Lab 06
Modeling of DC Motor

6.1 Objective

To create and validate mathematical models of the DC motor setup available in lab using first principles method and grey box modelling

6.2 Pre-Lab Task

Task 1: To find the transfer function relating DC motor speed and applied voltage

a) Derive the set of equations describing DC motor setup available in lab as depicted in Figure 1. This is an armature controlled DC motor, where R_m and L_m are resistance and self-inductance of armature coil inside and J_m, J_h and J_d are rotor moment of inertia, moment of inertia of hub (the silver colored metal disc used to mount the red load disc) and disc moment of inertia respectively. Assuming negligible damping, find the equations, which relate the voltage, applied to the motor to the resultant speed of the disc connected to the shaft. Let, K_m be the motor back-emf constant and K_t be the motor torque constant.

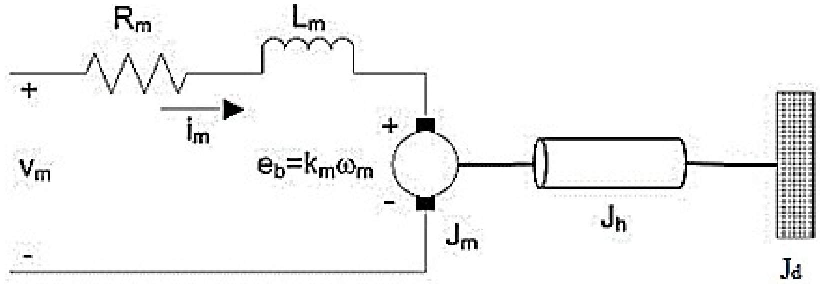

Figure 1: DC Motor with metallic disc connected through a hub

b) Find the corresponding transfer function between the speed and the input voltage of the motor from the equations derived in the previous part.

c) Assume that the motor inductance (i.e. L_m) is negligible. What is the transfer function between the speed and voltage of the motor now?

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Pre-lab
Task #1

Q, a) $J_{eq} = J_m + J_h + J_d$

Applying KVL.

$-V_m + R_m i_m + L_m i_m + e_d = 0$ — (i)

we know rotational motion is given by:

$T = J_{eq} \dot{\omega}_m$ — (ii)

: $\dot{\omega}$ = angular acceleration

→ Developed torque for steady state motor is:

$T = K_t I_m - T_f$: $T_f = 0$

$\boxed{T = K_t I_m}$ — (iii)

b) Transfer function: $\frac{\omega_m}{V_m}$

$I_m = \frac{T}{K_t} = \frac{J_{eq} s \omega_m}{K_t}$ → (a)

Substitute (a) in (i)

$$V_m = \left(\frac{J_{eq} s \omega_m}{K_t}\right)(R_m + L_m) + K_m \omega_m$$

$$V_m = \omega_m \left[\frac{J_{eq} s}{K_t}(R_m + L_m) + K_m\right]$$

$$\frac{\omega_m}{V_m} = \frac{K_t}{J_{eq} s R_m + J_{eq} s L_m + K_m K_t}$$

c) $L_m \approx 0$

$$\frac{\omega_m}{V_m} = \frac{K_t}{J_{eq} R_m s + K_m K_t}$$

6.3 Required Files

Table 1: Files and software modules required for lab experiment

LabVIEW with Control Design & Simulation	
QUBE Servo	**QNET DC Motor**
NI myRIO 15.0	NI DAQmx
QUBE-Servo-2.Ivproj	NI ELVISmx
QUBE-Servo 2 Modelling.vi with sub-VIs	QNET-Motor-Modeling.vi with sub-VIs

6.4 Introduction

In this lab, we are going to explore two different methods for developing models for dynamical systems. We know that a model of dynamical system is a mathematical relationship between the inputs and outputs. There are a number of ways for this description, such as differential or difference equations, transfer functions, state-space equations, or pole-zero gain models. We will focus on differential equation and transfer function models for now.

6.5 Modeling of DC Motor by First Principles

The first methodology is where we know the exact construction of our system entirely (e.g. Figure 1), including all the system parameters, and use physical principles (e.g. KCL, KVL, Newton's laws, etc.) to build our model. This is called "Modeling by First Principles".

The DC motor setups in the lab (shown in Figure 2) are direct-drive rotary servo systems. The DC motor shaft is connected to the load hub. The hub is a metal disc (silver colored) used to mount the load disc (red) and has a moment of inertia of J_h. A disc load is attached to the output shaft with a moment of inertia of J_d . J_m is the inertia of the rotor. The motor armature circuit schematic of the same is shown in Figure 1. The electrical and mechanical parameters for this motor setup can be found in the respective user manuals provided on Canvas LMS.

Figure 2: DC Motor setup available in lab

Task 2: To obtain the exact transfer function by applying first principles

Open the user manual of the respective setup (QNET or QUBE-Servo) and note down the electrical and mechanical parameters of the DC motor setup specified by the manufacturer. Based on the parameters in the user manual, find the exact transfer function (the expression for which has already been found in Pre Lab), between the applied voltage and disc speed, for the DC motor setup, ignoring the motor inductance.

Helpful fact: The moment of inertia of a disc about its pivot, with mass 'm' and radius 'r' is given by $\frac{1}{2}mr^2$.

Task #2

1. $J_h = 0.6 \times 10^{-6}$ kg-m²
2. $J_m = 4 \times 10^{-6}$ kg-m²
3. $J_d = \frac{1}{2} m r^2 = \frac{1}{2}(0.053)(0.0248)^2$

$J_d = 0.165 \times 10^{-4}$, $\boxed{J_{eq} = 0.209 \times 10^{-4}}$

4. $K_T = 0.042$ N-m/A
5. $K_m = 0.042$ V/rad/s
6. $R_m = 8.4\,\Omega$

$$G(s) = \frac{0.042}{s(8.4)(0.209 \times 10^{-4}) + (0.042)(0.042)}$$

$$G(s) = \frac{0.042}{0.000175s + 0.001764}$$

$$\boxed{G(s) = \frac{23.8}{0.0995s + 1}}$$

$K = 23.8$, $\tau = 0.09925$

6.6 Model Validation

You have now obtained a mathematical model to describe the actual hardware setup using manufacturer given data. If this model is accurate then given an input, it should be able to predict the corresponding output. However, before we trust this model and move on, we need to first verify the accuracy of our model. This process where you verify the accuracy of mathematical model is called "***Model Validation***".

Model validation can be carried out by applying same input to both; the model and actual process in open loop and then comparing the outputs. The output obtained from the simulation based on our mathematical model is compared against the output obtained from the actual process. Model validation is carried out for a range of inputs. The comparison results would indicate the accuracy of our constructed mathematical model. Based on the results, the model can then be adjusted by fine-tuning the modeling parameters.

Task 3: To validate the model obtained using first principles

1. To validate the model obtained in Task-2, use your VI from the last lab or the one provided on LMS and design a VI that applies a 1−3 V, 0.4 Hz square wave to the motor and reads the speed using the encoder. Do not forget the saturation block. Display both the voltage and motor speed on two separate waveform charts.
2. Implement the transfer function obtained in Task 2 in the same VI. Provide the same input to this transfer function as well. Plot its output on the same speed waveform chart used for the above sub task. Your output should look like Figure 2. Remember you can do this by bundling the two speed signals – one from the transfer function block and the other is the actual motor speed found using encoder reading, by "Bundle" block followed by "Cluster to array" and then sending the output to the waveform chart. Attach a capture of your waveform charts in your post lab submission. You can do this by right clicking on the waveform chart and exporting.

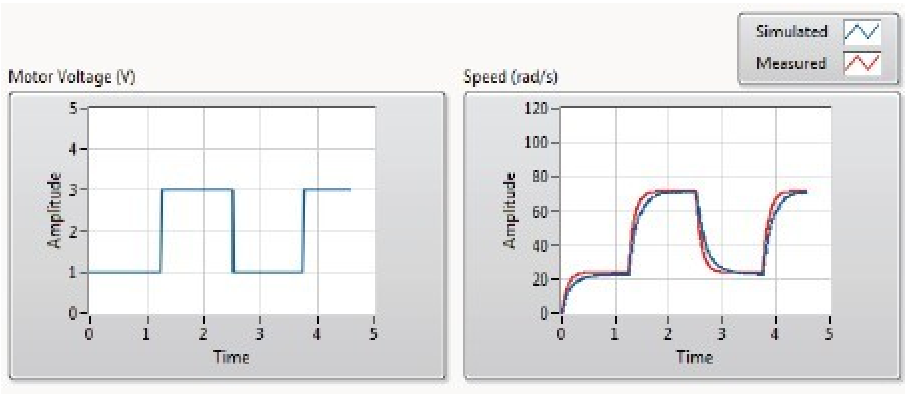
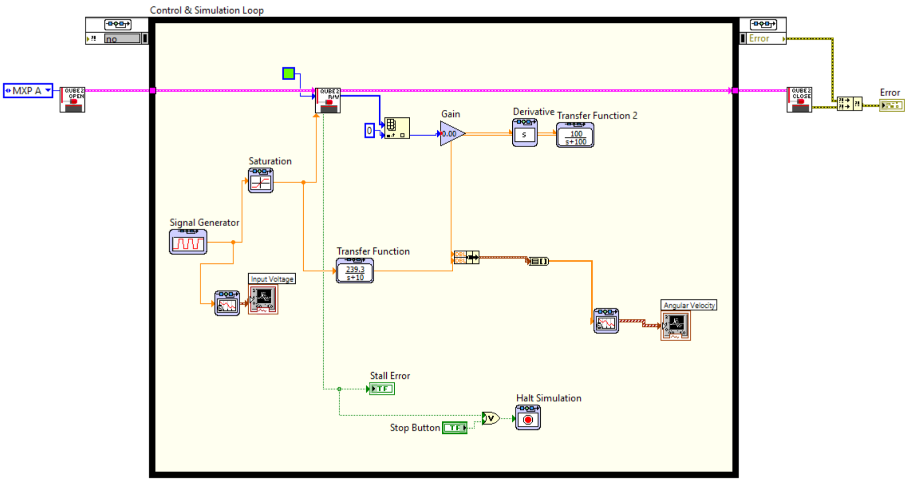
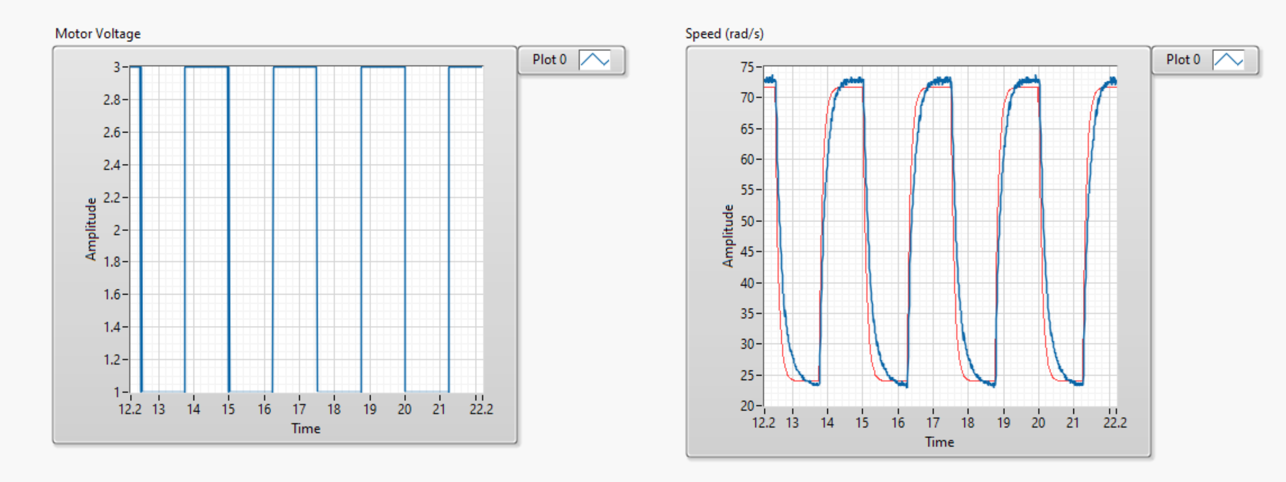


Figure 3: Measured and simulated responses of Motor

3. How well does the model obtained by first principles represent the actual system? If there are discrepancies between your model and actual system, what could be the source of these discrepancies?



Our output response matches the one given in figure 2. There is a very minor discrepancy here is that output from our model has a smaller rise time compared to the actual system but a smaller steady state value. These discrepancies can be due to mechanical wear of the motor, the bearings, and the shaft, all affecting how the motor accelerates and how it decelerates. Regardless, the simulated model fits the measured results quite well.



Legend:

Red line = measured speed, Blue line = simulated speed
Our model obtained by the first principle represents the system reasonably well.

6.7 System Identification

The previous methodology required us to know the exact construction of our system and the values of the system parameters (e.g. torque constant, motor resistance, rotor inertia, etc.). In the absence of complete knowledge of the internal structure and system parameters, a methodology called "System Identification" can be employed. System identification is the methodology of building mathematical models of dynamical systems using limited measurements of system's inputs and outputs, obtained experimentally. System identification can be divided into two types based on whether you have identified a specific structure for your model or not. The two types are grey-box modeling and black box modelling.

6.8 Grey-Box Modeling

In grey-box modeling, we know the internal structure for our process but we do not know the numerical values of the parameters in the model. Typically, one arrives at this structure from physical principles. How can we now estimate values of the system parameters? We can provide our system with various inputs and record the corresponding output responses. By using this input-output data, we can try to estimate the numerical values of the parameters in our model. Let us apply this idea to build a model of DC motor setup again in this lab. But we'll now assume that we have lost the user manuals and cannot find the values of the motor parameters.

Task 4: To represent the DC motor model in terms of lumped parameters

Can the transfer function obtained in the pre-lab, ignoring the motor inductance, be expressed in the form $\frac{K}{\tau s+1}$? If yes, what are the corresponding expressions for K and τ ?

- Yes, the transfer function expression then becomes:
  G(s) = 23.8 / (0.0995s + 1)
- K = 23.8 and T = 0.0995

Our eventual goal is to construct a model for the motor setup, and so we do not actually care about values of the individual parameters, but only about the coefficients of numerator and denominator in the transfer function. Thus, it is enough to lump the system parameters of the motor setup into two parameters K and τ, and subsequently find the values of these two parameters. We are going to use the ***bump test*** to find values of these lumped parameters.

6.9 Bump Test

The bump test is a simple test based on the step response of a stable system. A step input is given to the actual system and its experimental response is recorded. Using this response, we compute the values of the system parameters by comparing the actual response to what is theoretically expected from the system. As an example, consider a system given by the following transfer function:

$$\frac{Y(s)}{U(s)}=\frac{K}{\tau s+1}$$

If we provide this system with a step input, we expect to see a sample response as shown in Figure 4. The step input begins at time t_o. The input signal has a minimum value of u_{min} and a maximum value of u_{max}. The resulting output signal is initially at y_o. Once the step is applied, the output tries to follow it and eventually settles at its steady-state value y_{ss}. Let us try to find a mathematical expression for the output of the system to a step input, in time-domain. Say we have a step input with amplitude A with a time delay t_o. The s-domain representation of this step will be:

$$U(s)=\frac{Ae^{-st_o}}{s}$$

$$Y(s)=\frac{KAe^{-st_o}}{s(\tau s+1)}$$

We can now easily find the output of the system in time-domain by taking the inverse Laplace transform of the above equation, specifically:

$$y(t)=KA(1-e¿¿\frac{-t-t_o}{\tau})+y(t_o)¿$$

Here, K and τ are the unknown parameters of our model.

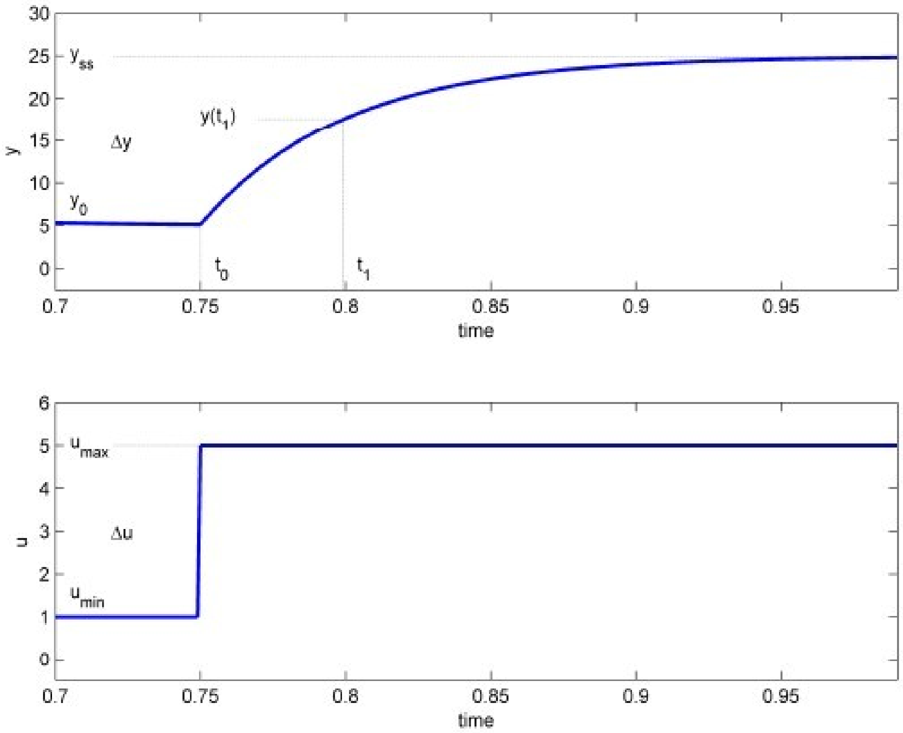


Figure 4: Input and output signal used in the bump test

Our goal in this modeling process is to estimate values of K and τ, which substituted in the expression of $y(t)$ result in a response that fits the experimental response obtained in Figure 3. If we play around with the expression of $y(t)$, we can find a nifty way of estimating these two parameters from the experimental response. Let us start by analyzing it for what happens at steady state or when $t \to \infty$.

$$\lim_{t\to\infty} y(t) = KA(1-0) + y(t_o)$$

¿$> \dfrac{y(\infty) - y(t_o)}{A} = K$

Thus, we can find K by finding the steady-state value of output, initial value of the output from the experimental response, initial and final values of the input, and using the expression:

$$K = \frac{\Delta y}{\Delta u}$$

For τ, substituting $t = t_o + \tau$ in the expression of $y(t)$

$$y(t_o+\tau) = KA\left(1 - e^{\frac{-\tau}{\tau}}\right) + y(t_o)$$

$$y(t_o+\tau) = \Delta y\left(1 - e^{-1}\right) + y(t_o)$$

$$y(t_o+\tau) = \Delta y(0.632) + y(t_o)$$

From the last expression, we can see τ is the time the system's response takes to reach 63.2% of Δy, after the application of step input. Thus, we can find the time t_1 from experimental response where the value of the output is $y(t_o) + 0.632\,\Delta y$, and then using $(\tau = t_1 - t_0)$ the value of time constant can be found.

Task 5: To find the lumped parameters of DC motor model using bump test

1. Based on the previously designed VIs, design a VI that applies a 2V step input to the motor and reads the disk speed using the encoder.
   Help with implementation: We want to be able to save the system's response from the beginning of the simulation until the end. We can do this by using a *Collector block from Control & Simulation | Simulation | Utilities palette* and then displaying the collected data in an *XY Graph* placed outside simulation loop, as shown in Figure 4. Add XY Graph from the Front Panel.

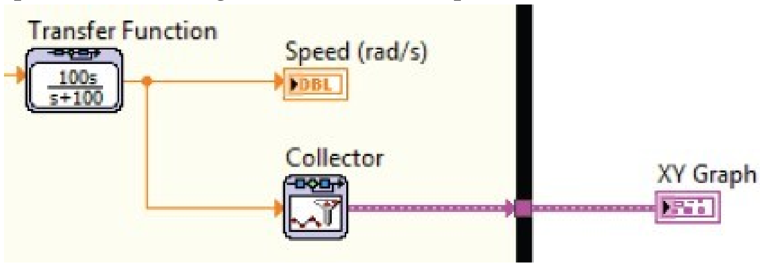
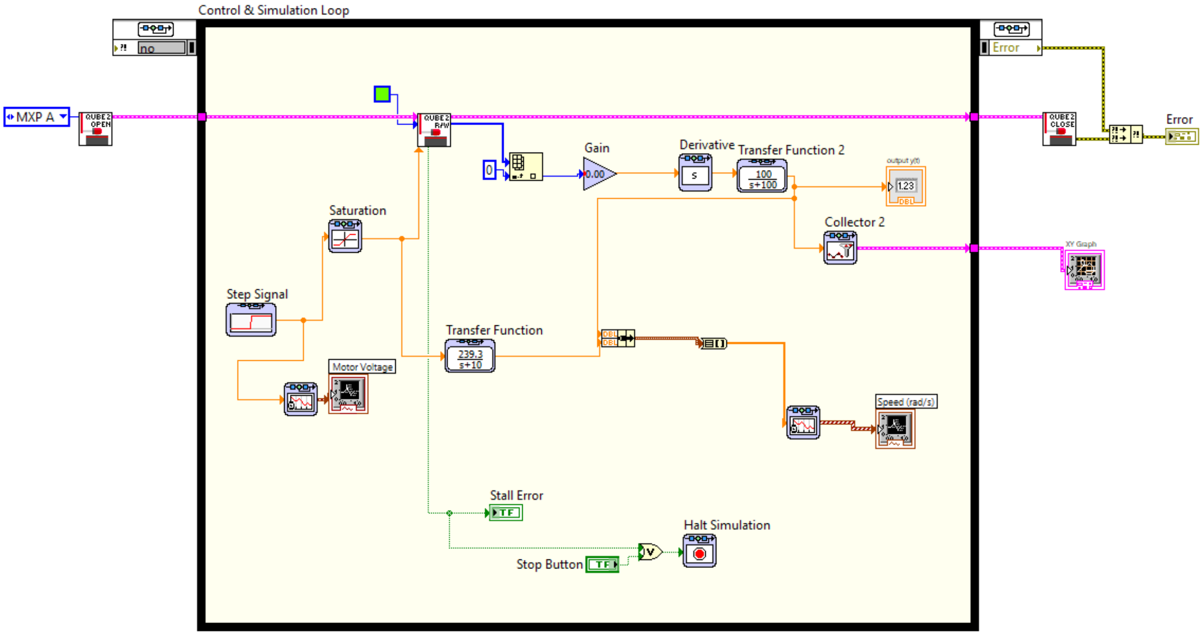
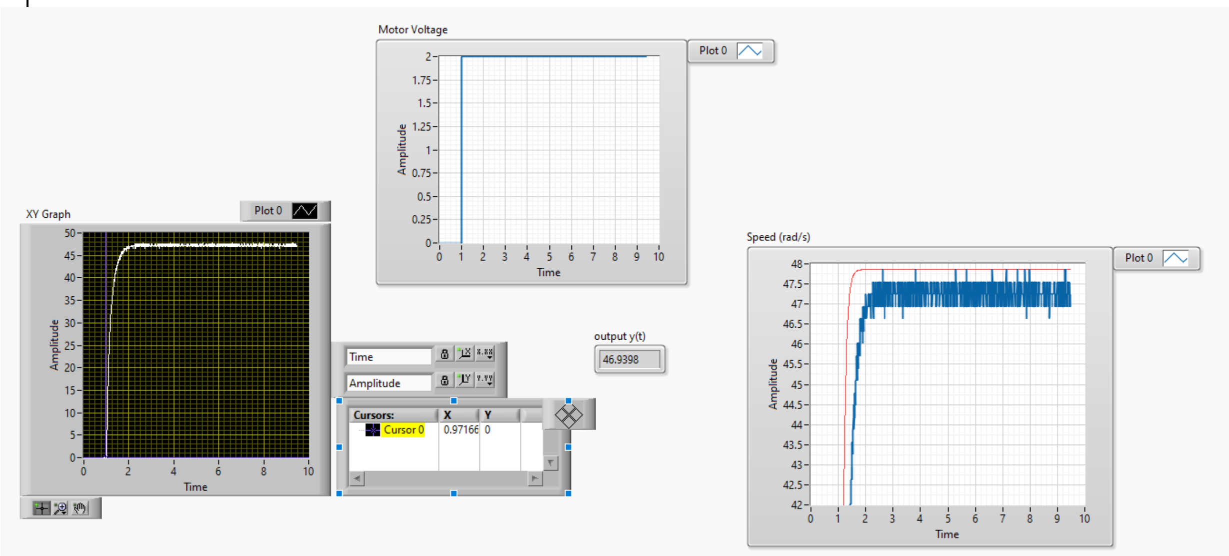


Figure 5: XY-graph with Collector

2. Plot the speed response and input voltage, and save the graphs for your post-lab submission. Use the response displayed in the XY Graph to find the values of K and τ.
   Implementation Help: Go to Graph properties> Cursors tab and add cursors. Right click the XY Graph, go to the Visible Items and select Cursor Legend, Scale Legend and Graph Palette. Using these cursors, find values of X and Y axes variables required to find K and τ.



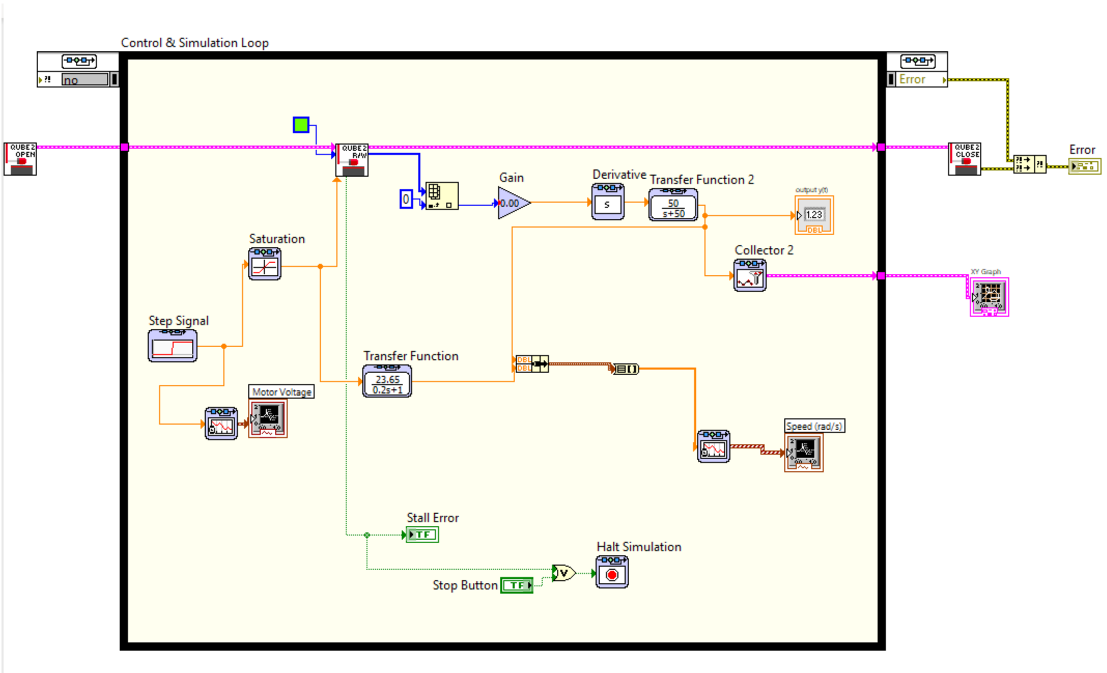


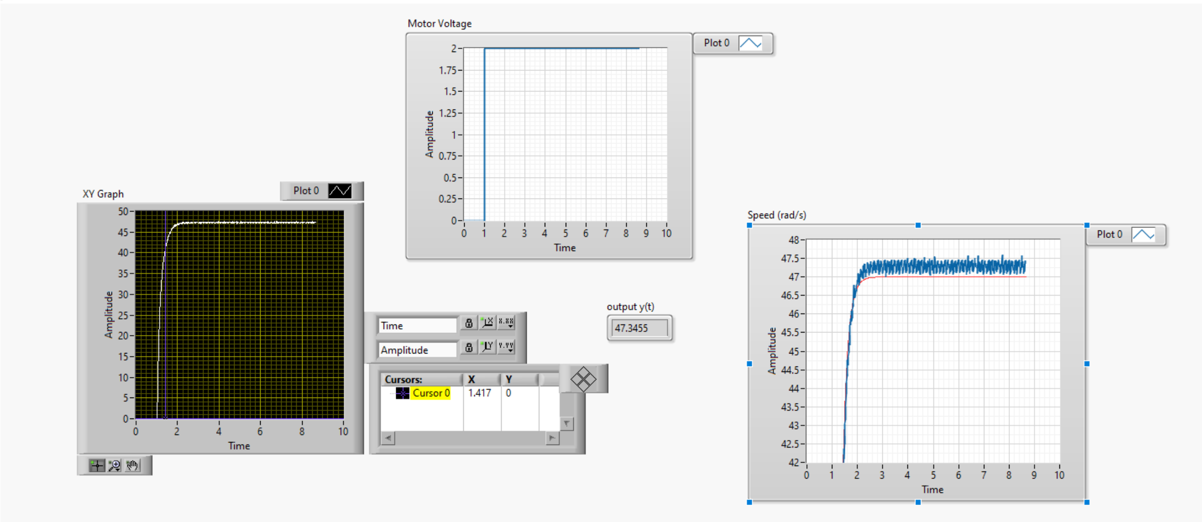
- K = 23.5 and Tau = 0.2 (simulated results)

Task 6: To validate the model obtained by grey-box modeling

1. To validate this model, modify the VI to simulate your mathematical model of the system with your computed values of K and τ. Display the response of DC Motor and response of Grey Box Model in one figure. Save the plots for submission.

Simulating the VI with K = 23.5 and Tau = 0.2 (simulated results from task 5)

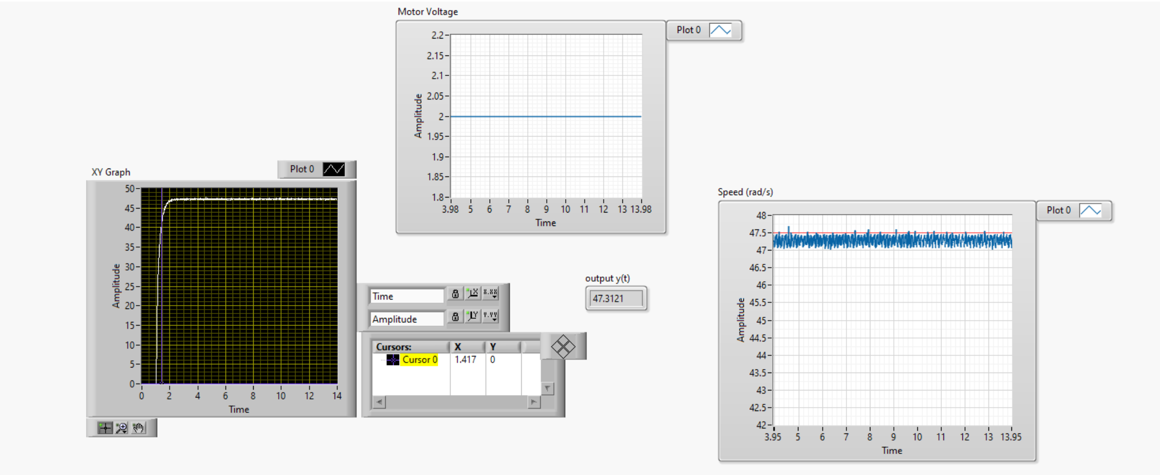




Here in the graph of speed we observe, that both the measured and simulated response have almost the same rise time i.e tau. But the steady state value that the simulated response reaches is larger than the steady state reached by the measured response. So, we have to increase K a bit to reduce the discrepancy, while keeping tau the same

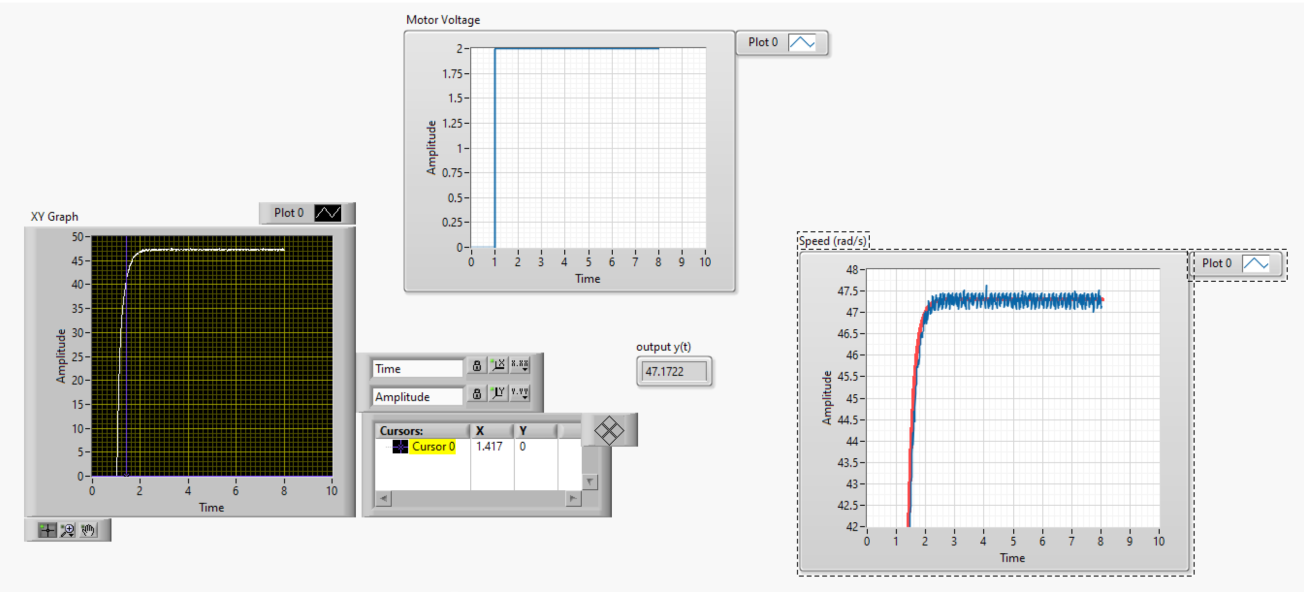
2. If there are any differences between measured and simulated responses, fine-tune the model parameters K and τ.

For K= 23.75, tau =0.2 (Fine-tuned):



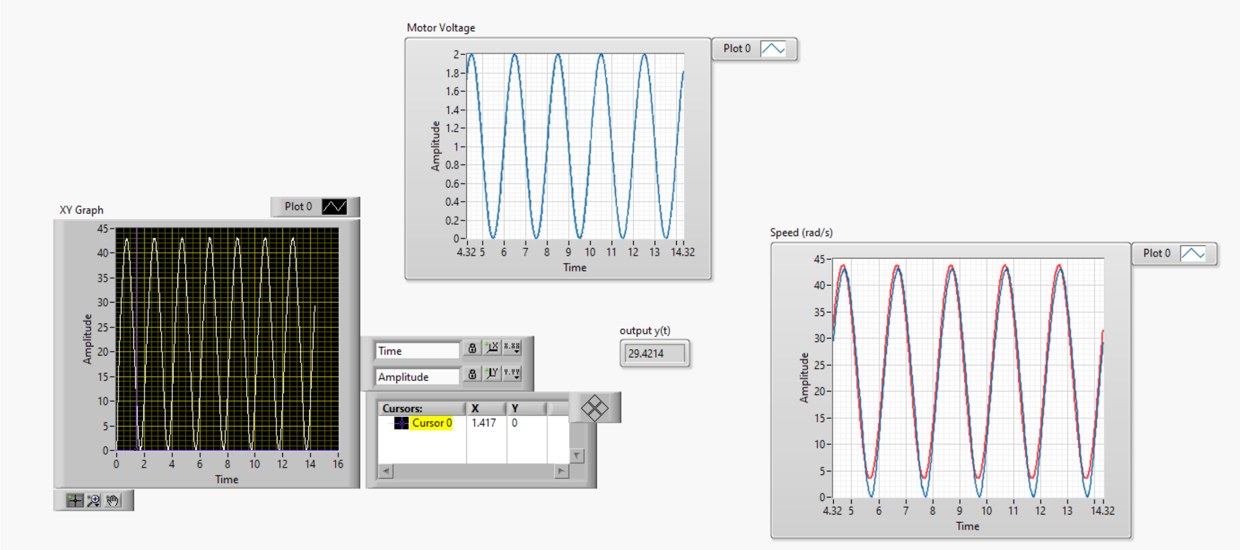
- Now the simulated results are higher than the actual system. So, we need to decrease the value of K.

For K=23.65, tau=2 (Fine-Tuned)



- Note that the actual system response and the simulated results are now perfectly coinciding and validating our model.

3. After fine-tuning, validate your model for other input signals such as sin or square wave.



4. How well does the model obtained through grey-box modeling represent the actual system? If there are discrepancies between your model and actual system, what could be the source of these discrepancies?

- The system works pretty much well with no huge errors for the step response.
- The rise time of both the signals Tau is same however, grey box modeling gives a slightly smaller value of steady state. To fine tune, we increase K.
- Reasons may be approximation in the calculations and also in measuring values through cursors in step supply.
- When a sine wave is applied to the system, we notice an offset in the modelled response and also some decrease in the amplitude.

6.10 Post-Lab Tasks

Task 7: Answer the following

1. Compare the two methods used in this experiment for obtaining models for systems.

Modeling by First Principles

- This type of mathematical modeling is used for system where we know exact construction of the system and hence can obtain the exact transfer function of the system using electrical and mechanical laws.
- For electromechanical systems, the electrical block is modelled using circuit laws with parameters such as voltage, current or inductance etc. similarly, the mechanical block is modelled either by making an electric equivalent of the systems or by using newton's laws or equilibrium conditions.

Grey Box Modeling

- This type of modeling is useful when we donot have the actual system and its parameters but rather an understanding of the internal working.
- This method thus involves both approaches i.e. theoretical modeling as well as experimental analysis simultaneously. Here, we test our fundamental model build through the available

knowledge about the system on different input signals, observe the results and analyze them rigorously to estimate the actual transfer function.

Comparison

- Some fine tuning is always done to achieve suitable results in the Grey Box modeling every time a different type of input is applied. This type of modeling is thus less precise as compared to the first principles method but provides a good way to estimate the results.
- When system is already known, first principles is preferable and when system is not know grey box modeling is the way to go.

2. Explain how we arrived at the formulas for estimating K andτ from the output. Do not just state the formulas.
   - For obtaining the value of K, we first find the steady state output and subtract the intia output value to get Δy. Similarly, we obtain change in input as Δu. Dividing the two i.e. Δy/ Δu gives us the value of K.
   - Tau is the time at the point when output signal reaches 63% of its steady state value after application of input signal. So, we first find the time when system reaches 63% of its steady state value (T_1) and subtract the time past before applying input (T_o). Hence, we have Tau = $T_1 - T_0$

3. If you are given a DC motor whose parameters are unknown to you, how would you obtain its model?
   - If we are assuming that we know the construction of the motor, we can use the first principles method to determine its transfer function. Once, we have that we can supply an input step signal and record the output. Now from the output-input characteristic graph, we can estimate the parameters in the transfer function.
   - If we assume system is unknown, then we need to use the grey block method where system is subjected to different inputs and observed outputs are recorded for analysis purposes. Here, we assume some basic model and estimate the values of the parameters from the wide range of data obtained through experimentation. Fine tuning continues until perfect results are achieved.

Task 8: To calculate moment of inertia for the given systems

Moments of inertia can be found by summing or integrating over every 'piece of mass' that makes up an object, multiplied by the square of the distance of each 'piece of mass' to the axis of rotation. Moment of inertia for a compound object is simply the sum of the moments of inertia for each individual object that makes up the compound object.

- Calculate the moment of inertia for the given uniformly shaped, rigid bodies

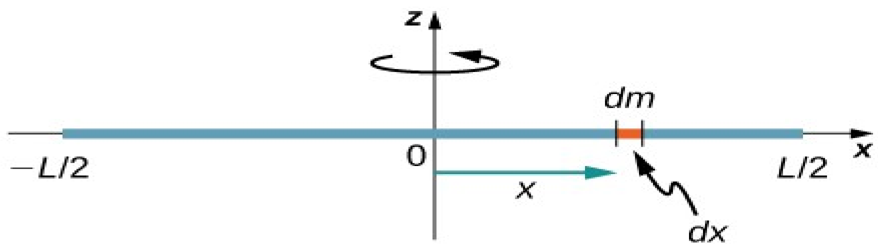
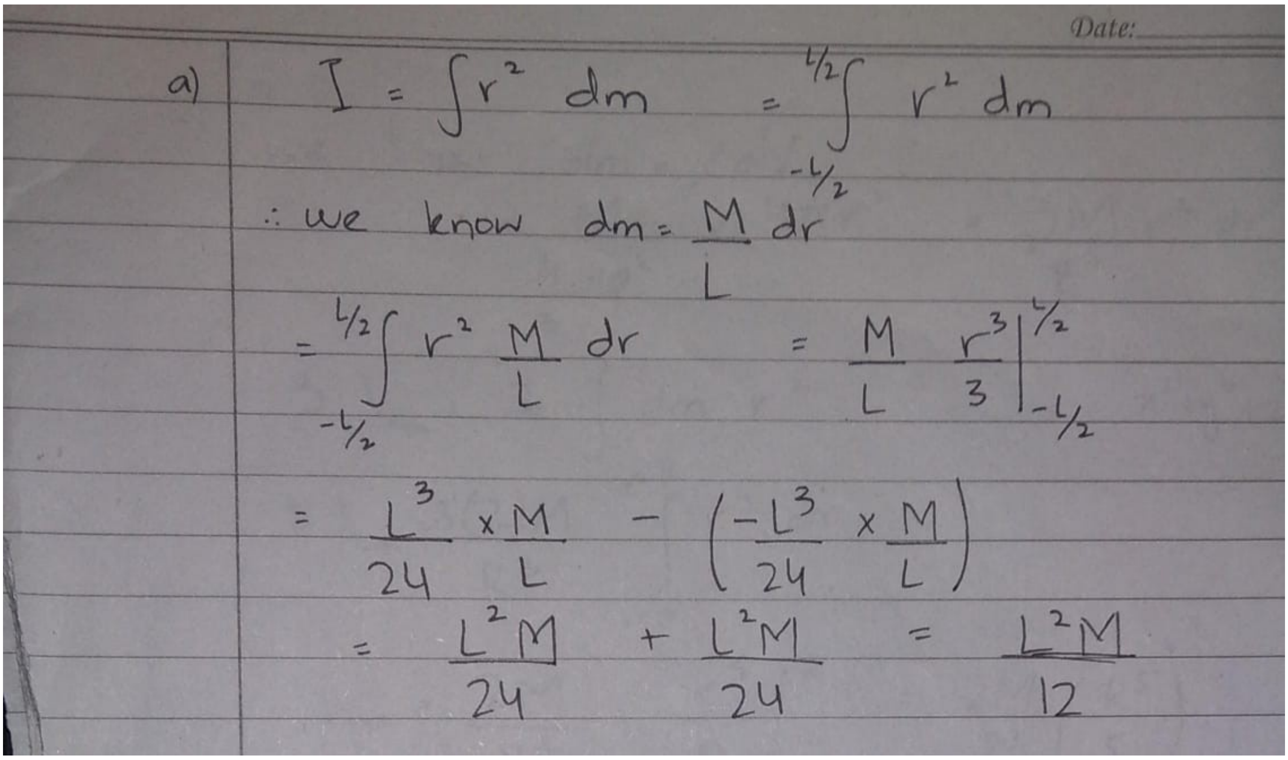


Figure 6: A uniform thin rod about an axis through the center of the rod.



Date:

a) $I = \int r^2 \, dm = \int_{-L/2}^{L/2} r^2 \, dm$

$\therefore$ we know $dm = \frac{M}{L} dr$

$= \int_{-L/2}^{L/2} r^2 \frac{M}{L} dr = \frac{M}{L} \frac{r^3}{3} \Big|_{-L/2}^{L/2}$

$= \frac{L^3}{24} \times \frac{M}{L} - \left(\frac{-L^3}{24} \times \frac{M}{L} \right)$

$= \frac{L^2 M}{24} + \frac{L^2 M}{24} = \frac{L^2 M}{12}$

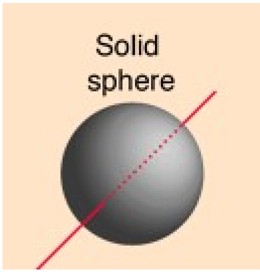
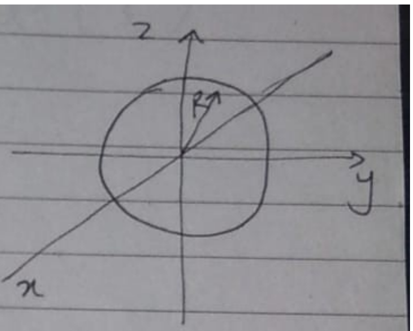


Figure 7: A solid sphere with mass ‘M’ and radius ‘R’ about central axis

b)



$$I_{cm,z} = \int_{sphere} dm\ (x^2+y^2)$$

$$I_{cm,y} = \int_{sphere} dm\ (x^2+z^2)$$

$$I_{cm,x} = \int_{sphere} dm\ (z^2+y^2)$$

$$I_{cm} = I_{cm,x} = I_{cm,y} = I_{cm,z}$$

$$3I_{cm} = I_{cm,x} + I_{cm,y} + I_{cm,z}$$

$$= \int_{sphere} dm\ 2(x^2+y^2+z^2)$$

Teacher's Sign.__________

Date:

we know $dm = \rho \, dV$

$$= \frac{M}{\frac{4}{3}\pi R^3} \times 4\pi r^2 = \frac{3M}{R^3} r^2 dr$$

$$3 I_{cm} = 2 \int dm \, r^2 \qquad \therefore x^2+y^2+z^2=r^2$$

$$\cancel{3} I_{cm} = \frac{\cancel{3}(2)M}{R^3} \int_{r=0}^{r=R} r^4 \, dr$$

$$I_{cm} = \frac{2M}{R^3} \left. \frac{r^5}{5} \right|_0^R = \frac{2M}{R^3}\left(\frac{R^5}{5}\right)$$

$$\boxed{I_{cm} = \frac{2M}{5} R^2}$$

- Calculate the moment of inertia for the given compound object

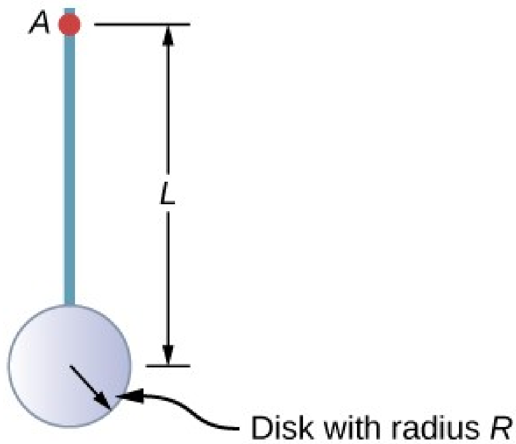


Figure 8: Compound object consisting of a disk with radius R at the end of a rod of length L. The axis of rotation is located at A.

) The system consists of a rod and disk.

$$J_{cm} = \frac{1}{12} mL^2$$ (for rod, as calculated in (a))

∴ using parallel axis theorem: J for rod rotating about an axis parallel to axis through center of mass is:

$$J_{total} = J_{cm} + md^2$$

∴ d = distance b/w axis + cm

$\boxed{\therefore \text{ here } d = L/2}$ so, $J_{total} = \frac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2$

$$\boxed{J_{total} = \frac{1}{3} m_r L^2}$$ (for rod) ∴ m_r = mass of rod

for disc: $J_{disc} = \int_{disc} r^2 \, dm$

$$dm = \rho \, dA = \frac{m}{\pi R^2}(2\pi r)dr = \frac{2M}{R^2} r \, dr$$

So, $J_{disc} = \int_0^R \frac{2M}{R^2} r^3 \, dr = \frac{2M}{R^2}\left(\frac{r^4}{4}\right)_0^R$

$$\boxed{J_{disc} = \frac{MR^2}{2}}$$

∴ applying parallel axis theorem:

$$J_{total\,2} = \frac{MR^2}{2} + M(L+R)^2$$

ii) $J_{system} = \left(J_{total}\right)_{rod} + \left(J_{total}\right)_{disc}$

$$= \frac{1}{3} m_r L^2 + \frac{1}{2} MR^2 + M(L+R)^2$$

∴ m_r = mass of rod

M = mass of disc.

Assessment Rubric

Lab 06
Modelling of DC Motor

Name: Afsah Hyder	**Student ID: ah07065**

Points Distribution

Task No.	LR2 Simulation	LR4 Data Collection	LR5 Results	LR 6 Calculations	LR 10 Analysis	AR 6 Class Participation
Task 1	-	-	-	-	12	-
Task 2	-	-	-	4	-	-
Task 3	6	-	8	-	6	-
Task 4	-	-	-	-	4	-
Task 5	6	8	4	4	-	-
Task 6	4	-	8	-	6	-
Task 7	-	-	-	-	8	-
Task 8	-	-	-	12	-	-
SEL	-	-	-	-	-	/20
Course Learning Outcomes	CLO 1					CLO 4
Total Points	/100					/20
	/120					

For details on rubrics, please refer to *Lab Evaluation Assessment Rubrics*.