_temp3 = pot_Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess
result_pot_temp4 = pot_Temp(np.repeat(Temp_0, len(interval_pressure)), np.repeat(A_guess
plt.plot(result_temp, interval_pressure, label='Temperature')
plt.plot(result_pot_temp1, interval_pressure, '--', label=r'$\theta$ $p_o$ = 1')
plt.plot(result_pot_temp2, interval_pressure, '--', label=r'$\theta$ $p_o$ = 10')
plt.plot(result_pot_temp3, interval_pressure, '--', label=r'$\theta$ $p_o$ = 100')
plt.plot(result_pot_temp4, interval_pressure, '--', label=r'$\theta$ $p_o$ = 500')
plt.ylabel(r'Units of pressure $p$')
plt.xlabel(r'Units of Temperature $T_{eq}$')
plt.ylim(1000, 0)
plt.title('Radiative Equilibrium Temperature as a function of pressure')
plt.grid(True)
plt.legend()
plt.show()
```

```
C:\Users\Jesus\Anaconda3\lib\site-packages\ipykernel_launcher.py:11: RuntimeWarning: divide by z
  # This is added back by InteractiveShellApp.init_path()
```



Notice that the temperature profile shown above depicts the temperature of the parcel increasing as height decreases. The potential temperature profile for four difference reference pressures P_o are shown as dotted lines. An increasing trend is shown along the radiative equilibirum temperature as pressure increases. Since pressure is inveresely proportional to height, where minimum pressure of the atmosphere is reached at a maximum height and maximum pressure is reached at the minimum height, all potential temperature profiles illustrate an increasing trend relative to increasing height. This results in an unstable equilibrium where an airparcel will flow towards lower pressure along the temperature profile given an initial perturbation. The instability results in the atmosphere being dominated by convection which is constrained by the top of the atmosphere. In this example we look at the case of the effective pressure being 500 Pa. The red arrows show the perturbation along the temperature profile. The red arrows show the adaibatic flow of the air parcel after the perturbation is set. Since the temperature profile of the gas is greater than the potential temperature profile, and since we mentioned earlier that there is an increasing trend in temperature and pressure, then the perturbation makes the air parcel unstable forcing it to flow to the top of the atmosphere in an adiabatic expansion process. Radiative energy transfer will only dominate the atmosphere below 500 Pa and convection will dominate the atmosphere above 500 Pa. Since particle density, and therefore pressure, is an exponential function dependent on height, then this implies that the majority of the atmosphere is dominated by convection. We can prove this by looking into the rate of change of the potential temperature relatived to pressure:

$$\frac{\partial\theta}{\partial P} = \frac{\partial T}{\partial P}\left(\frac{P}{P_o}\right)^{-\frac{2}{7}} - \frac{2}{7}T\left(\frac{P}{P_o}\right)^{-\frac{2}{7}-1}(\frac{1}{P_o})$$

$$\frac{\partial\theta}{\partial P} = \theta\left(\frac{1}{T}\frac{\partial T}{\partial P} - \frac{2}{7}(\frac{1}{P})\right)$$

Note that $\frac{\partial T}{\partial P} = \frac{3AT_oP}{4}(1+\frac{3}{2}AP^2)^{-\frac{3}{4}} = \frac{3ATP}{4(1+\frac{3}{2}AP^2)}$. Then:

$$\frac{\partial\theta}{\partial P} = \theta\left(\frac{3AP}{4(1+\frac{3}{2}AP^2)} - \frac{2}{7}(\frac{1}{P})\right)$$

$$\frac{\partial\theta}{\partial P} = T_o(1+\frac{3}{2}AP^2)^{1/4}\left(\frac{P}{P_o}\right)^{\frac{-2}{7}}\left(\frac{3AP}{4(1+\frac{3}{2}AP^2)} - \frac{2}{7}(\frac{1}{P})\right)$$

Notice how as P approaches a large magnitude, the rate of change of the potential energy and therefore the adiabatic rate of the air parcel reaches a net value of zero. As P approaches a small magnitude, the rate of change abruptly changes negative and thus implies a decreasing potential temperature as height increases. This proves that radiative energy transfer is the dominant source of energy transfer above a certain critical pressure, and convection is dominant source of energy transfer below this critical pressure.

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In [ ]:
```