Velocity Kinematics

EE366/CE366/CS380: Introduction to Robotics

Dr. Basit Memon

Electrical and Computer Engineering
Habib University

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- We know how to relate end-effector position and orientation to joint variables.
- What about velocity and acceleration?
- If a particle follows a curve $q(t) \in \mathbb{R}^3$, then it is easy to find its velocity, $v_q = \frac{dq(t)}{dt}$.

- What about a rigid body moving and described by homogeneous transformation $T(t) \in SE(3)$?
- $\dot{T}(t)$ doesn't work!
  - For one, $\dot{T}(t) \notin SE(3)$
  - We want some connection to our prior understanding of rotational and translational velocity.

How to express velocity of rigid body in motion?

- Trajectory of a rigid body in motion:

$$T_e(q(t)) = \begin{bmatrix} {}^0R_e(q(t)) & {}^0p_e(q(t)) \\ \mathbf{0}^T & 1 \end{bmatrix}$$

  where q is vector of joint variables.

- $v_e(t) = \dot{p}_e(t)$ is linear velocity, capturing the velocity of origin of the frame.

- Two ways to describe the evolution of orientation of frame:
  - ω related to the derivative of $R \in SO(3)$

$$\begin{bmatrix} v_e \\ \omega_e \end{bmatrix} = \mathbf{J}\dot{q}$$

J: Geometric Jacobian or Jacobian

  - $\dot{\alpha}$ related to the derivative of Euler angles

$$\begin{bmatrix} v_e \\ \dot{\alpha} \end{bmatrix} = J_a\dot{q}$$

J_a**: Analytical Jacobian**

Questions

- What is this ω? How is this related to R?
- Are v and ω a good description of velocity?
- How do we find velocity of any point on the rigid body?

The Jacobian haunts robotics! [2]

- Jacobian is used in:
  - Planning and execution of smooth trajectories
  - Determination of singular configurations
  - Transformation of forces and torques from end-effector to manipulator joints
  - Dynamic equations of motion
  - Manipulability

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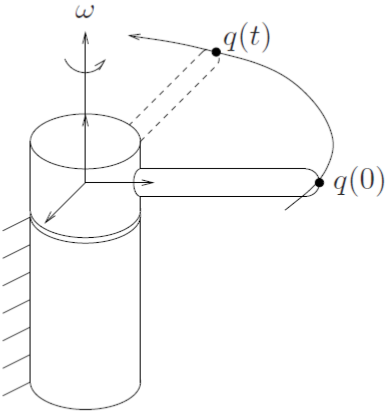
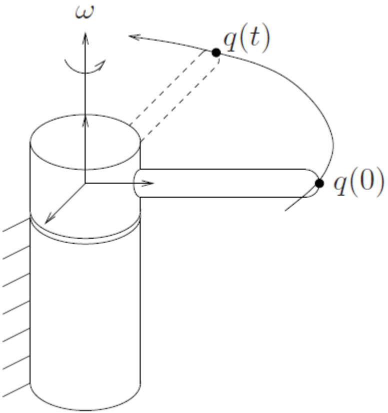
1. Interpretation of Angular Velocity: Fixed axis



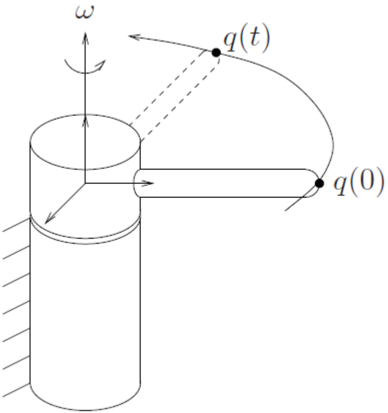
- Rigid body rotates about fixed axis, $\hat{k}$ at constant speed. What is angular velocity of each point?
- Each point moves in a circle with center on axis of rotation.
- Perpendicular from the point to axis sweeps same angle θ
- All points rotate with the same angular velocity.



- What about linear velocity of each point?
  - Consider the point q:

$$\dot{q}(t) = \omega \times q(t) = \dot{\theta}\hat{k} \times q(t)$$

- Linear velocity is different for each point.



- In fixed axis case, rotation is captured by rotation matrix $R_k(\theta)$. Rotation of every point is planar, and so $\omega = \dot{\theta}$.
- Angular velocity is the same, and so a property of attached coordinate frame or body and not of individual points.
- How to write ω when axis of rotation changes as rigid body moves?

- For two vectors, $a, b \in \mathbb{R}^3$,

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

- Cross-product by a can be written as a linear operator:

$$\begin{aligned} a \times b &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= S(a)\, b \end{aligned}$$

- $S(a)$ is a skew-symmetric matrix.

- An $n \times n$ matrix S is called skew symmetric iff $S^T + S = O$.

$$
\begin{aligned}
s_{ij} + s_{ji} &= 0 \\
\Rightarrow s_{ii} &= 0 \quad \text{Diagonal entries will be zero.} \\
s_{ij} &= -s_{ji} \quad \text{Off-diagonal entries will be negative of each other.}
\end{aligned}
$$

- 3×3 case:

$$
S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}
$$

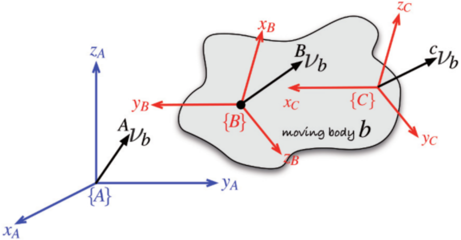


Figure: Rotational motion of a rigid body

- Attach frame b to the rigid-body, which moves along with it.
- ${}^aR_b(t) \in SO(3)$ is curve representing motion of body with respect to fixed frame a.
- Any point q of rigid-body follows a path:

$$\begin{aligned} {}^ap_q^a(t) &= {}^aR_b(t)\ {}^ap_q^a(0) \\ &= {}^aR_b(t)\ {}^bp_q^b \end{aligned}$$

Coordinates ${}^bp_q^b$ are fixed in the body frame.

Lemma-1

Given $R(t) \in SO(3)$, the matrices $\dot{R}(t)R^{-1}(t) \in \mathbb{R}^{3\times 3}$ and $R^{-1}(t)\dot{R}(t) \in \mathbb{R}^{3\times 3}$ are skew-symmetric.

- Velocity of point in fixed-frame coordinates is:

$$^{a}v_{q} = \frac{d}{dt}\, {}^{a}p_{q}(t) = {}^{a}\dot{R}_{b}(t)\; {}^{b}p_{q}.$$

- Deriving a more compact representation:

$$\begin{aligned}
{}^{a}v_{q} &= \underbrace{{}^{a}\dot{R}_{b}\, {}^{a}R_{b}^{-1}}_{S({}^{a}\omega_{b}^{a})}\, {}^{a}R_{b}\, {}^{b}p_{q} \\
&= S({}^{a}\omega_{b}^{a})\, {}^{a}R_{b}\, {}^{b}p_{q} \\
&= S({}^{a}\omega_{b}^{a})\, {}^{a}p_{q} \\
{}^{a}v_{q}(t) &= {}^{a}\omega_{b}^{a}(t) \times {}^{a}p_{q}(t)
\end{aligned}$$

- $R(t) \in SO(3)$ depends on one parameter θ.

$$R(t)R(t)^T = I$$

- Differentiating both sides wrt t:

$$\left[\frac{d}{dt}R\right] R(t)^T + R(t)\left[\frac{d}{dt}R^T\right] = O$$

- Define

$$S := \left[\frac{d}{dt}R\right] R(t)^T$$

- We have that, $S + S^T = O$, so S is a skew-symmetric matrix.

- Derivative of rotation matrix

$$S\,R(t) = \left[\frac{d}{dt}R\right] R(t)^T R(t) = \frac{d}{dt}R(t)$$

Angular velocity of a rigid body can generally be defined as:

- The vector in $\mathbb{R}^3$ equivalent to the skew-symmetric matrix $\dot{R}(t)R^{-1}(t)$.

Example-1: Rotating about x-axis by an angle θ

- $R = R_x(\theta(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & -\sin\theta(t) \\ 0 & \sin\theta(t) & \cos\theta(t) \end{bmatrix}.$
- $$S = \left[\frac{d}{d\theta}R\right] R(\theta)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta(t)\,\dot{\theta} & -\cos\theta(t)\,\dot{\theta} \\ 0 & \cos\theta(t)\,\dot{\theta} & -\sin\theta(t)\,\dot{\theta} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$
  $$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}$$
- $\omega = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

Example: Rotating about $\hat{k}$-axis by an angle θ

- If $R = R_k(\theta)$, then

$$\frac{d}{d\theta} R_k(\theta) = S(k) R_k(\theta).$$

  - HW: Verify this to be true using:

$$R_k(\theta) = I + S(k)\sin\theta + S^2(k)(1 - \cos\theta)$$
$$S^3(k) = -S(k)$$

Body viewed as rotating about an instantaneous velocity vector.

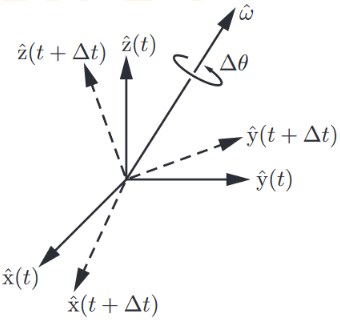


Figure: Instantaneous angular velocity vector

- Change in orientation at $t + \Delta t$ can be described as rotation about some $\hat{w}$ through the origin by $\Delta\theta$.
- In the limit, $\Delta\theta/\Delta t \to \dot{\theta}$ and angular velocity can be defined as:

$$\omega = \hat{w}\dot{\theta}.$$

Definition of ω coincides with instantaneous angular velocity.

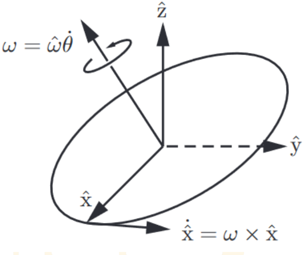


Figure: $\dot{\hat{x}} = \omega \times \hat{x}$

$$
\begin{aligned}
\dot{R} &= \begin{bmatrix} \omega \times \hat{x} & \omega \times \hat{y} & \omega \times \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} S(\omega)\hat{x} & S(\omega)\hat{y} & S(\omega)\hat{z} \end{bmatrix} \\
&= S(\omega) \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix} \\
&= S(\omega) R \\
\Rightarrow S(\omega) &= \dot{R} R^{-1}
\end{aligned}
$$

3. Velocity for rigid-body motion [1, Section 4.2]

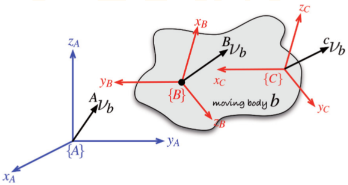


Figure: Motion of a rigid body

- Attach frame b to the rigid-body, which moves along with it.
- ${}^aT_b(t) \in SE(3)$ is curve representing motion of body with respect to fixed frame a.
- Any point q of rigid-body follows a path:

$$\begin{aligned} {}^ap_q^a(t) &= {}^aT_b(t)\ {}^bp_q^b \\ &= {}^aR_b(t)\ {}^bp_q + {}^ap_b^a(t) \end{aligned}$$

Spatial Velocity

The vector $\begin{bmatrix} {}^a v_{ab} \\ {}^a \omega_{ab} \end{bmatrix} \in \mathbb{R}^6$ is a good choice for representation of spatial velocity.

$$
\begin{aligned}
{}^a\dot{p}_q(t) &= {}^a\dot{R}_b(t)\ {}^b p_q + {}^a\dot{p}_b(t) \\
{}^a v_q(t) &= \underbrace{{}^a\dot{R}_b\, {}^aR_b^T}_{S({}^a\omega_{ab})}\ {}^aR_b\ {}^b p_q + \underbrace{{}^a\dot{p}_b(t)}_{{}^a v_{ab}} \\
&= {}^a\omega_b \times {}^a p_q + {}^a v_b
\end{aligned}
$$

Linear velocity

- Different for all points of body.

$$ {}^a v_q(t) = {}^a\omega_b \times {}^a p_q + {}^a v_b $$

- Distance ${}^a p_q^b$ is measured from the origin of $\{b\}$, but expressed in frame $\{a\}$.

Angular velocity

- ω is same for all points.
- $R(t)$ is rotation of body with respect to fixed frame $\{a\}$, expressed in frame $\{a\}$, i.e. ${}^a R_b^a$.
- S is the skew symmetric matrix $S = \dot{R}\,R^T$.
- The 3×1 vector associated with the matrix S is the angular velocity of body with respect to fixed frame $\{a\}$, expressed in frame $\{a\}$, i.e. ${}^a\omega_b^a$.

Change of reference for velocities

- Recall that the velocities are free vectors.

- We can express any velocity vector in the coordinates of any other frame by making use of rotation matrices. So, ${}^k\omega_{ab}$ is the angular velocity for the changing ${}^aR_b$ expressed relative to the frame k.

$$ {}^k\omega_{ab} = {}^kR_a \, {}^a\omega_{ab}. $$

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- n-link manipulator with joint variables, $q_1, \cdots, q_n$
- Angular velocity of end-effector frame, ${}^0\omega_{0n}$ be defined by

$$S\left({}^0\omega_{0,n}\right) = {}^0\dot{R}_n\,{}^0R_n^T$$

and linear velocity by

$${}^0v_n = {}^0\dot{p}_n$$

- Need expressions of the form:

$${}^0v_n = J_v\dot{q} \qquad \text{and} \qquad {}^0\omega_{0n} = J_\omega\dot{q}$$

- Combining the two: $\xi = J\,\dot{q}$, where J is a $6 \times n$ matrix

Claim: Addition of angular velocities

The angular velocity ${}^0\omega_{0,n}$ can be expressed as:

$$\begin{aligned} {}^0\omega_{0,n} &= {}^0\omega_{0,1} + {}^0\omega_{1,2} + {}^0\omega_{2,3} + \cdots + {}^0\omega_{n-1,n} \\ &= {}^0\omega_{0,1} + {}^0R_1\,{}^1\omega_{1,2} + {}^0R_2\,{}^2\omega_{2,3} + \cdots + {}^0R_{n-1}\,{}^{n-1}\omega_{n-1,n} \end{aligned}$$

where ${}^0\omega_{i,j}$ represents angular velocity corresponding to derivative of ${}^iR_j$ expressed in coordinates of 0 frame.

What is ${}^{i-1}\omega_{i-1,i}$?

- Frame i moves, when joint i moves.

- If joint i is revolute, then
  - joint variable $q_i = \theta_i$
  - axis of rotation is z_{i-1}
  - frame i rotates about the fixed axis, z_{i-1}

$$
\begin{aligned}
{}^{i-1}\omega_{i-1,i} &= \dot{q}_i \; {}^{i-1}z_{i-1} \\
&= \dot{q}_i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{q}_i \, \hat{k}
\end{aligned}
$$

- If joint i is prismatic, then
  - motion of frame i relative to $i-1$ is translation only

$$
{}^{i-1}\omega_{i-1,i} = 0
$$

What is ${}^{i-1}\omega_{i-1,i}$?

- Given that $\rho_i = 1$ if joint is revolute, and $\rho_i = 0$ if prismatic.

$$
\begin{aligned}
{}^0\omega_{0,n} &= {}^0\omega_{0,1} + {}^0R_1\,{}^1\omega_{1,2} + {}^0R_2\,{}^2\omega_{2,3} + \cdots + {}^0R_{n-1}\,{}^{n-1}\omega_{n-1,n} \\
&= \rho_1\dot{q}_1\hat{k} + \rho_2\dot{q}_2\,{}^0R_1\hat{k} + \rho_3\dot{q}_3\,{}^0R_2\hat{k} + \cdots + \rho_n\dot{q}_n\,{}^0R_{n-1}\hat{k} \\
&= \sum_{i=1}^{n} \rho_i\dot{q}_i\,{}^0R_{i-1}\hat{k}
\end{aligned}
$$

Deriving J_ω

$$
\begin{aligned}
{}^0\omega_{0,n} &= \rho_1 \hat{k}\, \dot{q}_1 + \rho_2\, {}^0R_1 \hat{k}\, \dot{q}_2 + \rho_3\, {}^0R_2 \hat{k}\, \dot{q}_3 + \cdots + \rho_n\, {}^0R_{n-1} \hat{k}\, \dot{q}_n \\
&= \begin{bmatrix} \rho_1\, {}^0R_0 \hat{k} & \rho_2\, {}^0R_1 \hat{k} & \cdots & \rho_n\, {}^0R_{n-1} \hat{k} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_n \end{bmatrix} \\
&= \begin{bmatrix} \rho_1\, {}^0z_0 & \rho_2\, {}^0z_1 & \cdots & \rho_n\, {}^0z_{n-1} \end{bmatrix} \dot{q} \\
&= J_\omega\, \dot{q}
\end{aligned}
$$

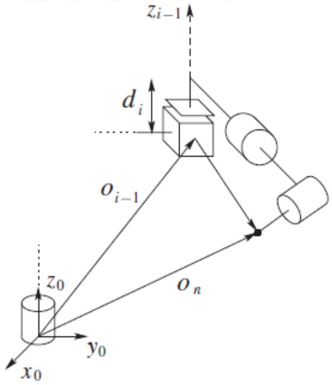
- Each column is 3×1 vector and J_ω is $3 \times n$.

- By chain rule,

$$ {}^0\dot{p}_n = \sum_{i=1}^{n} \frac{\partial\, {}^0 p_n}{\partial q_i}\, \dot{q}_i $$

- Thus, ith column of J_v is $\frac{\partial\, {}^0 p_n}{\partial q_i}$.
- Notice that this is the end-effector velocity, if all joints were fixed except i, and the joint i is actuated at unit velocity ($\dot{q}_i = 1$)
- Let's look at case of prismatic and revolute joint separately.

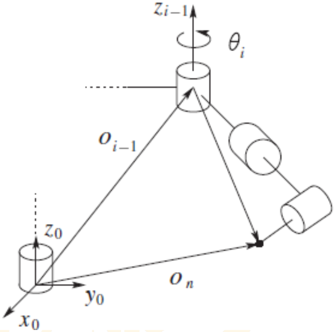
Deriving linear velocity for prismatic joint



- Assume all joints fixed except Prismatic Joint, i
- Joint imparts pure translation to end-effector parallel to axis z_{i-1}

$$^0\dot{p}_n = \dot{q}_i \, {}^0z_{i-1}$$

- So, ith column of J_v is ${}^0z_{i-1}$



- Assume all joints fixed except revolute Joint, i
- Linear velocity of end-effector is of form $\omega \times r$
- $\omega = \dot{\theta}_i\, z_{i-1}$ and $r = o_n - o_{i-1}$. So, the ith column of J_v is:

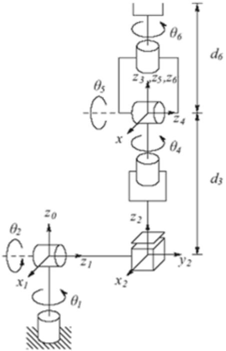
$$^{0}z_{i-1} \times (^{0}o_n - {}^{0}o_{i-1})$$

Deriving J

- $J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$

- ith column of J is:

$$J_i = \begin{cases} \begin{bmatrix} {}^0z_{i-1} \times \left({}^0p_n - {}^0p_{i-1}\right) \\ {}^0z_{i-1} \end{bmatrix}, & \text{if joint i is revolute,} \\ \begin{bmatrix} {}^0z_{i-1} \\ 0 \end{bmatrix}, & \text{if joint i is prismatic.} \end{cases}$$



- Last three axes are intersecting. $o_3 = o_4 = o_5 = o$, which makes life easier for us.
- The ith column of Jacobian is J_i. Then,

$$J_i = \begin{bmatrix} {}^0z_{i-1} \times (o_6 - o_{i-1}) \\ {}^0z_{i-1} \end{bmatrix}, \quad i = 1, 2$$

$$J_3 = \begin{bmatrix} {}^0z_2 \\ 0 \end{bmatrix}$$

$$J_i = \begin{bmatrix} {}^0z_{i-1} \times (o_6 - o) \\ {}^0z_{i-1} \end{bmatrix}, \quad i = 4, 5, 6$$

Example: Stanford manipulator

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

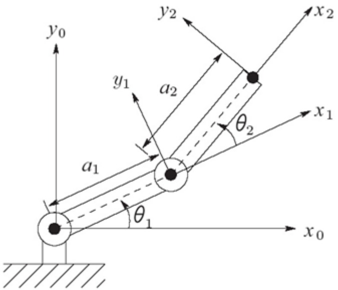
Example: Stanford manipulator

- ${}^{0}z_{i-1} = ?$
- ${}^{0}z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- Determine ${}^{0}T_{i-1} = A_1 A_2 \cdots A_{i-1}$.
  - ${}^{0}z_{i-1}$ is first three elements of third column of ${}^{0}T_{i-1}$.

Example: Stanford manipulator

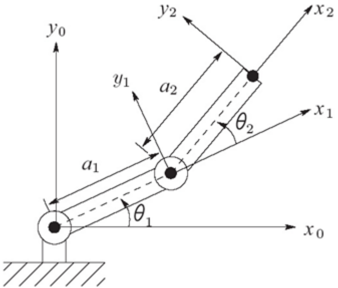
- ${}^0o_j = ?$
- ${}^0o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Determine ${}^0T_j$.
  - ${}^0o_j$ is first three elements of fourth column of ${}^0T_j$.

Example: Two-link planar manipulator



- $J(q) = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix}$
- $o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$
- $z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Example: Two-link planar manipulator



- $$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Skew Symmetric Matrices–More Properties

- S is linear operator, i.e.

$$S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$$

  for vectors $a, b \in \mathbb{R}^3$ and scalars $\alpha, \beta \in \mathbb{R}$

- $R \in SO(3)$ and $a, b \in \mathbb{R}^3$:

$$R(a \times b) = R(a) \times R(b)$$

- $R \in SO(3)$ and $a \in \mathbb{R}^3$:

$$R\,S(a)\,R^T = S(Ra)$$

Matrix representation of $S(a)$ in a coordinate frame rotated by R is the same as the skew-symmetric matrix $S(Ra)$ corresponding to the vector a rotated by R.

$$^0R_2(t) = {^0R_1(t)}\ {^1R_2(t)}$$

$$\underbrace{^0\dot{R}_2}_{S(^0\omega_{0,2})\,^0R_2} = \underbrace{^0\dot{R}_1}_{S(^0\omega_{0,1})\,^0R_1}\ {^1R_2} + {^0R_1}\ \underbrace{^1\dot{R}_2}_{S(^1\omega_{1,2})\,^1R_2}$$

$$S\left(^0\omega_{0,2}\right)\,^0R_2 = S\left(^0\omega_{0,1}\right)\ {^0R_2} + \underbrace{^0R_1\, S\left(^1\omega_{1,2}\right)\, {^0R_1}^T}_{S\left(^0R_1\,^1\omega_{1,2}\right)}\underbrace{^0R_1\ {^1R_2}}_{^0R_2}$$

$$\Rightarrow S\left(^0\omega_{0,2}\right) = S\left(^0\omega_{0,1}\right) + S\left(^0R_1\ {^1\omega_{1,2}}\right) = S\left(^0\omega_{0,1} + {^0\omega_{1,2}}\right)$$

$$^0\omega_{0,2} = {^0\omega_{0,1}} + {^0\omega_{1,2}}$$

- This can be extended to the case of $^0\omega_{0,n}$ as well

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- End-effector pose $X = \begin{bmatrix} p(q) \\ \alpha(q) \end{bmatrix}$, where d is the usual vector for position, but orientation α is in minimal coordinates. E.g., $\alpha = (\phi, \theta, \psi)^T$, the Euler angles.

- We want: $\dot{X} = J_\alpha(q)\,\dot{q}$. J_α is analytical jacobian.

Claim:

If $R = R_z(\psi)R_y(\theta)R_z(\phi)$, then $\dot{R}\,R^T = S(\omega)$ and

$$\omega = \begin{bmatrix} c_\phi s_\theta \dot{\psi} - s_\phi \dot{\theta} \\ s_\phi s_\theta \dot{\psi} + c_\phi \dot{\theta} \\ \dot{\phi} + c_\theta \dot{\psi} \end{bmatrix} {}^a$$

[a] Typo in Spong's book - ϕ and ψ are interchanged. See `AnalyticalJacobian.mlx`

Angular velocity in Euler angles [2, Section 3.6]

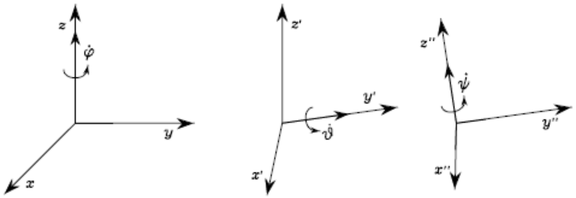

Figure: Rotational velocities of Euler angles. Source: [2]

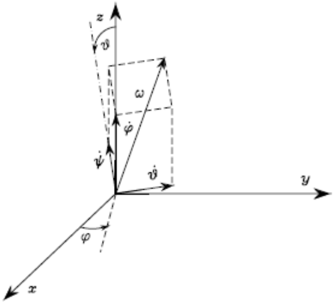

Figure: Composition of rotational velocities in ZYZ. Source: [2]

- Angular velocity due to first rotation:

$$ {}^{0}\omega_{01} = \dot{\phi} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

- Angular velocity due to second rotation:

$$ {}^{0}\omega_{12} = \dot{\theta}\ \underbrace{{}^{0}R_1}_{R_z(\phi)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} $$

- Angular velocity due to third rotation:

$$ {}^{0}\omega_{23} = \dot{\psi}\ {}^{0}R_1\ \underbrace{{}^{1}R_2}_{R_y(\theta)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

- Total velocity:

$$ {}^{0}\omega = {}^{0}\omega_{01} + {}^{0}\omega_{12} + {}^{0}\omega_{23} $$

$$\omega = \begin{bmatrix} c_\phi s_\theta \dot{\psi} - s_\phi \dot{\theta} \\ s_\phi s_\theta \dot{\psi} + c_\phi \dot{\theta} \\ \dot{\phi} + c_\theta \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & -s_\phi & c_\phi s_\theta \\ 0 & c_\phi & s_\phi s_\theta \\ 1 & 0 & c_\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = B(\alpha)\dot{\alpha}$$

$$J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{p} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & O \\ O & B(\alpha) \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & O \\ O & B(\alpha) \end{bmatrix} J_\alpha(q)\dot{q}$$

$$J_\alpha(q) = \begin{bmatrix} I & O \\ O & B^{-1}(\alpha) \end{bmatrix} J(q)$$

if $\det B(\alpha) \neq 0$

α at which $\det B(\alpha) = 0$: Representational singularities

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The position and orientation of the end-effector are related to the joint variables through

$$ {}^0T_n(q) = \begin{bmatrix} {}^0R_n(q) & {}^0p_n(q) \\ O & 1 \end{bmatrix} $$

- First three columns of transformation matrix, ${}^0R_n(q)$, relate orientation of end-effector to joint variables.
- Fourth column of transformation matrix, ${}^0p_n(q)$, relates position of end-effector to joint variables.

Summary – Relation between joint and end-effector velocities

$$\begin{bmatrix} {}^0v_n \\ {}^0\omega_n \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$
$$J_\omega = \begin{bmatrix} \rho_1\, {}^0z_0 & \rho_2\, {}^0z_1 & \cdots & \rho_n\, {}^0z_{n-1} \end{bmatrix}$$
$$J_v = \begin{bmatrix} c_1 & c_2 & \cdots & c_i & \cdots & c_n \end{bmatrix}$$

- ${}^0v_n$ and ${}^0\omega_n$ are linear and angular velocities of end-effector in 0-frame coordinates.
- ${}^0z_i$ is joint axis of joint $i+1$ in 0-frame coordinates.
- $\rho = 1$ for revolute joint and $\rho = 0$ for prismatic.

Summary – Relation between joint and end-effector velocities

$$\begin{bmatrix} {}^0v_n \\ {}^0\omega_n \end{bmatrix} = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

$$J_\omega = \begin{bmatrix} \rho_1\,{}^0z_0 & \rho_2\,{}^0z_1 & \cdots & \rho_n\,{}^0z_{n-1} \end{bmatrix}$$

$$J_v = \begin{bmatrix} c_1 & c_2 & \cdots & c_i & \cdots & c_n \end{bmatrix}$$

- $c_i = {}^0z_{i-1}$ if joint i is prismatic.
- $c_i = {}^0z_{i-1} \times \left({}^0o_n - {}^0o_{i-1}\right)$ if joint i is revolute.

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