Representing rigid motions

EE366/CE366/CS380: Introduction to Robotics

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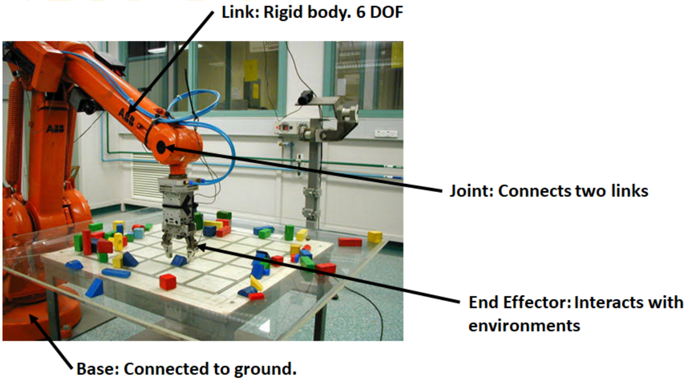


- Physical body of robot → Links (rigid) + Joints
- Configuration of robot = Configuration of all links

Configuration of rigid body

It is completely described by its position and orientation.

How to represent position and orientation of rigid body?

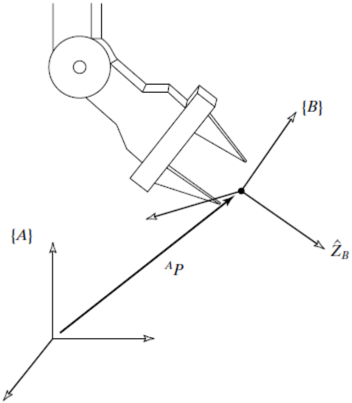


FIGURE 2.2: Locating an object in position and orientation.

Figure: Source: Introduction to Robot Mechanics and Control

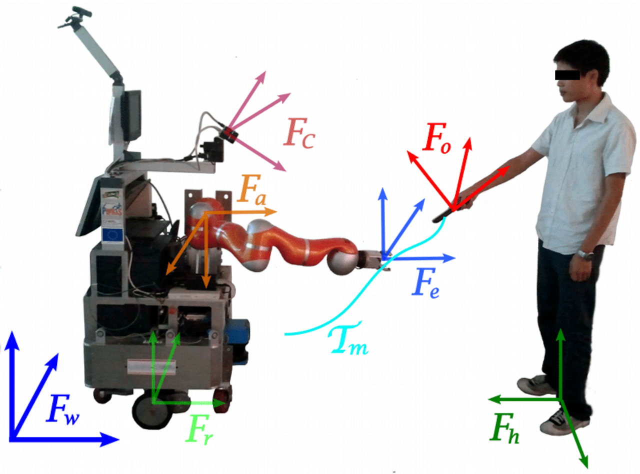
How to represent position and orientation of rigid body?

Attach reference frame to body and describe its pose with respect to a fixed reference frame.

Pose

The information of position and orientation collected together is called a **frame** or a **pose**.

Robot motion is all about reference frames relative to each other.



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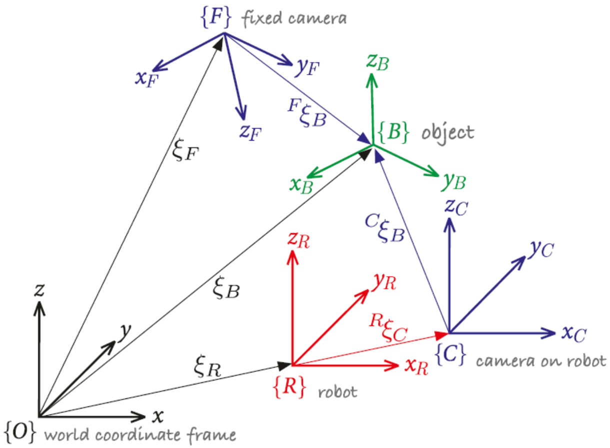

Figure: Source: Robotics, Vision, and Control

- Fixed Frame/World Reference: {S}
- Base frame: {B}

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- 2×1 position vector

$$^{A}P = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

- Superscript indicates coordinate axes or frame information.

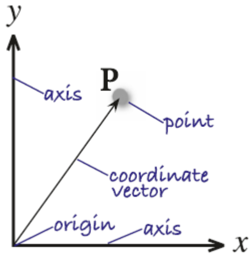

Figure: Source: Robotics, Vision, and Control

p exists in physical space and doesn't care about representation.

- $$^{A}P = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$
  $$^{B}P = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$
- Origin will have coordinates in different reference frames.
  $$^{A}O_B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$
  $$^{B}O_A = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

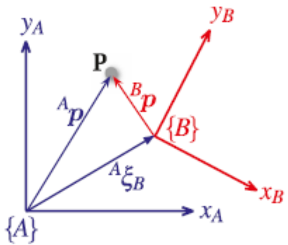

Figure: Source: Robotics, Vision, and Control

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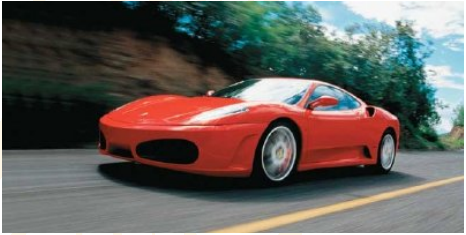
Representation of vectors is different. Vectors are free.



- Velocity of your car is measured as 45 km/h by the speed gun on Shuhada-e-ASF road.
- Every measurement requires context:
  - Unit system (e.g. meters, hour)
  - Number system (e.g. base 10)
  - Coordinate system (e.g. north, east)
  - Reference frame to which measurement is ascribed (e.g. car)
  - Reference system with respect to which measurement is made (e.g. speed gun)

Is coordinate system same as frame of reference? [?, Section 4.1]

- **Coordinate systems** are conventions for representation.
- A **reference frame** is a state of motion, which is linked to a moving body for convenience.
- We use laws of physics to convert among frames, while laws of physics hold regardless of coordinate system.

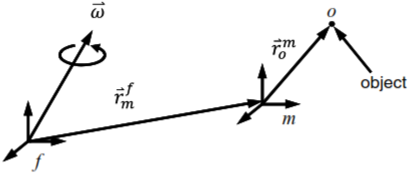
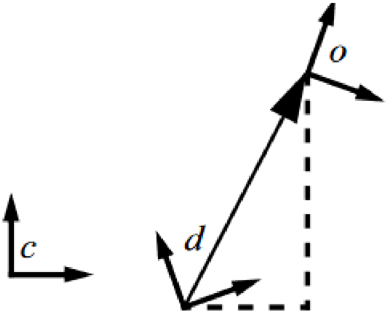


Figure: Transformation between frames

$${}^{c}r_{o}^{d}$$



r: physical quantity / property
o: object possessing property
d: object whose state of motion serves as datum
c: object whose coordinate system is used to express result

Describing the orientation of frame is not straightforward. [1]

- Angle θ of body frame wrt some fixed frame.
  - Discontinuity at $\theta = 0$
  - Doesn't scale well to 3D case.
- Specify the coordinate vectors for the axes of body frame with respect to the fixed frame.
  - Less simple, but scales well.

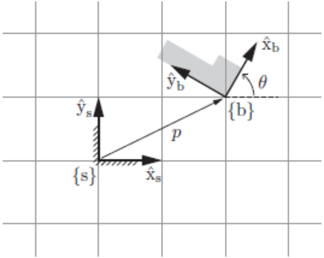


Figure: Source: Modern Robotics

- $\hat{x}_b = \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s$
- $\hat{y}_b = -\sin\theta\, \hat{x}_s + \cos\theta\, \hat{y}_s$
- Orientation representation

$$
\begin{aligned}
{}^sR_b &= \begin{bmatrix} {}^s\hat{x}_b & {}^s\hat{y}_b \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_b \cdot \hat{x}_s & \hat{y}_b \cdot \hat{x}_s \\ \hat{x}_b \cdot \hat{y}_s & \hat{y}_b \cdot \hat{y}_s \end{bmatrix}
\end{aligned}
$$

- Called the **Rotation matrix**.

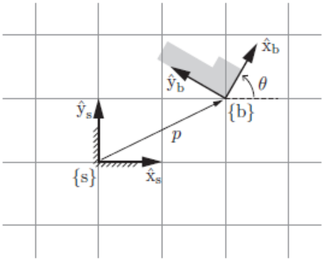


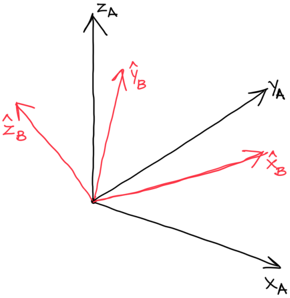
Figure: Source: Modern Robotics

Pair $({}^{s}P, {}^{s}R_b)$ can describe pose of a reference frame.

- Dof of planar end-effector (rigid body) is 3, but we're using 6 numbers here!
- Any rotation matrix, $R \in \mathbb{R}^{2\times 2}$ with columns c_i, has 3 constraints.
  - Each column is a unit vector, i.e. $||c_i|| = 1, \text{ for } i \in \{1, 2\}$.
  - Two columns are orthogonal to each other, i.e. $c_1^T c_2 = 0$.

Spatial orientation can also be described by rotation matrix.

- We want to describe the orientation of frame $\{B\}$ with respect to frame $\{A\}$, i.e. ${}^A R_B$



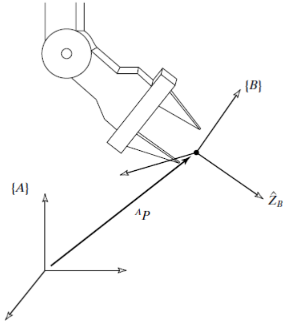


FIGURE 2.2: Locating an object in position and orientation.

Figure: Source: Introduction to Robot Mechanics and Control

Spatial orientation can also be described by rotation matrix.

$$
\begin{aligned}
{}^{A}R_B &= \begin{bmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{bmatrix}
\end{aligned}
$$

How many constraints are there?

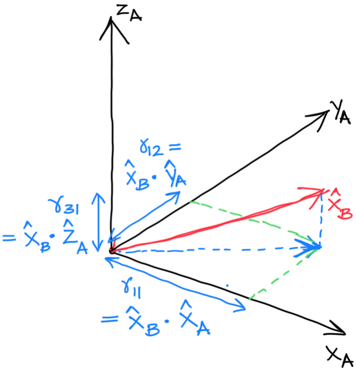
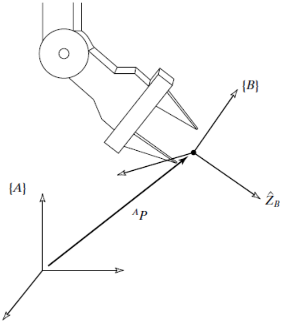


Figure: Finding coordinates of $\hat{x}_B$ in frame $\{A\}$

- Implicit representation with 12 numbers.



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Convention is to use right-handed frames.

- The $\hat{x}$, $\hat{y}$, and $\hat{z}$ of right-handed reference frame are aligned with index finger, middle finger, and thumb respectively.

- Positive rotation along an axis is in direction the fingers of right-hand curl when thumb is pointed along axis.

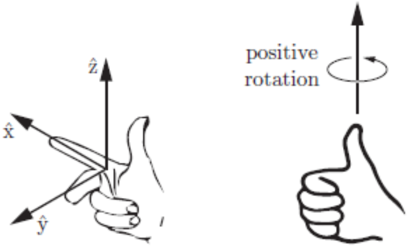
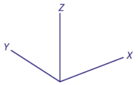


Figure: Source: Modern Robotics

Spatial rotations require description of both angle and axis.

- Right-hand rule for rotation
- $\pi/2$ about x-axis?
- $\pi/2$ about z-axis?



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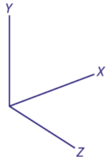
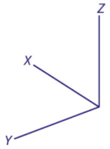
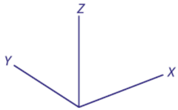




Example Rotation in 3D

- What is rotation matrix for rotation of $\pi/2$ about x-axis?
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$



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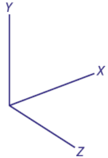

Canonical Rotation Matrices

- $$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Special notation – subscript for axis of rotation; angle of rotation θ

What do other rotation matrices mean?

- Rotation about a non-Cartesian axis.
- Which axis? By what angle? More on this later.

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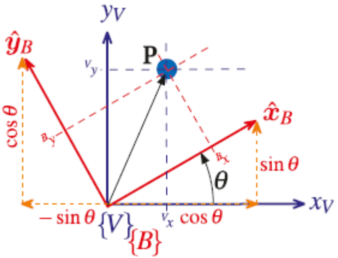
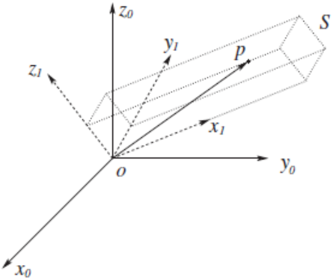
Let $P = B_x\,\hat{x}_B + B_y\,\hat{y}_B$. What are its coordinates in frame $\{V\}$?

$$
\begin{aligned}
{}^V P &= \begin{bmatrix} P\cdot\hat{x}_V \\ P\cdot\hat{y}_V \end{bmatrix} \\
&= \begin{bmatrix} (u\,\hat{x}_B + w\,\hat{y}_B)\cdot\hat{x}_V \\ (u\,\hat{x}_B + w\,\hat{y}_B)\cdot\hat{y}_V \end{bmatrix} \\
&= \begin{bmatrix} u\,\hat{x}_B\cdot\hat{x}_V + w\,\hat{y}_B\cdot\hat{x}_V \\ u\,\hat{x}_B\cdot\hat{y}_V + w\,\hat{y}_B\cdot\hat{y}_V \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_B\cdot\hat{x}_V & \hat{y}_B\cdot\hat{x}_V \\ \hat{x}_B\cdot\hat{y}_V & \hat{y}_B\cdot\hat{y}_V \end{bmatrix}\begin{bmatrix} u \\ w \end{bmatrix} \\
&= {}^V R_B\,{}^B P
\end{aligned}
$$



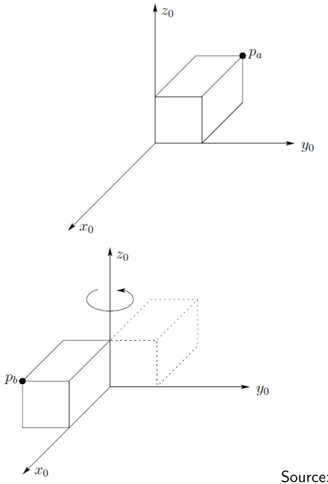
Let ${}^1p = u\,\hat{x}_1 + v\,\hat{y}_1 + w\,\hat{z}_1$.

$$
\begin{aligned}
{}^0p &= \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
&= {}^0R_1 \; {}^1p
\end{aligned}
$$



R can also describe rigid-body rotation. [1]

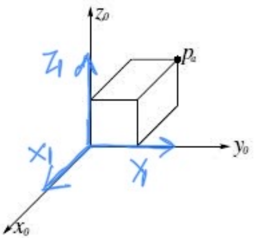
- Box has rotated about z_0 by π.
- Point p_a has moved to p_b.
- We can find coordinates of p_b in the same reference frame 0, if we know p_a and R capturing rotation.

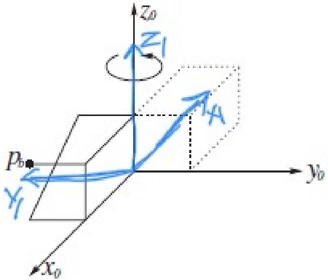


Source: Robot Modeling and Control

- Let's attach a body frame $\{1\}$ to rigid body coincident with existing frame $\{0\}$.
- As body rotates, body frame rotates along with it and we can write ${}^0R_1$.
- ${}^1p_b = {}^0p_a$
-

$$
\begin{aligned}
{}^0p_b &= {}^0R_1\,{}^1p_b \\
&= {}^0R_1\,{}^0p_a
\end{aligned}
$$





How could R cause rotation, in general?

- Any rotation matrix R gives coordinates of rotated frame in a fixed frame, i.e.

$$R = \begin{bmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \end{bmatrix}$$

- This same matrix R also rotates vector, e.g. if ${}^{A}\hat{x}_B = (a_1, a_2, a_3)^T$, then we already know that R moves the vector $(1, 0, 0)^T$ in $\{A\}$ to $(a_1, a_2, a_3)^T$ in $\{A\}$.

- Given an arbitrary vector $p = (p_x, p_y, p_z)^T$ in $\{A\}$, we'll have that

$$\begin{aligned} Rp &= R\left(p_x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + R\left(p_y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + R\left(p_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= p_x\, {}^{A}\hat{x}_B + p_y\, {}^{A}\hat{y}_B + p_z\, {}^{A}\hat{z}_B \end{aligned}$$

Three interpretations of a rotation matrix

- Orientation of transformed coordinate frame with respect to a fixed coordinate frame.
- Coordinate transformation relating the coordinates of a point in two different frames.
- Operator acting on a vector and rotating it to give another vector in same coordinate frame.

The MATLAB script in course LMS module illustrates these interpretations in the planar case.

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Pose of a reference frame can be described by the pair (d, R).

- Recall that a pair (d, R) where $d \in \mathbb{R}^3$ and $R \in SO(3)$ can describe the pose of any coordinate frame with respect to a fixed coordinate frame.
  - $d \in \mathbb{R}^{3\times 1}$ is a vector that tells us where to place the origin of frame $\{B\}$ in terms of coordinates of frame $\{A\}$.
  - The first column of R tells us how to draw the x-axis of frame $\{B\}$ in terms of coordinates of frame $\{A\}$, and so on.
- But it has more power than that.

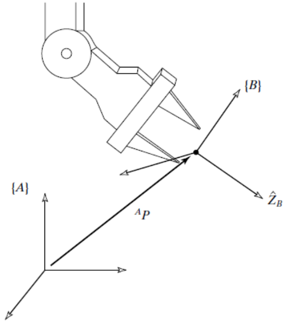
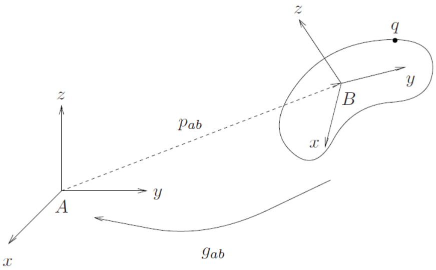


FIGURE 2.2: Locating an object in position and orientation.

Figure: Source: Introduction to Robot Mechanics and Control

Rigid Motions



- Any rigid motion is a pure translation together with a pure rotation.
- Rigid motion is equivalent to the pair (d, R) where $d \in \mathbb{R}^3$ and $R \in SO(3)$.

Special Euclidean Group – $SE(n)$

$$SE(n) = \{(d, R) : d \in \mathbb{R}^n, R \in SO(n)\}$$

- $SE(3)$ is set of all rigid motions.

- We're given coordinates of P in $\{B\}$ coordinate system,

$$^{B}P = u\,\hat{x}_B + w\,\hat{y}_B,$$

and want to know ^{A}P.

- Imagine a frame $\{V\}$ with same orientation as $\{A\}$ but origin coinciding with $\{B\}$.

$$^{V}P = {}^{V}R_B\, {}^{B}P$$

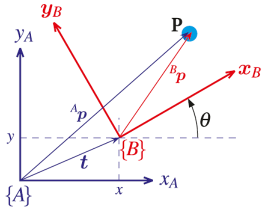
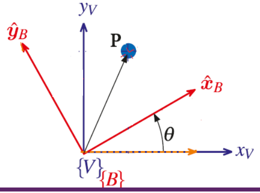


Figure: Source: Robotics, Vision and Control



(d, R) pair can also change coordinates representation of P.

- $\begin{bmatrix} {}^A P_x \\ {}^A P_y \end{bmatrix} = \begin{bmatrix} {}^V P_x \\ {}^V P_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$
- ${}^A P = {}^V R_B \, {}^B P + t$
- ${}^A P = {}^A R_B \, {}^B P + {}^A O_B$

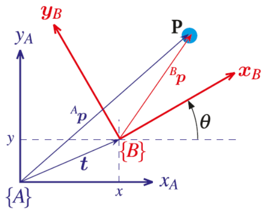


Figure: Source: Robotics, Vision and Control

Homogeneous Transformation

- ${}^{A}P = {}^{A}R_B\, {}^{B}P + {}^{A}O_B$
- $\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_B & {}^{A}O_B \\ \mathbf{O} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$
- 4×4 matrix is called **homogeneous transformation,** ${}^{A}T_B$.
- $\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$ is **homogeneous representation** of vector ${}^{B}P$.

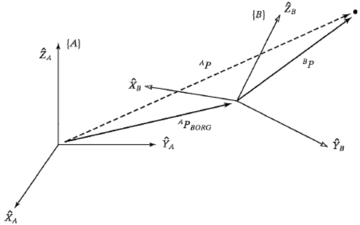


Figure: Source: Intro to Robotics, Mechanics and Control

Three interpretations of (d, R)

1. Configuration of rigid body.
2. Coordinate transformation relating coordinates of point in two different arbitrary frames.
3. Acts on points or vectors in a rigid body to displace them.

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- Say we have three frames, $\{A\}$, $\{B\}$, and $\{C\}$, and a point p.

$$
\begin{aligned}
{}^{A}p &= {}^{A}R_{B}\,{}^{B}p \\
{}^{B}p &= {}^{B}R_{C}\,{}^{C}p \\
\Rightarrow {}^{A}p &= {}^{A}R_{B}\,{}^{B}R_{C}\,{}^{C}p
\end{aligned}
$$

But,

$$
{}^{A}p = {}^{A}R_{C}\,{}^{C}p
$$

So,

$$
{}^{A}R_{C} = {}^{A}R_{B}\,{}^{B}R_{C}
$$

Homogeneous transformations can also be compounded.

$$
\begin{aligned}
{}^{B}p &= {}^{B}T_{C}\, {}^{C}p \\
{}^{A}p &= {}^{A}T_{B}\, {}^{B}p \\
&= {}^{A}T_{B}\, {}^{B}T_{C}\, {}^{C}p
\end{aligned}
$$

But, ${}^{A}p = {}^{A}T_{C}\, {}^{C}p$. This holds for any frames A, B, and C and arbitrary p.

$$
\begin{aligned}
\Rightarrow {}^{A}\mathbf{T_C} &= {}^{A}\mathbf{T_B}\, {}^{B}\mathbf{T_C} \\
&= \begin{bmatrix} {}^{A}R_{B}\, {}^{B}R_{C} & {}^{A}R_{B}\, {}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \mathbf{0} & 1 \end{bmatrix}
\end{aligned}
$$

What do transforms get us?

- What is ${}^{T}T_{G}$?
- ${}^{B}T_{T}$ is known.
- ${}^{B}T_{S}$ is known.
- ${}^{S}T_{G}$ is known.

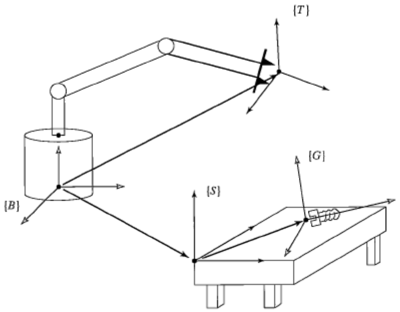

Figure: Source: Intro to Robotics, Mechanics and Control

- ${}^{B}T_{T}\,{}^{T}T_{G} = {}^{B}T_{S}\,{}^{S}T_{G}$
- ${}^{T}T_{G} = {}^{B}T_{T}^{-1}\,{}^{B}T_{S}\,{}^{S}T_{G}$

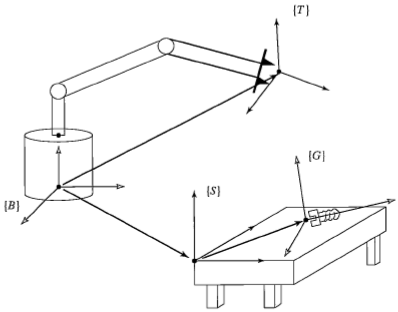

Figure: Source: Intro to Robotics, Mechanics and Control

What is inverse of homogeneous transformation, ${}^{A}T_{B}$?

Table of Contents

[1] **Mark W Spong, Seth Hutchinson, and Mathukumalli Vidyasagar.**
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