Homework 1
Solution

Assigned on January 23, 2024
Due on February 2, 2024

Learning Outcomes:

After this homework, you should be able to:

(a) manipulate rotation matrices and homogeneous transformations to describe frames, change coordinates, or carry out rigid body motions;

(b) use composition of homogeneous transforms to model physical situations involving a collection of rigid objects;

(c) convert between implicit and explicit representations of orientation;

Tasks

(*Modern Robotics-2.26)* The tip coordinates for the two-link planar 2R robot of Figure 1 are given by: Problem 1 CLO1-C2

$$x = 2\cos\theta_1 + \cos(\theta_1 + \theta_2)$$
$$y = 2\sin\theta_1 + \sin(\theta_1 + \theta_2).$$

20 points

(a) What is the robot's configuration space (C-space)?

(b) What is the robot's workspace (i.e., the set of all points reachable by the tip)?

(c) If a pen is placed at the end of the 2R arm and used to draw on paper, then what is a suitable task space for the robot? Why?

(d) Suppose infinitely long vertical barriers are placed at $x = 1$ and $x = -1$. What is the free C-space of the robot (i.e., the portion of the C-space that does not result in any collisions with the vertical barriers)?

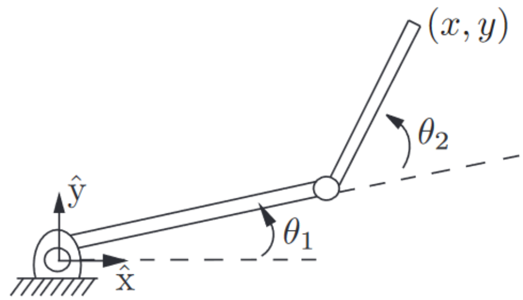

Figure 1: A 2R planar arm

(e) A not-so-clever designer claims that he can make the workspace of any planar open chain larger simply by increasing the length of the last link. Explain the fallacy behind this claim.

Solution 1

(a) A possible C-space for this robot is the two-dimensional space formed by θ_1 and θ_2. Without any constraints, $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 < 2\pi$, and the configuration space has the shape of a donut[1]. Every point on the surface of the donut is a particular (θ_1, θ_2) corresponding to a configuration of this robot.

(b) The set of all points reachable by the tip of a robot is called its workspace. Since this is a planar robot, we know that the workspace of this robot is a subset of the plane or $\mathbb{R}^2$. A way for finding the workspace of a robot is to start with the last link, find the curve traced by the last link due to motion of the joint, move this curve according to the possible motion of the second-last link to trace a surface, and so on. From the accompanying livescript, we can see that the workspace is an annular region, with outer radius of three and inner radius of 1.

(c) A suitable task space for the robot would be the paper, a specific subset of $\mathbb{R}^2$, on which the robot will be drawing.

(d) The free C-space is the subset of θ_1, θ_2 that results in the end-effector position between $x = -1$ and $x = 1$. We can see from Figure 2 that if the end-point of link 1 falls below point A in the figure, then the manipulator collides with the barrier on the right. From

[1] A donut is formally called a torus in the mathematical discipline of Topology, which is about the shape of spaces.

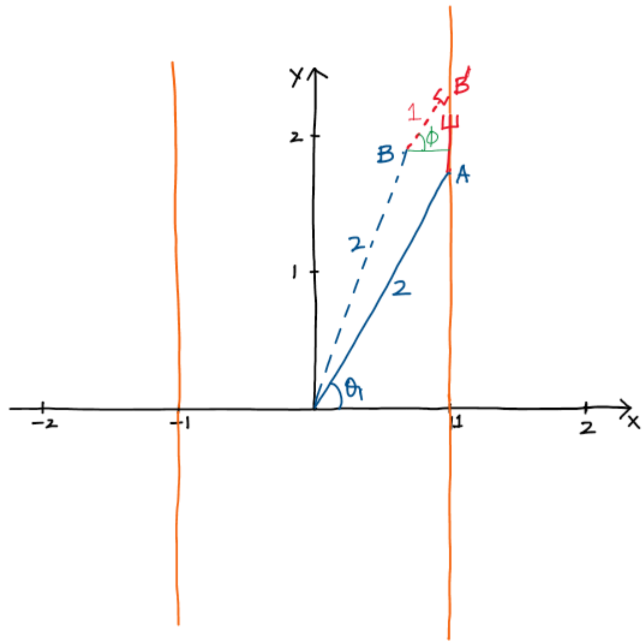

Figure 2: Motion of 2R manipulator constrained by barriers at $x = -1$ and $x = 1$

the bottom right triangle, we have that

$$\cos\theta_1 = \frac{1}{2}$$
$$\implies \theta_1 = 60^\circ$$

Using symmetry, we can then argue that $\theta_1 \in [60^\circ, 120^\circ] \cup [-120^\circ, -60^\circ]$ to avoid collision with the barriers on both sides.

To determine the limits of θ_2, we can consider the manipulator drawn with dashed lines in Figure 2. Now considering the smaller triangle formed by BB', the horizontal line, and the barrier line,

$$\cos\phi = \frac{1 - 2\cos\theta_1}{1}$$
$$\implies \phi = \arccos(1 - 2\cos\theta_1),$$

where $\theta_1 \in [60^\circ, 90^\circ]$.

$$\implies \theta_2 = \arccos(1 - 2\cos\theta_1) - \theta_1$$

Again by symmetry, $\theta_2 \in [\phi - \theta_1, 2\pi - \phi]$ whenever θ_1 is in the first or fourth quadrant, i.e. $\theta_1 \in [60°, 90°] \cup [-90°, -60°]$. When θ_1 is in the second or third quadrant, then $\phi = \arccos(1 + 2\cos\theta_1)$ and $\theta_2 \in [0°, 180° - (\theta_1 + \phi)] \cup [\theta_1 + \phi, 360°]$. The resulting free C-space is shown in Figure 3.

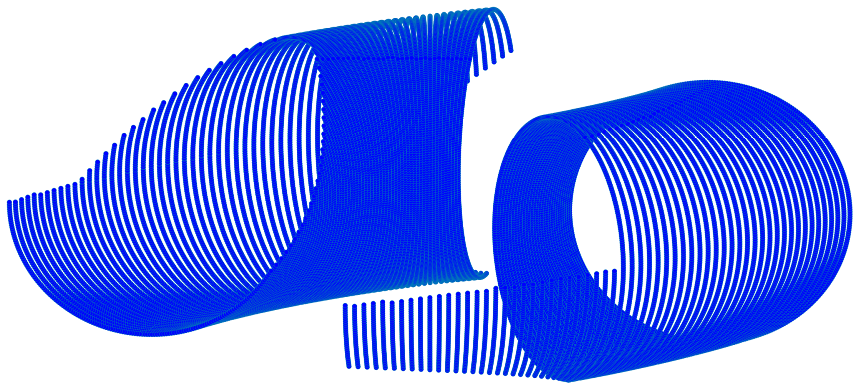

Figure 3: Free C-space of 2R manipulator with barriers

(e) It is correct that for a manipulator theoretically once could increase the workspace by increasing the length of the last link. Consider the example of 2R manipulator with link lengths l_1 and l_2. Then,

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ x^2 + y^2 &= l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2 \end{aligned}$$

We know that $-1 \leq \cos\theta_2 \leq 1$,

$$\implies (l_1 - l_2)^2 \leq x^2 + y^2 \leq (l_1 + l_2)^2$$

Thus, the workspace is always an annulus with outer radius $l_1 + l_2$ and inner radius $|l_1 - l_2|$. The area of this workspace is $\pi(l_1 + l_2)^2 - \pi(l_1 - l_2)^2 = 4\pi l_1 l_2$. Increasing l_2, increases the area of the workspace. However, the inner circle also extends farther and farther out, i.e. the points closer to base become more and more inaccessible. This also assumes that all joints will be able to rotate 360° and does not account for possible physical limits of the joints, e.g. increasing the length of link 2 may result in it colliding with the base of the manipulator and in order to prevent that the range of allowable angles for θ_2 will have to be reduced, which would further decrease the workspace. Additionally, this increase in the length of last link may also introduced singularities in the interior of the workspace (more on that later).

Problem 2
CLO1-C3
30 points

This is a MATLAB-based problem where you'll get to play around with construction of homogeneous transformations.

(a) Find the 4×4 homogeneous transformation that corresponds to the motion of the red square in Figure 4 to the blue square. You can accurately read the coordinates of the

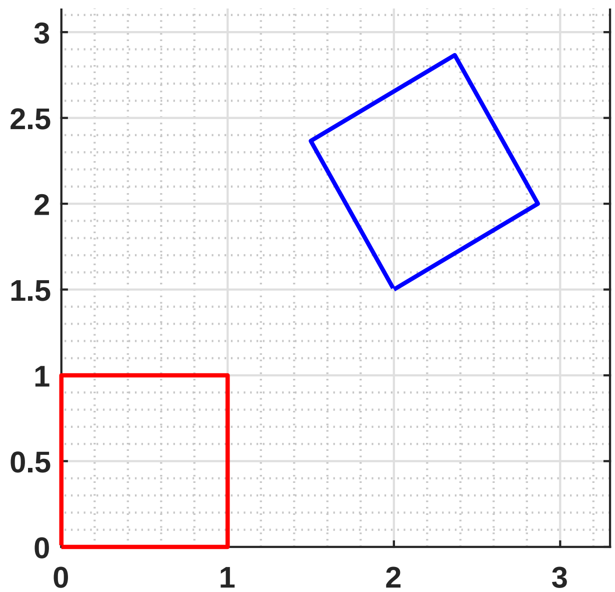

Figure 4: Red square is rotated and translated to obtain blue square

different points in the figure by opening the provided MATLAB figure file: `hw1-q2.fig`. The MATLAB functions `atan2` and `atan2d` may be of help.

(b) Verify that your found transformation is indeed correct by applying your transformation to the points of red square in MATLAB. Each point of the red square can be viewed as a vector, which is displaced by your homogeneous transformation. A MATLAB file `hw1_q2` is provided to you, you'll apply your transformation in the code at the indicated location, and if it is correct then the script should recreate Figure 4.

Points of the red square are provided in the matrix `pts`, where each column is a point of the square. Your transformation will operate on this matrix and generate a new matrix `ptsBlue`, where column i are the coordinates of displaced point i of `pts`.

(c) Say we attach a frame to the blue square as shown in Figure 5, where x and y are the yellow and purple axes respectively. What is the pose of this frame?

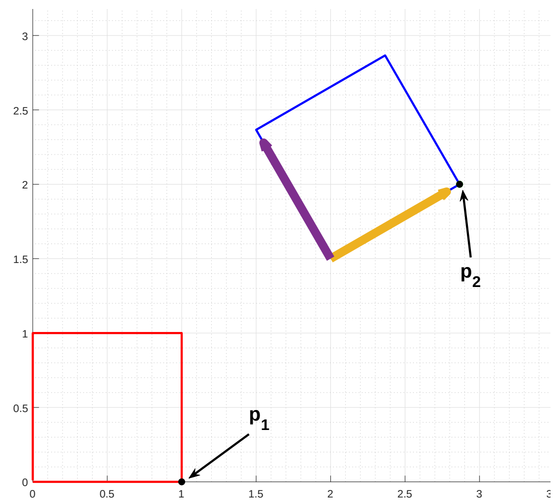

Figure 5: There are two frames: red and blue. There are two physical points: p_1 and p_2

(d) Say ${}^{R}p_i$ and ${}^{B}p_i$ are the coordinates of the points p_i, marked in Figure 5, in the red and blue frames respectively. Find numeric values of all four possible coordinates of the two points.

(e) How can you obtain coordinates ${}^{B}p_1$, given the coordinates ${}^{B}p_2$ and the homogeneous transformation of (a)?

(f) How can you obtain coordinates ${}^{B}p_2$, given the coordinates ${}^{R}p_2$ and the homogeneous transformation of (a)?

Solution 2

(a) By observation, the red square has been translated by 2 units along the x-axis and by 1.5 units along the y-axis. Additionally, the z-axis should be out of the page to have a right-handed frame, so the red square has also been rotated about this z-axis in the counter-clockwise direction. The angle of rotation can be determined by looking at the coordinates of bottom-right corner of the blue square in the MATLAB figure, i.e. $(2.866, 2)$. This corresponds to an angle of $\arctan 2(2 - 1.5, 2.866 - 2) = 30°$. The corresponding rotation matrix is:

$${}^{R}R_B = R_z(30°) = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the homogeneous transformation is

$$^{R}T_{B} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 2 \\ \sin 30^\circ & \cos 30^\circ & 0 & 1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This can also be written as a composition of two operations, i.e. homogeneous transformation corresponding to a translation followed by a rotation:

$$^{R}T_{B} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that you cannot reverse the order of these operations because multiplying on the right corresponds to motion with respect to current axes. So, if you rotate first and then translate along x-axis by 2 units then it corresponds to translation along the new x-axis.

(b) One way to implement this in MATLAB is:

```
R = eul2rotm([deg2rad(30),0,0]);
H = [R, [2;1.5;0]; zeros(1,3), 1];
ptsBlue = H*pts;
```

(c) Since the frame is attached to the blue square in such a way that the frame would have been aligned with the red frame before motion, so the homogeneous transformation found in (a) also serves the purpose of defining the pose of the blue frame with respect to the red frame.

(d) We can write the coordinates of both points in the red frame by placing datatips in the provided MATLAB figure, and finding the corresponding coordinates. They are:

$$^{R}p_1 = (1, 0, 0)$$
$$^{R}p_2 = (2.866, 2, 0)$$

Coordinates of p_2 in the blue frame are easy:

$$^{B}p_2 = (1, 0, 0)$$

For p_1, we have to find the coordinates of head of the position vector from origin of blue frame to p_1. Coordinates are its projections onto the x and y axes of the blue frame. We can write the vector in the red frame and then changes its representation using rotation matrix:

$$^{B}p_1 = {}^{B}R_R \begin{bmatrix} 1-2 \\ 0-1.5 \\ 0 \end{bmatrix} = (-1.62, -0.8, 0)$$

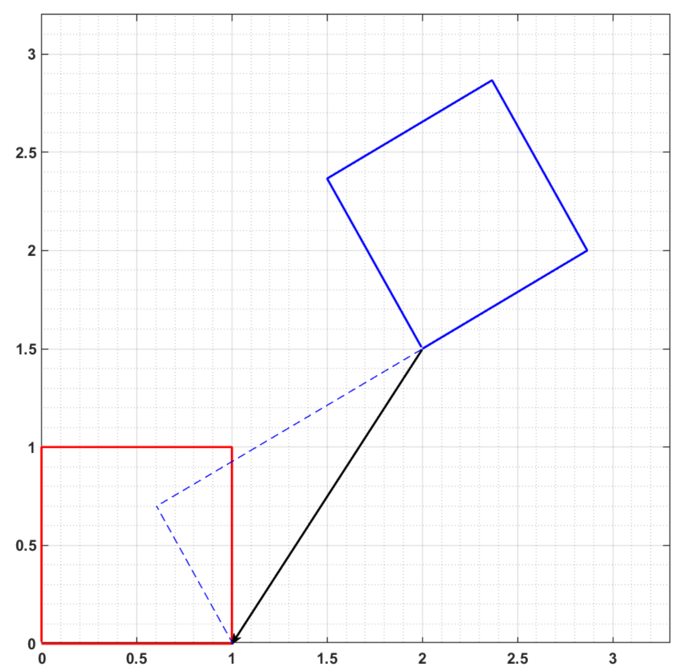

Figure 6: Project the position vector in black on to the axes of the blue frame (dashed lines)

(e) The transformation from (a) is $^{R}T_B$, symbolically, where R and B represent the frames of the red and blue squares respectively. In this problem, the coordinate frame stays the same but we wish to move the point p_2 to a new location p_1. This motion could be due to the motion of the blue square to the red square, which is encapsulated in the transformation $^{B}T_R$. Thus,

$$\begin{aligned} ^{B}p_1 &= {}^{B}T_R \, {}^{B}p_2 \\ &= {}^{R}T_B^{-1} \, {}^{B}p_2 \end{aligned}$$

(f) The transformation from (a) is ${}^{R}T_{B}$, symbolically, where R and B represent the frames of the red and blue squares respectively. Given the coordinates ${}^{R}p_2$, we can change the coordinates to another frame using:

$$\begin{aligned} {}^{B}p_2 &= {}^{B}T_{R}\,{}^{R}p_2 \\ &= {}^{R}T_{B}^{-1}\,{}^{R}p_2 \end{aligned}$$

Given a fixed frame $\{0\}$ and a moving frame $\{1\}$ initially aligned with $\{0\}$, Problem 3 CLO1-C3

(a) perform the following sequence of rotations on $\{1\}$ 15 points

1. Rotate $\{1\}$ about the $\{0\}$ frame $\hat{x}$-axis by α; call this new frame $\{2\}$.
2. Rotate $\{2\}$ about the $\{0\}$ frame $\hat{y}$-axis by β; call this new frame $\{3\}$.
3. Rotate $\{3\}$ about the $\{0\}$ frame $\hat{z}$-axis by γ; call this new frame $\{4\}$.

What is the final orientation ${}^{0}R_4$?

(b) Suppose that the third step above is replaced by the following: "Rotate $\{3\}$ about the $\{3\}$ frame $\hat{z}$-axis by γ; call this new frame $\{4\}$." What is the final orientation ${}^{0}R_4$?

(c) What will be the sequence of three rotations about the current axes, starting with frame $\{1\}$, which will result in the exact same final orientation ${}^{0}R_4$ as in (b)? i.e. complete the following:

1. Rotate $\{1\}$ about the $\{1\}$ frame ______-axis by ______; call this new frame $\{2\}$.
2. Rotate $\{2\}$ about the $\{2\}$ frame ______-axis by ______; call this new frame $\{3\}$.
3. Rotate $\{3\}$ about the $\{3\}$ frame ______-axis by ______; call this new frame $\{4\}$.

Solution 3

(a) Let $R_x(\alpha)$, $R_y(\beta)$, and $R_z(\gamma)$ represent the canonical rotations matrices for rotation about the x, y, and z axes respectively. Then, in this case

$${}^{0}R_4 = R_z(\gamma)\,R_y(\beta)\,R_x(\alpha),$$

as rotation about the fixed (world) axes correspond to multiplication on the left.

(b) In this case, we have

$$\begin{aligned} {}^{0}R_2 &= R_x(\alpha) \\ {}^{0}R_3 &= R_y(\beta)\,R_x(\alpha) \\ {}^{0}R_4 &= R_y(\beta)\,R_x(\alpha)\,R_z(\gamma) \end{aligned}$$

(c) If we rotate about the current axes only, we can easily accomplish it by executing rotations in the reverse order than the one specified in the question. This can easily by seen by reading the expression for ${}^0R_4$ in (a) from left to right, i.e.

1. Rotate {1} about the {1} frame $\hat{y}$-axis by β; call this new frame {2}.
2. Rotate {2} about the {2} frame $\hat{x}$-axis by α; call this new frame {3}.
3. Rotate {3} about the {3} frame $\hat{z}$-axis by γ; call this new frame {4}.

Problem 4
CLO1-C3
10 points

Two satellites are circling the Earth as shown in Figure 7. Frames {1} and {2} are rigidly attached to the satellites in such a way that their $\hat{x}$-axes always point toward the Earth. Satellite 1 moves at a constant angular speed v_1, while satellite 2 moves at a constant angular speed v_2. To simplify matters, ignore the rotation of the Earth about its own axis.

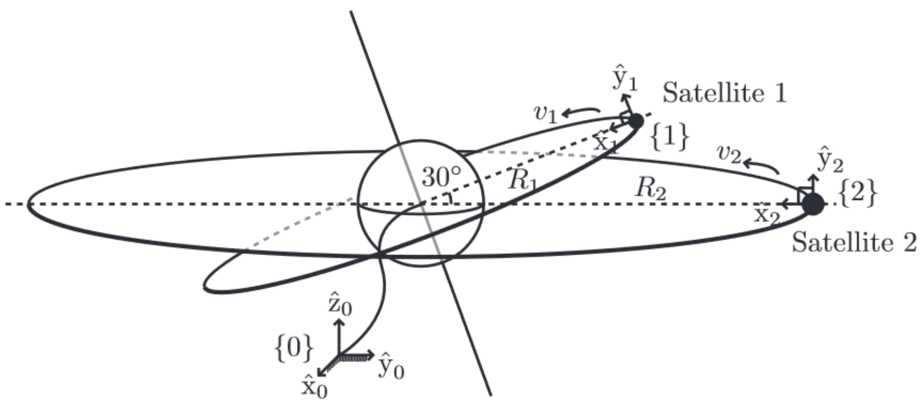

Figure 7: Two satellites circling the Earth

The fixed frame {0} is located at the center of the Earth. Figure 7 shows the position of the two satellites at $t = 0$.

(a) Derive ${}^0T_1$ and ${}^0T_2$ as a function of t.

(b) Using your previous results, find ${}^1T_2$ as a function of t.

Solution 4

We can obtain these transformations by carrying out rotations and translations in the right order.

(a) To obtain ${}^0T_2$, let's first rotate about $\hat{z}_0$ by $-90°$, rotate about $\hat{z}_0$ by $\theta_2 = v_2 t$, translate along the new $\hat{x}$ by $-R_2$, and then rotate about this new $\hat{x}$ by 90°.

$$
{}^0T_2 = \begin{bmatrix} \cos(\theta_2 - 90^\circ) & -\sin(\theta_2 - 90^\circ) & 0 & 0 \\ \sin(\theta_2 - 90^\circ) & \cos(\theta_2 - 90^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -R_2 \\ 0 & \cos(90^\circ) & -\sin(90^\circ) & 0 \\ 0 & \sin(90^\circ) & \cos(90^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin\theta_2 & 0 & -\cos\theta_2 & -R_2\sin\theta_2 \\ -\cos\theta_2 & 0 & -\sin\theta_2 & R_2\cos\theta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

To obtain ${}^0T_1$, rotate about $\hat{x}_0$ by 30°, rotate about new $\hat{z}$ by -90°, rotate about new $\hat{z}$ by $\theta_1 = v_1 t$, translate along the new $\hat{x}$ by $-R_1$, and then rotate about this new $\hat{x}$ by 90°.

$$
{}^0T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(30^\circ) & -\sin(30^\circ) & 0 \\ 0 & \sin(30^\circ) & \cos(30^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta_1 & 0 & -\cos\theta_1 & -R_1\sin\theta_1 \\ -\cos\theta_1 & 0 & -\sin\theta_1 & R_1\cos\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin\theta_1 & 0 & -\cos\theta_1 & -R_1\sin\theta_1 \\ -\frac{\sqrt{3}}{2}\cos\theta_1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\sin\theta_1 & \frac{\sqrt{3}}{2}R_1\cos\theta_1 \\ -\frac{1}{2}\cos\theta_1 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta_1 & \frac{1}{2}R_1\cos\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(b)

$$
{}^1T_2 = {}^0T_1^{-1}\,{}^0T_2
$$

$$
= \begin{bmatrix} \sin\theta_1\sin\theta_2 + \frac{\sqrt{3}}{2}\cos\theta_1\cos\theta_2 & -\frac{1}{2}\cos\theta_1 & \frac{\sqrt{3}}{2}\cos\theta_1\sin\theta_2 - \cos\theta_2\sin\theta_1 \\ \frac{1}{2}\cos\theta_2 & \frac{\sqrt{3}}{2} & \frac{1}{2}\sin\theta_2 \\ \frac{\sqrt{3}}{2}\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 & -\frac{1}{2}\sin\theta_1 & \cos\theta_1\cos\theta_2 + \frac{\sqrt{3}}{2}\sin\theta_2\sin\theta_1 \\ 0 & 0 & 0 \end{bmatrix}.
$$

$$
\begin{matrix} R_1 - R_2\sin\theta_1\sin\theta_2 - \frac{\sqrt{3}}{2}R_2\cos\theta_1\cos\theta_2 \\ -\frac{1}{2}R_2\sin\theta_2 \\ R_2\cos\theta_1\sin\theta_2 - \frac{\sqrt{3}}{2}R_2\sin\theta_1\cos\theta_2 \\ 1 \end{matrix} \Bigg]
$$

I'll be adding this by tomorrow. Working on some starter code. Problem 5 CLO1-C3

Answer the following questions individually: Problem 6 15 points CLO1-C2

10 points

(a) How many hours did each of you spend on this homework and specifically state your contribution in this homework assignment? Answer as accurately as you can, as this will be used to structure next year's class.

(b) Do you have any specific advice for students attempting this homework next year?

(c) Each group member is to provide a self-reflection in the form of a note or a concept map. This requires you to reflect on your learning in relation to each of the outcomes stated at the beginning of this document.

Some questions that may help in this regard are: Have I achieved this outcome? What do I currently understand about content related to this outcome? How does it help me understand or build any robot? Do I have unanswered questions? What went wrong? How can I enable myself to achieve this outcome? What could I do to know more or enhance my skills in this context?

Grading:

To obtain maximal score for each question, make sure to elaborate and include all the steps.

Further exploration

The notion of modeling robots as links and joint is widely used not only in robotics but for modeling other mechanical structures as well. In fact, robotics software have agreed to used a common XML file format for describing the construction of any robot, called the **'Universal Robot Description Format (URDF)'**. If interested, you can explore this further using the following resources:

- Section 4.2, Modern Robotics
- Creating a robot in MATLAB SimMechanics using URDF (`https://www.mathworks.com/help/sm/ug/import-a-urdf-model.html`)
- URDF in ROS (`https://articulatedrobotics.xyz/ready-for-ros-7-urdf/`)
- URDF in ROS (`https://ocw.tudelft.nl/course-lectures/2-2-1-introduction-to-urdf/`)

You'll notice that two of the links above refer to ROS (`https://www.ros.org/`). The **Robot Operating System (ROS)** is a collection of software components, interfaces, and tools for building robots. ROS has gained near ubiquity in the present world and most of present-day robots are built on ROS. You can find out more about ROS by watching this video (`https://vimeo.com/639236696`). The basic module for motion utilized by ROS for any robot is a transform tree. You would be right if you guess that this transform tree is essentially based on homogeneous transformations. The following resources are a starting point for transform trees in ROS:

- `https://www.mathworks.com/help/ros/ug/access-the-tf-transformation-tree-in-ros.html`
- `https://articulatedrobotics.xyz/ready-for-ros-6-tf/`