Homework 5

Table of Contents

Problem 1

Use the method of splitting a trajectory into multiple segments to find a trajectory that satisfies the following conditions:

Solution:

The trajectory has four segments: (i) from t=0 to t=2; (ii) t=2 to t=4; (iii) t=4 to t=7.5; (iv) t=7.5 to t=10. Given that we're provided four constraints for the first segment, we can use a cubic polynomial for it. So, let

$$q(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0, \qquad \forall t \in [0, 2].$$

Incorporating the constraints, we have:

$$q(0) = 5^\circ \quad \Rightarrow \quad a_0 = 5.$$

$$q'(0) = 0 \quad \Rightarrow \quad a_1 = 0$$
$$q''(0) = 0 \quad \Rightarrow \quad a_2 = 0$$

$$q(2) = 15 \quad \Rightarrow \quad 8a_3 + 4a_2 + 2a_1 + a_0 = 8a_3 + 0 + 0 + 5 = 15 \quad \Rightarrow \quad a_3 = 5/4.$$

First Segment

```
syms t a_3 a_2 a_1 a_0
```

```
q1(t) = a_3*t^3+a_2*t^2+a_1*t+a_0
```

q1(t) = $a_3\,t^3 + a_2\,t^2 + a_1\,t + a_0$

```
v1(t) = diff(q1(t),t);
a1(t) = diff(v1(t),t);
coeff1 = solve({q1(0)==5, q1(2)==15, v1(0)==0, a1(0)==0},[a_0, a_1, a_2, a_3])
```

```
coeff1 = struct with fields:
    a_0: 5
    a_1: 0
    a_2: 0
    a_3: 5/4
```

```
q1(t) = subs(q1(t),[a_0, a_1, a_2, a_3],[coeff1.a_0, coeff1.a_1, coeff1.a_2, coeff1.a_3])
```

q1(t) =

$$\frac{5\,t^3}{4} + 5$$

```
v1(t) = subs(v1(t),[a_0, a_1, a_2, a_3],[coeff1.a_0, coeff1.a_1, coeff1.a_2, coeff1.a_3])
```

v1(t) =

$$\frac{15\,t^2}{4}$$

```
a1(t) = subs(a1(t),[a_0, a_1, a_2, a_3],[coeff1.a_0, coeff1.a_1, coeff1.a_2, coeff1.a_3])
```

a1(t) =

$$\frac{15\,t}{2}$$

Fourth Segment

The constraints are:

$q(7.5) = 65$

$q(10) = 100$

$q'(10 = 0$

$q''(10) = 0$

```
syms t a_3 a_2 a_1 a_0
q4(t) = a_3*t^3+a_2*t^2+a_1*t+a_0
```

q4(t) = $a_3\,t^3 + a_2\,t^2 + a_1\,t + a_0$

```
v4(t) = diff(q4(t),t);
a4(t) = diff(v4(t),t);
coeff4 = solve({q4(7.5)==65, q4(10)==100, v4(10)==0, a4(10)== 0},[a_0, a_1, a_2, a_3])
```

```
coeff4 = struct with fields:
    a_0: -2140
    a_1: 672
```

```
    a_2: -336/5
    a_3: 56/25
```

Thus, the polynomial is:

```
q4(t) = subs(q4(t),[a_0, a_1, a_2, a_3],[coeff4.a_0, coeff4.a_1, coeff4.a_2, coeff4.a_3])
```

q4(t) =

$$\frac{56\,t^3}{25} - \frac{336\,t^2}{5} + 672\,t - 2140$$

```
v4(t) = subs(v4(t),[a_0, a_1, a_2, a_3],[coeff4.a_0, coeff4.a_1, coeff4.a_2, coeff4.a_3])
```

v4(t) =

$$\frac{168\,t^2}{25} - \frac{672\,t}{5} + 672$$

```
a4(t) = subs(a4(t),[a_0, a_1, a_2, a_3],[coeff4.a_0, coeff4.a_1, coeff4.a_2, coeff4.a_3])
```

a4(t) =

$$\frac{336\,t}{25} - \frac{672}{5}$$

Second Segment

There are only two provided constraints for this segment:

$$q(2) = 15$$
$$q(4) = 35$$

We can achieve this by a line, but let's use a cubic polynomial for smoothness.

```
syms t a_3 a_2 a_1 a_0 c
q2(t) = a_3*t^3+a_2*t^2+a_1*t+a_0
```

q2(t) = $a_3\,t^3 + a_2\,t^2 + a_1\,t + a_0$

We'll use extra constraints to ensure that the velocity is also continuous. So, the constraints are:

$$q_2'(2) = q_1'(2)$$
$$q_2'(4) = q_3'(4)$$

```
v2(t) = diff(q2(t),t);
a2(t) = diff(v2(t),t);
coeff2 = solve({q2(2)==15, q2(4)==35, v2(2)==v1(2), a_3==c},[a_0, a_1, a_2, a_3])
```

```
coeff2 = struct with fields:
    a_0: - 16*c - 25
    a_1: 20*c + 25
    a_2: - 8*c - 5/2
    a_3: c
```

Thus, the polynomial is:

```
q2(t) = subs(q2(t),[a_0, a_1, a_2, a_3],[coeff2.a_0, coeff2.a_1, coeff2.a_2, c])
```

q2(t) =

$$c\,t^3+\left(-8\,c-\frac{5}{2}\right)t^2+(20\,c+25)\,t-16\,c-25$$

```
v2(t) = subs(v2(t),[a_0, a_1, a_2, a_3],[coeff2.a_0, coeff2.a_1, coeff2.a_2, c])
```

v2(t) =

$$20\,c+3\,c\,t^2-2\,t\left(8\,c+\frac{5}{2}\right)+25$$

```
a2(t) = subs(a2(t),[a_0, a_1, a_2, a_3],[coeff2.a_0, coeff2.a_1, coeff2.a_2, c])
```

a2(t) = $6\,c\,t-16\,c-5$

Third Segment

There are again only two constraints for this segment:

$$q(4)=35$$
$$q(7.5)=65$$

```
syms t a_3 a_2 a_1 a_0
q3(t) = a_3*t^3+a_2*t^2+a_1*t+a_0
```

q3(t) = $a_3\,t^3+a_2\,t^2+a_1\,t+a_0$

The remaining two constraints could be imposed in terms of continuity of the velocity and acceleration, i.e.

$$q_3'(4)=q_2'(4)$$
$$q_3''(4)=q_2''(4)$$

```
v3(t) = diff(q3(t),t);
a3(t) = diff(v3(t),t);
coeff3 = solve({q3(4)==35, q3(7.5)==65, v3(7.5)== v4(7.5), v3(4) == v2(4)},[a_0, a_1, a_2, a_3
```

```
coeff3 = struct with fields:
    a_0: - (3600*c)/49 - 89575/343
    a_1: (1860*c)/49 + 62451/343
    a_2: - (304*c)/49 - 12608/343
    a_3: (16*c)/49 + 836/343
```

Thus, the polynomial is:

```
q3(t) = subs(q3(t),[a_0, a_1, a_2, a_3],[coeff3.a_0, coeff3.a_1, coeff3.a_2, coeff3.a_3])
```

q3(t) =

$$\left(\frac{16\,c}{49}+\frac{836}{343}\right)t^3+\left(-\frac{304\,c}{49}-\frac{12608}{343}\right)t^2+\left(\frac{1860\,c}{49}+\frac{62451}{343}\right)t-\frac{3600\,c}{49}-\frac{89575}{343}$$

Note that we're now left with one unknown. All our requirements are satsified for any value of c. So let's choose it to be zero.

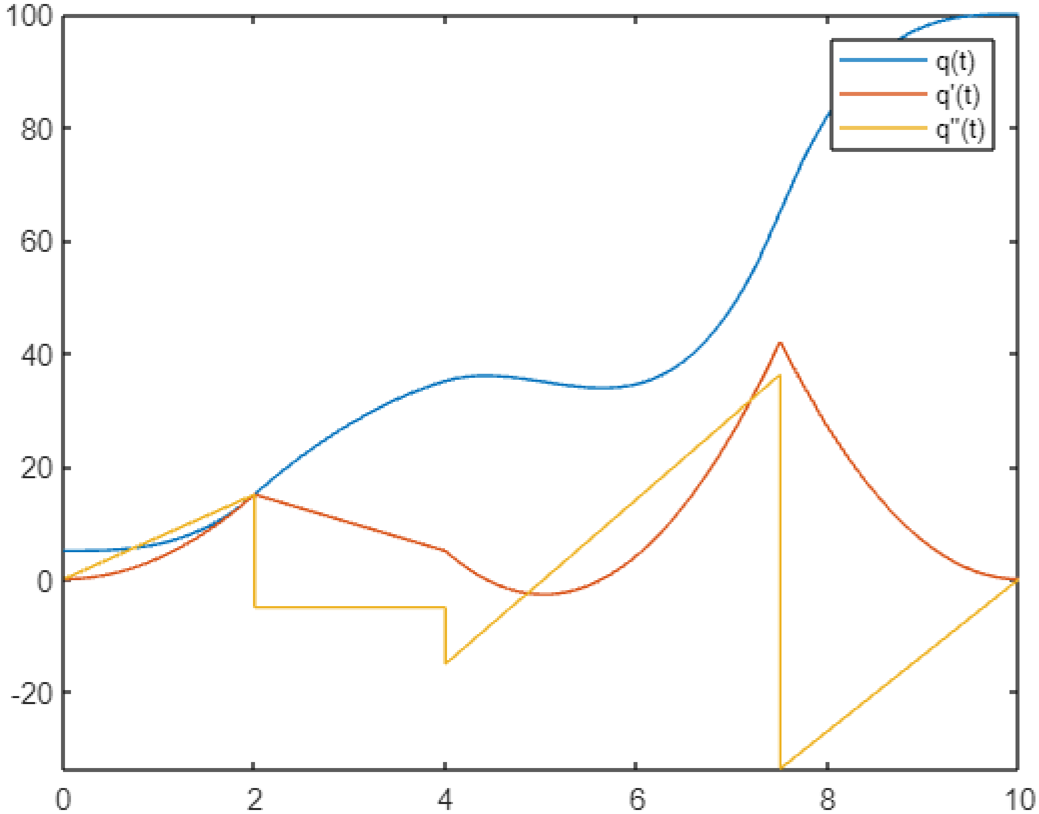
Complete trajectory

```
q2(t) = subs(q2(t),c,0);
q3(t) = subs(q3(t),c,0);
syms q(t)
q(t) = piecewise(0 <= t <2, q1(t), 2<= t < 4, q2(t), 4<= t< 7.5, q3(t), 7.5<= t <= 10, q4(t))
```

q(t) =

$$\begin{cases} \frac{5\,t^3}{4}+5 & \text{if } t\in[0,2) \\ -\frac{5\,t^2}{2}+25\,t-25 & \text{if } t\in[2,4) \\ \frac{836\,t^3}{343}-\frac{12608\,t^2}{343}+\frac{62451\,t}{343}-\frac{89575}{343} & \text{if } t\in\left[4,\frac{15}{2}\right) \\ \frac{56\,t^3}{25}-\frac{336\,t^2}{5}+672\,t-2140 & \text{if } t\in\left[\frac{15}{2},10\right] \end{cases}$$

```
h(1) = fplot(q,[0,10]);
hold on;
h(2) = fplot(diff(q(t),t),[0,10]);
h(3) = fplot(diff(diff(q(t),t),t),[0,10]);
legend(h, 'q(t)', "q'(t)", "q''(t)");
hold off;
```



Problem 2

Solution 2(a)

The path is comprised of two semicircular segments. The first segment is from point P1 = (1.5,0.5) to the point on the y-axis (0,0.5). This can be expressed as:

$$X_1(s) = 0.75\cos s + 0.75$$
$$Y_1(s) = -0.75\sin s + 0.5$$

where $s \in [0, \pi]$.

Similarly, the second segment of the path is:

$$X_2(s) = 0.75\cos(s - \pi) - 0.75$$
$$Y_2(s) = 0.75\sin(s - \pi) + 0.5,$$

where $s \in [\pi, 2\pi]$.

Let's verify that the path is correct by plotting it:

```
syms X(s) Y(s)
X(s) = piecewise(0<= s <= pi, 0.75*cos(s)+.75,pi<= s<= 2*pi,.75*cos(s-pi)-.75)
```

```
X(s) =
```

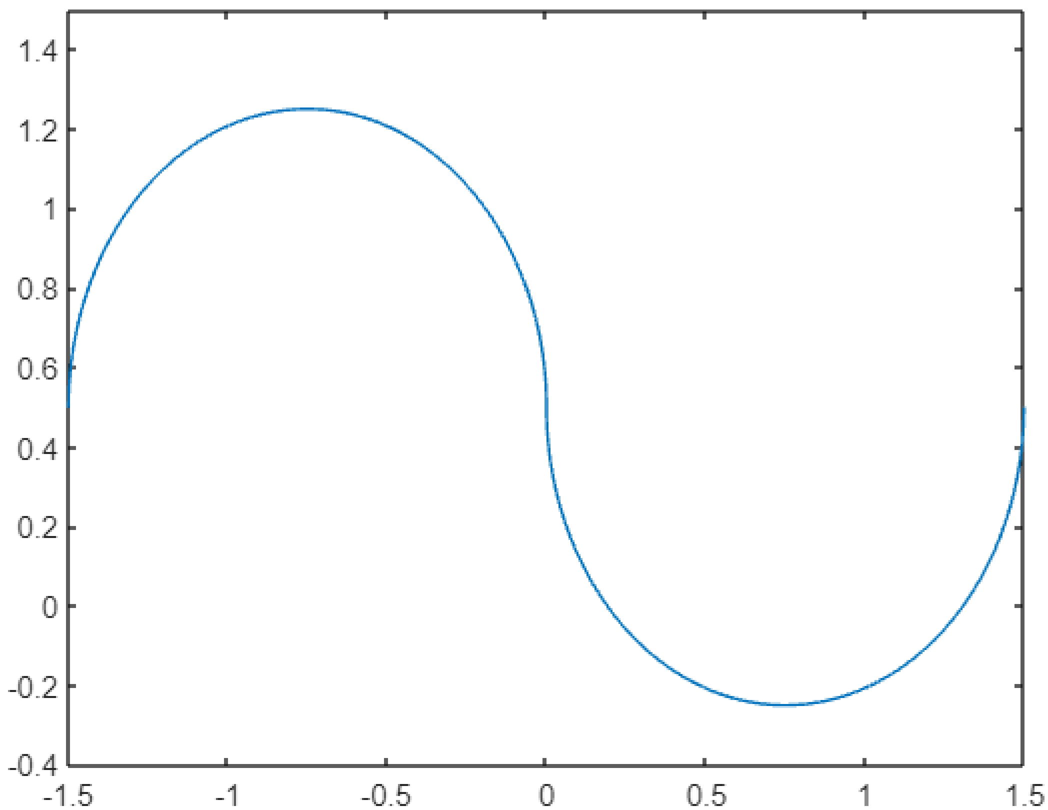
$$X(s) = \begin{cases} \frac{3\cos(s)}{4} + \frac{3}{4} & \text{if } s \in [0, \pi] \\ -\frac{3\cos(s)}{4} - \frac{3}{4} & \text{if } s \in [\pi, 2\pi] \end{cases}$$

```
Y(s) = piecewise(0<= s <= pi, -0.75*sin(s)+.5,pi<= s<= 2*pi,.75*sin(s-pi)+.5)
```

```
Y(s) =
```

$$Y(s) = \begin{cases} \frac{1}{2} - \frac{3\sin(s)}{4} & \text{if } s \in [0, 2\pi] \end{cases}$$

```
fplot(X,Y,[0,2*pi])
axis([-1.5 1.5 -0.4 1.5])
```



To convert it to a trajectory we need s(t). Let it be written as two line segments, i.e.

$s_1(t) = a_1 t + a_0,$

where $t \in [0, T/2]$ and T is the total time of the trajectory. Since the two segments are identical and velocity is constant, one segment will require $T/2$ time. The second segment will be defined in exactly the same way. We have the constraints that $s_1(0) = 0$ and $s_1(T/2) = \pi$. So,

$s_1(t) = \frac{2\pi}{T} t.$

The second segment is $s_2(t) = a_1 t + a_0,$ over $t \in [T/2, T]$. The constraints are that $s_2(T/2) = \pi$ and $s_2(T) = 2\pi$. The coefficients can be determined as:

```
syms a_1 a_0 t T;
s2(t) = a_1*t+a_0;
coeff = solve({s2(T/2)==pi, s2(T)==2*pi},[a_0, a_1]);
s2(t) = subs(s2(t),[a_0,a_1],[coeff.a_0,coeff.a_1])
```

```
s2(t) =
```

$$\frac{2\,\pi\,t}{T}$$

Thus,

$s(t) = \frac{2\pi}{T} t$, for $t \in [0, T]$.

The final constraint is that the velocity of trajectory should be 1, i.e.

$$\sqrt{\dot{X}^2(t) + \dot{Y}^2(t)} = 1, \qquad \forall t \in [0, T].$$

We have that

$$X(t) = \begin{cases} \frac{3}{4}\left[1 + \cos\left(\frac{2\pi t}{T}\right)\right] & \text{if} \quad t \in [0, T/2] \\ -\frac{3}{4}\left[1 + \cos\left(\frac{2\pi t}{T}\right)\right] & \text{if} \quad t \in [T/2, T] \end{cases}$$

and

$$Y(t) = \frac{1}{2} - \frac{3}{4}\sin\left(\frac{2\pi t}{T}\right), \quad \forall t \in [0, T].$$

Correspondingly,

$$\dot{X}(t) = \begin{cases} -\frac{3\pi}{2T}\sin\left(\frac{2\pi t}{T}\right) & \text{if} \quad t \in [0, T/2] \\ \frac{3\pi}{2T}\sin\left(\frac{2\pi t}{T}\right) & \text{if} \quad t \in [T/2, T] \end{cases}$$

and

$$\dot{Y}(t) = -\frac{3\pi}{2T}\cos\left(\frac{2\pi t}{T}\right), \quad \forall t \in [0, T].$$

Thus,

$$\sqrt{\dot{X}^2(t)+\dot{Y}^2(t)} = \frac{3\pi}{2T} = 1 \quad \Rightarrow \quad T = \frac{3\pi}{2}.$$

The final trajectory is:

$$X(t) = \begin{cases} \frac{3}{4}\left[1+\cos\left(\frac{4t}{3}\right)\right] & \text{if} \quad t \in [0, 3\pi/4] \\ -\frac{3}{4}\left[1+\cos\left(\frac{4t}{3}\right)\right] & \text{if} \quad t \in [3\pi/4, 3\pi/2] \end{cases}$$

$$Y(t) = \frac{1}{2} - \frac{3}{4}\sin\left(\frac{4t}{3}\right), \quad \forall t \in [0, 3\pi/2].$$

```
syms t X(t) Y(t);
Y(t) = 1/2-3/4*sin(4*t/3);
X(t) = piecewise(0<=t<=3*pi/4,3/4+3/4*cos(4*t/3), 3*pi/4<=t<=3*pi/2,-3/4-3/4*cos(4*t/3));
```

Solution 2(b)

T has already been found in (a).

Solution 2(c)

We know closed-form solution for IK of this manipulator from the slides. The elbow-down solution is

$$\theta_1(t) = \arctan 2(Y(t), X(t)) - \arccos\left(\frac{l_1^2+X^2+Y^2-l_2^2}{2l_1\sqrt{X^2+Y^2}}\right)$$

$$\theta_2(t) = \pi - \arccos\left(\frac{l_1^2+l_2^2-X^2-Y^2}{2l_1l_2}\right).$$

The elbow-up solution is:

$$\theta_1(t) = \arctan 2(Y(t), X(t)) + \arccos\left(\frac{l_1^2+X^2+Y^2-l_2^2}{2l_1\sqrt{X^2+Y^2}}\right)$$

$$\theta_2(t) = -\pi + \arccos\left(\frac{l_1^2+l_2^2-X^2-Y^2}{2l_1l_2}\right).$$

Let's try to use the elbow-down solution. Substituting $X(t), Y(t)$ from the previous part in this expression:

```
syms t theta_1(t) theta_2(t)
l1 = 1;
l2 = 1;
theta_1(t) = atan2(Y(t),X(t)) - acos((l1^2+X(t)^2+Y(t)^2-l2^2)/(2*l1*sqrt(X(t)^2+Y(t)^2)))
```

```
theta_1(t) =
```

$$\begin{cases} -\sigma_2 + \mathrm{angle}\left(\sigma_3 - \sigma_1 + \frac{3}{4} + \frac{1}{2}\,\mathrm{i}\right) & \text{if } t \in \left[0, \frac{3\,\pi}{4}\right] \\ -\sigma_2 + \mathrm{angle}\left(-\sigma_3 - \sigma_1 - \frac{3}{4} + \frac{1}{2}\,\mathrm{i}\right) & \text{if } t \in \left[\frac{3\,\pi}{4}, \frac{3\,\pi}{2}\right] \end{cases}$$

where

$$\sigma_1 = \frac{3\,\sin\left(\frac{4\,t}{3}\right)\,\mathrm{i}}{4}$$

$$\sigma_2 = \mathrm{acos}\left(\frac{\sqrt{\left(\sigma_3 + \frac{3}{4}\right)^2 + \left(\frac{3\,\sin\left(\frac{4\,t}{3}\right)}{4} - \frac{1}{2}\right)^2}}{2}\right)$$

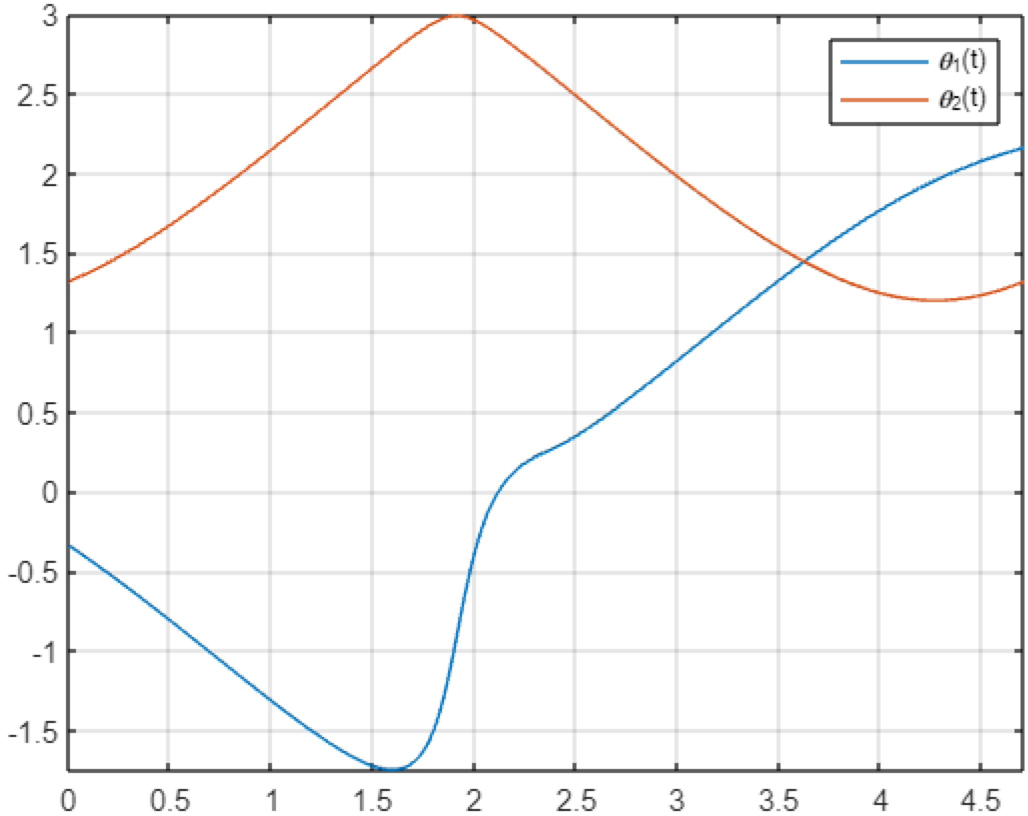
$$\sigma_3 = \frac{3\,\cos\left(\frac{4\,t}{3}\right)}{4}$$

```
theta_2(t) = pi-acos((l1^2+l2^2-X(t)^2-Y(t)^2)/(2*l1*l2))
```

```
theta_2(t) =
```

$$\begin{cases} \pi - \mathrm{acos}\left(1 - \frac{\left(\frac{3\,\sin\left(\frac{4\,t}{3}\right)}{4} - \frac{1}{2}\right)^2}{2} - \frac{\left(\frac{3\,\cos\left(\frac{4\,t}{3}\right)}{4} + \frac{3}{4}\right)^2}{2}\right) & \text{if } t \in \left[0, \frac{3\,\pi}{2}\right] \end{cases}$$

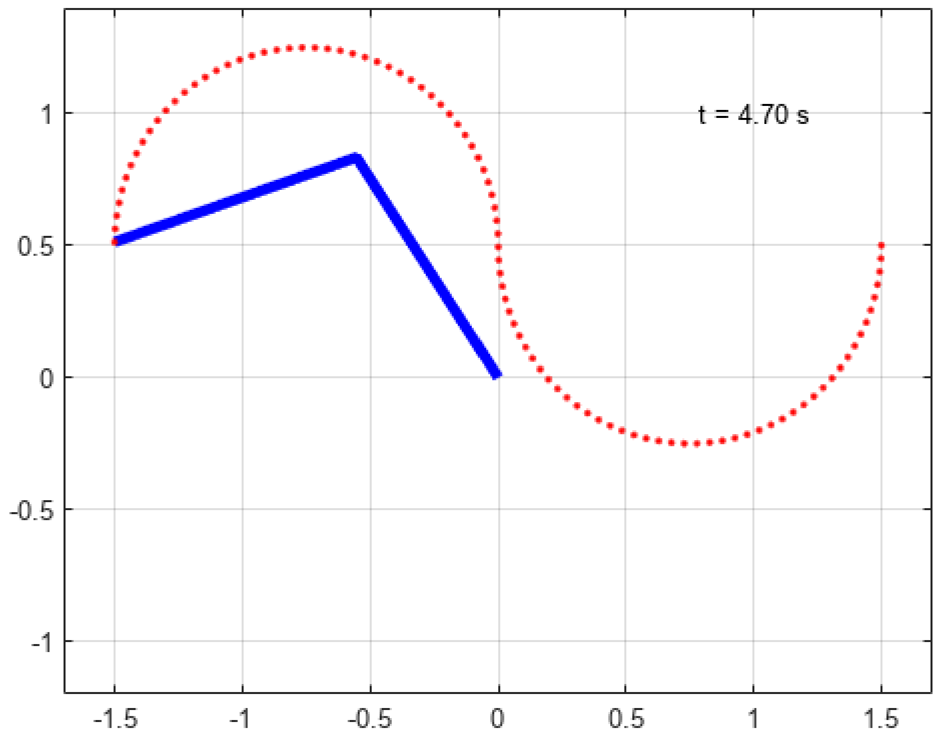
```
clear h;
h(1) = fplot(theta_1,[0,3*pi/2]);
hold on;
h(2) = fplot(theta_2,[0,3*pi/2]);
xlim([0.00 4.71]);
ylim([-1.75 3.00]);
grid on;
legend(h,'\theta_1(t)','\theta_2(t)');
```



Animation for the manipulator tracing the trajectory

```
ti = 0:0.05:3*pi/2;
x1 = l1*cos(theta_1(ti));
x2 = x1+l2*cos(theta_1(ti)+theta_2(ti));
y1 = l1*sin(theta_1(ti));
y2 = y1+l2*sin(theta_1(ti)+theta_2(ti));

for i = 1:length(ti)
    if i == 1
        figure;
        % Plot the manipulator at its current configuration and store its
        % handle.
        hrobot = plot([0 x1(i) x2(i)], [0 y1(i) y2(i)], 'LineWidth',4, 'Color', 'blue');
        % Plot the tip of the robot and store the handle.
        hold on;
        htip = plot(x2(i),y2(i),'r.');
        hold off;
        % Turn on the grid and box
        grid on;
        box on;
        % Set axis limits
        axis([-1.7 1.7 -1.2 1.4])
```

```
        % Put text on the plot to show how much time has elapsed.  The text
        % is centered.
        htime = text(1,1,1,sprintf('t = %.2f s',ti(i)),'horizontalAlignment','center');
    else
        set(hrobot,'xdata',[0 x1(i) x2(i)],'ydata',[0 y1(i) y2(i)]);
        set(htip,'xdata',x2(1:i),'ydata',y2(1:i))
        set(htime,'string', (sprintf('t = %.2f s',ti(i))));
    end
    pause(0.1);
end
```



Solution 2(d)

Let's differentiate $\theta_1(t)$ and $\theta_2(t)$ to obtain the joint angular velocities:

```
syms w1(t) w2(t)
w1(t) = simplify(expand(diff(theta_1(t),t)))
```

w1(t) =

$$\begin{cases} -\dfrac{2\sqrt{3}\left(22\,\sigma_8+33\,\sigma_9+\sigma_1+36\,{\sigma_8}^2+\sigma_5+\sigma_3-\sigma_2-18\right)}{\sigma_4} & \text{if } t\in\left(0,\dfrac{3\,\pi}{4}\right) \\ -\dfrac{2\sqrt{3}\left(22\,\sigma_8+33\,\sigma_9+\sigma_1+36\,{\sigma_8}^2-\sigma_5-\sigma_3+\sigma_2-18\right)}{\sigma_4} & \text{if } t\in\left(\dfrac{3\,\pi}{4},\dfrac{3\,\pi}{2}\right) \end{cases}$$

where

$$\sigma_1=\frac{15\sin\left(\frac{8\,t}{3}\right)}{2}$$

$$\sigma_2=2\,\sqrt{3}\,\sigma_9\,\sigma_6\,\sigma_7$$

$$\sigma_3=3\,\sqrt{3}\,\sigma_8\,\sigma_6\,\sigma_7$$

$$\sigma_4=3\,\sigma_6\,(9\,\sigma_8-6\,\sigma_9+11)^{3/2}$$

$$\sigma_5=3\,\sqrt{3}\,\sigma_6\,\sigma_7$$

$$\sigma_6=\sqrt{2\,\sigma_9-3\,\sigma_8+7}$$

$$\sigma_7=\sqrt{9\,\sigma_8-6\,\sigma_9+11}$$

$$\sigma_8=\cos\left(\frac{4\,t}{3}\right)$$

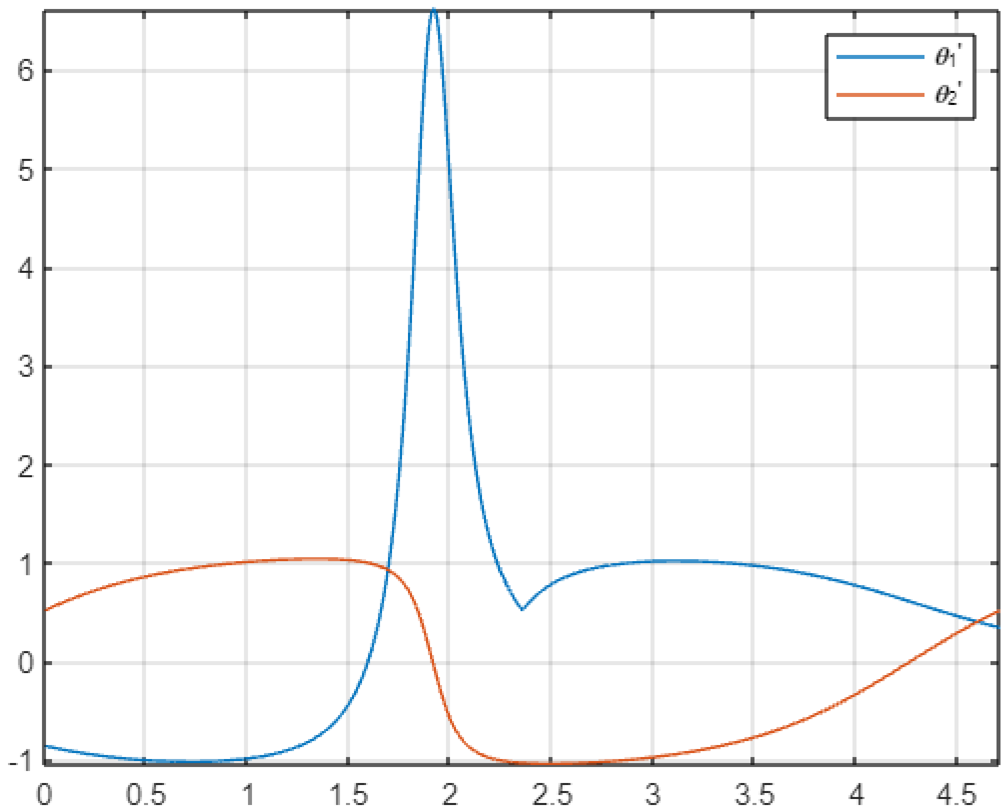
$$\sigma_9=\sin\left(\frac{4\,t}{3}\right)$$

```
w2(t) = simplify(diff(theta_2(t),t))
```

w2(t) =

$$\begin{cases} \dfrac{4\,\sqrt{13}\,\sin\left(\frac{4\,t}{3}+\operatorname{atan}\left(\frac{2}{3}\right)\right)}{\sqrt{256-\left(6\sin\left(\frac{4\,t}{3}\right)-9\cos\left(\frac{4\,t}{3}\right)+5\right)^2}} & \text{if } t\in\left(0,\dfrac{3\,\pi}{2}\right) \end{cases}$$

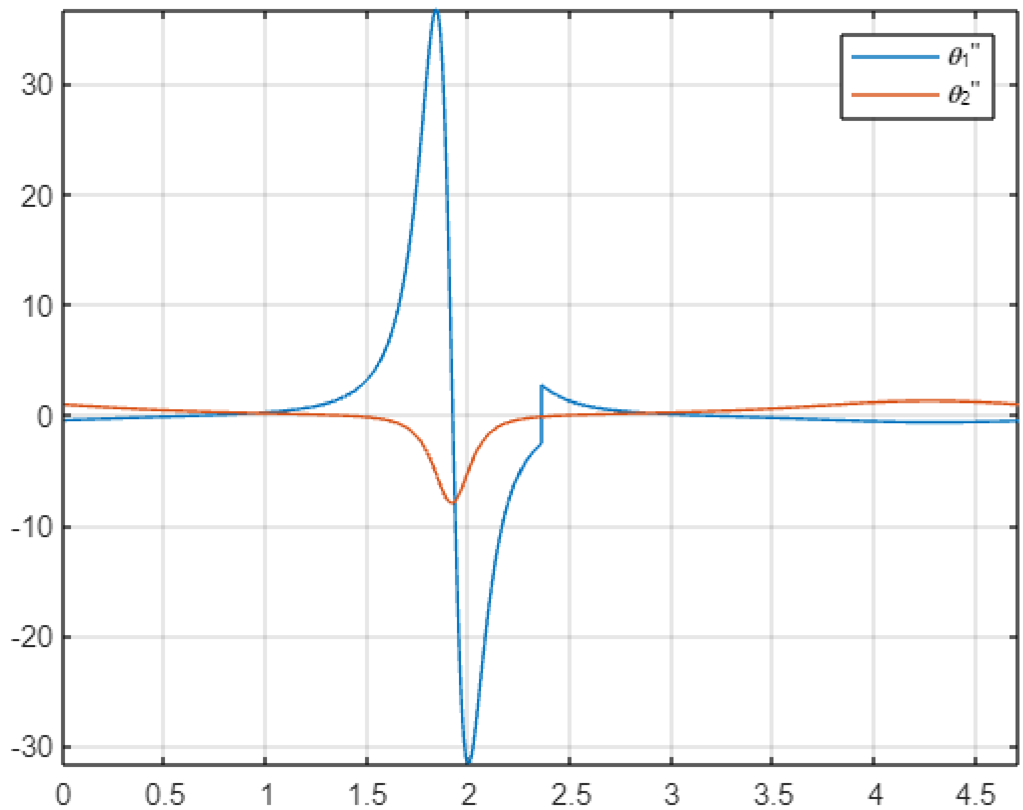
```
clear h;
figure;
h(1) = fplot(w1,[0,3*pi/2]);
hold on;
h(2) = fplot(w2,[0,3*pi/2]);
grid on;
hl = legend(h,"\theta_1'","\theta_2'");
```



We can see from the speed plot that of joint 1 that there is a spike. This spike happens at the time when the manipulator is switching from elbow-down to elbow-up solution. If the actuator limits don't allow this speed then the trajectory's speed will have to be reduced. We can determine the value and position of the maximum from the plot:

Solution 2(e)

```
syms a1(t) a2(t)
a1(t) = diff(w1(t),t);
a2(t) = diff(w2(t),t);
clear h;
figure;
h(1) = fplot(a1,[0,3*pi/2]);
hold on;
h(2) = fplot(a2,[0,3*pi/2]);
grid on;
hl = legend(h,"\theta_1''","\theta_2''");
```



Problem 3

Solution (a)

The general model of a manipulator as provided in the slides is:

$$J(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + B_v\,\dot{q} + g(q) = \tau.$$

For the 2R manipulator, the mass-matrix, coriollis matrix, and gravity terms are:

$$J(q) = \begin{pmatrix} m_1 l_1^2 + m_2(l_1^2 + 2l_1 l_2 \cos q_2 + l_2^2) & m_2(l_1 l_2 \cos q_2 + l_2^2) \\ m_2(l_1 l_2 \cos q_2 + l_2^2) & m_2 l_2^2 \end{pmatrix}$$

$$C(q,\dot{q}) = \begin{pmatrix} -2m_2 l_1 l_2 \sin q_2\;\dot{q}_2 & \dot{q}_2 \\ m_2 l_1 l_2 \sin q_2\;\dot{q}_1 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} (m_1 + m_2) l_1 g \cos q_1 + m_2 g l_2 \cos(q_1 + q_2) \\ m_2 g a_2 \cos(q_1 + q_2) \end{pmatrix}$$

Let's create these matrices numerically with the provided parameter values. Since the friction values are not provided, let's ignore those terms.

```
m1=1;
m2=1;
```

```
l1=1;
l2=1;
g=9.8;
J(t) = [m1*l1^2+m2*(l1^2+2*l1*l2*cos(theta_2(t))+l2^2), m2*(l1*l2*cos(theta_2(t))+l2^2);
    m2*(l1*l2*cos(theta_2(t))+l2^2), m2*l2^2];
C(t) = [-2*m2*l1*l2*sin(theta_2(t))*w2(t), w2(t);
    m2*l1*l2*sin(theta_2(t))*w1(t),  0];
G(t) = [(m1+m2)*l1*g*cos(theta_1(t))+m2*g*l2*cos(theta_1(t)+theta_2(t));
    m2*g*l2*cos(theta_1(t)+theta_2(t))];
```

So the required torque by each joint to execute our desired trajectory, as a function of time, is computed below:

```
tau(t) = simplify(J(t)*[a1(t);a2(t)]+C(t)*[w1(t);w2(t)]+G(t))
```

```
tau(t) =
```

$$\left(\begin{cases} \frac{98\cos(\sigma_{42}+\sigma_{44})}{5} - \sigma_{11} + \sigma_9 - \sigma_{12} + \frac{\sigma_4 + 344224\sqrt{3}\,\sigma_{53} + 270480\sqrt{3}\,\sigma_{46} + \sigma_1 - 394095\sqrt{3} + \sigma_6 -}{} \\ \frac{98\sin(\sigma_{42}+\sigma_{41})}{5} - \sigma_{10} + \sigma_9 - \sigma_{12} + \frac{\sigma_4 + 344224\sqrt{3}\,\sigma_{53} + 270480\sqrt{3}\,\sigma_{46} + \sigma_1 - 394095\sqrt{3} + \sigma_6 -}{} \end{cases}\right.$$

$$\begin{cases} \frac{\sigma_5 + 247040\sqrt{3}\,\sigma_{53} + 205584\sqrt{3}\,\sigma_{46} + \sigma_1 - 278031\sqrt{3} + \sigma_7 - 370560\sqrt{3}\,\sigma_{52} - 8}{\sigma_{22}} \\ \frac{\sigma_5 + 247040\sqrt{3}\,\sigma_{53} + 205584\sqrt{3}\,\sigma_{46} + \sigma_1 - 278031\sqrt{3} + \sigma_7 - 370560\sqrt{3}\,\sigma_{52} - 8}{\sigma_{22}} \end{cases}$$

where

$$\sigma_1 = 5400\sqrt{3}\,\sin\left(\frac{16\,t}{3}\right)$$

$$\sigma_2 = 5355\sqrt{3}\,\cos\left(\frac{16\,t}{3}\right)$$

$$\sigma_3 = \left(\frac{3\,\pi}{4}, \frac{3\,\pi}{2}\right)$$

$$\sigma_4 = 72864\sqrt{3}\,\sin(4\,t)$$

$$\sigma_5 = 61824\sqrt{3}\,\sin(4\,t)$$

$$\sigma_6 = 14256\sqrt{3}\,\cos(4\,t)$$

$$\sigma_7 = 12096\sqrt{3}\,\cos(4\,t)$$

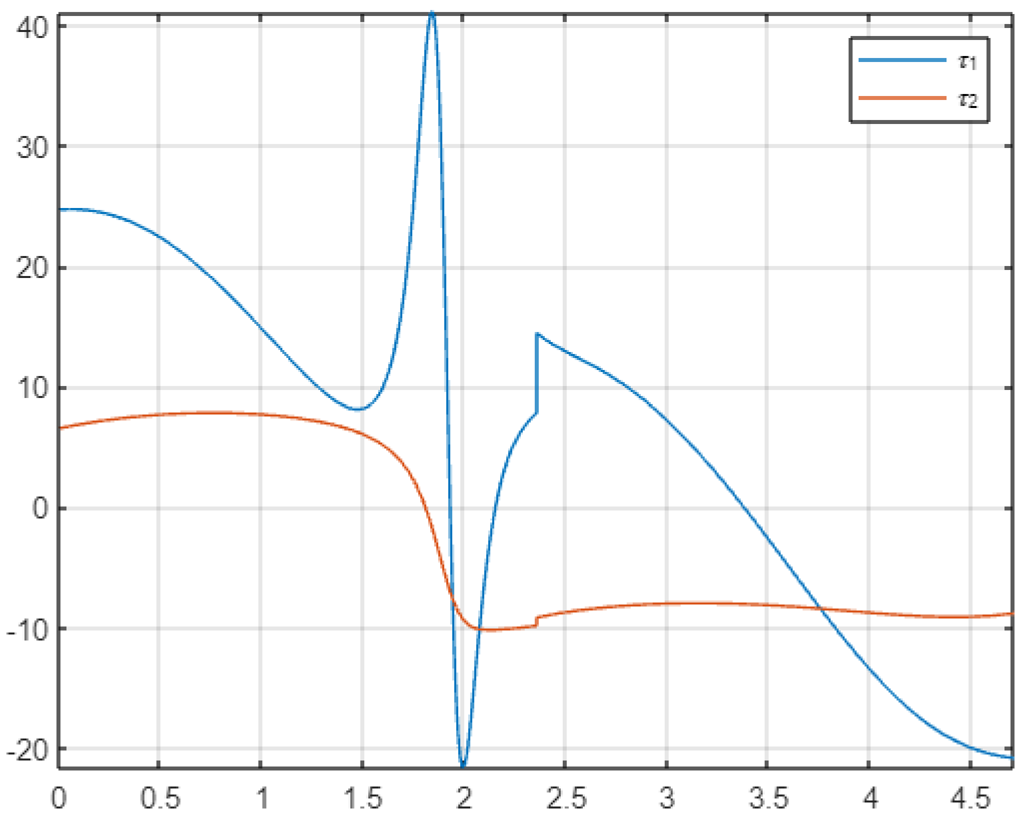
$$\sigma_8 = \left(0, \frac{3\,\pi}{4}\right)$$

$$\sigma_9 = (\sigma_{40} + \sigma_{39})\left(\frac{9\,\sigma_{52}}{16} - \frac{3\,\sigma_{53}}{8} + \frac{11}{16}\right)$$

$$\sigma_{10} = \frac{49\sin(\sigma_{42} + \sigma_{41} + \sigma_{43})}{5}$$

$$\sigma_{11} = \frac{49\cos(\sigma_{42} + \sigma_{44} + \sigma_{43})}{5}$$

```
torque = tau(t);
clear h;
figure;
h(1) = fplot(torque(1),[0,3*pi/2]);
hold on;
h(2) = fplot(torque(2),[0,3*pi/2]);
grid on;
hl = legend(h,"\tau_1","\tau_2");
```



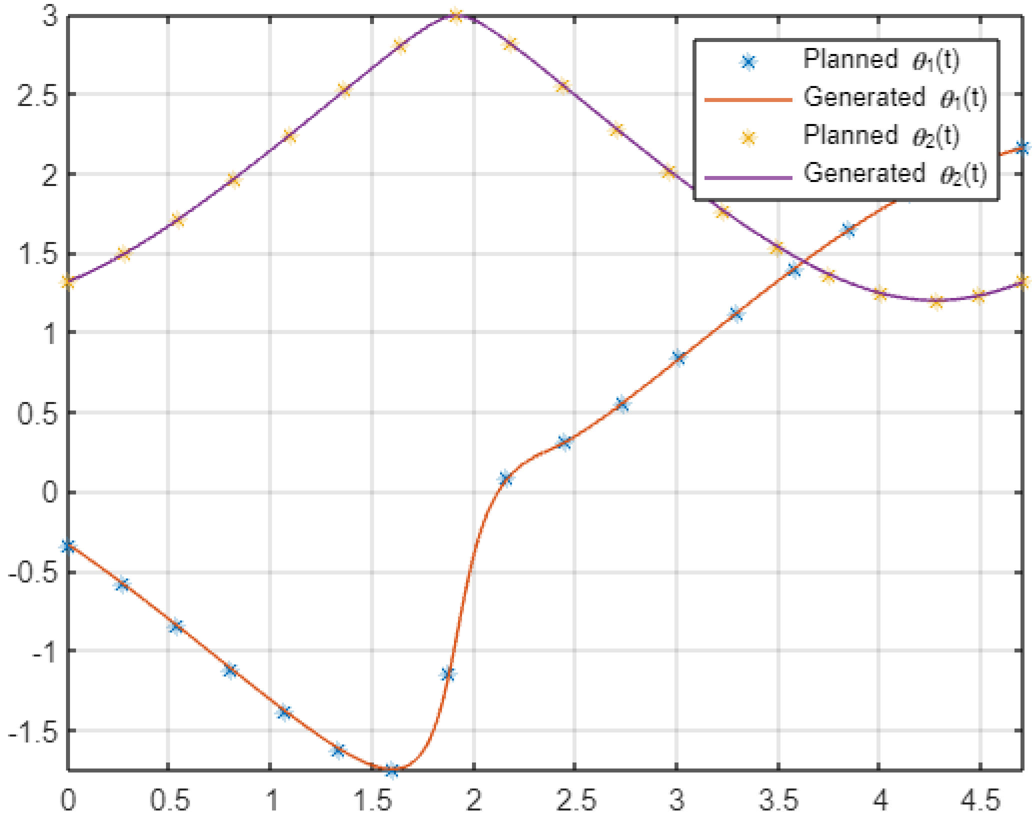
Solution 3(b)

We'll use ode45 to simulate the system. Our state will be

$$q(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{pmatrix}$$

```
% Define initial conditions and time span
syms t;
clear q;
tspan = [1e-12 3*pi/2-1e-12];
q0 = double([theta_1(0); theta_2(0); limit(w1,t,0,'right'); limit(w2,t,0,'right')]);
```

```
% Define ODE function
dqdt = @(ti,q) ([q(3); q(4);
    2*m2*l1*l2*sin(q(2))*q(4)*q(3)-q(4)^2-(m1+m2)*l1*g*cos(q(1))-m2*g*l2*cos(q(1)+q(2));
    -m2*l1*l2*sin(q(2))*q(3)^2-m2*g*l2*cos(q(1)+q(2))]+[0;0;double(subs(torque(1),t,ti));double

% Define the mass matrix
M = @(ti,q)[1 0 0 0; 0 1 0 0; 0 0 m1*l1^2+m2*(l1^2+2*l1*l2*cos(q(2))+l2^2) m2*(l1*l2*cos(q(2))-
% Set the mass-matrix option of ode
options = odeset('Mass',M,'RelTol', 1e-10, 'AbsTol', 1e-12);
% Solve the ODE
[ti,q] = ode45(dqdt, tspan, q0,options);

% Plotting and comparing the results to the ones obtained earlier
figure;
clear h;
h(1) = fplot(theta_1,[0,3*pi/2],'*');
hold on;
h(2) = plot(ti,q(:,1));
h(3) = fplot(theta_2,[0,3*pi/2],'*');
h(4) = plot(ti,q(:,2));

xlim([0.00 4.71]);
ylim([-1.75 3.00]);
grid on;
legend(h,'Planned \theta_1(t)','Generated \theta_1(t)','Planned \theta_2(t)', 'Generated \theta
```



Solution 3(c)

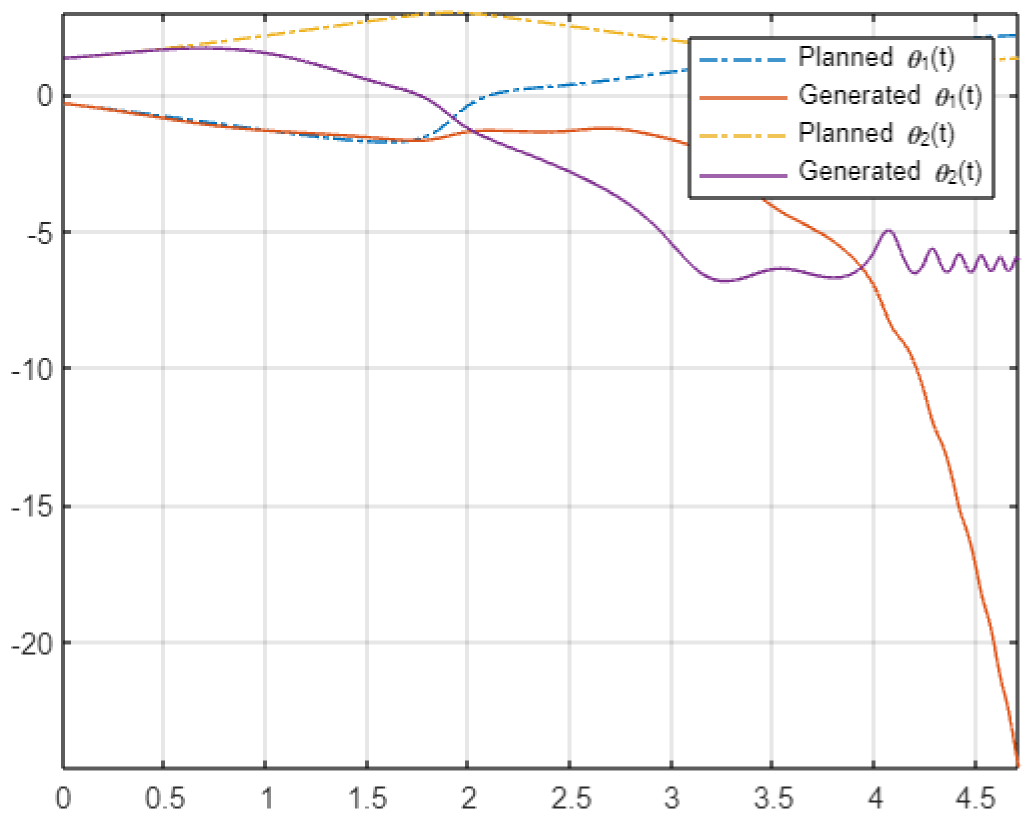
Let's increase the masses by 10%:

```
m1 = 1.1;
m2 = 1.1;
```

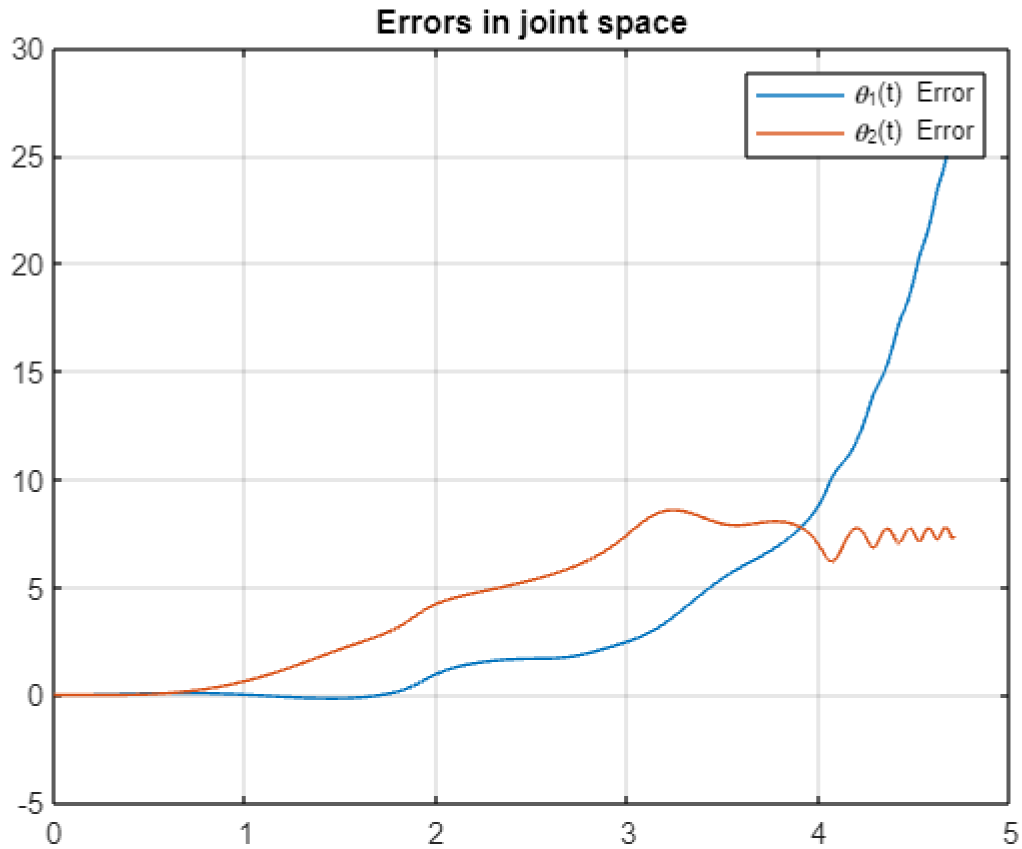
Now let's simulate the system again:

```
% Define ODE function
dqdt = @(ti,q) ([q(3); q(4);
    2*m2*l1*l2*sin(q(2))*q(4)*q(3)-q(4)^2-(m1+m2)*l1*g*cos(q(1))-m2*g*l2*cos(q(1)+q(2));
    -m2*l1*l2*sin(q(2))*q(3)^2-m2*g*l2*cos(q(1)+q(2))]+[0;0;double(subs(torque(1),t,ti));double

% Define the mass matrix
M = @(ti,q)[1 0 0 0; 0 1 0 0; 0 0 m1*l1^2+m2*(l1^2+2*l1*l2*cos(q(2))+l2^2) m2*(l1*l2*cos(q(2))-
% Set the mass-matrix option of ode
options = odeset('Mass',M);
% Solve the ODE
[ti,q] = ode45(dqdt, tspan, q0,options);

% Plotting and comparing the results to the ones obtained earlier
figure;
clear h;
```

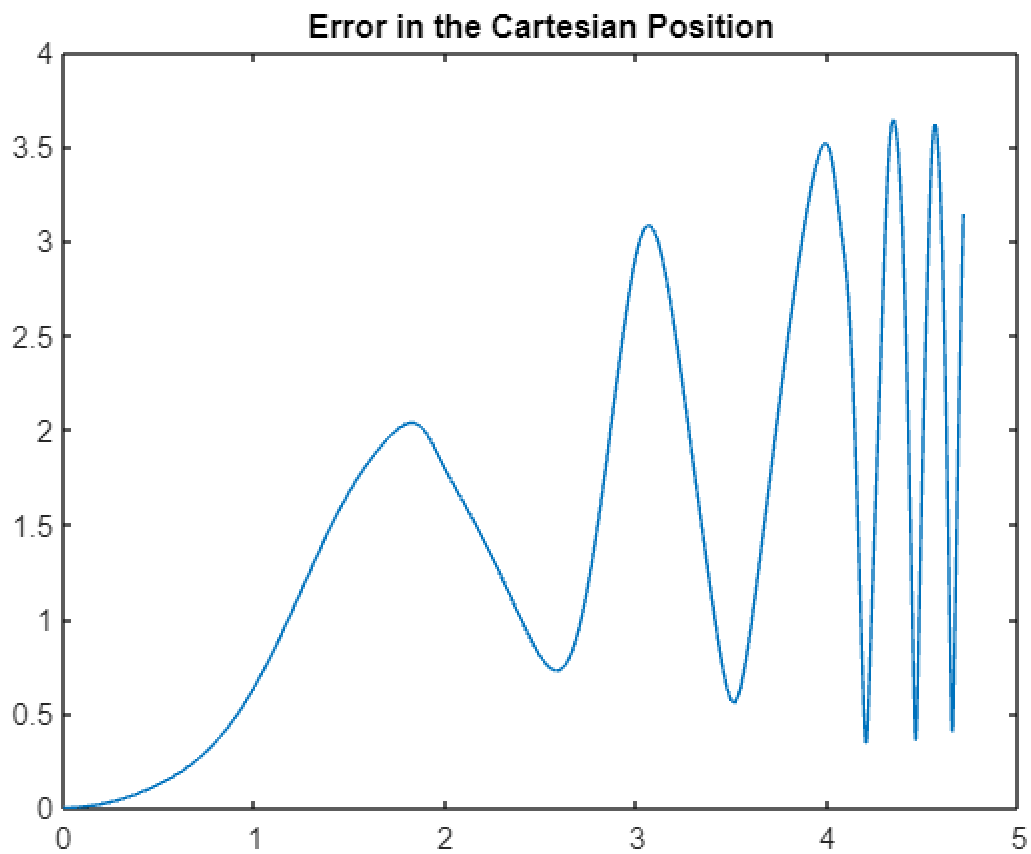
```
h(1) = fplot(theta_1,[0,3*pi/2],'-.');
hold on;
h(2) = plot(ti,q(:,1));
h(3) = fplot(theta_2,[0,3*pi/2],'-.');
h(4) = plot(ti,q(:,2));
grid on;
legend(h,'Planned \theta_1(t)','Generated \theta_1(t)','Planned \theta_2(t)', 'Generated \theta
```



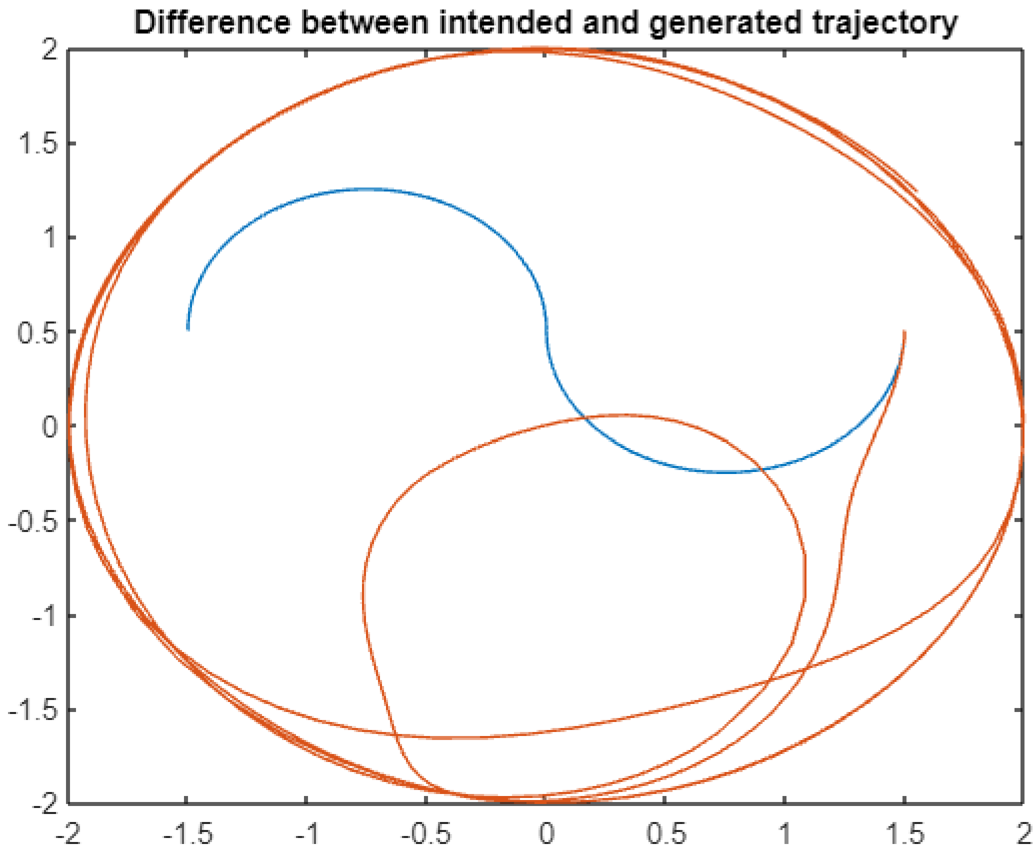

```
% Error in joint space
q1_error = double(theta_1(ti))-q(:,1);
q2_error = double(theta_2(ti))-q(:,2);
figure;
clear h;
h(1) = plot(ti,q1_error);
hold on;
h(2) = plot(ti,q2_error);
grid on;
legend(h,'\theta_1(t) Error','\theta_2(t) Error');
title('Errors in joint space');
```



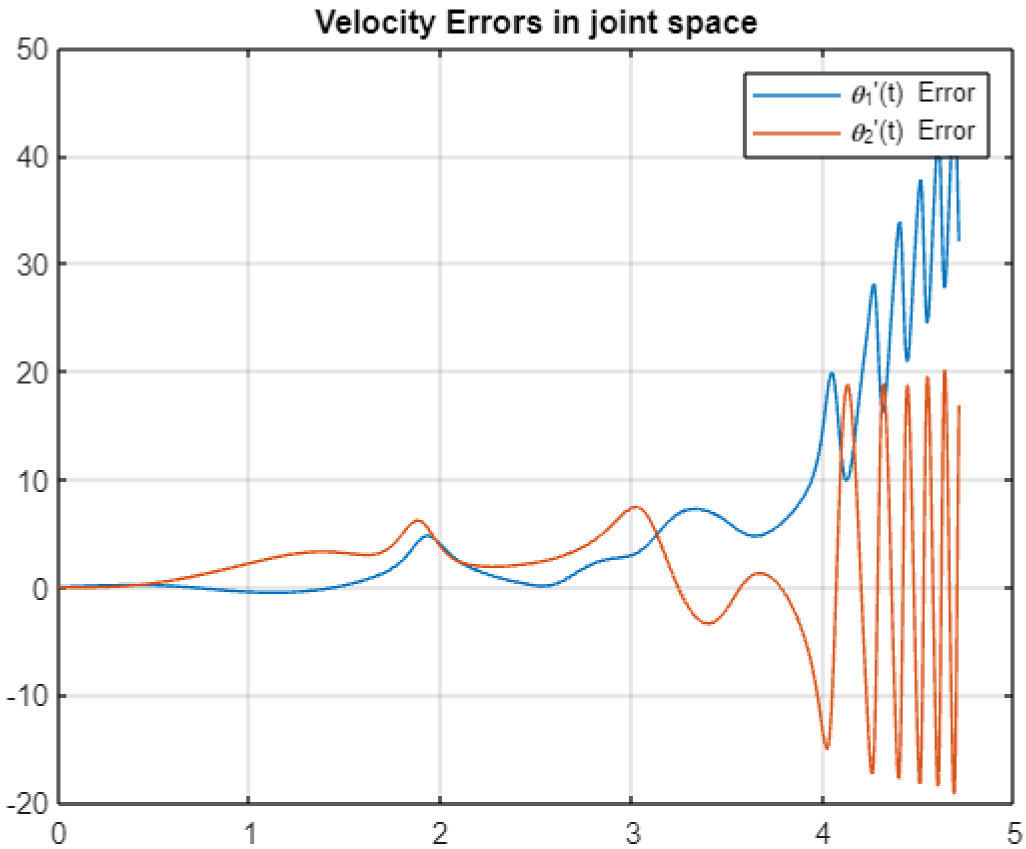

```
% Error in Cartesian Space
x_planned = double(l1*cos(theta_1(ti))+l2*cos(theta_1(ti)+theta_2(ti)));
y_planned = double(l1*sin(theta_1(ti))+l2*sin(theta_1(ti)+theta_2(ti)));
x_gen = l1*cos(q(:,1))+l2*cos(q(:,1)+q(:,2));
y_gen = l1*sin(q(:,1))+l2*sin(q(:,1)+q(:,2));
error_cart = sqrt((x_planned-x_gen).^2+(y_planned-y_gen).^2);

% Plotting
figure;
plot(ti,error_cart);
title('Error in the Cartesian Position');
```




```
figure;
plot(x_planned,y_planned);
hold on;
plot(x_gen,y_gen);
title('Difference between intended and generated trajectory');
```




```
% Velocity error in the joint space
w1_error = double(w1(ti))-q(:,3);
w2_error = double(w2(ti))-q(:,4);
figure;
clear h;
h(1) = plot(ti,w1_error);
hold on;
h(2) = plot(ti,w2_error);
grid on;
legend(h,"\theta_1'(t) Error","\theta_2'(t) Error");
title('Velocity Errors in joint space');
```



Solution 3(d)

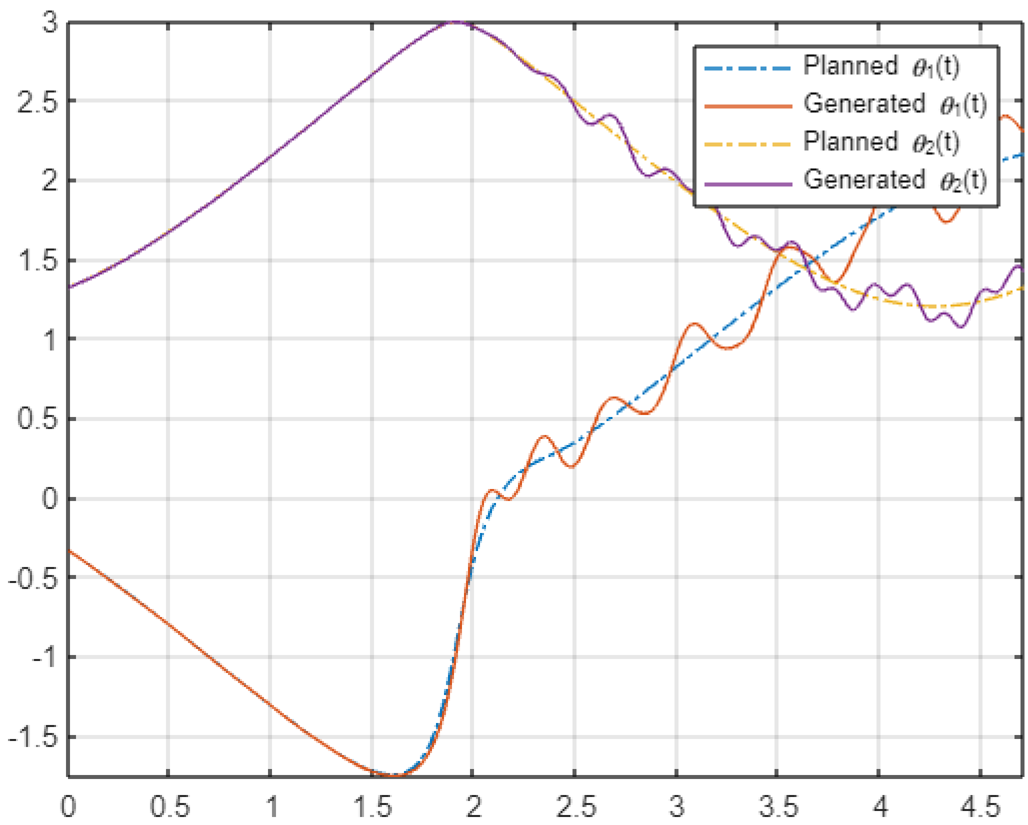
Recall that for the computed torque controller we'll set our control, i.e. torque in this case as:

$$\tau = C(q,\dot{q})\,\dot{q} + g(q) + J(q)\,(\ddot{q}_d + \text{Feedback Controller})$$

The mass value that you'll be using for computing these terms will be 1kg, which is what you believe to be the mass. In reality, the mass is 10% higher and we'll see if the controller is able to compensate for that.

```
m1 = 1.1;
m2 = 1.1;
l1 = 1;
l2 = 1;
g = 9.8;

% Define ODE function
dqdt = @(ti,q) ([q(3); q(4);
    2*m2*l1*l2*sin(q(2))*q(4)*q(3)-q(4)^2-(m1+m2)*l1*g*cos(q(1))-m2*g*l2*cos(q(1)+q(2));
    -m2*l1*l2*sin(q(2))*q(3)^2-m2*g*l2*cos(q(1)+q(2))]+[0;0;controller(ti,q,theta_1, theta_2, a

% Define the mass matrix
M = @(ti,q)[1 0 0 0; 0 1 0 0; 0 0 m1*l1^2+m2*(l1^2+2*l1*l2*cos(q(2))+l2^2) m2*(l1*l2*cos(q(2))-
% Set the mass-matrix option of ode
options = odeset('Mass',M);
```

```
% Solve the ODE
[ti,q] = ode45(dqdt, tspan, q0,options);

% Plotting and comparing the results to the ones obtained earlier
figure;
clear h;
h(1) = fplot(theta_1,[0,3*pi/2],'-.');
hold on;
h(2) = plot(ti,q(:,1));
h(3) = fplot(theta_2,[0,3*pi/2],'-.');
h(4) = plot(ti,q(:,2));
grid on;
legend(h,'Planned \theta_1(t)','Generated \theta_1(t)','Planned \theta_2(t)', 'Generated \theta
```




```
function tau_computed = controller(t,q,theta_1, theta_2, a1, a2)
    m1 = 1;
    m2 = 1;
    l1 = 1;
    l2 = 1;
    g=9.8;
    persistent eint;
    persistent tprev;
```

```
    if isempty(eint)
        eint = [0;0];
        tprev = 0;
    end

    % This is compensating the corriolis and gravity terms
    tau_computed = [-2*m2*l1*l2*sin(q(2))*q(4)*q(3)+q(4)^2+(m1+m2)*l1*g*cos(q(1))+m2*g*l2*cos(q
                    m2*l1*l2*sin(q(2))*q(3)^2+m2*g*l2*cos(q(1)+q(2))];
    % Add the feedforward controller
    tau_computed = tau_computed + [m1*l1^2+m2*(l1^2+2*l1*l2*cos(q(2))+l2^2)*double(a1(t))+m2*(l
                                   m2*(l1*l2*cos(q(2))+l2^2)*double(a1(t))+ m2*l2^2*double(a2(t
    % Add the feedback controller part
    Kp = 600;
    Ki = 100;
    e = [double(theta_1(t))-q(1); double(theta_2(t))-q(2)];
    dt = t-tprev;
    eint = eint + e*dt;
    tau_computed  = tau_computed + Kp*e + Ki*eint;
    tprev = t;
end
```