Taylor Series

It is the generalization of Maclaurin series.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

at $x = a$

(Maclaurin series is a special case of Taylor Series when $a = 0$).

Let

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots$$

$$f''(x) = 2a_2 + 6a_3(x-a) + \dots$$

$$f'''(x) = 6a_3 + \dots$$

Now,

$f(a) = a_0 \Rightarrow a_0 = f(a)$

$f'(a) = a_1 \Rightarrow a_1 = f'(a)$

$f''(a) = 2a_2 \Rightarrow a_2 = \frac{1}{2} f''(a)$

$f'''(a) = 6a_3 \Rightarrow a_3 = \frac{1}{6} f'''(a)$.

Now,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

Obtain Taylor series for $f(x) = \ln x$ at $x = 1$.

$f(x) = \ln x$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f'''(x) = 2/x^3$

$f^{(4)}(x) = -6/x^4$.

At $x = 1$,

$f(1) = 0$.

$f'(1) = 1$

$f''(1) = -1$

$f'''(1) = 2$

$f^{(4)}(1) = -6$.

∵ The Taylor series for the given function is:

$$\ln x = 0 + (x-1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6)$$

$$\Rightarrow \ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

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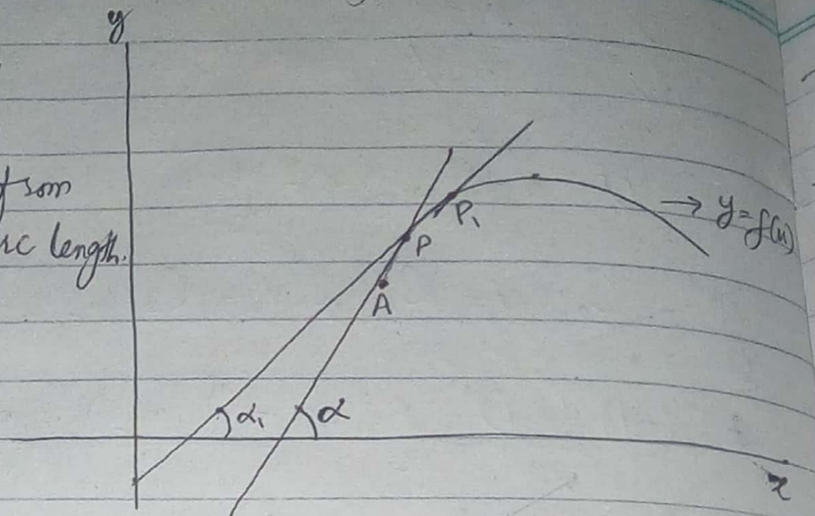
Curvature And Radius Of Curvature

A is fixed point.
P is arbitrary point.
The portion of curve from A to P is called arc length.



The portion of curve
K → also denoted by greek letter (Kapa)

$$K = \left| \frac{d\alpha}{ds} \right|_{P(x,y)}$$

↓
Curvature of the curve.

Curvature is a rate at which tangent line to a curve turns is called its curvature.

The curvature of straight line is zero. It is only curve at which curvature is zero otherwise for other curves the value of curvature is non-zero.

→ For a straight line, curvature K at any point is zero.
At any point where $K \neq 0$, the reciprocal of curvature is called radius of curvature ρ

$K = \frac{1}{\rho}$ $K = \frac{1}{r}$ $\rho = r$ $\rho = 1/K$

$y = f(x) \rightarrow$ One variable or single-variable function.

$z = f(x, y)$
$u = g(x, y, z)$ multivariable functions or functions of several variables.

For a circle radius of curvature is equal to its radius r.

The curvature K of circle is $1/r$.

$y = f(x)$ explicit form

$f(x, y) = 0$ Implicit form

$x = f(t),\ y = g(t)$ parametric form

$f(r, \theta) = 0$ Polar form

Explicit form:

$$\rho = \frac{\left[1 + [f'(x)]^2\right]^{3/2}}{|f''(x)|}$$

Implicit form:

$$\rho = \frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{|f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2|}$$

Derivatives of single variable functions are ordinary derivatives.

$$y' = f'(x).$$

Partial derivatives are used for multivariable functions. let $z = f(x, y)$.

$$\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}$$

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Q Determine radius of curvature of the curve

$x^2y = a(x^2+y^2)$ at point $(2a, 2a)$.

Solution:

$x^2y - a(x^2+y^2) = 0$.

$f_x = 2xy - 2ax \Rightarrow 2(2a)(2a) - 2(2a)a$

$\Rightarrow 8a^2 - 4a^2 \Rightarrow 4a^2$.

$f_y = x^2 - 2ay \Rightarrow (2a)^2 - 2a(2a) \Rightarrow 0$

$f_{xx} = 2y - 2a \Rightarrow 2(2a) - 2a = 2a$

$f_{yy} = -2a$ ~~$\Rightarrow 2(2a)$~~ $\Rightarrow -2a$.

$f_{xy} = 2x \Rightarrow 2(2a) \Rightarrow 4a$.

$f_{yx} = 2x \Rightarrow 4a$.

$$\rho = \frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{\left|f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2\right|}$$

$2^2 \times$

$$= \frac{\left[(4a^2)^2 + 0\right]^{3/2}}{\left|2a(0)^2 - 2(4a^2)(0)(4a) + (-2a)(4a^2)^2\right|}$$

$$= \frac{(16a^4)^{3/2}}{\left|(-4a) \times 16a^4\right|}$$

$16a^4$

$4a^2$

$64a^6$

$$= (16a^4)^{3/2} / 32a^5$$

$$\frac{64a^6}{64a^5}$$

$$\frac{64a^6}{32a^5}$$

$\Rightarrow$ $2a$.

(3) For a curve in parametric form $x = f(t)$, $y = g(t)$!

$$\rho = \frac{\left[(f'(t))^2 + (g'(t))^2\right]^{3/2}}{\left|f'(t)\,g''(t) - g'(t)\,f''(t)\right|}$$

(4) For a curve in polar form $r = f(\theta)$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

Q. Determine radius of curvature of the curve $r = 2\cos 2\theta$ at $\theta = \frac{\pi}{2}$

Sol:

$$r = 2\cos 2\theta$$

$$\frac{dr}{d\theta} = -4\sin 2\theta, \quad \frac{d^2r}{d\theta^2} = -8\cos 2\theta$$

At $\theta = \frac{\pi}{2}$;

$$r = 2\cos 2\left(\frac{\pi}{2}\right) = -2$$

$$\frac{dr}{d\theta} = -4\sin 2\left(\frac{\pi}{2}\right) = 0$$

$$\frac{d^2r}{d\theta^2} = -8\cos 2\left(\frac{\pi}{2}\right) = 8.$$

Now,

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$= \frac{\left[(-2)^2 + (0)^2\right]^{3/2}}{\left|(-2)^2 + 2(0)^2 - (-2)(8)^1\right|}$$

$$= \frac{(4)^{3/2}}{|4 + 16|}$$

~~8 4 2~~

~~132 66 33~~

$$\rho = 8/20 = 2/5$$

Q Find radius of curvature of the curve $x = a(t - \sin t)$, $y = a(1-\cos t)$ at any point.

$\therefore$ $x = a(t - \sin t)$. $y = a(1-\cos t)$.

$f'(x) = a - a\cos t$ $f'(y) = a\sin t$

$f''(x) = a\sin t$ $f''(y) = a\cos t$

$$\rho = \frac{[(f'(t))^2 + (g'(t))^2]^{3/2}}{|f'(t)\,g''(t) - g'(t)f''(t)|}$$

$$= \cancel{\frac{[(a\sin t)^2 + (a\cos t)^2]^{3/2}}{|}}$$

$$= \frac{[(a - a\cos t)^2 + (a\sin t)^2]^{3/2}}{|(a-a\cos t)(a\cos t) - (a\sin t)(a\sin t)|}$$

$$= \frac{[a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t]^{3/2}}{|a^2\cos t - a^2\cos^2 t - a^2\sin^2 t|}$$

$$= \frac{[a^2 - 2a^2\cos t + a^2]^{3/2}}{|a^2\cos t - a^2|}$$

$$= \frac{[2a^2 - 2a^2\cos t]^{3/2}}{|a^2(\cos t - 1)|}$$

$$= \frac{[2a^2(1-\cos t)]^{3/2}}{|a^2(\cos t - 1)|} = \frac{2\sqrt{2}\,a^3\sqrt{1-\cos t}}{a^2}$$

$$= 2\sqrt{2}\,a\sqrt{1-\cos t}$$

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$$\ell = 2\sqrt{2}\, a \sqrt{1-\left(1-2\sin^2\frac{t}{2}\right)}$$

$$= 2\sqrt{2}\, a \sqrt{2\sin^2\frac{t}{2}}$$

$$\boxed{\ell = 4a \sin\frac{t}{2}}$$

Maxima And Minima for $f(x)$.

Stationary points / critical points

$f'(x) = 0$

Maximum ~~function~~ point
Minimum point — Turning point
Point of inflexion

Increasing Function:

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

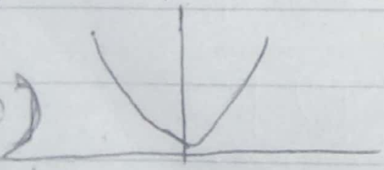
example: e^x, $x > 0$, straight line with +ve slop
$y = x^3$

Decreasing Function:

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

example: e^{-x}, $x > 0$, straight line with -ve slop.
$y = -x^3$

$y = x^2$, $x \in \mathbb{R}$.



$[-\infty, 0)$ decreasing increasing $[0, \infty]$.

Minimum Point:- A point from which function changes itself from decreasing to increasing value.

Maximum Point:- A point from which function changes itself from increasing to decreasing value.

No change in the behaviour of function

Point of Inflexion:- It is neither ~~po~~ minimum point or maximum point.
If at the left of the point of inflexion the value of function is decreasing then it is also decreasing at its right.

=> e^x has no stationary point.

=> $y = x^3$ has stationary point at $x = 0$.
and this stationary point is point of inflexion.

1st derivative Test

let $x = a$ be a stationary point of $f(x)$

	$f'(x)$		
Before $x=a$	at $x=a$	after $x=a$	
$-$	0	$+$	minimum
$+$	0	$-$	maximum
$-$	0	$-$	point of inflexion
$+$	0	$+$	point of inflexion

* +ve slope indicates increasing function.
* -ve slope indicates decreasing function.

Second Derivative Test

=> If $f''(a) > 0$, $x = a$ is minimum point

=> If $f''(a) < 0$, $x = a$ is maximum point.

=> If $f''(a) = 0$, the test is inconclusive.

Q) Find and classify the stationary points of the function.

$$f(x) = -\cancel{x}(2-x)^4$$

$$\cancel{f'(x) = -2 \times 4(2-x)^3 x}$$

Solⁿ To find stationary points, we solve $f'(x) = 0$ for x.

$$f'(x) = 4(2-x)^3 = 0$$

=> $(2-x)^3 = 0$

=> $2-x = 0$

=> $x = 2$

To classify this stationary point, we apply second derivative test.

$$f''(x) = -12(2-x)^2$$

At $x = 2$, $f''(x) = 0$.

The test is inconclusive. So, we apply the first derivative test.

$f'(1.9) > 0$ and $f'(2.1) < 0$

As the function changes from increasing to decreasing function about the stationdy point $x = 2$, therefore $x = 2$ is a maximum point.

Q $y = x^3 - 9x^2 + 24x + 3$, Find the overall minimum and maximum value of this function in the range $[0, 5]$. 12

Solution:

$$f'(x) = 3x^2 - 18x + 24.$$

$$3x^2 - 18x + 24 = 0.$$

$$3x^2 - 12x - 6x + 24 = 0$$

$$\Rightarrow 3x(x-4) - 6(x-4) = 0$$

$$\Rightarrow (3x-6)(x-4) = 0.$$

Either

$3x - 6 = 0$ $\quad$ $x - 4 = 0$

$\Rightarrow x = 2$ $\quad$ $\Rightarrow x = 4.$

$$f''(x) = 6x - 18.$$

At $x = 2$, $f''(x) < 0$

Maximum point.

$$f''(x) = 6x - 18$$

At $x = 4$, $f''(4) > 0.$

Minimum point

maximum / minimum point → value of x.

maximum / minimum value → function value.

Stationary points : $x = 2$, $x = 4$

local maximum point (at $x = 2$) — local minimum point (at $x = 4$)

At $x = 2$,

$$y = (2)^3 - 9(2)^2 + 24(2) + 3$$

$$\Rightarrow y = 8 - 36 + 48 + 3$$

$$\Rightarrow y = 23$$

At $x = 4$,

$$y = (4)^3 - 9(4)^2 + 24(4) + 3$$

$$\Rightarrow y = 19$$

At $x = 0$,

$$y = 3$$

At $x = 5$,

$$y = (5)^3 - 9(5)^2 + 24(5) + 3$$

$$\Rightarrow y = 23$$

Overall minimum value of given function is 3 which occurs at $x = 0$.

Overall maximum value for the function is 23 which occurs at $x = 2$ and $x = 5$.

Q The power delivered to the load resistance R_L for a circuit is given by.

$$P = \frac{25 R_L}{(2000 + R_L)^2}$$

Show that maximum power delivered to the load resistance occurs for $R_L = 2000$.

$$P = \frac{25 R_L}{(2000 + R_L)^2}$$

$$P = \frac{25 R_L}{(2000 + R_L)^2}$$

$$P' = \frac{(2000+R_L)^2 (25) - 25 R_L \times 2(2000+R_L)}{(2000 + R_L)^4}$$

$$\Rightarrow P' = \frac{25(2000 + R_L)^2 - 50 R_L (2000 + R_L)}{(2000 + R_L)^4}$$

$$\frac{25(2000 + R_L)^2 - 50 R_L (2000 + R_L)}{(2000 + R_L)^4} = 0$$

$$\Rightarrow \frac{25(2000 + R_L) - 50 R_L}{(2000 + R_L)^3} = 0$$

$$\Rightarrow 50,000 + 25 R_L - 50 R_L = 0.$$

$$\Rightarrow 50,000 = 25 R_L$$

$$\Rightarrow R_L = 2000.$$

$R_L = 2000$.

Stationary point : $R_L = 2000$.

~~R_L~~ $P'(1999) \cancel{>} 0$ and $P'(2001) \cancel{<} 0$.

$R_L = 2000$ is a maximum point.

Maxima and Minima For Functio Of Two Variables

Stationary points / critical points

- Maximum point
- Minimum point
- Saddle point (same as point of inflexion)

To find the stationary points for $z = f(x, y)$ Determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and equate to zero both these partial derivatives to get a system of two equations in x and y solving the equation simultaneously we obtain value of x and y which gives the stationary point.

In this case, stationary point is in the form of an ordered pair in which the first value of x and the second is of y.

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(x, y) or (x_1, y_1) (x_2, y_2)

(2) To classify the stationary points

1. Find second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$ and evaluate these at each stationary point.

2. Evaluate $\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$ at each stationary point.

discriminant

3. a) if $\Delta > 0$, the stationary point is a saddle point.

b) If $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$, the stationary point is a maximum point.

c) if $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$, the stationary point is a minimum point.

d) if $\Delta = 0$, the test is inconclusive.

→ This is often also called as second partial derivative test.

Q Find and classify the stationary points for the function $z = (x-1)^2 + (y-2)^2$.

For stationary point $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

$2(x-1) = 0$ $\quad\quad$ $2(y-2) = 0$

$\Rightarrow x = 1$ $\quad\quad$ $\Rightarrow y = 2$

Thus the stationary point is $(1, 2)$

$\frac{\partial^2 z}{\partial x^2} = 2, \; \frac{\partial^2 z}{\partial y^2} = 2, \; \frac{\partial^2 z}{\partial x \partial y} = 0.$

$\Delta = 0 - 4$

$\Delta = -4$

As $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$, this implies that $(1, 2)$ is a minimum point.

$$\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$$

$= (-6)^2 - 6(2) \cdot 6(2)$

$= 36 - 144$

$= -108$

As $\Delta < 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$ so $(2, 2)$ is a minimum point.

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Q) Find and classify the stationary points for the function $z = x^3 - 6xy + y^3$.

For stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

$3x^2 - 6y = 0$ $\qquad$ $3y^2 - 6x = 0$

$\Rightarrow x^2 - 2y = 0 \quad \dots (1)$ $\qquad$ $\Rightarrow y^2 - 2x = 0 \quad \dots (2)$

$\Rightarrow \quad y = \frac{x^2}{2} \quad \dots (3)$

Substituting it in eq (2)

$\frac{x^4}{4} - 2x = 0.$

$\Rightarrow \quad x^4 - 8x = 0$

$\Rightarrow \quad x(x^3 - 8) = 0.$

Either

$x = 0$ $\qquad$ $x = 2.$

$\Rightarrow \quad y = 0$ $\qquad$ $\Rightarrow \quad y = 2.$

Stationary point : $(0,0)$, $(2,2)$

$\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2}$ $6x$, $\frac{\partial^2 z}{\partial x \partial y} = -6.$

$\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$

$= (-6)^2 - 6(0).6(0)$

$= 36$

As $\Delta > 0$ so $(0,0)$ is a saddle poi

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Q. An open rectangular box is to have volume of $32\,m^3$. Find dimensions of the box so that the total surface area is minimum.

Let x, y, z be the dimensions of the box.
Volume of the box is $V = 32\,m^3$.

$$\therefore \quad xyz = 32 \quad \dots (1)$$

Surface area of the box is given by:

$$S = xy + 2xz + 2yz \quad \dots (2)$$

From (1), $z = \frac{32}{xy}$. Substituting it in (2), we obtain,

16 + 16 + 16

$$S = xy + 2x \cdot \frac{32}{xy} + 2y \cdot \frac{32}{xy}$$

$$\Rightarrow \quad S = xy + \frac{64}{y} + \frac{64}{x} \quad \dots (3)$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} \qquad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$y - \frac{64}{x^2} = 0 \quad \dots (4) \qquad x - \frac{64}{y^2} = 0 \quad \dots (5)$$

$$\Rightarrow \quad x^2 y - 64 = 0 \qquad \Rightarrow \quad xy^2 - 64 = 0.$$

$$\Rightarrow \quad x = \frac{64}{y^2}$$

$$x^2y - 64 = 0$$

$$\Rightarrow \frac{64\times 64}{y^4}\cdot y - 64 = 0$$

$$\Rightarrow \frac{64}{y^3} - 1 = 0$$

$$\Rightarrow 64 - y^3 = 0$$

$$\Rightarrow y = 4$$

Using in eq (5),

$$x - \frac{64}{16} = 0$$

$$\Rightarrow x = 4$$

$\therefore$ (4, 4) is a stationary point for the function in (3).

$$\frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}, \quad \frac{\partial^2 S}{\partial x \partial y} = 1$$

At (4, 4),

$$\frac{\partial^2 S}{\partial x^2} = 2, \quad \frac{\partial^2 S}{\partial y^2} = 2, \quad \frac{\partial^2 S}{\partial x \partial y} = 1$$

Now

$$\Delta = (1)^2 - (2)(2) = -3 < 0$$

As $\Delta < 0$ and $\frac{\partial^2 S}{\partial x^2} > 0$, hence (4, 4) is a minimum point.

$$z = \frac{32}{xy} = 2$$

Required dimensions are 4m, 4m and 2m.
Minimum surface area is:

$$S = 48 m^2.$$

Differentials Of A Function

=> Gives us the approximate changes in the function.

$$y = f(x)$$

$\Delta \to$ exact change

For an independent variable.

$$dx = \Delta x$$

For dependent variable.

$$dy \approx \Delta y$$

$$dy = f'(x)\, dx$$

$f'(x)$ → differential coefficient

dx → differential.

$$z = f(x,y)$$

Total differential

$$dz = \frac{\partial f}{\partial x} \underbrace{dx}_{=\Delta x} + \frac{\partial f}{\partial y} \underbrace{dy}_{=\Delta y}$$

$$w = g(x, y, z)$$

$$dw = \frac{\partial g}{\partial x} \underbrace{dx}_{\Delta x} + \frac{\partial g}{\partial y} \underbrace{dy}_{\Delta y} + \frac{\partial g}{\partial z} \underbrace{dz}_{\Delta z}$$

Purpose of differentials is to calculate approximate changes.

(1) The power P consumed in a resistor is given by $P = \frac{V^2}{R}$ watts. Determine the approximate change in power when V increases by 5% and R decreases by 0.5%. If the original values of V and R are 50 Volts and 12.5 ohms respectively.

$50 \times \frac{5}{\cancel{100}} = 2.5$ ~~52.5~~ $\frac{5}{10 \times 10} \times \frac{12.5}{10}$

Solution

$$\because P = \frac{V^2}{R}$$

$$\Rightarrow dP = \frac{\partial P}{\partial V} dV + \frac{\partial P}{\partial R} dR \quad \ldots (1)$$

$$\frac{\partial P}{\partial V} = \frac{2V}{R}, \quad \frac{\partial P}{\partial R} = \frac{-V^2}{R^2}, \quad dV = \Delta V, dR = \Delta$$

Now, $V = 50$ Volts, $R = 12.5$ ohms

$\Delta V = 5\%$ of $50 = 2.5$ Volts.

$\Delta R = -0.5\%$ of $12.5 = -0.0625$ ohm

$$\frac{\partial P}{\partial V} = \frac{2 \times 50}{12.5} = 8, \quad \frac{\partial P}{\partial R} = -16$$

2.500

Substituting these values in (1), we get

$dP = (8)(2.5) + (-16)(-0.0625)$

$\Rightarrow dP = 21$ Watts

$\therefore$ Power increases approximately by 21 Watts

(2) $f_r = \frac{1}{2\pi\sqrt{LC}}$ represents the resonant frequency of a series connected circuit containing inductance L and capacitance C. Determine the approximate percentage change in f_r when L is decreased by 3% and C is increased by 5%.

$\therefore f_r = \frac{1}{2\pi\sqrt{LC}}$

$\Rightarrow df_r = \frac{\partial f_r}{\partial L} dL + \frac{\partial f_r}{\partial C} dC \ldots (1)$

$\frac{\partial f_r}{\partial L} = \frac{-1}{L \cdot 4\pi\sqrt{LC}}, \quad \frac{\partial f_r}{\partial C} = \frac{-1}{C \times 4\pi\sqrt{LC}}$

$dL = \Delta L, \quad dC = \Delta C.$

Now, $\Delta L = -3\%$ of $L = -0.03L$

$\Delta C = 5\%$ of $C = 0.05C$

$df_r = \frac{-(-0.03)}{4\pi\sqrt{LC}} + \frac{(-0.05)}{4\pi\sqrt{LC}}$

$\Rightarrow df_r = \frac{1}{2\pi\sqrt{LC}} \times \frac{(0.03 - 0.05)}{2}$

$$df_r = \frac{-0.01}{2\pi\sqrt{LC}}$$

$\Rightarrow df_r = -1\% f_r$.

$\therefore$ frequency decreases approximately by 1%.

Integration :- (Antiderivative)

$$\frac{d}{dx} f(x) = F(x)$$

$$\Rightarrow \int F(x)\,dx = f(x) + C$$

arbitrary constant

constant of integration

$\rightarrow$ indefinite integral

Two types of integration:

$\Rightarrow$ Indefinite integral

$$\int F(x)\,dx = f(x) + C$$

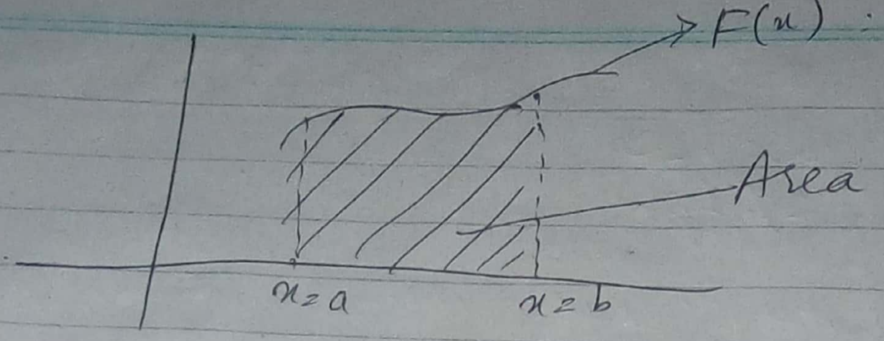
$\Rightarrow$ <u>Definite integral</u>: It results in a definite value.

$$\int_a^b F(x)\,dx = \left[f(x)+C\right]_a^b = (f(b)+C) - (f(a)+C)$$
$$= f(b) - f(a) = A$$

limits of integration

Geometrical meaning: Area under the curve bounded b/w x-axis and vertical lines. Vertical lines are the limits of integral.

$x^{-?}$



$\frac{1}{\sqrt{x}}$

$f(x)$	$\int f(x)\,dx$
0	C
k	$kx + C$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln x + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$\ln \sec x$ / $-\ln \cos x$
$\cot x$	$\ln \sin x$ / $-\ln \operatorname{cosec} x$
$\sec x$	$\ln(\sec x + \tan x)$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x)$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1} x$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + C$$

$$\int f(x) \cdot f'(x)\, dx = \frac{[f(x)]^2}{2} + C$$

$$\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + C$$

I L A T E

Integration By Parts

$$\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\left(\int v\,dx\right)dx.$$

Evaluate.

① $\int x \sin x \, dx$.

let $u = x$; $v = \sin x$.

$$\therefore \int x \sin x = x\int \sin x - \int \frac{dx}{dx}\left(\int \sin x\,dx\right)dx$$

$$= -x\cos x - \int (1)(-\cos x)\,dx.$$

$$= -x\cos x + \sin x + C.$$

② $\int x \ln x$

let $u = \ln x$, $v = x$.

$$\therefore \int x \ln x = \ln x \int x\,dx - \int \frac{d\ln x}{dx}\left(\int x\,dx\right)dx$$

$$= \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C.$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

(3) $\int \ln x \, dx =$

$u = \ln x \quad , \quad v = 1$

$\therefore \int \ln x \, dx = \ln x \int 1 \, dx - \int \frac{d \ln x}{dx} \left(\int 1 \, dx \right) dx$

$= \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx$

$= x \ln x - x + C$

(4) $\int \tan^{-1} x \, dx$

$u = \tan^{-1} x \quad , \quad v = 1$

$\therefore \int \tan^{-1} x \, dx = \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(5) $\int e^{ax} \sin bx \, dx$

$\therefore$ let $u = \sin bx \quad , \quad v = e^{ax}$

Let $I = \int e^{ax} \sin bx \, dx = \sin bx \int e^{ax} \, dx - \int \frac{d \sin bx}{dx} \left(\int e^{ax} dx \right)$

$= \sin bx \cdot \frac{e^{ax}}{a} - \int \cos bx (b) \frac{e^{ax}}{a} \, dx$

$I = \frac{1}{a} \sin bx \, e^{ax} - \frac{b}{a} \int \cos bx \, e^{ax} \, dx$

Again integration by parts.

$1+\frac{b^2}{a^2}$

$\frac{a^2+b^2}{a^2}$

$$I = \frac{1}{a}\sin bx\, e^{ax} - \frac{b}{a}\left[\cos bx\int e^{ax}dx - \int\frac{d\cos bx}{dx}\left(\int e^{ax}dx\right)dx\right]$$

$$= \frac{1}{a}\sin bx\, e^{ax} - \frac{b}{a}\left[\cos bx.\frac{e^{ax}}{a} + \int \sin bx.b.\frac{e^{ax}}{a}dx\right]$$

$$= \frac{1}{a}\sin bx\, e^{ax} - \frac{b}{a^2}\cos bx\, e^{ax} - \frac{b^2}{a^2}\int e^{ax}\sin bx\, dx$$

$$I = \frac{1}{a}\sin bx\, e^{ax} - \frac{b}{a^2}\cos bx\, e^{ax} - \frac{b^2}{a^2} I.$$

$$\Rightarrow I + \frac{b^2}{a^2} I = \frac{1}{a}\sin bx\, e^{ax} - \frac{b}{a^2}\cos bx\, e^{ax}$$

$$\Rightarrow I = \left(\frac{\cancel{a^2}}{a^2+b^2}\right)\left[\frac{e^{ax}}{\cancel{a^2}}\left(a\sin bx - b\cos bx\right)\right]$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + C$$

$$\int e^{ax}\cos bx\, dx.$$

let

$$I = \int e^{ax}\cos bx\, dx.$$

Using integration by parts.

$$\therefore I = \cos bx\int e^{ax}dx - \int\frac{d\cos bx}{dx}\left(\int e^{ax}dx\right)dx$$

$$= \cos bx.\frac{e^{ax}}{a} + \int \sin bx.b.\frac{e^{ax}}{a}dx.$$

$$= \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a}\int \sin bx\, e^{ax}dx.$$

dy $\cancel{\frac{dy}{dx}}$

Again integration by parts

$$I = \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a}\left[\sin bx \int e^{ax}dx - \int \frac{d\sin bx}{dx}\left(\int e^{ax}dx\right)dx\right.$$

$$\Rightarrow I = \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a}\left[\sin bx \cdot \frac{e^{ax}}{a} - \int \cos bx \cdot b \cdot \frac{e^{ax}}{a}dx\right.$$

$$\Rightarrow I = \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a^2}\sin bx\, e^{ax} - \frac{b^2}{a^2}\int \cos bx\, e^{ax}dx$$

$$\Rightarrow I + \frac{b^2}{a^2}I = \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a^2}\sin bx\, e^{ax}$$

$$\Rightarrow I = \frac{\cancel{a^2}}{a^2+b^2}\cdot\frac{e^{ax}}{\cancel{a^2}}\left[a\cos bx + b\sin bx\right]$$

$$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + C.$$

$$-\frac{1}{2(x^2+1)} \qquad \frac{1}{2(x^2+1)} \qquad (x^2+1)^{-2}$$

$$\int \frac{x^2}{(x^2+1)^2}\, dx$$

$$\Rightarrow \int x \cdot \frac{x}{(x^2+1)^2}\, dx$$

$$\Rightarrow x \cdot \int \frac{x}{(x^2+1)^2}\, dx - \int \left(1 \cdot \int \frac{x}{(x^2+1)^2}\, dx \right) dx$$

$$\Rightarrow x \left[-\frac{1}{2} \cdot \frac{1}{x^2+1} \right] - \frac{1}{2} \int \left(\frac{2x}{(x^2+1)^2}\, dx \right) dx$$

$$\Rightarrow \frac{-x}{2(x^2+1)} + \frac{1}{2} \int \frac{1}{x^2+1}\, dx$$

$$\Rightarrow \frac{-x}{2(x^2+1)} + \frac{1}{2} \tan^{-1} x + C$$

Integration By Completing Square Me...

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\int \frac{1}{x^2 - 8x + 65}\, dx \qquad y^2 + 7^2$$

$$\Rightarrow \int \frac{1}{x^2 - 2 \cdot x \cdot 4 + 16 + 49}\, dx$$

$$\Rightarrow \int \frac{1}{(x-4)^2 + 7^2}\, dx$$

$$\Rightarrow \frac{1}{7^2} \int \frac{1}{\left(\frac{x-4}{7}\right)^2 + 1}\, dx$$

let

$$y = \frac{x-4}{7}$$

$$dy = \frac{1}{7}\,dx$$

$$\frac{1}{7}\cdot\frac{1}{7}\int \frac{1}{\frac{(x-4)^2}{7^2}+1}\,dx$$

$$\Rightarrow \frac{1}{7}\int \frac{dy}{y^2+1}$$

$$\Rightarrow \frac{1}{7}\tan^{-1} y + C$$

$$\Rightarrow \frac{1}{7}\tan^{-1}\left(\frac{x-4}{7}\right) + C$$

$$\int \frac{1}{\sqrt{5x-6-x^2}}\,dx$$

$$-\left(x^2-5x+6\right)$$

$$-\left(x^2-2\times 5\cdot\frac{1}{2}+\left(\frac{5}{2}\right)^2\right.$$

$$+6$$

$$-\left(x-\frac{1}{2}\right)^2+6-2$$

$$-$$

$$\Rightarrow \int \frac{1}{\sqrt{-6-(x^2-5x)}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{-6-\left(x^2-5x+\frac{25}{4}-\frac{25}{4}\right)}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{-6-\left(x-\frac{5}{2}\right)^2+\frac{25}{4}}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{1}{4}-\left(\cancel{x}\frac{-5}{2}\right)^2}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{1}{4}-\left(\frac{2x-5}{2}\right)^2}}\,dx$$

$$\Rightarrow \int \frac{1}{1/2}\times\frac{1}{\sqrt{1-(2x-5)^2}}\,dx$$

$$\Rightarrow 2\int \frac{1}{\sqrt{1-(2x-5)^2}}\,dx$$

let

$$y = 2x - 5$$
$$\Rightarrow dy = 2\,dx$$

~~Ext~~
Now

$$\int \frac{1}{\sqrt{1-y^2}}\,dy$$
$$\Rightarrow \cancel{\int \frac{1}{\sqrt{1-(2x-5}}} \quad \sin^{-1} y + C$$
$$\Rightarrow \sin^{-1}(2x-5) + C.$$

Integration By Partial Fractions

⇒ Used to integrate rational function.
⇒ Purpose: Give simplified form of rational funct

Case I: Distinct linear factors in the denominator.

$$\frac{x}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1} \quad \ldots (i)$$

where A and B are unknown constant

From (i),

$$x = A(2x+1) + B(x-1).$$

Substituting $x = 1$, $1 = 3A \Rightarrow A = 1/3$

Substituting $x = -\frac{1}{2}$, $-\frac{1}{2} = -\frac{3}{2}B \Rightarrow B = 1/3$

$$\therefore \frac{x}{(x-1)(2x+1)} = \frac{1/3}{x-1} + \frac{1/3}{2x+1}$$

$$\therefore \int \frac{x}{(x-1)(2x+1)}\,dx = \frac{1}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{2x+1}$$

$$= \frac{1}{3}\ln(x-1) + \frac{1}{6}\ln(2x+1) + c$$

Case II :- Repeated linear factors in the denominator

$$\frac{2x-1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad \ldots(1)$$

where A, B and C are unknown constants.

From (1), $2x-1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \ldots(2)$

Substituting $x=1$,

$1 = 3B \Rightarrow B = 1/3.$

Substituting $x=-2$,

$-5 = 9C \Rightarrow C = -5/9.$

Simplifying eq (2), we obtain -

$2x-1 = A(x^2+x-2) + B(x+2) + C(x^2-2x+1)$

$\Rightarrow 2x-1 = (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)$

Comparing coefficients of x^2,

$\Rightarrow 0 = A+C$

$\Rightarrow A = -C$

$\Rightarrow A = 5/9.$

$$\therefore \frac{2x-1}{(x-1)^2(x+2)} = \frac{5/9}{x-1} + \frac{1/3}{(x-1)^2} - \frac{5/9}{x+2}$$

$x^2 - 2$

$x^2 + 3x + 9$

$$\therefore \int \frac{2x-1}{(x-1)^2(x+2)} = \frac{5}{9}\int \frac{dx}{(x-1)} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{5}{9}\int \frac{dx}{x+2}$$

$$= \frac{5}{9}\ln(x-1) + \frac{1}{3}\cdot\frac{(-1)}{x-1} - \frac{5}{9}\ln(x+2)$$

$$= \frac{5}{9}\ln\left(\frac{x-1}{x+2}\right) - \frac{1}{3(x-1)} + C.$$

Case III: Distinct irreducible quadratic factor

$$\frac{x}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$\frac{x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

Case IV: Repeated irreducible quadratic factors

$$\frac{3x-1}{(x+3)(x^2+2)^2} = \frac{A}{x+3} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

$\cos x = -1$

$x = \cos^{-1}(-$

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$$\int \frac{\sin x}{(1+\cos x)(2+\cos x)}\, dx$$

let

$$y = \cos x$$

$$\Rightarrow dy = -\sin x\, dx$$

$$\Rightarrow -dy = \sin x\, dx$$

$$\frac{1}{(1+y)(2+y)} = \frac{A}{1+y} + \frac{B}{2+y}$$

$$\Rightarrow 1 = A(2+y) + B(1+y)$$

Put $y = -2$,

$$1 = -B \Rightarrow B = -1$$

Put $y = -1$

$$1 = A \Rightarrow A = 1$$

Now,

$$\frac{1}{(1+y)(2+y)} = \frac{1}{1+y} - \frac{1}{2+y}$$

$$\Rightarrow \int \frac{-dy}{(1+y)(2+y)} = \int \frac{-dy}{1+y} - \int \frac{-dy}{2+y}$$

$$= -\int \frac{dy}{1+y} + \int \frac{dy}{2+y}$$

$$= -\ln(1+y) + \ln(2+y)$$

$$= \ln\left(\frac{2+y}{1+y}\right) + C$$

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$$\int \frac{\sin x \, dx}{(1+\cos x)(2+\cos x)} = \ln\left(\frac{2+\cos x}{1+\cos x}\right) + C.$$

$$\int \frac{\tan x}{1+\cos x} \, dx$$

$$\int \frac{\sin x \, dx}{\cos x(1+\cos x)}$$

let

$$y = \cos x$$

$$\Rightarrow \quad dy = -\sin x \, dx$$

$$\Rightarrow \quad -dy = \sin x \, dx$$

Now,

$$\frac{1}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y}$$

$$1 = A(1+y) + By$$

Put $y = 0$.

$$\Rightarrow \quad 1 = A \quad \Rightarrow \quad A = 1$$

Put $y = -1$.

$$\Rightarrow \quad 1 = -B \quad \Rightarrow \quad B = -1$$

Now,

$$\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{1+y}$$

$$\Rightarrow \int \frac{-dy}{y(1+y)} = -\int \frac{dy}{y} + \int \frac{dy}{1+y}$$

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$$\int \frac{-dy}{y(1+y)} = -\ln y + \ln(1+y) + C$$

$$\Rightarrow \int \frac{-dy}{y(1+y)} = \ln\left(\frac{1+y}{y}\right) + C$$

$$\Rightarrow \int \frac{\sin x \, dx}{\cos x(1+\cos x)} = \ln\left(\frac{1+\cos x}{\cos x}\right) + C$$

$$\Rightarrow \int \frac{\tan x \, dx}{1+\cos x} = \ln\left(\frac{1+\cos x}{\cos x}\right) + C$$

☆ $$\int \frac{1}{(x^2+1)^2} \, dx$$

$$\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$1 = (Ax+B)(x^2+1) + (Cx+D)$$

$$\Rightarrow 1 =$$

$$\int \frac{dx}{(1+x^2)^2}$$

let

$$x = \tan\theta \Rightarrow \theta = \tan^{-1}(x)$$
$$dx = \sec^2\theta \, d\theta$$

Now,

$$\int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^2}$$

$$\Rightarrow \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2}$$

$$\Rightarrow \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$\Rightarrow \int \frac{d\theta}{\sec^2\theta}$$

$$\Rightarrow \int \cos^2\theta \, d\theta$$

$$\Rightarrow \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\Rightarrow \frac{1}{2}\int d\theta + \frac{1}{2\times 2}\int 2\cos 2\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\sin\theta\cos\theta + C$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\frac{\tan\theta}{\sec^2\theta} + C$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\frac{x}{1+x^2} + C$$

Reduction Formula

$$\int \sin^n x\, dx = \int \sin^{n-1} x \cdot \sin x\, dx.$$

let

$$u = \sin^{n-1} x, \quad v = \sin x.$$

$$\int \sin^n x\, dx$$

$$\Rightarrow -\sin^{n-1} x \cdot \cos x \int (n-1) \sin^{n-2} x \cdot \cos x(-\cos x)\, dx$$

$$\Rightarrow -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$$

$$\Rightarrow -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, (1-\sin^2 x)\, dx$$

$$\Rightarrow -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

$$\Rightarrow n\left(\int \sin^n x\, dx\right) = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx$$

$$\Rightarrow \int \sin^n x = \frac{-\cos x \sin^{n-1} x}{n} + \frac{(n-1)}{n} \int \sin^{n-2} x\, dx$$

Such a formula which involves an integral of n^{th} power and connects it with integral of reduced power of same function.

$$\int \cos^n x \cdot dx$$

$$\Rightarrow \int \cos^{n-1} x \cos x \, dx$$

$$\Rightarrow \cos^{n-1} x \int \cos x \, dx - \int \frac{d\cos^{n-1} x}{dx} \left(\int \cos x \, dx \right) dx$$

$$\Rightarrow \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x (-\sin x) (\sin x) \, dx$$

$$\Rightarrow \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx.$$

$$\Rightarrow \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$\Rightarrow \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx \; (n-1) \int \cos^n x \, d$$

$$\Rightarrow n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, d$$

$$\Rightarrow \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} \int \cos^{n-2} x \, dx$$

$$\int \tan^n x \, dx$$

$$\Rightarrow \int \tan^{n-2} x \cdot \tan^2 x \, dx$$

$$\Rightarrow \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$\Rightarrow \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx.$$

$$\Rightarrow \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$\int \cot^n x\,dx = \int \cot^{n-2} x \cot^2 x\,dx.$$

$$= \int \cot^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$$

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$$\int \sec^n x \, dx$$

$$\int \sec^{n-2} x \sec^2 x \, dx$$

$$\Rightarrow \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \cdot \tan x \, dx$$

$$\Rightarrow \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$\Rightarrow \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$\Rightarrow \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x \, dx - (n-2) \int \sec^n x \, dx$$

$$\Rightarrow \int \sec^n x \, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x \, dx - (n-2) \int \sec^n x \, dx$$

$$\Rightarrow \int \sec^n x \, dx + n \int \sec^n x \, dx - 2 \int \sec^n x \, dx = \tan x \sec^{n-2} x + (n-2) \int \sec^{n-2} x \, dx$$

$$\Rightarrow \int \sec^n x \, dx = \frac{\tan x \sec^{n-2} x}{(n-1)} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x \, dx$$

$$\int \operatorname{cosec}^n x \, dx$$

$$\Rightarrow \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$$

$$\Rightarrow \operatorname{cosec}^{n-2} x \int \operatorname{cosec}^2 x \, dx - \int \frac{d \operatorname{cosec}^{n-2} x}{dx} \left(\int \operatorname{cosec}^2 x \, dx \right) dx$$

$$\Rightarrow -\cot x \operatorname{cosec}^{n-2} x - \int (n-2) \operatorname{cosec}^{n-3} x (-\operatorname{cosec} x \cot x)(-\cot x) \, dx$$

$$\Rightarrow -\cot x \operatorname{cosec}^{n-2} x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$$

$$-\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$\Rightarrow \int \operatorname{cosec}^n x \, dx = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^n x \, dx + (n-2)\int \operatorname{cosec}^{n-2} x \, dx$$

$$\Rightarrow \int \operatorname{cosec}^n x \, dx + n\int \operatorname{cosec}^n x \, dx - 2\int \operatorname{cosec}^n x \, dx = -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \, dx$$

$$\Rightarrow \int \operatorname{cosec}^n x \, dx = \frac{-\cot x \operatorname{cosec}^{n-2} x}{(n-1)} + \frac{(n-2)}{n-1}\int \operatorname{cosec}^{n-2} x \, dx.$$

$$(n-1)\int \sin^{n-2} x \left(1 - \sin^2 x\right) dx.$$

$$(n-1)\int \left(\sin^{n-2} x - \sin^n x\right) dx$$

$$(n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx.$$

$$\int_{-a}^{a} f(x)\, dx = 0 \text{ if } f(x) \text{ is an odd function.}$$

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx \text{ if } f(x) \text{ is}$$

an even function

sec^n

10-Feb-2020 # APPLICATIONS OF DEFINITE INTEGRALS

(1) The p.d. b/w boundaries a and b of an electric field is given by. $V = \int_a^b \frac{Q}{2\pi r \epsilon_0 \epsilon_r} dr$. Find V given that $a = 10$, $b = 20$, $Q = 2\mu C$, $\epsilon_0 = 8.85 \times 10^{-12}$ and $\epsilon_r = 2.77$.

Solution

$$V = \int_a^b \frac{Q}{2\pi r \epsilon_0 \epsilon_r} dr$$

$$\Rightarrow V = \frac{Q}{2\pi \epsilon_0 \epsilon_r} \int_a^b \frac{1}{r} dr.$$

$$\Rightarrow V = \frac{Q}{2\pi \epsilon_0 \epsilon_r} \times \ln r \Big|_a^b$$

$$\Rightarrow V = \frac{Q}{2\pi \epsilon_0 \epsilon_r} (\ln b - \ln a).$$

$$\Rightarrow V = \frac{Q}{2\pi \epsilon_0 \epsilon_r} (\ln b - \ln a).$$

$$\Rightarrow V = \frac{2 \times 10^{-6}}{2 \times 3.1415 \times 8.85 \times 10^{-12} \times 2.77} (\ln 20 - \ln 1$$

$$\Rightarrow V = \frac{2 \times 10^{-6}}{154.024 \times 10^{-12}} \times (2.9957 - 2.3026$$

$$\Rightarrow V = 0.01298 \times 10^{6} (0.6931)$$

$$\Rightarrow V = 9 \times 10^{-3+6} = 9 \times 10^{3}$$

$$\Rightarrow \boxed{V = 9\,kV}$$

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(2) The average value of a complex voltage waveform is given by:

$$V_{av} = \frac{1}{\pi}\int_0^{\pi}(10\sin\omega t + 3\sin 3\omega t + 2\sin 5\omega t)\, d(\omega t)$$

Evaluate V_{av} correct to 2 decimal places

Solution:

$$V_{av} = \frac{1}{\pi}\int_0^{\pi}(10\sin\omega t + 3\sin 3\omega t + 2\sin 5\omega t)\, d(\omega t)$$

$$\Rightarrow V_{av} = \frac{1}{\pi}\left[-10\cos\omega t - \frac{3\cos 3\omega t}{3} - \frac{2\cos 5\omega t}{5}\right]_0^{\pi}$$

$$\Rightarrow V_{av} = \frac{1}{\pi}\left[\left\{-10\cos\pi - \cos 3\pi - \frac{2}{5}\cos 5\pi\right\} - \left\{-10\cos 0 - \cos 0 - \frac{2}{5}\cos 0\right\}\right]$$

$$\Rightarrow V_{av} = \frac{1}{\pi}\left[10 + 1 + \frac{2}{5} - \left\{-10 - 1 - \frac{2}{5}\right\}\right]$$

$$\Rightarrow V_{av} = \frac{1}{\pi}\left[20 + 2 + 4/5\right].$$

$$\Rightarrow V_{av} = \frac{1}{3.1415}\left[22 + \frac{4}{5}\right]$$

$$\Rightarrow V_{av} = \frac{1}{3.1415}\left[22.8\right]$$

$$\Rightarrow V_{av} = 7.2577$$

$$\Rightarrow \boxed{V_{av} = 7.26\ V}$$

root mean square

Mean and r.m.s values.

Consider a function $y = f(x)$ defined on $[a, b]$. Mean value $\bar{y} = \frac{1}{b-a}\int_a^b y\, dx$.

r.m.s. value $= \sqrt{\frac{1}{b-a}\int_a^b y^2\, dx}$

A sinusoidal voltage is given by $V = 100 \sin \omega t$ volts. Find its mean value and r.m.s value over half a cycle.

Solution

$$\bar{V} = \frac{1}{\pi - 0}\int_0^\pi V\, d(\omega t).$$

$$\Rightarrow \bar{V} = \frac{1}{\pi}\int_0^\pi 100 \sin\omega t\, d(\omega t)$$

$$\Rightarrow \bar{V} = \frac{1}{\pi}\left[-100 \cos\omega t\right]_0^\pi$$

$$\Rightarrow \bar{V} = \frac{1}{\pi}\left[-100\{-1-1\}\right]$$

$$\Rightarrow \bar{V} = \frac{1}{\pi}(+200)$$

$$\Rightarrow \bar{V} = \frac{+200}{3.1418}$$

$$\Rightarrow \boxed{\bar{V} = 63.664 \text{ Volts}}$$

$A = \int_a^n y$

$$r.m.s\ value = \sqrt{\frac{1}{\pi - 0}\int_0^{\pi} v^2\, d(\omega t)}$$

$$\Rightarrow r.m.s\ value = \sqrt{\frac{1}{\pi}\int_0^{\pi} (100 \sin\omega t)^2\, d(\omega t)}$$

$$= \sqrt{\frac{1}{\pi} \times 10000 \int_0^{\pi} \sin^2 \omega t\, d(\omega t)}$$

$$= \sqrt{\frac{1}{\pi} \times 10000 \int_0^{\pi} \left(\frac{1 - \cos 2\omega t}{2}\right) d\omega t}$$

$$= \sqrt{\frac{10000}{\pi} \int_0^{\pi} \frac{1}{2}\, d\omega t - \int_0^{\pi} \frac{\cos 2\omega t}{2}\, d\omega t}$$

$$= \sqrt{\frac{10000}{\pi} \left\{ \frac{1}{2}\left[\omega t\right]_0^{\pi} - \frac{1}{4}\left[\sin 2\omega t\right]_0^{\pi} \right\}}$$

$$= \sqrt{\frac{10000}{\cancel{\pi}} \left\{ \frac{1}{2} \cancel{\pi} \right\}}$$

$$= \sqrt{5000}$$

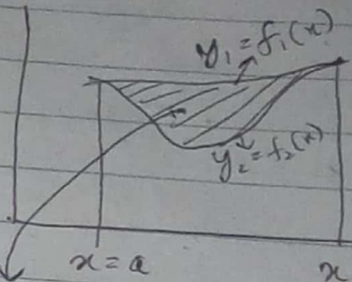
$$= 70.71 \text{ volts}$$

Area Between The Curves

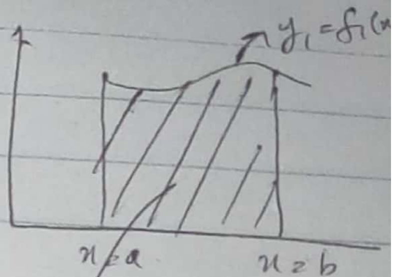
Determine the area enclosed between the curves $y = x^2 + 1$ and $y = 7 - x$.

Solution

$y = x^2 + 1 \quad \ldots (1)$
$y = 7 - x \quad \ldots (2)$



$\text{Area} = \int_a^b [f_1(x) - f_2(x)]\,d$



$\text{Area} = \int_a^b f_1(x)\,d$

$7 - x = x^2 + 1$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow x^2 + 3x - 2x - 6 = 0$

$\Rightarrow x(x+3) - 2(x+3) = 0$

$\Rightarrow (x-2)(x+3) = 0$

Either

$x - 2 = 0$ $\Rightarrow x = 2$

$x + 3 = 0$ $\Rightarrow x = -3.$

Put the value of x in eq (2)

$y = 7 - 2$
$= 5$

$y = 7 - (-3)$
$= 7 + 3$
$= 10$

$(2, 5)$

$(-3, 10)$

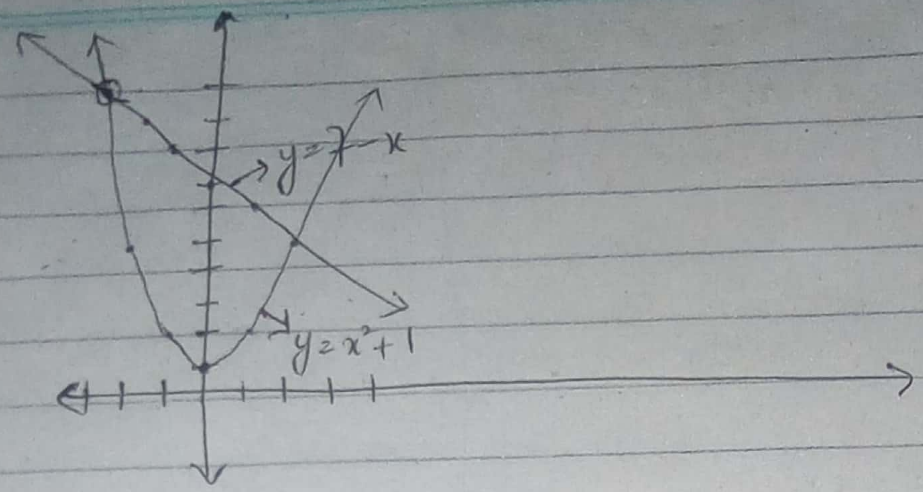
The points of intersection $(-3, 10)$ & $(2,$

$y = x^2 + 1$

x	-3	-2	-1	0	1	2
y	10	5	2	1	2	5

$y = 7 - x$

x	-3	-2	-1	0	1	2
y	10	9	8	7	6	5



$$Area = \int_{-3}^{2} [(7-x) - (x^2+1)]\,dx$$

$$= \int_{-3}^{2} [7 - x - x^2 - 1]\,dx$$

$$= \int_{-3}^{2} [-x^2 - x + 6]\,dx.$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_{-3}^{2}$$

$$\Rightarrow \left\{-\frac{(2)^3}{3} - \frac{(2)^2}{2} + 6(2)\right\} - \left\{-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3)\right\}$$

$$\Rightarrow -\frac{8}{3} - \frac{4}{2} + 12 - \frac{27}{3} + \frac{9}{2} + 18$$

$$\Rightarrow 30 + \frac{5}{2} - \frac{35}{3}$$

$$\Rightarrow 30 + 2.5 - 11.667$$

$$\Rightarrow 20.833 \text{ sq. units.}$$

-8 8

Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis.

$y = x^3 - 2x^2 - 8x$ ... (i)

$y = 0$ (For x-axis) ... (ii)

$x^3 - 2x^2 - 8x = 0$

$\Rightarrow x(x^2 - 2x - 8) = 0$

Either

$x = 0$

$x^2 - 2x - 8 = 0$

$\Rightarrow x^2 - 4x + 2x - 8 = 0$

$\Rightarrow x(x-4) + 2(x-4) = 0$

$\Rightarrow (x+2)(x-4) = 0$.

Either

$x+2 = 0$ $\quad$ $x-4=0$

$\Rightarrow x = -2$ $\quad$ $\Rightarrow x = 4$.

The points of intersection are $(0,0), (-2,0), (4,0)$

The curve intersects x-axis at $x = -2, 0, 4$.

$y = x^3 - 2x^2 - 8x$

x	-2	-1	0	1	2	3	4		
y	0	5	0	-9	-16	-15	0		

$$\text{Area} = \int_{-2}^{0} (x^3 - 2x^2 - 8x)\,dx + \int_{0}^{4} [0 - (x^3 - 2x^2 - 8x)]\,dx$$

$$= \int_{-2}^{0} (x^3 - 2x^2 - 8x)\,dx - \int_{0}^{4} (x^3 - 2x^2 - 8x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{2}{3}x^3 - 4x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - \frac{2}{3}x^3 - 4x^2\right]_{0}^{4}$$

$$\left[\left\{\frac{(0)^4}{4}-\frac{2}{3}(0)^3-4(0)^2\right\}-\left\{\frac{(-2)^4}{4}-\frac{2}{3}(-2)^3-4(-2)^2\right\}\right]$$

$$-\left[\left\{\frac{(4)^4}{4}-\frac{2}{3}(4)^3-4(4)^2\right\}-\left\{\frac{(0)^4}{4}-\frac{2}{3}(0)^3-4(0)^2\right\}\right]$$

$$\Rightarrow \left[-4+\frac{16}{3}+16\right]-\left[\frac{256}{4}-\frac{128}{3}-64\right]$$

$$\Rightarrow -4-\frac{16}{3}+16-\frac{\cancel{256}}{4}+\frac{128}{3}+\cancel{64}$$

$$\Rightarrow \cancel{56}+\cancel{48}+16-\cancel{64}$$

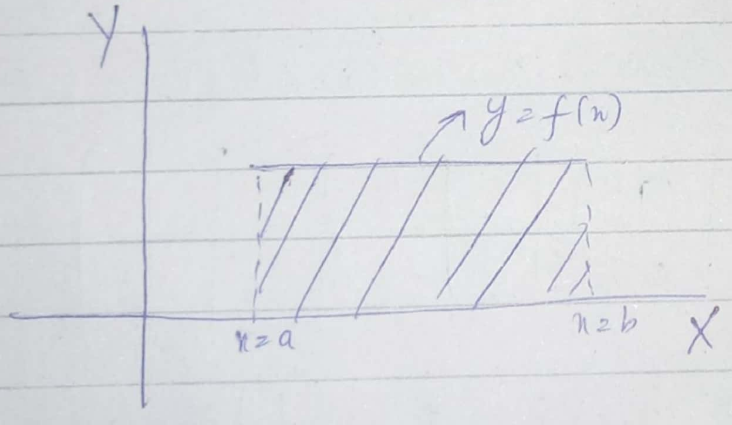
$$\Rightarrow \cancel{56}$$

$$\Rightarrow -4-\frac{16}{3}+16+\frac{128}{3}$$

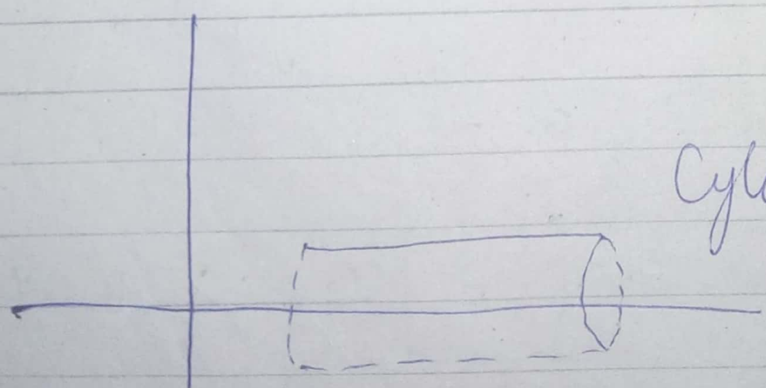
$$\Rightarrow -4-5.33+16+42.66$$

$$\Rightarrow \boxed{49.33 \text{ sq units.}}$$

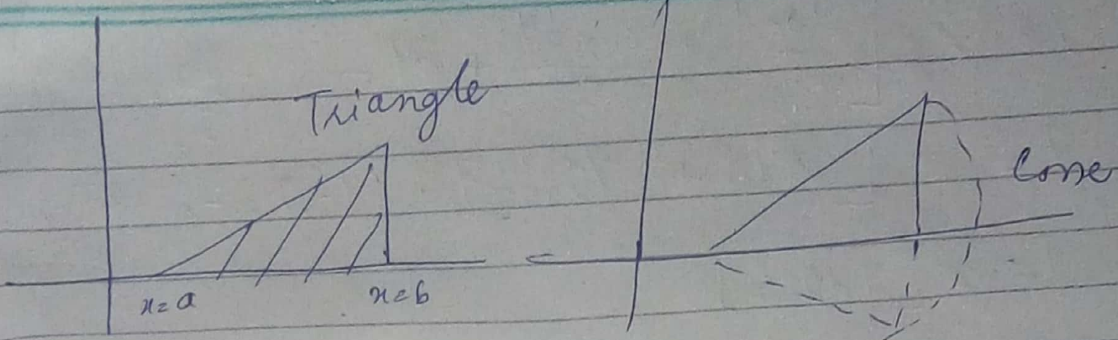
Volume Of Solid Of Revolution



Rectangle.



Cylinderical.



The Volume of solid of object is given by:

$$V = \int_a^b \pi y^2 \, dx.$$

For vertical objects:

$$V = \int_c^d \pi x^2 dy \rightarrow x = f(y)$$

Q The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits $x = 1$ and $x = 4$. Determine the volume of solid of revolution produced.

$$V = \int_b^a \pi y^2 \, dx.$$

$$\Rightarrow V = \int_b^a \pi (x^2+4)^2 \, dx$$

$$\Rightarrow V = \int_1^4 \pi (x^4 + 8x^2 + 16) \, dx.$$

$$\Rightarrow V = \pi \left[\frac{x^5}{5} + \frac{8}{3} x^3 + 16x \right]_1^4$$

$$\Rightarrow V = \pi \left\{ \left[\frac{(4)^5}{5} + \frac{8}{3}(4)^3 + 16(4) \right] - \left[\frac{1}{5} + \frac{8}{3} + 16 \right] \right.$$

$$V = \pi\left[\frac{1024}{5} + \frac{8 \times 64}{3} + 64 - \frac{1}{5} - \frac{8}{3} - 16\right]$$

$$\Rightarrow V = \pi\left[204.8 + 170.667 + 64 - 0.2 - 2.667 - 16\right]$$

$$\Rightarrow V = \pi[420.6]$$

$$\Rightarrow \boxed{V = 420.6\pi \text{ cubic units.}}$$

Improper Integral

(1) Interval of integration is infinite.

$$\int_{-\infty}^{b} f(x)\,dx, \quad \int_{a}^{\infty} f(x)\,dx \quad \& \quad \int_{-\infty}^{\infty} f(x)\,dx$$

(2) The integrand is infinite / undefined at a point in the finite interval of integration.

$$\int_{0}^{1} \ln x\,dx, \quad \int_{-2}^{2} \frac{1}{\sqrt{x-2}}\,dx, \quad \int_{-2}^{2} \frac{1}{x+1}\,dx$$

(3) Improper integral

Convergent → Value exists

Divergent → Value doesnot exist

$+\sec^n x$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_{t}^{b} f(x)\,dx$$

$$\int_{a}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{a}^{t} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} f(x)\,dx + \lim_{t\to\infty} \int_{0}^{t} f(x)\,dx$$

Evaluate:

$$\int_{a}^{\infty} e^{-x}\,dx.$$

$$\Rightarrow \lim_{t\to\infty} \int_{0}^{t} e^{-x}\,dx.$$

$$\Rightarrow \lim_{t\to\infty} \left[-e^{-x}\right]_0^t.$$

$$\Rightarrow \lim_{t\to\infty} \left[(-e^{-t}) - (-e^{0})\right].$$

$$\Rightarrow \lim_{t\to\infty} (1 - e^{-t})$$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1.$$

The integral is convergent.

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Evaluate :

$$\int_{-\infty}^{0} \frac{dx}{1+x^2}$$

$$\Rightarrow \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{1+x^2}$$

$$\Rightarrow \lim_{t\to-\infty} \left(\tan^{-1}x\right)_t^0$$

$$\Rightarrow \lim_{t\to-\infty} \left[\tan^{-1}0 - \tan^{-1}t\right]$$

$$\Rightarrow \lim_{t\to-\infty} \left(0 - \tan^{-1}t\right)$$

$$\Rightarrow -\left(-\pi/2, \cancel{-}3\pi/2\right)$$

$$\Rightarrow \pi/2, 3\pi/2, \ldots$$

① $\int_{0}^{\infty} e^{-x} \sin x \, dx$

② $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

③ $\int_{-\infty}^{\infty} \frac{x}{\sqrt{x^2+2}} \, dx$

Pg: 186, 187
Ex: 15, 16, 17

(2) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$\Rightarrow \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$\Rightarrow \lim_{t \to -\infty}$

Improper integral Of Second Kind.

$\int_a^b f(x)\,dx \qquad a \leq x \leq b$.

$\Rightarrow$ If $f(x)$ is undefined at $x = a$,

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

$\Rightarrow$ If $f(x)$ is undefined at $x = b$,

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

$\Rightarrow$ If $f(x)$ is undefined at $x = c \in (a,b)$ i.e. $a < c < b$. Evaluate.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

Vector Calculas

17-Feb-2020

1. Gradient
2. Divergence
3. Curl
4. Directional derivative

Del operator/ vector differentiation operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

symbol: nebla

Scalar Functions: Function that is not associated with direction.

Vector Functions: That functions which are assigned with a direction.

Scalar Functions → (output is scalar)

$z = f(x,y) = \sin xy,\ x^2 - 2xy + y^2,\ \sin t - 2te^{-t}.$

Vector Functions → (output is vector)

$\sin t\,\hat{i} - \cos t\,\hat{j} + 2\hat{k},$

$[2x^2y, 4xy, 8xz^2] = 2x^2y\,\hat{i} + 4xy\,\hat{j} + 8xz^2\,\hat{k}$

Gradient: Consider a scalar function ~~(f, x, y~~ $f(x,y,z)$. Gradient of the function f, denoted by $\vec{\nabla f}$, is defined as

$$\nabla f = \text{grad} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f(x,y,z)$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Gradient of scalar function is vector function.
of the same variables

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$\int e^{x}\,dx$

$1-2n^2$

(2) Divergence:

Q Given $\phi = x^2 + xyz$, find its gradient.

Solution:

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2 + xyz).$$

$$\Rightarrow \nabla\phi = \frac{\partial}{\partial x}(x^2+xyz)\hat{i} + \frac{\partial}{\partial y}(x^2+xyz)\hat{j} + \frac{\partial}{\partial z}(x^2+xyz)\hat{k}$$

$$\Rightarrow \nabla\phi = (2x+yz)\hat{i} + (xz)\hat{j} + (xy)\hat{k}.$$

Q Given $\phi = z - x^{1/2} - y^{3/2}$, find its gradient, where $x, y > 0$. Also find the maximum slope at the point $(4,9,1)$ and the direction of maximum slope.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(z - x^{1/2} - y^{3/2}\right).$$

$$\Rightarrow \nabla\phi = \frac{\partial}{\partial x}\left(z-x^{1/2}-y^{3/2}\right)\hat{i} + \frac{\partial}{\partial y}\left(z-x^{1/2}-y^{3/2}\right)\hat{j} + \frac{\partial}{\partial z}\left(z-x^{1/2}-y^{3/2}\right)$$

$$\Rightarrow \nabla\phi = \left(-\frac{1}{2}x^{-1/2}\right)\hat{i} + \left(-\frac{3}{2}y^{1/2}\right)\hat{j} + (1)\hat{k}.$$

$$\Rightarrow \nabla\phi = -\frac{1}{2}x^{-1/2}\hat{i} - \frac{3}{2}y^{1/2}\hat{j} + \hat{k}.$$

At point $P(4,9,1)$

$$(\nabla\phi)_P = -\frac{1}{2}(4)^{-1/2}\hat{i} - \frac{3}{2}(9)^{1/2}\hat{j} + \hat{k}.$$

$$= -\frac{1}{4}\hat{i} - \frac{9}{2}\hat{j} + \hat{k}$$

$$\therefore \quad |\nabla\phi| = \sqrt{\left(-\tfrac{1}{4}\right)^2 + \left(-\tfrac{9}{2}\right)^2 + (1)^2}$$

$$= \sqrt{\frac{1}{16} + \frac{81}{4} + 1}$$

$$= \sqrt{\frac{1 + 324 + 16}{16}}$$

$$= \sqrt{341}/4.$$

$$= 4.61.$$

$81 \times 4 = 324$, $324 + 16 = 340$

Hence, maximum slope at point $(4, 9, 1)$ is 4.61

And, direction of maximum slope is

$$\frac{(\nabla\phi)_P}{|(\nabla\phi)_P|} = \frac{1}{4.61}\left(-\frac{1}{4}\hat{i} - \frac{9}{2}\hat{j} + \hat{k}\right).$$

$$= -\frac{1}{18.44}\hat{i} - \frac{9}{9.22}\hat{j} + \frac{1}{4.61}\hat{k}.$$

$$= 0.05\hat{i} - 0.978\hat{j} + 0.21\hat{k}.$$

Divergence and Curl:

Given a vector function $\vec{f}(x, y, z)$

$$\vec{f}(x,y,z) = [f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)]$$

divergence of $\vec{f}$, denoted by $\operatorname{div}\vec{f}$, is defined as

$$\operatorname{div}\vec{f} = \nabla.\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Curl of $\vec{f}$, denoted by curl $\vec{f}$, is defined as

$$\text{curl } \vec{f} = \nabla \times \vec{f}$$

$$\Rightarrow \text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}.$$

(result will be a vector)

Q) For $\vec{f} = [2xy^2, 4y^2z^2, 8z^3]$, find $\nabla \cdot \vec{f}$ and $\nabla \times \vec{f}$.

Solution:

$$\nabla \cdot \vec{f}$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\Rightarrow \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\Rightarrow \frac{\partial(2xy^2)}{\partial x} + \frac{\partial(4y^2z^2)}{\partial y} + \frac{\partial(8z^3)}{\partial z}$$

$$\Rightarrow 2y^2 + 8yz^2 + 24z^2.$$

$\nabla \times \vec{f}$

$$\Rightarrow \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$\Rightarrow \left(\frac{\partial(8z^3)}{\partial y} - \frac{\partial(4y^2z^2)}{\partial z}\right)\hat{i} - \left(\frac{\partial(8z^3)}{\partial x} - \frac{\partial(2xy^2)}{\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial(4y^2z^2)}{\partial x} - \frac{\partial(2xy^2)}{\partial y}\right)\hat{k}$$

$$\Rightarrow -8y^2z\hat{i} + 0\cancel{\frac{\partial}{\partial y}}\hat{j} - 4xy\hat{k}.$$

$$\Rightarrow -8y^2z\hat{i} - 4xy\hat{k}$$

4

Show that $\nabla . \nabla \times \vec{f} = 0$.

let $\vec{f} = \vec{f}(x, y, z) = [f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)]$

$\nabla . \nabla \times \vec{f} = 0$

Taking L.H.S.

$\nabla . \nabla \times \vec{f}$

$$\Rightarrow \nabla \cdot \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}\right]$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i}\right.$$

$$\left. - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}\right]$$

$$\left(\frac{\cancel{\partial^2 f_3}}{\cancel{\partial x \partial y}} - \frac{\partial^2 f_2}{\partial x \partial z}\right) - \left(\frac{\cancel{\partial^2 f_3}}{\cancel{\partial x \partial y}} - \frac{\partial^2 f_1}{\partial y \partial z}\right) + \left(\frac{\cancel{\partial^2 f_2}}{\cancel{\partial x \partial z}} - \frac{\cancel{\partial^2 f_1}}{\cancel{\partial y \partial z}}\right)$$

$\Rightarrow$ 0.

(4) Directional derivative of a function $\phi(x,y,z)$ at the point P in the direction of $\vec{a}$ is given by

$$(\nabla \phi)_P \cdot \hat{a}$$

Find directional derivative of the function $\phi = 4x^2y^2z^2$ at the point $(1,2,1)$ in the direction of $\hat{i}+\hat{j}+\hat{k}$.

$\phi = 4x^2y^2z^2$ point $P(1,2,1)$

$\vec{a} = \hat{i}+\hat{j}+\hat{k}$.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) f(x,y,z).$$

$$\Rightarrow \nabla\phi = \frac{\partial(4x^2y^2z^2)}{\partial x}\hat{i} + \frac{\partial(4x^2y^2z^2)}{\partial y}\hat{j} + \frac{\partial(4x^2y^2z^2)}{\partial z}$$

$$\Rightarrow \nabla\phi = 8xy^2z^2\hat{i} + 8x^2yz^2\hat{j} + 8x^2y^2z\hat{k}$$

At $P(1,2,1)$.

$$(\nabla\phi)_P = 8(1)(2)^2(1)^2 + 8(1)^2(2)(1)^2 + 8(1)^2(2)^2($$

$$= 32\hat{i} + 16\hat{j} + 32\hat{k}.$$

~~= 80~~

6
+1
8

Physical meaning
divergence
curl, directional
derivative

$$\hat{a} = \frac{\vec{a}}{a}$$

$$\Rightarrow \hat{a} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{(1)^2+(1)^2+(1)^2}}$$

$$\Rightarrow \hat{a} = \frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k}).$$

Now,

$$(\nabla\phi)_P \cdot \hat{a}$$

$$\Rightarrow 32\hat{i}+16\hat{j}+32\hat{k} \cdot \left(\frac{\hat{i}}{\sqrt{3}}+\frac{\hat{j}}{\sqrt{3}}+\frac{\hat{k}}{\sqrt{3}}\right)$$

$$\Rightarrow \frac{32}{\sqrt{3}}+\frac{16}{\sqrt{3}}+\frac{32}{\sqrt{3}}$$

$$\Rightarrow 18.4756+18.4756+9.2379$$

$\Rightarrow$ 46.19 units.

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