

EE 366/CE 366/CS 380 Mid-term Exam 2023 Solution

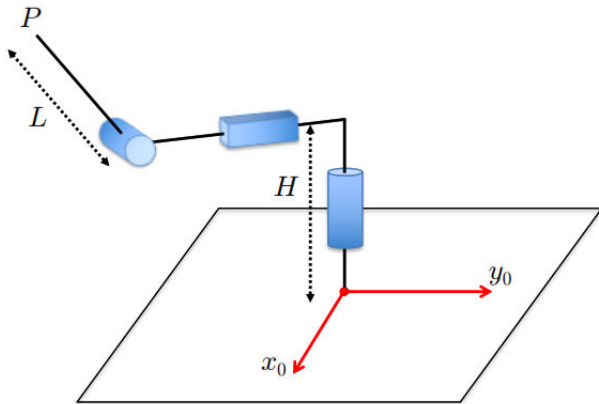
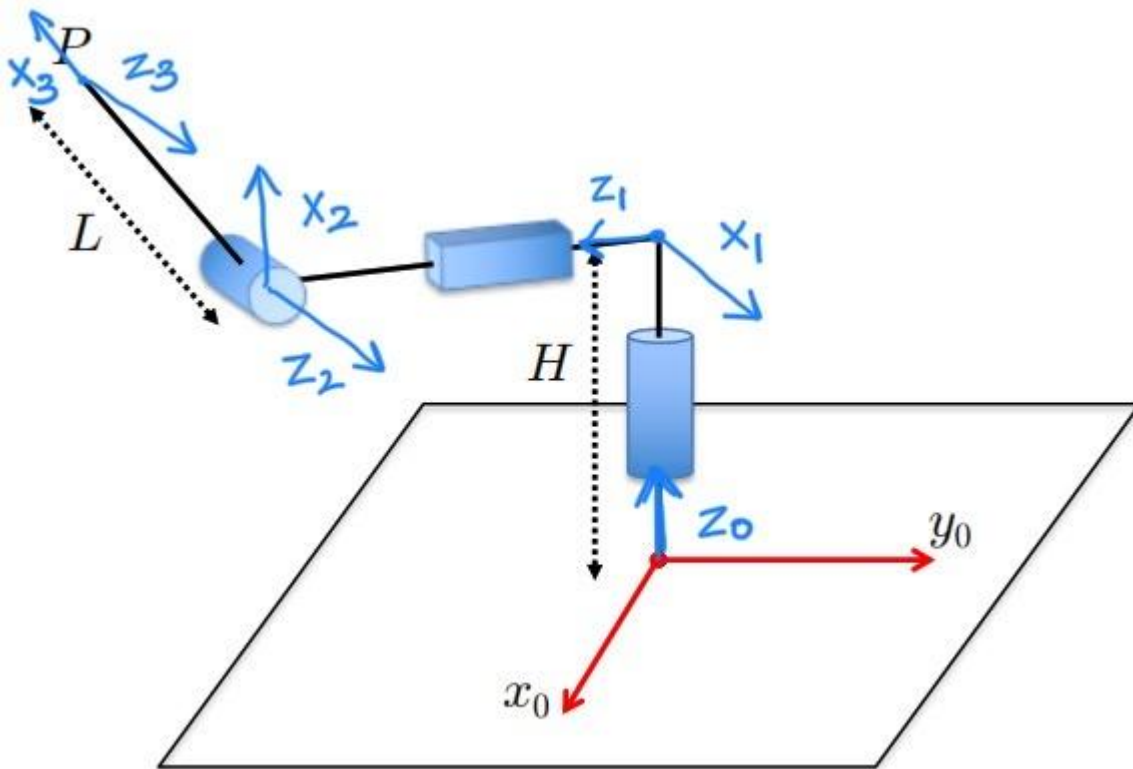


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Question (a) DH Frame Assignment and Parameters

Complete the assignment of Denavit-Hartenberg (DH) frames and fill in the associated table of parameters. Assign the last frame by carefully considering the remaining parts of this question. Moreover, the frame assignment should be such that all constant DH parameters are non-negative and the value of the joint variables q_i are strictly positive in the shown configuration.



Link	θ	d	a	α
1	θ_1	H	0	90°
2	90°	d_2	0	90°
3	θ_3	0	L	0

Question (b) Forward Kinematics

Find the coordinates of the point P in the (x_0, y_0, z_0) frame.

We can obtain coordinates using forward kinematics.

$$T_{01}(t) =$$

$$\begin{pmatrix} \cos(\theta_1(t)) & 0 & \sin(\theta_1(t)) & 0 \\ \sin(\theta_1(t)) & 0 & -\cos(\theta_1(t)) & 0 \\ 0 & 1 & 0 & H \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{12}(t) =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2(t) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{23}(t) =$$

$$\begin{pmatrix} \cos(\theta_3(t)) & -\sin(\theta_3(t)) & 0 & L \cos(\theta_3(t)) \\ \sin(\theta_3(t)) & \cos(\theta_3(t)) & 0 & L \sin(\theta_3(t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$T_{03}(t) =$

$$\begin{pmatrix} \sin(\theta_1(t)) \sin(\theta_3(t)) & \cos(\theta_3(t)) \sin(\theta_1(t)) & \cos(\theta_1(t)) & \sin(\theta_1(t)) d_2(t) + L \sin(\theta_1(t)) \sin(\theta_3(t)) \\ -\cos(\theta_1(t)) \sin(\theta_3(t)) & -\cos(\theta_1(t)) \cos(\theta_3(t)) & \sin(\theta_1(t)) & -\cos(\theta_1(t)) d_2(t) - L \cos(\theta_1(t)) \sin(\theta_3(t)) \\ \cos(\theta_3(t)) & -\sin(\theta_3(t)) & 0 & H + L \cos(\theta_3(t)) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, coordinates of P are

P =

$$\begin{pmatrix} \sin(\theta_1(t)) d_2(t) + L \sin(\theta_1(t)) \sin(\theta_3(t)) \\ -\cos(\theta_1(t)) d_2(t) - L \cos(\theta_1(t)) \sin(\theta_3(t)) \\ H + L \cos(\theta_3(t)) \end{pmatrix}$$

Question (c) Inverse Kinematics

If $H = 2$, $L = 2$, can this manipulator reach the coordinates $P = (2.6, -1.5, 3)$? If yes, find all possible configurations in which you can achieve this point.

Geometric Approach

Given $P = (x, y, z)$, let us first find θ_3 geometrically, by projecting onto the $x_2 - y_2$ plane,

$$L \cos \theta_3 = z - H$$

$$\Rightarrow \cos \theta_3 = \frac{z - H}{L}$$

$$\theta_3 = \arctan 2 \left(\pm \sqrt{1 - \cos^2 \theta_3}, \cos \theta_3 \right) = \arctan 2 \left(\pm \sqrt{\frac{L^2 - (z - H)^2}{L^2}}, \frac{z - H}{L} \right)$$

If the desired point is in the upper half above the horizontal length of the manipulator, then $z > H$ and $\frac{z - H}{L} > 0$.

Consequently,

Note that $\theta_3 \in (-180^\circ, 180^\circ]$. There is going to be a positive solution, θ_3^+ , and a negative solution, θ_3^- .

From the same projection, we can find d_2 :

$$d_2 + L \sin \theta_3 = \sqrt{x^2 + y^2}$$

$$\Rightarrow d_2 = \sqrt{x^2 + y^2} - L \sin \theta_3$$

Note that for θ_3^+ , there is a possibility that we could get a negative d_2 , when $\sqrt{x^2 + y^2} < L \sin \theta_3$. This will be the case for points close to the center of the manipulator. We'll address this case after finding θ_1 .

By projecting onto the $x_0 - y_0$ plane,

$$\theta_1 = 90^\circ + \alpha$$

$$\alpha = \arctan 2(y, x)$$

For the special case, $\{\theta_3^+, d_2 < 0\}$, we can rotate the first joint to move the manipulator to the opposite side, i.e.

$$\alpha' = \alpha + 180^\circ$$

$$\Rightarrow \theta_1' = 90^\circ + \alpha' = 270^\circ + \alpha = -(90^\circ - \alpha)$$

Correspondingly,

$$d_2' = -\sqrt{x^2 + y^2} - L \sin \theta_3^-$$

So, the two solutions, given (x, y, z) , are:

Solution number	Conditions	θ_3	d_2	θ_1
1	None	θ_3^-	$\sqrt{x^2 + y^2} - L \sin \theta_3^-$	$90^\circ + \alpha$
2	$\sqrt{x^2 + y^2} \geq L \sin \theta_3^+$	θ_3^+	$\sqrt{x^2 + y^2} - L \sin \theta_3^+$	$90^\circ + \alpha$
2	$\sqrt{x^2 + y^2} < L \sin \theta_3^+$	θ_3^-	$-\sqrt{x^2 + y^2} - L \sin \theta_3^-$	$270^\circ + \alpha$

For the given P,

```
theta3_1 = 60.0000
theta3_2 = -60.0000
d2_1 = 1.2696
d2_2 = 4.7337
theta1 = 60.0184
```

Analytical Approach

IK for this problem may be easier with the analytical approach. We have the following equations:

$$d_2 \sin \theta_1 + L \sin \theta_1 \sin \theta_3 = x \quad (1)$$

$$-d_2 \cos \theta_1 - L \cos \theta_1 \sin \theta_3 = y \quad (2)$$

$$H + L \cos \theta_3 = z \quad (3)$$

Finding θ_3

From (3), we can easily find out θ_3 :

$$\cos \theta_3 = \frac{z - H}{L}$$

$$\theta_3 = \arctan 2 \left(\pm \sqrt{1 - \cos^2 \theta_3}, \cos \theta_3 \right) = \arctan 2 \left(\pm \sqrt{\frac{L^2 - (z - H)^2}{L^2}}, \frac{z - H}{L} \right)$$

Finding d_2

Now, let's square (1) and (2), and then add them:

$$x^2 + y^2 = d_2^2 \sin^2 \theta_1 + d_2^2 \cos^2 \theta_1 + L^2 \sin^2 \theta_3 \sin^2 \theta_1 + L^2 \sin^2 \theta_3 \cos^2 \theta_1 + 2d_2 L \sin \theta_3 \sin^2 \theta_1 + 2d_2 L \sin \theta_3 \cos^2 \theta_1$$

$$x^2 + y^2 = d_2^2 + L^2 \sin^2 \theta_3 + 2d_2 L \sin \theta_3$$

$$0 = d_2^2 + (2L \sin \theta_3) d_2 + (L^2 \sin^2 \theta_3 - x^2 - y^2)$$

This is a quadratic in d_2 , so

$$d_2 = \frac{-2L \sin \theta_3 \pm \sqrt{4L^2 \sin^2 \theta_3 - 4(L^2 \sin^2 \theta_3 - x^2 - y^2)}}{2}$$

$$d_2 = -L \sin \theta_3 \pm \sqrt{x^2 + y^2}$$

For θ_3^+ , the two possible solutions for d_2 , are:

$$d_2^1 = -L \sin \theta_3^+ + \sqrt{x^2 + y^2}$$

$$d_2^2 = -L \sin \theta_3^+ - \sqrt{x^2 + y^2}$$

But, note that $\sin \theta_3^+ > 0$, which means that $d_2^2 < 0$, and so invalid. Furthermore, d_2^1 is valid only when

$$\sqrt{x^2 + y^2} \geq L \sin \theta_3^+.$$

For θ_3^- , we have that $\sin \theta_3^- < 0$ and $\sin \theta_3^- = -\sin \theta_3^+$, and the two solutions are:

$$d_2^3 = -L \sin \theta_3^- + \sqrt{x^2 + y^2}$$

$$d_2^4 = -L \sin \theta_3^- - \sqrt{x^2 + y^2}$$

From these, d_2^4 is only valid when $\sqrt{x^2 + y^2} < L \sin \theta_3^+$. Note that this is the condition when no valid solution is obtained from θ_3^+ . So, the solutions up till this point are:

$$\begin{cases} \left(\cdot, \sqrt{x^2 + y^2} - L \sin \theta_3^+, \theta_3^+ \right) \text{ and } \left(\cdot, \sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) & \text{if } \sqrt{x^2 + y^2} \geq L \sin \theta_3^+ \\ \left(\cdot, -\sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) \text{ and } \left(\cdot, \sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) & \text{if } \sqrt{x^2 + y^2} < L \sin \theta_3^+ \end{cases}$$

Finding θ_1

Finally for θ_1 , (1) and (2) are:

$$(d_2 + L \sin \theta_3) \sin \theta_1 = x$$

$$-(d_2 + L \sin \theta_3) \cos \theta_1 = y$$

Thus,

$$\theta_1 = \begin{cases} \arctan 2(x, -y) & \text{if } d_2 + L \sin \theta_3 > 0 \\ \arctan 2(-x, y) & \text{if } d_2 + L \sin \theta_3 < 0 \end{cases}$$

and solutions are:

$$\begin{cases} \left(\arctan 2(x, -y), \sqrt{x^2 + y^2} - L \sin \theta_3^+, \theta_3^+ \right) \text{ and } \left(\arctan 2(x, -y), \sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) & \text{if } \sqrt{x^2 + y^2} \geq L \sin \theta_3^+ \\ \left(\arctan 2(-x, y), -\sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) \text{ and } \left(\arctan 2(x, -y), \sqrt{x^2 + y^2} - L \sin \theta_3^-, \theta_3^- \right) & \text{if } \sqrt{x^2 + y^2} < L \sin \theta_3^+ \end{cases}$$

Question (d) Jacobian

We wish to move the point P at a velocity (0.5,0.5,-0.5) m/s. At what rate will we have to move the joints?

We can find J_v by differentiating $P(t)$:

$$\mathcal{J}(t) =$$

$$\begin{pmatrix} \cos(\theta_1(t)) d_2(t) + L \cos(\theta_1(t)) \sin(\theta_3(t)) & \sin(\theta_1(t)) & L \cos(\theta_3(t)) \sin(\theta_1(t)) \\ \sin(\theta_1(t)) d_2(t) + L \sin(\theta_1(t)) \sin(\theta_3(t)) & -\cos(\theta_1(t)) & -L \cos(\theta_1(t)) \cos(\theta_3(t)) \\ 0 & 0 & -L \sin(\theta_3(t)) \end{pmatrix}$$

To answer this question we can find the joint velocities at the current configuration, given the end-effector velocities by the expression $\dot{q} = J^{-1}(\theta_1, d_2, \theta_3) v$. Substituting the values of the parameters and $(\theta_1, d_2, \theta_3)$ obtained from part (c),

$J_n(t) =$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{13}{20} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{13\sqrt{3}}{20} + \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\sqrt{3} \end{pmatrix}$$

Now, let's invert this matrix and multiply it with the provided velocity vector $[0.5, 0.5, -0.5]^T$:

```
ans = 3x1
    0.2253
   -0.1057
    0.2887
```

As we can see, the joint rates of the first and third revolute joints are less than 0.3, and so the specified velocity is achievable.

Question (e) Angular Velocity

If the point P is moving at the velocity specified in the previous part, does it have a non-zero angular velocity? If yes, find the velocity, and if no, provide justification.

Yes, it has an angular velocity, which will be the same as the angular velocity of the last link. Since it is a chain of rigid bodies, we can find the angular velocity of last link by taking the vector sum of frame-frame angular velocities, i.e.

$$\omega_{03}^0 = \omega_{01}^0 + \omega_{12}^0 + \omega_{23}^0$$

$$\omega_{03}^0 = \dot{\theta}_1 z_0^0 + 0 + \dot{\theta}_3 z_2^0$$

Let's now find the entries in this expression. We know that $z_0^0 = [0 \ 0 \ 1]^T$. We can find z_2^0 by finding third column of T_{02} :

$T_{02}(t) =$

$$\begin{pmatrix} 0 & \sin(\theta_1(t)) & \cos(\theta_1(t)) & \sin(\theta_1(t)) d_2(t) \\ 0 & -\cos(\theta_1(t)) & \sin(\theta_1(t)) & -\cos(\theta_1(t)) d_2(t) \\ 1 & 0 & 0 & H \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\omega_{03}^0 = \dot{\theta}_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\theta}_3 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \end{bmatrix}$$

Substituting $\theta_1 = 60^\circ$ from (c) and $(\dot{\theta}_1, \dot{\theta}_3) = (0.22, 0.29)$ from (d), we have:

$$\begin{aligned} \mathbf{w}_3 &= \begin{bmatrix} 3 \times 1 \\ 0.1450 \\ 0.2511 \\ 0.2200 \end{bmatrix} \end{aligned}$$

Question (f) Singularities

Find the singular configurations (if any) for this manipulator and argue about the velocity directions that cannot be instantaneously achieved in those configurations.

We can find singular configurations by finding determinant of Jacobian and equating it to zero. The determinant is:

$$\det(\mathbf{J}) = L^2 \sin^2(\theta_3(t)) + d_2(t) L \sin(\theta_3(t))$$

Since $L \neq 0$, this determinant will be zero under one of two conditions:

1. $\theta_3 = 0$ or $\theta_3 = 180^\circ$: At this configuration, it is not possible to instantaneously move vertically in the z_0 direction.
2. $d_2 = -L \sin \theta_3$: This translates to the point P being on the z_0 axis. At this singularity, the manipulator cannot move instantaneously orthogonal to the prismatic joint or in the x_1 direction. This can also be seen by rewriting the Jacobian in the coordinates of frame 1, as shown below. In this case we can see that velocity in the \dot{x}_1 direction is zero irrespective of the joint velocities if $d_2 + L \sin \theta_3 = 0$.

$$\det(\mathbf{J}) =$$

$$\begin{pmatrix} d_2(t) + L \sin(\theta_3(t)) & 0 & 0 \\ 0 & 0 & -L \sin(\theta_3(t)) \\ 0 & 1 & L \cos(\theta_3(t)) \end{pmatrix}$$