Trajectory Planning

EE366/CE366/CS380: Introduction to Robotics

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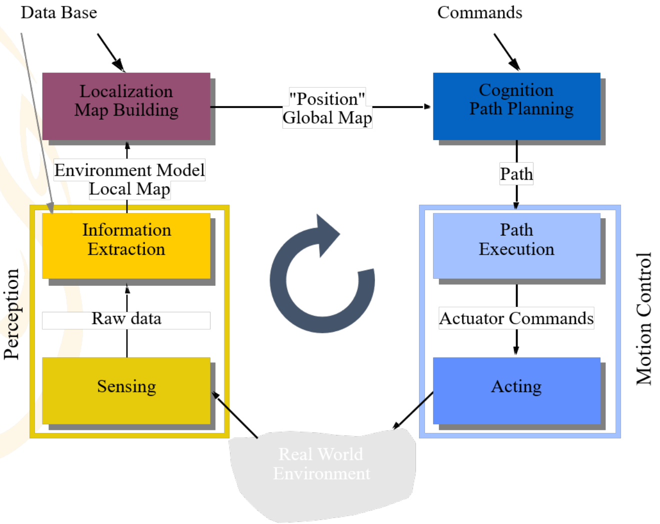
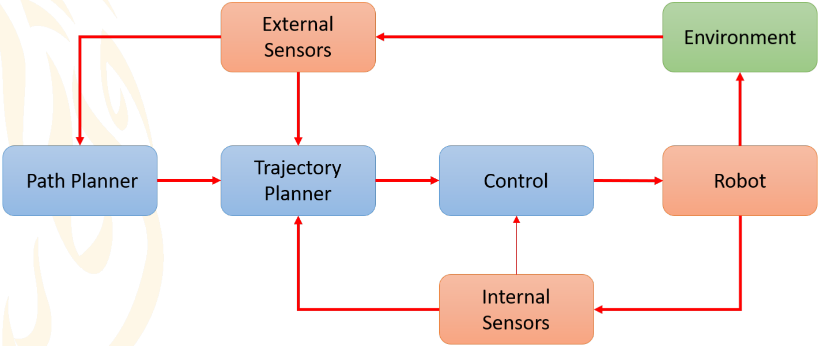
Functional Decomposition of a Robot



The Robot Motion Control System



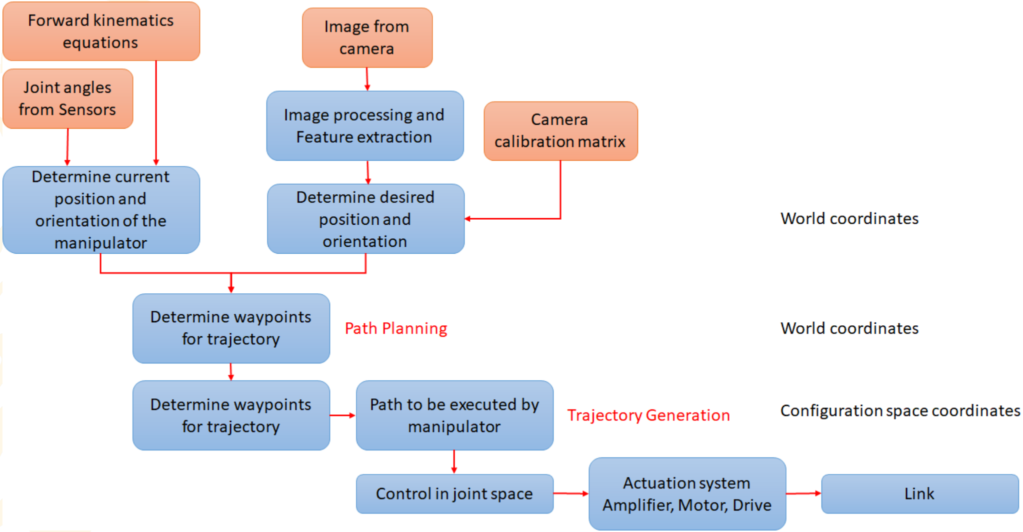
What are we missing?

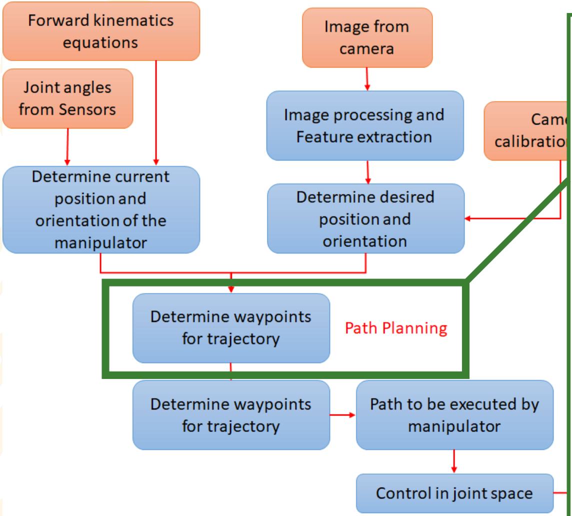
- The FK and IK problems solved are point problems, dependent only on intrinsic geometry of robot.

- Given an initial and final configuration of robot, what could happen in between?
  - Transition should not make robot collide with obstacles, boundary of the map, or itself, at minimum.

  - Transition should require the actuators to exert generalized torques that don't violate their saturation limits, at minimum.

  - What else? Minimum time; Shortest distance; Avoiding singularities

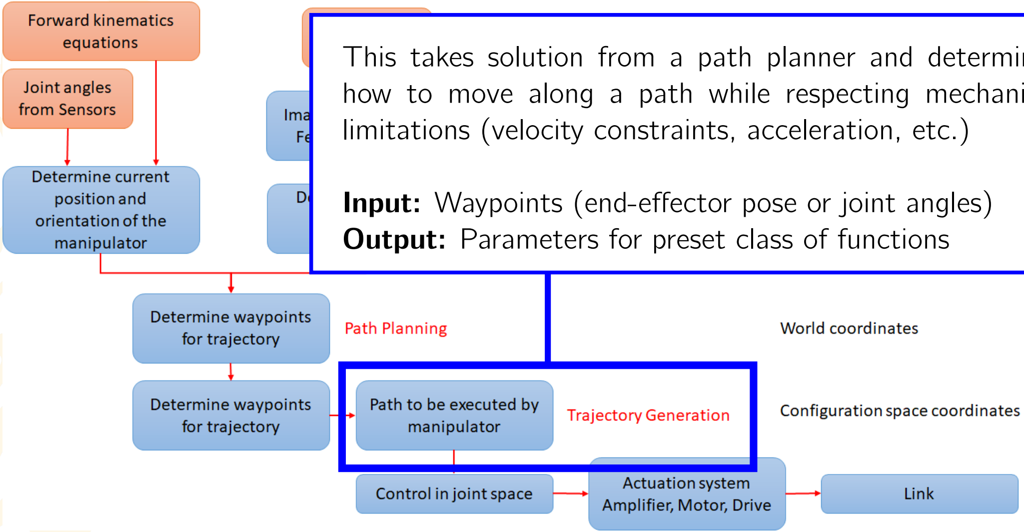

Forward kinematics equations
Joint angles from Sensors
Image from camera
Image processing and Feature extraction
Camera calibration matrix
Determine current position and orientation of the manipulator
Determine desired position and orientation
World coordinates
Determine waypoints for trajectory
Path Planning
World coordinates
Determine waypoints for trajectory
Path to be executed by manipulator
Trajectory Generation
Configuration space coordinates
Control in joint space
Actuation system Amplifier, Motor, Drive
Link



This is the purely geometric problem of finding a collision-free path between starting configuration and goal configuration, without concern for dynamics, duration of motion, constraints. It is also sometimes called piano mover's problem.

Input: Initial and final configurations, Map
Output: Waypoints

What is **trajectory** planning?



This takes solution from a path planner and determines how to move along a path while respecting mechanical limitations (velocity constraints, acceleration, etc.)

Input: Waypoints (end-effector pose or joint angles)
Output: Parameters for preset class of functions

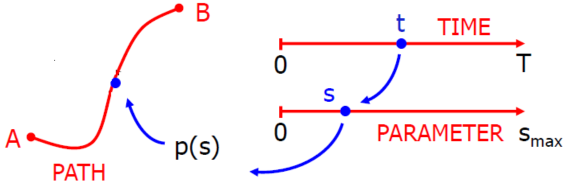
Trajectory vs Path

Path

Geometric description of sequence of configurations or end-effector poses achieved by the robot.

Trajectory

The specification of the robot configurations or end-effector poses as a function of time is called a trajectory. It is combination of path and timing law.



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Trajectory Planning and Generation

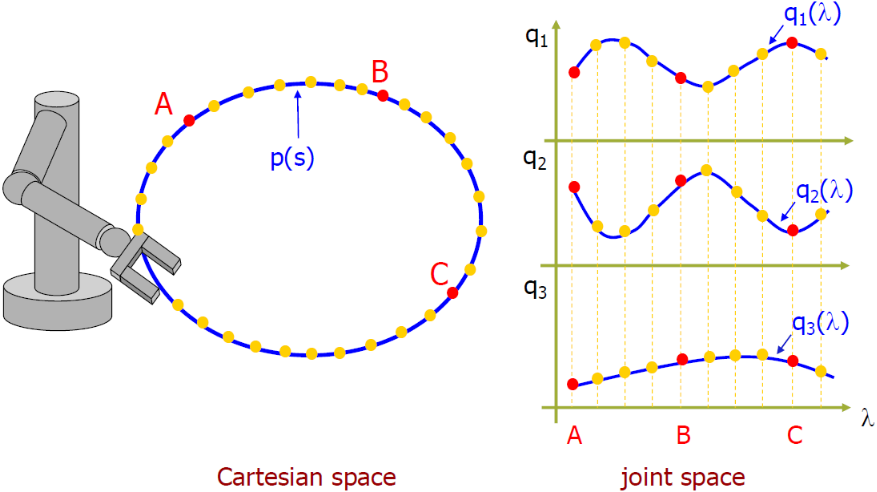
Sequence of poses or configurations
Constraints:
Velocity, Acceleration
Actuator limits
Time
Path constraints

$\longrightarrow$ Trajectory Planner $\longrightarrow$ Trajectories for controller

Where does input to trajectory planner come from?

- User typically specifies initial and desired final pose, because of complexity.
- An automatic path planner will also only provide a sequence of points along desired path (via points).
- Paths for industrial robots in static situations are specified by physically leading the robot through the desired motion with a teach pendant, the so-called **teach and playback mode**.

IK is used to transform trajectories to joint space.



IK is used to transform trajectories to joint space.

1. Sequence of poses in Cartesian space (yellow points)
2. Interpolation in Cartesian space ($p(s)$)
3. Sampling and IK to move to joint space (red points)
4. Sequence of configurations in joint space
5. Interpolation in Joint space ($q(\lambda)$)

What other constraints does trajectory planner have?

- Timing law is not typically specified, but total trajectory time.
- Constraints on maximum velocities and accelerations, and specific values at points of interest.
- Saturation limits of actuators shouldn't be violated.

Point-to-Point Motion

Given: Initial and Final configuration in joint space.

Desired properties of trajectories

- Joints positions and velocities, at least, should be continuous.
- Smooth trajectories are desirable.
  - Rough, jerky motions tend to cause increased wear on the mechanism and cause vibrations by exciting resonances in the manipulator.
- Trajectory generation happens at run-time, so generator shouldn't be computationally intensive.

Trajectories for point-point motion

- $q(t)$: Scalar Joint Variable

- Constraints:

$$q(t_0) = q_0$$
$$\dot{q}(t_0) = v_0$$
$$q(t_f) = q_f$$
$$\dot{q}(t_f) = v_f$$

- Other possible constraints:

$$\ddot{q}(t_0) = \alpha_0$$
$$\ddot{q}(t_f) = \alpha_f$$

- Infinitely many trajectories passing through these constraints.

- Choose parameterized family of functions, e.g. polynomials based on number of constraints.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\begin{aligned} q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\ v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 \\ q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{aligned}$$

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}$$

- Determinant of coefficient matrix is $(t_f - t_0)^4$
  - Unique solution exists for every non-zero time interval.

Cubic Polynomial Trajectories have a jump in acceleration.

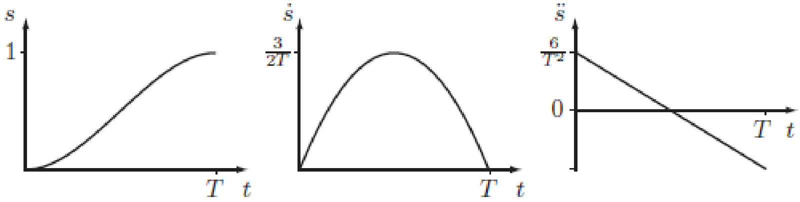


Figure: Plots of q, $\dot{q}$, $\ddot{q}$ for cubic time scaling

- In cubic case, there is jump in acceleration at $t = 0$ and $t = t_f$.
  - Infinite jerk $\rightarrow$ Possible vibrations.

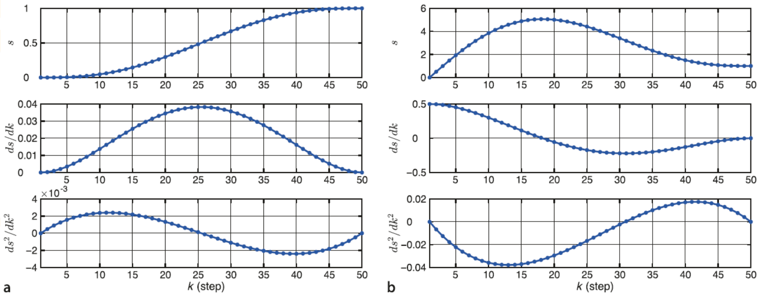
Position, Velocity, Acceleration constraints: Quintic Polynomials

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\begin{aligned}
q_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 + a_5 t_0^5 \\
v_0 &= a_1 + 2a_2 t_0 + 3a_3 t_0^2 + 4a_4 t_0^3 + 5a_5 t_0^4 \\
\alpha_0 &= 2a_2 + 6a_3 t_0 + 12a_4 t_0^2 + 20a_5 t_0^3 \\
q_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \\
v_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4 \\
\alpha_f &= 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
\end{aligned}$$

$$\begin{bmatrix}
1 & t_0 & t_0^2 & t_0^3 & t_0^4 & t_0^5 \\
0 & 1 & 2t_0 & 3t_0^2 & 4t_0^3 & 5t_0^4 \\
0 & 0 & 2 & 6t_0 & 12t_0^2 & 20t_0^3 \\
1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\
0 & 1 & 2t_f & 3t_f^2 & 4t_0^3 & 5t_0^4 \\
0 & 0 & 2 & 6t_0 & 12t_f^2 & 20t_f^3
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
=
\begin{bmatrix} q_0 \\ v_0 \\ \alpha_0 \\ q_f \\ v_f \\ \alpha_f \end{bmatrix}$$

Issues with Quintic Polynomial Trajectories



- Velocity is less than maximum most of time.
- For non-zero initial velocity, trajectory overshoots the end values.

- Common for control of motors.
- Velocity profile is trapezoidal – constant acceleration phase (a), constant velocity phase (v), constant decceleration phase
- The q profile is concatenation of parabola, linear, and parabola.

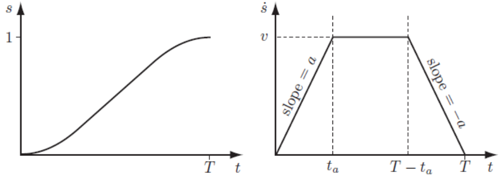


Figure: Plots of s, $\dot{s}$ for trapezoidal time scaling

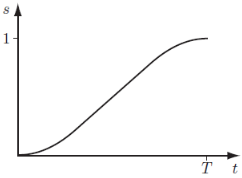
LSPB – First Segment

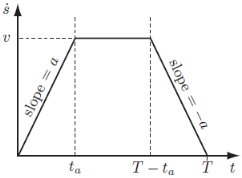
- **Assuming, motion is possible**
- For simplicity: $t_0 = 0,\ t_f = T,$
  $\dot{q}(t_0) = \dot{q}(t_f) = 0$

$$q(t) = a_0 + a_1 t + a_2 t^2$$
$$\Rightarrow \dot{q}(t) = a_1 + 2a_2 t$$

For $0 \le t \le t_a = \frac{v}{a}$,

$$\dot{q}(t_a) = v$$
$$\Rightarrow \dot{q}(t) = \frac{v}{t_a} t = at,$$
$$\ddot{q}(t) = a,$$
$$q(t) = q_0 + \frac{at^2}{2}$$



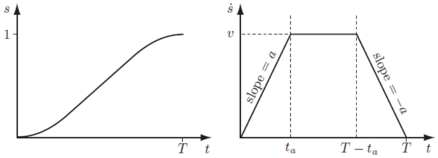


- **Assuming, motion is possible**

For $t_a \leq t \leq T - t_a$,

$$\ddot{q}(t) = 0,$$
$$\dot{q}(t) = v,$$
$$q(t) = q(t_a) + v(t - t_a)$$

$$\begin{aligned} q(t) &= q_0 + \frac{at_a^2}{2} + v(t - t_a) \\ &= q_0 + vt + \frac{v^2}{2a} - \frac{v^2}{a} \\ &= q_0 + vt - \frac{v^2}{2a} \end{aligned}$$



- **Assuming, motion is possible**

For $T - t_a \le t \le T$,

$$q(t) = a_0 + a_1 t + a_2 t^2$$
$$\dot{q}(t) = a_1 + 2a_2 t$$
$$\ddot{q}(t) = 2a_2$$

$$\Rightarrow q(t) = q_f - \frac{aT^2}{2} + aTt - \frac{at^2}{2}$$

Constraints

$$\ddot{q}(t) = -a$$
$$\Rightarrow a_2 = -\frac{a}{2}$$
$$\dot{q}(T) = 0$$
$$a_1 + 2a_2 T = 0 \Rightarrow a_1 = aT$$
$$q(T) = q_f$$
$$\Rightarrow a_0 + a_1 T + a_2 T^2 = q_f$$
$$\Rightarrow a_0 = q_f - aT^2 + \frac{aT^2}{2}$$

There is still one more constraint!

$$q_2(T - t_a) = q_3(T - t_a)$$

$$q_0 - \frac{v^2}{2a} + v(T - t_a) = q_f - \frac{aT^2}{2} + aT(T - t_a) - \frac{a(T - t_a)^2}{2}$$

$$q_f - q_0 = \frac{a^2T^2 - v^2}{2a} + (T - t_a)(v - aT + aT/2 - at_a/2)$$

Substituting $t_a = v/a$,

$$q_f - q_0 = -\frac{v^2 - a^2T^2}{2a} - \frac{1}{2a}(v - aT)(v - aT)$$

$$q_f - q_0 = -\frac{1}{2a}(v - aT)(v + aT + v - aT) = \frac{v(aT - v)}{a}$$

Only two of v, a, and T can be independently chosen.

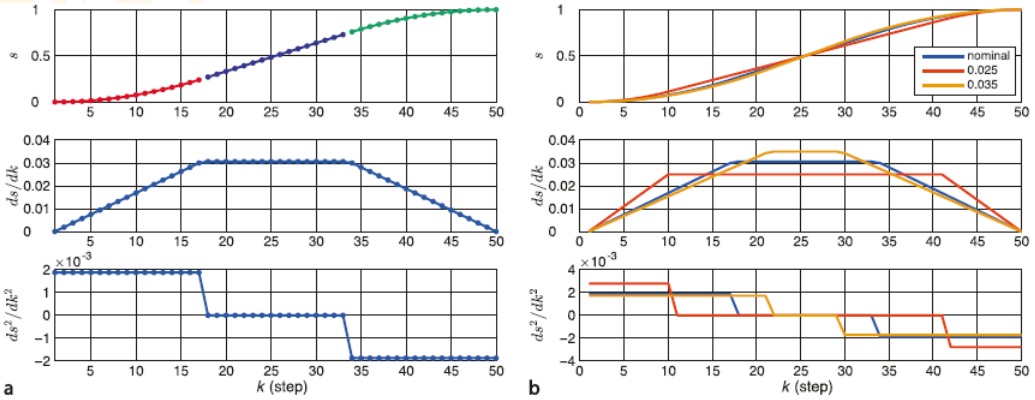
Constraint.

$$q_f - q_0 = \frac{v(aT - v)}{a}$$

$$T = \frac{v}{a} + \frac{q_f - q_0}{v}$$

$$a = \frac{v^2}{vT - (q_f - q_0)}$$

$$v = \frac{1}{2}\left(aT - \sqrt{a}\sqrt{aT^2 - 4(q_f - q_0)}\right)$$


1
0.5
0
s
5
10
15
20
25
30
35
40
45
50
0.04
0.03
0.02
0.01
0
ds/dk
×10^-3
2
1
0
−1
−2
ds^2/dk^2
k (step)
a
nominal
0.025
0.035
4
2
−2
−4
b

LSPB – When is motion possible?

- We assumed that

$$0 < t_a \leq \frac{T}{2}$$

- Or, specified velocity and time must satisfy the following relation or motion is not possible:

$$\frac{q_f - q_0}{T} < v \leq \frac{2(q_f - q_0)}{T}.$$

- Or, specified time and acceleration must satisfy the following relation or motion is not possible:

$$4|q_f - q_0| \leq aT^2.$$

- Or, specified acceleration and velocity must satisfy the following relation or motion is not possible:

$$\frac{v^2}{a} \leq (q_f - q_0).$$

LSPB – Choose parameters in one of three ways:

- Choose v and T such that $q_f - q_0 < vT \le 2(q_f - q_0)$,

$$a = \frac{v^2}{vT - (q_f - q_0)}$$

- Choose v and a such that $\frac{v^2}{a} \le (q_f - q_0)$,

$$T = \frac{v}{a} + \frac{q_f - q_0}{v}.$$

- Choose a and T such that $4|q_f - q_0| \le aT^2$,

$$v = \frac{1}{2}\left(aT - \sqrt{a}\sqrt{aT^2 - 4(q_f - q_0)}\right)$$

Minimum-time trajectories or Bang-bang trajectories

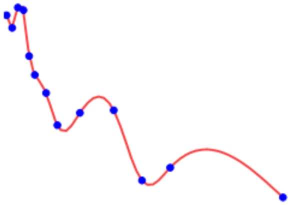
- Minimize T, given a.
- Special case of LSPB.
- Keep acceleration at maximum till a suitable switching time, t_s, at which time switch to maximum decceleration.
- $v = \frac{aT}{2}$, and so

$$T = 2\sqrt{\frac{q_f - q_0}{a}}$$

Sequence of Points/ Via Points

Given: Initial, Final, and some intermediate configurations in joint space.

Fitting a common polynomial

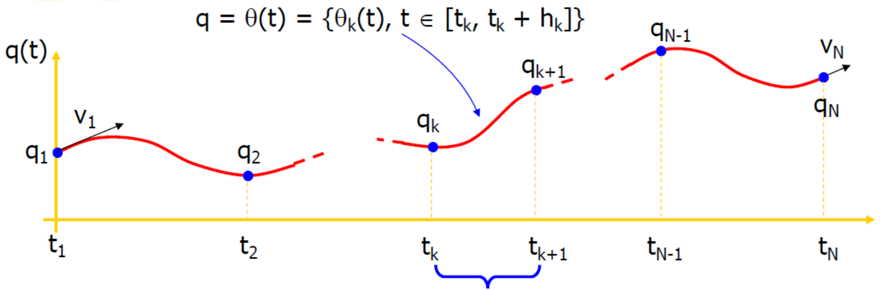


- Trajectory should pass through a sequence of configurations at specified times
- **Position Constraints:** $q(t_0) = q_0$, $q(t_1) = q_1, \cdots, q(t_{n-1}) = q_{n-1}$
- There could be velocity or acceleration constraints at end points or even intermediate points.
- If there are N constraints in total, choose an $N-1$ degree polynomial and solve for its coefficients.

Fitting a polynomial through via points

- **Advantage:** The velocity and acceleration are guaranteed to be continuous at intermediate points.
- **Disadvantages:**
  - Oscillatory behavior increases with order of polynomial.
  - Polynomial coefficients depend on all points. So, if one point changes all coefficients are recomputed.
  - Numerical accuracy decreases as order increases.
- **Alternate:** We can build trajectory using low-order polynomials between specified points.

Low-order polynomials between points



- Fit a cubic or quintic polynomial between any two successive via points
- Velocities and accelerations may be specified at intermediate points or not.

Cubic Spline

- Any $q(t)$ segment is:

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad t \in [t_k, t_{k+1}]$$

- Four unknowns for each cubic segment.
- $q(t_k)$ and $q(t_{k+1})$ are known.
- Two coefficients left undetermined.
- If $\dot{q}(t_k)$ and $\dot{q}(t_{k+1})$ are also given, then remaining two coefficients can be determined.

Cubic Spline

- Any $q(t)$ segment is:

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad t \in [t_k, t_{k+1}]$$

- If only $\dot{q}(t_0)$ and $\dot{q}(t_f)$ are given, but via points velocities are not specified

- We can impose continuity constraints:

$$\dot{q}_k(t_k) = \dot{q}_{k+1}(t_k)$$
$$\ddot{q}_k(t_k) = \ddot{q}_{k+1}(t_k)$$

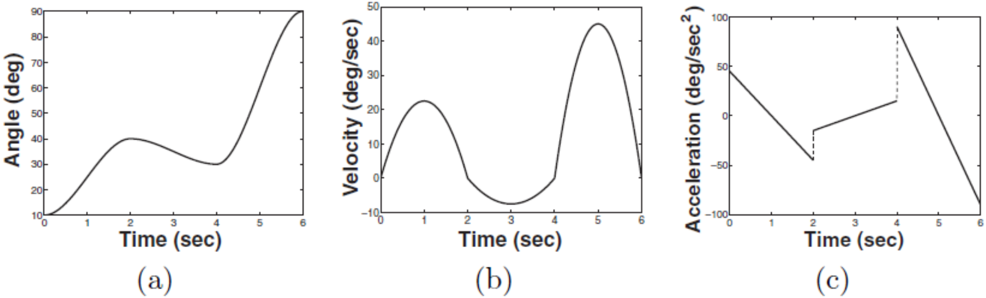
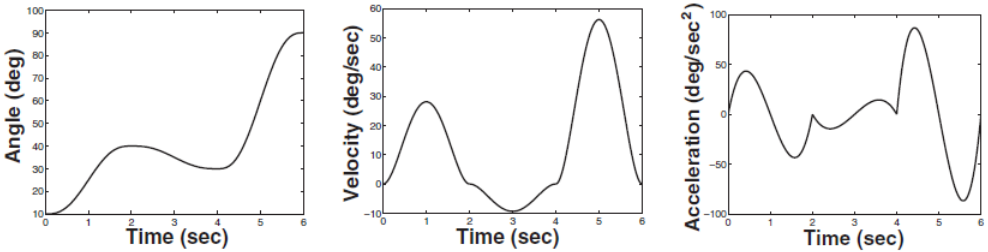


Figure 7.23: (a) Cubic spline trajectory made from three cubic polynomials. (b) Velocity profile for multiple cubic polynomial trajectory. (c) Acceleration profile for multiple cubic polynomial trajectory.

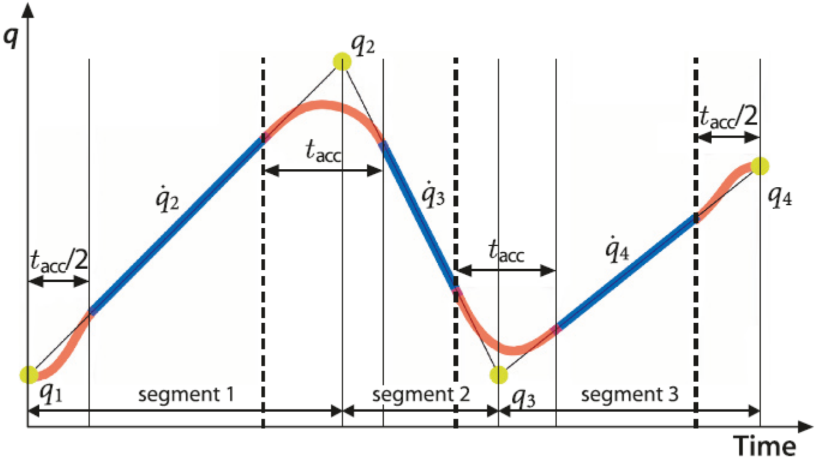
Quintic Spline between 4 via points



(a) (b) (c)

Figure 7.24: (a) Trajectory with multiple quintic segments. (b) Velocity profile for multiple quintic segments. (c) Acceleration profile for multiple quintic segments.

Linear segments with parabolic blends



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Joint Space vs Task Space

- The path points could be specified in the joint space, i.e. in terms of joint variables, or in task space (Cartesian space/Operational Space), i.e. in terms of pose (position and orientation of end-effector).

Should you plan in joint space or task space? [1, Section 4.1]

- Trajectory planning can happen in joint space or task space.
  - Trajectory planning in the task space allows for easy accommodation of any path constraints, e.g. obstacles.
  - Trajectory planning in the joint space allows for easy handling of kinematic singularities, redundant dof.

Trajectory Generation in Task Space

- We now have a sequence of poses – position and orientation
- The same techniques (polynomials, LSPB) can be used to interpolate position
- The same techniques can be used to interpolate orientation described in Euler angles form.
- Linear interpolation applied to rotation matrices will result in intermediate matrices that are not necessarily rotation matrices. So, Euler angle representation is used.

Geometric Problems with Cartesian Paths

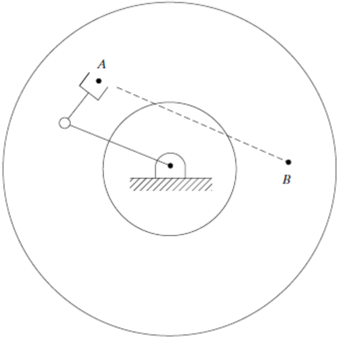

Figure: Intermediate points unreachable. Source: Craig

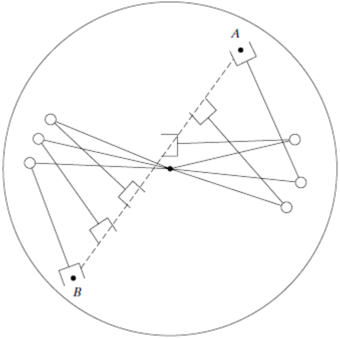

Figure: High joint rates near singularities. Source: Craig

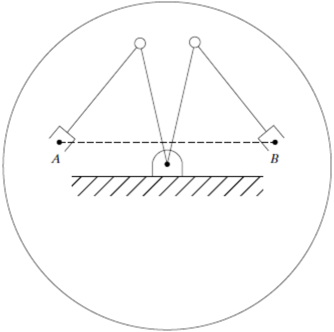

Figure: Start and goal reachable in different solutions.
Source: Craig

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[1] **Bruno Siciliano, Lorenzo Sciavicco, Luigi Villani, and Giuseppe Oriolo.**
Robotics: modelling, planning and control.
Springer Science & Business Media, 2010.