Homework 3
Solution

Assigned on March 4, 2024
Due on March 11, 2024

Learning Outcomes:

After this homework, you should be able to:

(a) Appreciate the existence of multiple solutions for inverse kinematics problem;

(b) Determine the inverse kinematics mapping of manipulators.

Tasks

Problem 1
CLO2-C4

$2\times5+15+5$ points

(a) Given a desired (x, y), how many solutions are there to the inverse kinematics of the arm shown in Figure 1? Treat the cases of position at boundary of workspace and in the interior of workspace separately.

(b) If an orientation of the end effector is also specified, how many solutions are there? Treat the cases of position at boundary of workspace and in the interior of workspace separately.

(c) Use the geometric approach to find solutions for both cases, i.e. (i) only (x, y) is provided, and (ii) (x, y, ϕ) is provided.

(d) The manipulator of this question is installed on a bomb-disposal robot as shown in Figure 2. Your task is to move the bomb, the red box, from its present location to the containment unit, the container in black, safely, i.e. it is required that the bomb remains in its current orientation throughout the motion. The motion will be carried out in three steps - move straight up, move left, and move straight down.

The provided script `bombDisposal.m` requires you to provide it values of the joint variables at different instants in time for the entire motion. It will then set up the display of the robot and play the motion corresponding to the provided joint variables. The required motion is captured in `bombDisposal.mp4`

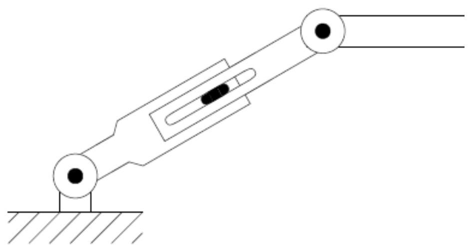

Figure 1: Three link planar robot

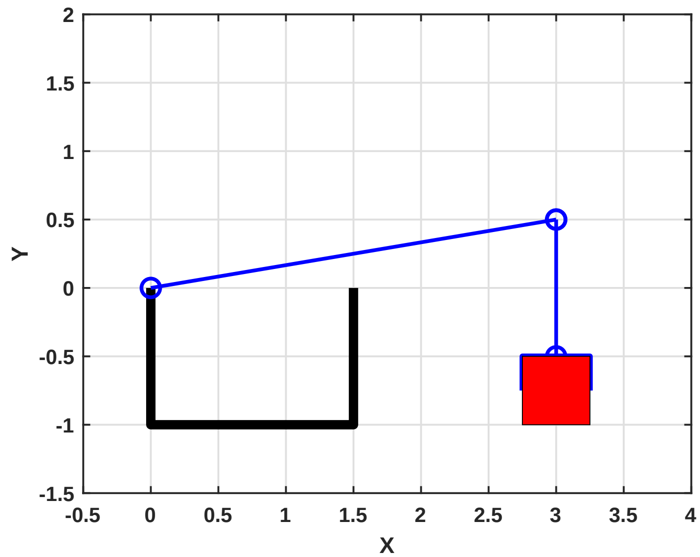

Figure 2: The Bomb Disposal Robot

Solution 1 The reachable workspace is the annular region between the two circles. The outer circle corresponds to $d_2 = d_{max}$, $\theta_3 = 0°$, while the inner circle corresponds to $d_2 = d_{min}$, $\theta_3 = 180°$.

1(a)

A purely geometric argument can be made here. The two links and line segment from origin to end-effector form three sides of a triangle. For a two-link manipulator, the lengths of all three sides are provided which results in a unique triangle (See Figure 4). The triangle could then be located on either side of line segment between origin and end-effector, resulting in elbow-up and elbow-down configurations. On the other hand, for the manipulator of this question only two lengths are fixed, i.e. l_3 and $\sqrt{x^2 + y^2}$. An infinite number of triangles can be constructed in this case while keeping these two lengths fixed, as illustrated in Figure 4.

However, for end-effector positions at the boundary of the reachable space, there is only one possible solution, as $d = d_{max}$ or $d = d_{min}$ and $\theta_3 = 0°$ or $\theta_3 = 180°$ respectively are fixed.

Alternate Explanation: Notice that if d is fixed, then this a 2-link planar arm. For a 2-link

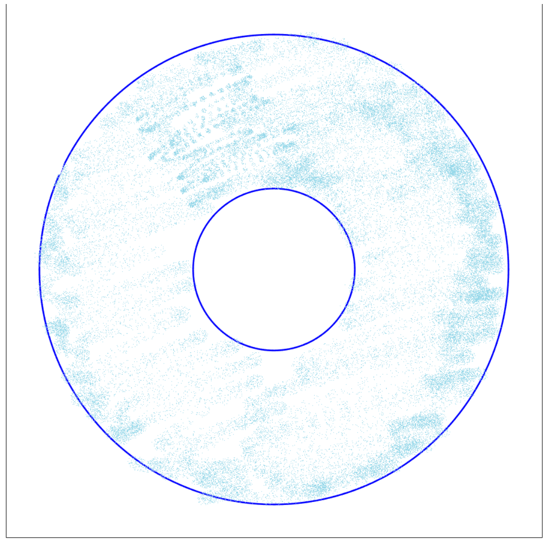

Figure 3: Workspace for Q1

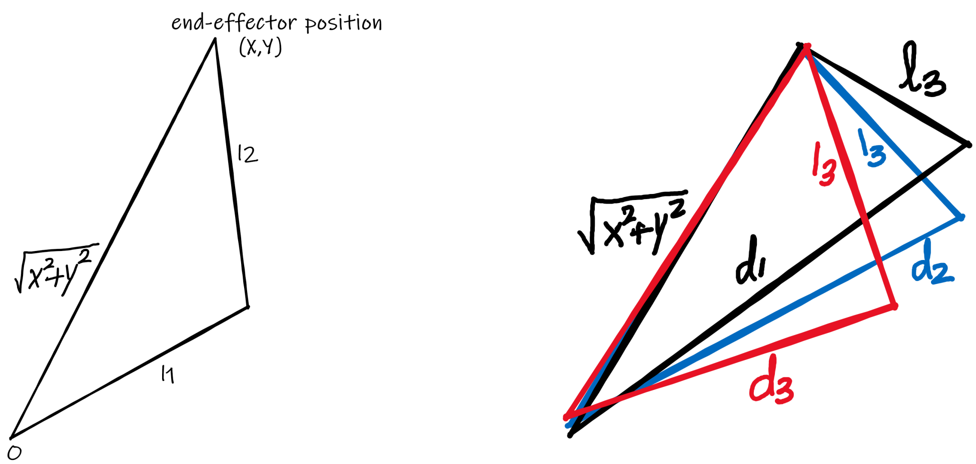

Figure 4: (Left) Two-link manipulator. (Right) Manipulator of Figure 1

planar arm,

$$x = d\cos\theta_1 + l_3\cos(\theta_1+\theta_3)$$
$$y = d\sin\theta_1 + l_3\sin(\theta_1+\theta_3)$$
$$x^2+y^2 = d^2 + l_3^2 + 2dl_3\cos\theta_3$$

Assuming that joint 1 can rotate 360°, the above equation should hold for any reachable (x, y). Written differently,

$$\cos\theta_3 = \frac{x^2 + y^2 - l_3^2 - d^2}{2dl_3}$$

Assuming that joint 3 can rotate 360°,

$$-1 \le \frac{x^2 + y^2 - l_3^2 - d^2}{2dl_3} \le 1$$

or,

$$(d - l_3)^2 \le x^2 + y^2 \le (d + l_3)^2$$

From this we can see that if point (x, y) is in reachable space, then any

$$\sqrt{x^2 + y^2} - l_3 \le d \le \sqrt{x^2 + y^2} + l_3$$

yields an inverse solution. So, there possibly could be infinite inverse kinematic solutions.

1(b) If an orientation, ϕ, is also specified, then we have an additional constraint, i.e.

$$\theta_1 + \theta_3 = \phi.$$

Let's continue with the geometric argument. All the provided information is illustrated in Figure 5. Both the endpoints of the line $\sqrt{x^2 + y^2}$ are fixed. Since the angle ϕ is known the direction in which l_3 is to be draw is also known, and correspondingly both endpoints of l_3 are fixed. Thus, the triangle is fixed and correspondingly there is only one solution.

For end-effector positions at the boundary of reachable workspace, a solution may not exist. For example, if $d = d_{max}$ then $\theta_3 = 0°$, and $\phi = \theta_1$. So a solution will only exist in this case if $\phi = \arctan(y, x)$ is true.

1(c) **Orientation is specified.**

Algebraic Approach: In case the orientation is specified, the algebraic approach yields a simple solution.

$$x = d\cos\theta_1 + l_3\cos(\theta_1 + \theta_3) \tag{1}$$
$$y = d\sin\theta_1 + l_3\sin(\theta_1 + \theta_3) \tag{2}$$
$$\phi = \theta_1 + \theta_3 \tag{3}$$

Substituting (3) in (1) and (2):

$$x = d\cos\theta_1 + l_3\cos(\phi)$$
$$y = d\sin\theta_1 + l_3\sin(\phi)$$
$$\Rightarrow x - l_3\cos(\phi) = d\cos\theta_1 \tag{4}$$
$$y - l_3\sin(\phi) = d\sin\theta_1 \tag{5}$$

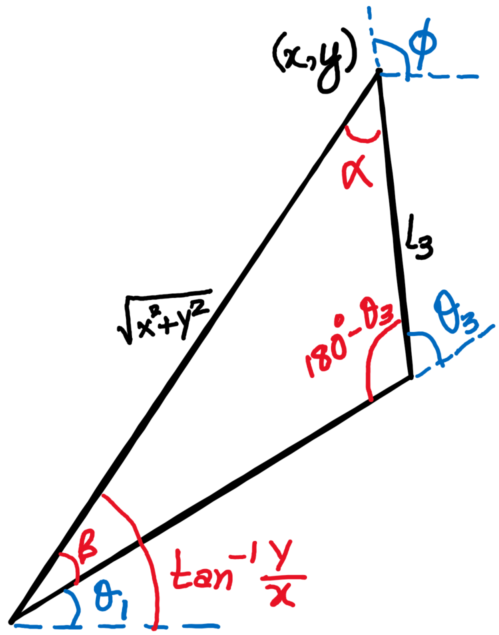

Figure 5: Diagram of manipulator for 1

Squaring (4) and (5), and adding:

$$(x - l_3\cos\phi)^2 + (y - l_3\sin\phi)^2 = d^2$$
$$\Rightarrow \sqrt{(x - l_3\cos\phi)^2 + (y - l_3\sin\phi)^2} = d \tag{6}$$

From (4) and (5):

$$\theta_1 = \arctan 2\left(\frac{y - l_3\sin\phi}{d}, \frac{x - l_3\cos\phi}{d}\right) \tag{7}$$

Finally,

$$\theta_3 = \phi - \theta_1. \tag{8}$$

Geometric Approach: We'll geometrically construct the manipulator and as a consequence determine the values of the joint variables. We're provided the end-effector position (x, y). Let this point be E, as shown in Figure 6. We're also provided the angle ϕ. Link 3 is represented by the line segment DE in the figure and it should make the angle ϕ with the

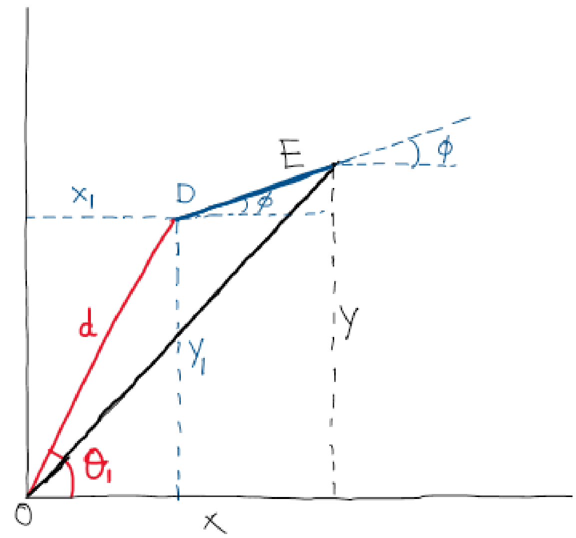

Figure 6: RPR IK when (x, y, ϕ) are provided.

horizontal. Let $D = (x_1, y_1)$. Then,

$$\begin{aligned} x &= x_1 + l_3 \cos\phi \\ y &= y_1 + l_3 \sin\phi \\ \Rightarrow x_1 &= x - l_3 \cos\phi \\ y_1 &= y - l_3 \sin\phi \end{aligned}$$

Since the length of the line segment OD is d and it makes an angle θ_1 with the horizontal, we have:

$$\begin{aligned} d &= \sqrt{x_1^2 + y_1^2} \\ \theta_1 &= \arctan 2(y_1, x_1) \end{aligned}$$

Finally,

$$\theta_3 = \phi - \theta_1$$

Orientation is not specified. We'll refer to Figure 5. In the case when orientation is not specified, we have many possible values for d. Theoretically, we can find all inverse kinematic solutions in this case too, and the value of d is bounded by

$$max(d_{min}, \sqrt{x^2 + y^2} - l_3) \leq d \leq min(\sqrt{x^2 + y^2} + l_3, d_{max})$$

Once a value of d is chosen, we'll obtain two solutions for (θ_1, θ_3) for each value of d. See class notes for details.

$$\theta_3 = \arctan 2(\sin\theta_3, \cos\theta_3)$$

where,

$$\cos\theta_3 = \frac{x^2 + y^2 - l_3^2 - d^2}{2dl_3}$$
$$\sin\theta_3 = \pm\sqrt{1 - \cos^2\theta_3}.$$

And,

$$\theta_1 = \arctan 2(y, x) - \arctan 2(k_2, k_1)$$

where,

$$k_1 = d + l_3\cos\theta_3$$
$$k_2 = L_3\sin\theta_3$$

Problem 2
CLO2-C3
25 points

Gen3 robots, shown in Figure 7, are a popular series of industrial robots from Kinova Robotics. They offer two robotic manipulators - a 6 dof and 7 dof one, under this title. In this task, you'll obtain closed-form expressions for all possible inverse kinematics solutions of the 6 DOF Kinova Gen3 robot. You can use either the algebraic or geometric approach. The schematics and dimensions of this robot can be located on page 62 of its user guide.

A second part to this question will be added soon.

Solution 2

The IK solution are obtained according to the DH frames assignments made in Figure 8. As illustrated in the figure, origins of frames 0, 1, and 2 are aligned while the rest are aligned with each other. The corresponding DH parameters are:

Link	a_i	α_i	d_i	θ_i
1	0	90°	d_1	θ_1
2	a_2	0°	0	θ_2
3	0	90°	$-d_3$	θ_3
4	0	-90°	d_4	θ_4
5	0	90°	0	θ_5
6	0	0°	d_6	θ_6

The values of these parameters from the specification sheet are: $d_1 = 156.4 + 128.4$, $a_2 = 410$, $d_3 = 1$, $d_4 = 208.4 + 105.9$, and $d_6 = 105.9 + 61.5$

Figure 7: Kinova Gen3 6 DOF Arm

1. Kinematic Decoupling. Since the manipulator has a spherical wrist, we can kinematically decouple the wrist from the arm. For the IK, we're provided:

$$^0T_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & x \\ r_{21} & r_{22} & r_{23} & y \\ r_{31} & r_{32} & r_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The coordinates of the wrist center, i.e. the origin of frames 4 and 5, can be obtained in the 0 frame as:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x - d_6 r_{13} \\ y - d_6 r_{23} \\ z - d_6 r_{33} \end{bmatrix}.$$

2. Inverse Position Kinematics. We'll determine the values of the first three joint variables, given the position of the wrist center. Both the algebraic and geometric approach are provided here.

2a. Algebraic Approach. Based on the DH parameters, we can determine $^0T_4$ as O_4 is the

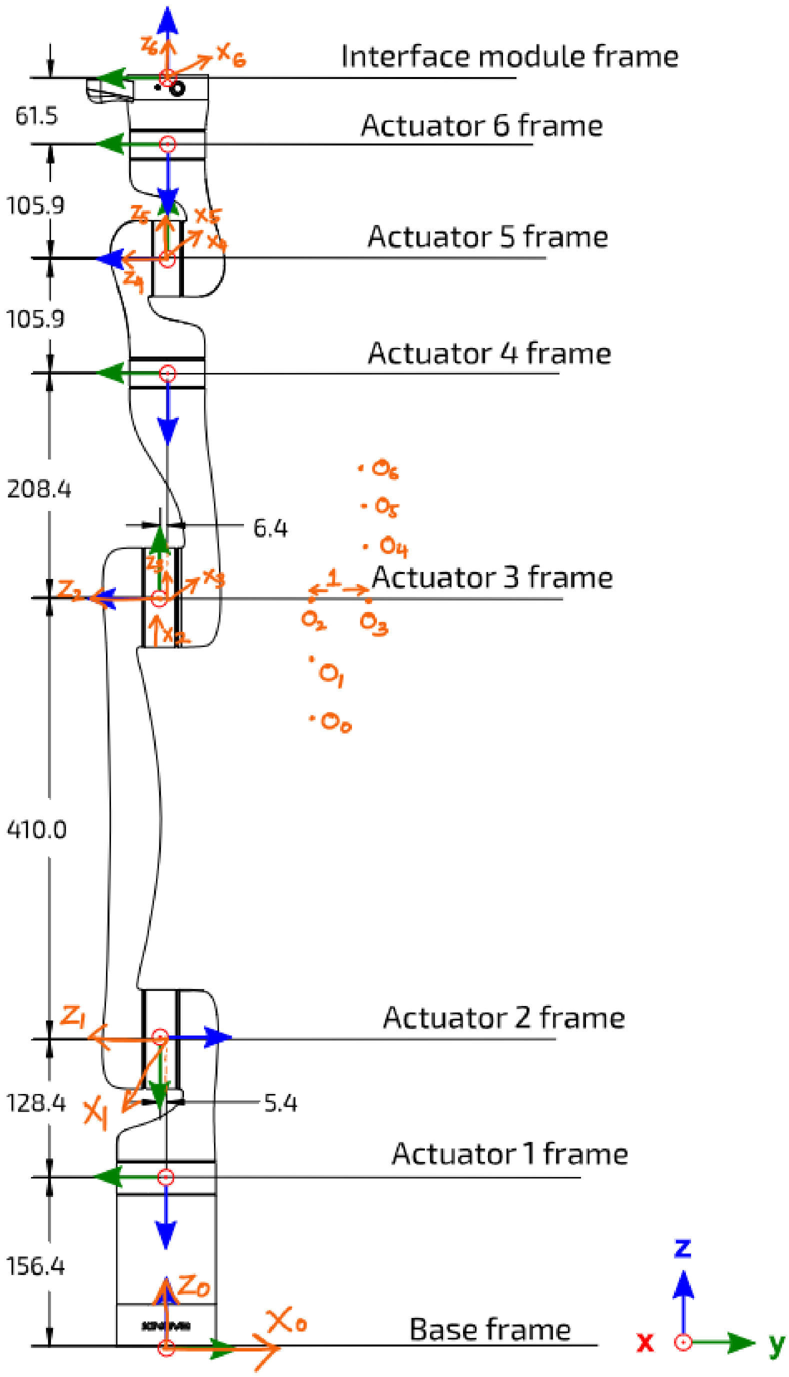

Figure 8: DH Frame Assignment of Kinova Gen3

wrist center.

$$\begin{aligned} x_c &= (d_4 \sin(\theta_2 + \theta_3) + a_2 \cos\theta_2)\cos\theta_1 - d_3 \sin\theta_1 \\ y_c &= (d_4 \sin(\theta_2 + \theta_3) + a_2 \cos\theta_2)\sin\theta_1 + d_3 \cos\theta_1 \\ z_c &= d_1 - d_4 \cos(\theta_2 + \theta_3) + a_2 \sin\theta_2 \end{aligned}$$

Let

$$\begin{aligned} k\cos\psi &= d_4 \sin(\theta_2 + \theta_3) + a_2 \cos\theta_2 \\ k\sin\psi &= d_3. \end{aligned}$$

This is simply using polar coordinates to represent a pair of real numbers. Note that $k \geq 0$. Then,

$$\begin{aligned} x_c &= k\cos(\theta_1 + \psi) \\ y_c &= k\sin(\theta_1 + \psi). \\ \Rightarrow \theta_1 + \psi &= \arctan 2(y_c, x_c) \end{aligned}$$

From above,

$$\begin{aligned} \sin\psi &= \frac{d_3}{k} \\ \cos\psi &= \pm\sqrt{1 - \frac{d_3^2}{k^2}} \\ \Rightarrow \psi^{1,2} &= \arctan 2\left(\frac{d_3}{k}, \pm\sqrt{1 - \frac{d_3^2}{k^2}}\right) \\ k &= \sqrt{x_c^2 + y_c^2} \\ \Rightarrow \theta_1 &= \arctan 2(y_c, x_c) - \arctan 2\left(\frac{d_3}{k}, \pm\sqrt{1 - \frac{d_3^2}{k^2}}\right) \end{aligned}$$

The two solutions for θ_1 are:

$$\begin{aligned} \Rightarrow \theta_1^1 &= \arctan 2(y_c, x_c) - \arctan 2\left(\frac{d_3}{k}, \sqrt{1 - \frac{d_3^2}{k^2}}\right) \\ \theta_1^2 &= \arctan 2(y_c, x_c) - 180^\circ + \arctan 2\left(\frac{d_3}{k}, \sqrt{1 - \frac{d_3^2}{k^2}}\right) \end{aligned}$$

We can find solutions for θ_3 as follows:

$$\begin{aligned} (k\cos\psi)^2 + (z_c - d_1)^2 &= a_2^2 + d_4^2 + 2a_2 d_4 \sin\theta_3 \\ \Rightarrow \sin\theta_3 &= \frac{(k\cos\psi)^2 + (z_c - d_1)^2 - a_2^2 - d_4^2}{2a_2 d_4} := u \\ \cos\theta_3 &= \pm\sqrt{1 - u^2} \end{aligned}$$

The two solutions of θ_3 are:

$$\theta_3^{a,b} = \arctan 2(u, \pm\sqrt{1-u^2})$$

Notice that this is the same technique used for the 2R arm in class. θ_2 can now be determined following the same strategy:

$$\begin{aligned} z_c - d_1 &= (a_2 + d_4 \sin\theta_3)\sin\theta_2 - d_4\cos\theta_3\cos\theta_2 \\ k\cos\psi &= (a_2 + d_4\sin\theta_3)\cos\theta_2 + d_4\cos\theta_3\sin\theta_2 \end{aligned}$$

Define $l \geq 0$ and ϕ such that,

$$\begin{aligned} z_c - d_1 &= l\cos\phi\sin\theta_2 - l\sin\phi\cos\theta_2 = l\sin(\theta_2 - \phi) \\ k\cos\psi &= l\cos\phi\cos\theta_2 + l\sin\phi\sin\theta_2 = l\cos(\theta_2 - \phi) \\ \Rightarrow \theta_2 &= \arctan 2(z_c - d_1, k\cos\psi) + \phi \end{aligned}$$

The corresponding l and ϕ are defined as:

$$\begin{aligned} l\cos\phi &= a_2 + d_4\sin\theta_3 \\ l\sin\phi &= d_4\cos\theta_3 \\ \Rightarrow l &= \sqrt{(a_2 + d_4\sin\theta_3)^2 + (d_4\cos\theta_3)^2} \\ \phi &= \arctan 2(d_4\cos\theta_3, a_2 + d_4\sin\theta_3) \end{aligned}$$

Consequently,

$$\theta_2^{a,b} = \arctan 2(z_c - d_1, k\cos\psi) + \arctan 2(d_4\cos\theta_3^{a,b}, a_2 + d_4\sin\theta_3^{a,b}).$$

In the solutions of both θ_2 and θ_3, the angle ψ is being used that has two solutions. Substituting ψ^1 in these expressions will yield two solution pairs $(\theta_2^{1a}, \theta_3^{1a})$ and $(\theta_2^{1b}, \theta_3^{1b})$.

2b. Geometric Approach. A 3D view of the first three joints is attempted in Figure 9. The manipulator projected onto the $x_0 - y_0$ plane is shown in Figure 10. From this figure, it is clear that:

$$\theta_1^1 = \arctan 2(y_c, x_c) - \beta$$

where,

$$\begin{aligned} \sin\beta &= \frac{d_3}{\sqrt{x_c^2 + y_c^2}} \\ \Rightarrow \beta &= \arcsin\left(\frac{d_3}{\sqrt{x_c^2 + y_c^2}}\right). \end{aligned}$$

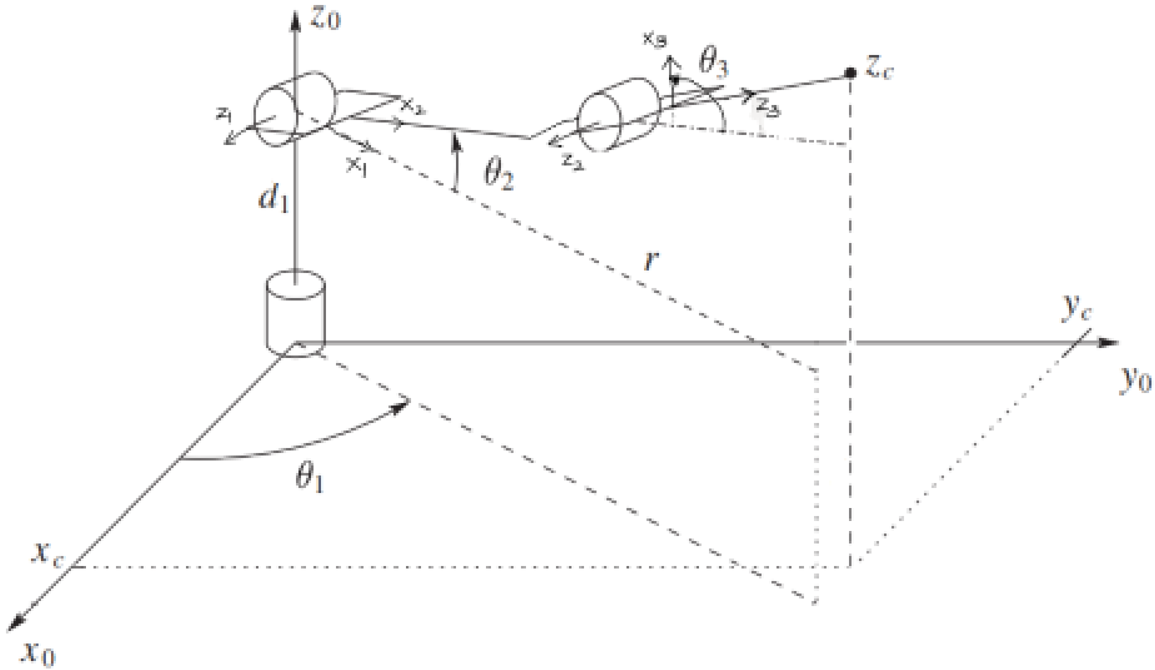

Figure 9: First 3 joints of Gen3 along with DH frames

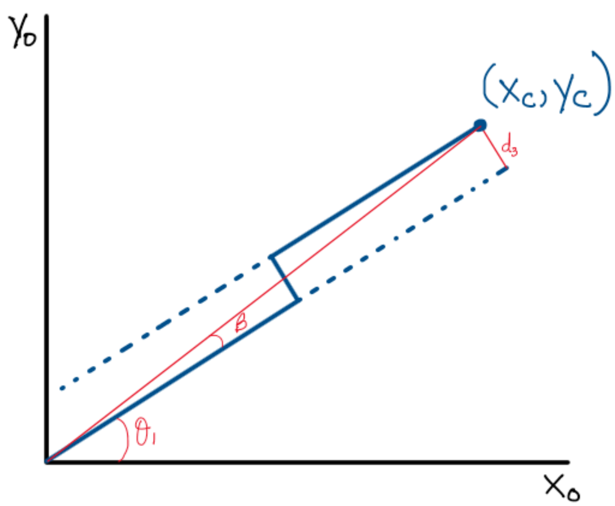

Figure 10: Projection of Gen3 on the $x_0 - y_0$ plane

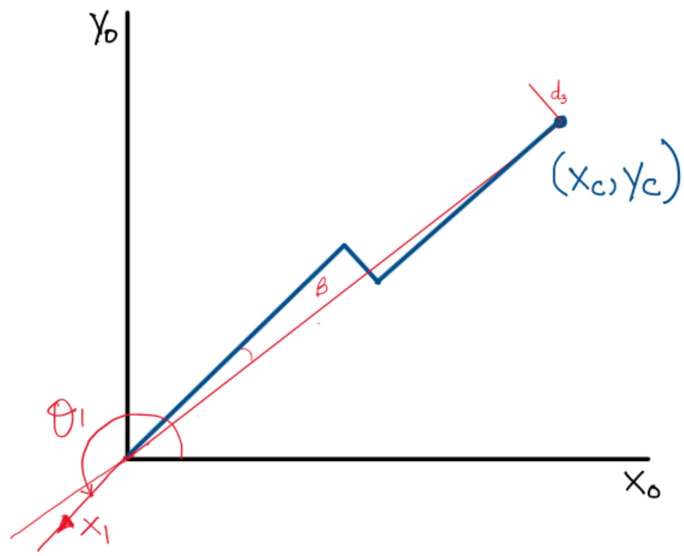

Figure 11: The other solution of θ_1

The other solution is obtained by rotating the arm to the other side and then flipping θ_2. This is illustrated in Figure 11.

$$\theta_1^2 = \arctan 2(y_c, x_c) + 180^\circ + \beta.$$

To find θ_2 and θ_3, the arm is projected onto the $x_1 - y_1$ plane, as shown in Figure 12. Following the class notes geometric approach for 2R arm, the solutions are:

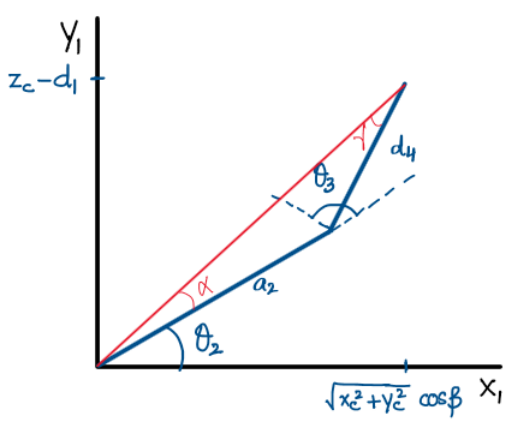

Figure 12: Projection of Gen3 on $x_1 - y_1$ plane

$$\theta_2^{1a} = \arctan 2\left(z_c - d_1, \sqrt{x_c^2 + y_c^2}\cos\beta\right) - \arccos\left(\frac{a_2^2 + (x_c^2 + y_c^2)\cos^2\beta + (z_c - d_1)^2 - d_4^2}{2a_2\sqrt{(x_c^2 + y_c^2)\cos^2\beta + (z_c - d_1)^2}}\right)$$
$$\theta_3^{1a} = 180^\circ - \arccos\left(\frac{a_2^2 + d_4^2 - (x_c^2 + y_c^2)\cos^2\beta - (z_c - d_1)^2}{2a_2 d_4}\right) + 90^\circ$$

The other solution is:

$$\theta_2^{1a} = \arctan 2\left(z_c - d_1, \sqrt{x_c^2 + y_c^2}\cos\beta\right) + \arccos\left(\frac{a_2^2 + (x_c^2 + y_c^2)\cos^2\beta + (z_c - d_1)^2 - d_4^2}{2a_2\sqrt{(x_c^2 + y_c^2)\cos^2\beta + (z_c - d_1)^2}}\right)$$
$$\theta_3^{1a} = -180^\circ + \arccos\left(\frac{a_2^2 + d_4^2 - (x_c^2 + y_c^2)\cos^2\beta - (z_c - d_1)^2}{2a_2 d_4}\right) + 90^\circ$$

The solutions for rotated θ_1 are:

$$\theta_2^{2a,2b} = 180^\circ - \theta_2^{1a,1b}$$
$$\theta_3^{2a,2b} = -\theta_3^{1a,1b}.$$

3. Inverse Orientation Kinematics. See class slides.

Problem 3
CLO2-C4

30 points

In this task, you'll find closed-form expressions for all the inverse kinematics solutions for the Universal Robotics' UR5e arm, introduced to you in the previous homework and being utilized for RoboCup ARM Challenge 2024. As you've determined in the previous homework, this arm does not employ a spherical wrist thereby eliminating the possibility of decoupling the kinematics as shown in class. You'll have to explore yourself the possibilities of any decoupling and strategies for solving inverse kinematics problem.

Problem 4
CLO2-C2

5+10 points

Answer the following questions individually:

(a) How many hours did each of you spend on this homework and specifically state your contribution in this homework assignment? Answer as accurately as you can, as this will be used to structure next year's class.

(b) Do you have any specific advice for students attempting this homework next year?

(c) **This question has been revised compared to the previous homework assignments.** Each group member is to provide their reflections as answers to each of the following questions. You are expected to be precise in your responses.

1. Explain each of the outcomes, stated at the beginning of this document, in your own words.

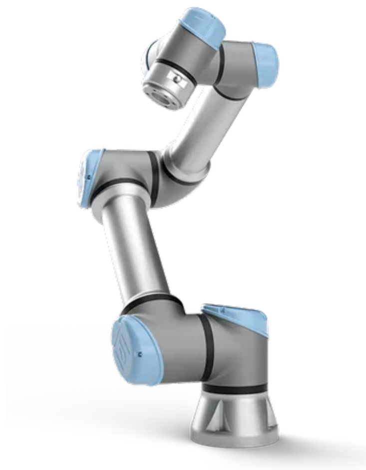

Figure 13: Universal Robotics UR5E Arm

2. Why is it important for you to achieve each of the outcomes in relation to understanding or building any robot?
3. What do you currently understand about content related to these outcomes? Do you have unanswered questions?
4. Have you achieved these outcomes? What went wrong? How will you enable yourself to achieve these outcomes? What could you do to know more or enhance your skills in this context?