Homework 4
Solution

Assigned on March 27, 2024
Due on April 9, 2024

Learning Outcomes:

After this homework, you should be able to:

(a) Determine the angular velocities of rigid bodies in motion;

(b) Determine the Jacobian of manipulators;

(c) Determine the singularities of manipulators, allowing singularity decoupling.

Tasks

A design for a new amusement park ride is shown in Figure 1. The rider sits at the location indicated by the moving frame $\{b\}$. The fixed frame $\{s\}$ is attached to the top shaft connected to the walls, as shown, and does not move. The mentioned dimensions are $R = 10$ m and $L = 20$ m, and the two joints each rotate at a constant angular velocity of 1 rad/s.

Problem 1
CLO1-C3

5+10+10+5+5 points

(a) Suppose $t = 0$ at the instant shown in the figure. What is ${}^sT_b$, the configuration of frame $\{b\}$, as seen from the fixed frame $\{s\}$, at time t?

(b) What is the angular velocity, ${}^s\omega_{sb}$?

(c) What is the linear velocity, ${}^sv_{sb}$?

(d) What is the spatial velocity vector in body coordinates, $\{b\}$?

(e) What is the linear velocity of the rider?

Solution 1

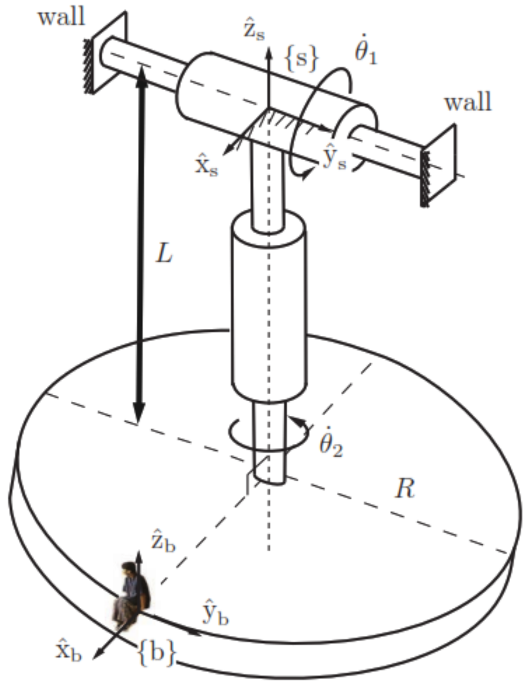

Figure 1: Amusement park ride

(a) Let's assign a frame $\{a\}$ such that its origin is at the center of the disc and its orientation at $t = 0$ is the same as frame $\{s\}$. We'll find the transformation ${}^sT_b$, by finding the two transformations ${}^sT_a$ and ${}^aT_b$ and using the fact that

$$ {}^sT_b = {}^sT_a\,{}^aT_b. $$

The transformation ${}^sT_a$ is only dependent on θ_1 and the transformation ${}^aT_b$ is only dependent on θ_2, making the process simpler.

It is easy to see that ${}^sT_a$ at time t is obtained by rotating frame $\{s\}$ about $\hat{y}_s$ by $\theta_1(t)$ radians, and then translating along $-\hat{z}_a$ by L units. Given that $\dot{\theta}_1 = 1$ rad/s, the angle of rotation at time t is $\theta_1(t) = \int_0^t \dot{\theta}_1(u)du = t$ radians. Thus,

$$ {}^sT_a = \begin{bmatrix} \cos t & 0 & \sin t & 0 \\ 0 & 1 & 0 & 0 \\ -\sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & 0 & \sin t & -L\sin t \\ 0 & 1 & 0 & 0 \\ -\sin t & 0 & \cos t & -L\cos t \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

For ${}^aT_b$ can be obtained by rotating about $\hat{z}_a$ by $\theta_2(t)$, followed by translating along current $\hat{x}$ by R. Thus,

$$^{a}T_{b} = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & R \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 & R\cos t \\ \sin t & \cos t & 0 & R\sin t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally,

$$^{s}T_{b} = \begin{bmatrix} \cos^2 t & -\cos t \sin t & \sin t & R\cos^2 t - L\sin t \\ \sin t & \cos t & 0 & R\sin t \\ -\cos t\sin t & \sin^2 t & \cos t & -L\cos t - R\cos t \sin t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) We can obtain this in two ways.
Method-1: The first way is to use $\dot{R}$, i.e.

$$\begin{aligned} S(^{s}\omega_{sb}) &= {}^{s}\dot{R}_{b}\,{}^{s}R_{b}^{T} \\ &= \begin{bmatrix} -2\cos t\sin t & \sin^2 t - \cos^2 t & \cos t \\ \cos t & -\sin t & 0 \\ \sin^2 t - \cos^2 t & 2\cos t \sin t & -\sin t \end{bmatrix} \begin{bmatrix} \cos^2 t & \sin t & -\cos t \sin t \\ -\cos t \sin t & \cos t & \sin^2 t \\ \sin t & 0 & \cos t \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\cos t & 1 \\ \cos t & 0 & -\sin t \\ -1 & \sin t & 0 \end{bmatrix} \end{aligned}$$

Comparing this to a canonical skew-symmetric matrix,

$$^{s}\omega_{sb} = \begin{bmatrix} \sin t \\ 1 \\ \cos t \end{bmatrix} \text{ rad/s.}$$

Method-2: The second way is to determine taking a vector sum of angular velocities between successive frames, i.e.

$$^{s}\omega_{sb} = {}^{s}\omega_{sa} + {}^{s}R_{a}\,{}^{a}\omega_{ab}$$

From the provided data,

$$\begin{aligned} &= (1 \text{ rad/s}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} (1 \text{ rad/s}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sin t \\ 1 \\ \cos t \end{bmatrix} \text{ rad/s.} \end{aligned}$$

(c) There are again two ways to go about this.
Method-1: We could differentiate the fourth column of ${}^sT_b$ to obtain ${}^sv_{sb}$, i.e.

$$\begin{aligned} {}^sv_{sb} &= {}^s\dot{o}_b \\ &= \begin{bmatrix} -L\cos t - R\sin 2t \\ R\cos t \\ L\sin t - R + 2R\sin^2 t \end{bmatrix} \end{aligned}$$

Method-2: We can use cross-product to find it, i.e.

$$\begin{aligned} {}^sv_{sb} &= {}^s\omega_{sb} \times {}^so_b \\ &= \begin{bmatrix} \sin t \\ 1 \\ \cos t \end{bmatrix} \times \begin{bmatrix} R\cos^2 t - L\sin t \\ R\sin t \\ -L\cos t - R\cos t\sin t \end{bmatrix} \\ &= \begin{bmatrix} -L\cos t - R\sin 2t \\ R\cos t \\ L\sin t - R + 2R\sin^2 t \end{bmatrix} \end{aligned}$$

(d) The spatial velocity vector in body coordinates is obtained by changing the coordinates using ${}^bR_s$, i.e.

$$\begin{aligned} \begin{bmatrix} {}^b\omega_{sb} \\ {}^bv_{sb} \end{bmatrix} &= \begin{bmatrix} {}^bR_s & 0 \\ 0 & {}^bR_s \end{bmatrix} \begin{bmatrix} {}^s\omega_{sb} \\ {}^sv_{sb} \end{bmatrix} \\ &= \begin{bmatrix} \sin t \\ \cos t \\ 1 \\ -L\cos t \\ R + L\sin t \\ -R\cos t \end{bmatrix} \end{aligned}$$

(e) Since the rider is seated at the origin of the body frame, their linear velocity is ${}^bv_{sb}$ in the body-frame coordinates and ${}^sv_{sb}$ in the fixed-frame coordinates.

Problem 2
CLO1-C3
20 points

Given $R \in SO(3)$, $a, b \in \mathbb{R}^3$, and $S \in \mathbb{R}^{3\times 3}$, a skew-symmetric matrix, prove the following identities:

(a) $R(a \times b) = Ra \times Rb$, where $\times$ represents cross-product and Ra is matrix multiplication.

(b) $R\,S(a)\,R^T = S(Ra)$

The two identities are dependent on each other, so you have to prove at least one of them without using the other.

Solution 2

(a) Let the rotation matrix $R = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$, where c_1, c_2, and c_3 are columns of R. Let $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$ and $b = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$. Then,

$$\begin{aligned} Ra &= a_1 c_1 + a_2 c_2 + a_3 c_3 \\ Rb &= b_1 c_1 + b_2 c_2 + b_3 c_3 \\ \Rightarrow Ra \times Rb &= (a_1 c_1 + a_2 c_2 + a_3 c_3) \times (b_1 c_1 + b_2 c_2 + b_3 c_3) \end{aligned}$$

By distributive property of cross-product,

$$= (a_1 c_1 + a_2 c_2 + a_3 c_3) \times b_1 c_1 + (a_1 c_1 + a_2 c_2 + a_3 c_3) \times b_2 c_2 + (a_1 c_1 + a_2 c_2 + a_3 c_3) \times b_3 c_3$$

Since cross-product is anti-commutative,

$$= -b_1 c_1 \times (a_1 c_1 + a_2 c_2 + a_3 c_3) - b_2 c_2 \times (a_1 c_1 + a_2 c_2 + a_3 c_3) - b_3 c_3 \times (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

By distributive property and scalar compatibility of cross-product,

$$\begin{aligned} &= -a_1 b_1 (c_1 \times c_1) - a_2 b_1 (c_1 \times c_2) - a_3 b_1 (c_1 \times c_3) - a_1 b_2 (c_2 \times c_1) - a_2 b_2 (c_2 \times c_2) \\ &\quad - a_3 b_2 (c_2 \times c_3) - a_1 b_3 (c_3 \times c_1) - a_2 b_3 (c_3 \times c_2) - a_3 b_3 (c_3 \times c_3) \end{aligned}$$

Recall that the columns of a rotation matrix are orthonormal, so $c_i \times c_i = 0$, $c_1 \times c_2 = c_3$, $c_2 \times c_3 = c_1$, and $c_3 \times c_1 = c_2$. Thus,

$$\begin{aligned} &= -a_2 b_1 c_3 + a_3 b_1 c_2 + a_1 b_2 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_2 b_3 c_1 \\ &= (a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3 \\ &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \\ &= R(a \times b) \end{aligned}$$

(b) Let $b \in \mathbb{R}^{3 \times 1}$ be an arbitrary vector. Then,

$$\begin{aligned} S(Ra)b &= Ra \times b \\ &= Ra \times R(R^T b) \end{aligned}$$

By the property in (a),

$$= R(a \times R^T b)$$
$$S(Ra)b = RS(a)R^T b$$

Since the above equality is true for arbitrary b, we have that

$$S(Ra) = RS(a)R^T$$

Problem 3
CLO2-C3

30 points

In this problem, you'll obtain the Jacobian for the Universal Robotics' UR5e arm from the previous homework assignments. You're welcome to directly use homogeneous transformation derived for this arm in your previous work, without re-derivation, for the purposes of writing the Jacobian.

(a) Use the position of the end-effector in the base frame, as provided by the homogeneous transformation, ${}^0T_6$, to calculate the 3×6 linear velocity Jacobian J_v for this manipulator.

(b) Use the expressions derived in terms of the origins, O_i, and the z-axes, $\hat{z}_i$, to now write the same linear velocity Jacobian J_v. Your answer should be the same as the previous part.

(c) Find the 3×6 angular velocity Jacobian, J_ω, for this manipulator.

(d) Write the complete Jacobian matrix, J.

(e) Determine all the singular configurations of this arm, if any. You can use the idea of reassigning the frames to decouple the position and orientation parts of the Jacobian, and easily derive the singularities. A discussion of the singular configurations of this arm is located at this link.

(f) Sketch the arm in each singular configuration and indicate the velocity direction(s) that cannot be instantaneously achieved in that configuration.

Solution 3 See the associated `hw4.mlx` file for the solution of parts a, b, c, and d. For (e), this robot does not have a spherical wrist that will allow us to simplify our determinant finding process. The best we can do is align O_6 with O_5 as the linear velocity field of O_6 is identical to O_5. Since, we're using MATLAB we can also computationally determine the determinant of our Jacobian, which is determined to be the following by MATLAB:

$$\begin{aligned}
\det(J) &= -\frac{a_2 a_3 \sin\theta_5}{2}\,[-a_2\sin(\theta_2+\theta_3) + a_2\sin(\theta_2-\theta_3) - a_3\sin(\theta_2+2\theta_3) + a_3\sin\theta_2 \\
&\qquad -d_5\cos(\theta_2+\theta_4) + d_5\cos(\theta_2+2\theta_3+\theta_4)] \\
&= -\frac{a_2 a_3 \sin\theta_5}{2}\,[-2a_2\cos\theta_2\sin\theta_3 - a_3\sin(\theta_2+\theta_3+\theta_3) + a_3\sin(\theta_2+\theta_3-\theta_3) \\
&\qquad -d_5\cos(\theta_2+\theta_4+\theta_3-\theta_3) + d_5\cos(\theta_2+\theta_4+\theta_3+\theta_3)] \\
&= -\frac{a_2 a_3 \sin\theta_5}{2}\,[-2a_2\cos\theta_2\sin\theta_3 - 2a_3\cos(\theta_2+\theta_3)\sin\theta_3 - 2d_5\sin(\theta_2+\theta_3+\theta_4)\sin\theta_3] \\
&= a_2 a_3\sin\theta_3\sin\theta_5\,[a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) + d_5\sin(\theta_2+\theta_3+\theta_4)]
\end{aligned}$$

A necessary and sufficient condition for singular configurations is that the determinant of Jacobian is zero. In this case, this is possible in one or more of the following cases:

1. $\sin\theta_3 = 0 \implies \theta_3 = n\pi$ (Elbow Singularity)
2. $\sin\theta_5 = 0 \implies \theta_5 = n\pi$ (Wrist Singularity)
3. $a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) + d_5\sin(\theta_2+\theta_3+\theta_4) = 0$ (Shoulder Singularity)

Since the Jacobian loses its rank at each of these conditions, the robot will be unable to instantaneously generate some linear velocities or angular velocities independently. This can be identified by studying the Jacobian. In order to make the expressions simpler, we can express the Jacobian in frame 1 instead of frame 0 by multiplying it with ${}^1R_0$. These directions are identified in MATLAB with the help of appropriate manipulations of the Jacobian.

Problem 4
CLO2-C2
5+10 points

Answer the following questions individually:

(a) How many hours did each of you spend on this homework and specifically state your contribution in this homework assignment? Answer as accurately as you can, as this will be used to structure next year's class.

(b) Do you have any specific advice for students attempting this homework next year?

(c) **This question has been revised compared to the previous homework assignments.** Each group member is to provide their reflections as answers to each of the following questions. You are expected to be precise in your responses.

   1. Explain each of the outcomes, stated at the beginning of this document, in your own words.
   2. Why is it important for you to achieve each of the outcomes in relation to understanding or building any robot?

3. What do you currently understand about content related to these outcomes? Do you have unanswered questions?
4. Have you achieved these outcomes? What went wrong? How will you enable yourself to achieve these outcomes? What could you do to know more or enhance your skills in this context?