Rotations

EE366/CE366/CS380: Introduction to Robotics

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January 24, 2023

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What is inverse of homogeneous transformation, ${}^{A}T_{B}$?

$$
\begin{aligned}
{}^{B}\mathbf{T_A} &= ?\\
{}^{B}T_A\,{}^{A}T_B &= I\\
\begin{bmatrix} {}^{B}R_A\,{}^{A}R_B & {}^{B}R_A\,{}^{A}P_{BORG} + {}^{B}P_{AORG} \\ \mathbf{O} & 1 \end{bmatrix} &= \begin{bmatrix} I & \mathbf{O} \\ \mathbf{O} & 1 \end{bmatrix}\\
{}^{B}R_A &= {}^{A}R_B^{-1}\\
{}^{B}P_{AORG} &= -\,{}^{B}R_A\,{}^{A}P_{BORG}\\
&= -\,{}^{A}R_B^{-1}\,{}^{A}P_{BORG}
\end{aligned}
$$

Let $R \in \mathbb{R}^{3\times 3}$ be a rotation matrix and $c_1, c_2, c_3 \in \mathbb{R}^3$ be its columns.

- Columns are mutually orthonormal:

$$c_i^T c_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

- In matrix form,

$$RR^T = R^T R = I$$

- It follows that

$$\det R = \pm 1.$$

Inverse of R

Inverse of rotation matrix is its transpose.

Special Orthogonal Group – $SO(n)$

The set of $n \times n$ real-valued matrices

$$SO(n) = \{R \in \mathbb{R}^{n\times n} | R^T R = RR^T = I \text{ and } \det R = +1\}$$

What's magical about $SO(3)$ and $SE(3)$?

- Every configuration of a rigid body that is free to only rotate relative to a fixed frame can be identified with $R \in SO(3)$.
- $SO(3)$ is group, closed under multiplication, i.e. the product of any two rotation matrices is another rotation matrix.

- Every configuration of a rigid body can be identified with $T \in SE(3)$.
- $SE(3)$ is group, closed under multiplication, i.e. the product of any two homogeneous transformations is another homogeneous transformation.

What's magical about $SO(3)$ and $SE(3)$?

- Inverse of rotation matrix is a rotation matrix. In fact,

$$R^{-1} = R^T$$

- Product is associative.

- Inverse of homogeneous transformation is another homogeneous transformation.

- Product is associative.

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- $$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Special notation – subscript for axis of rotation; angle of rotation θ

Order of rotation matters in 3D.

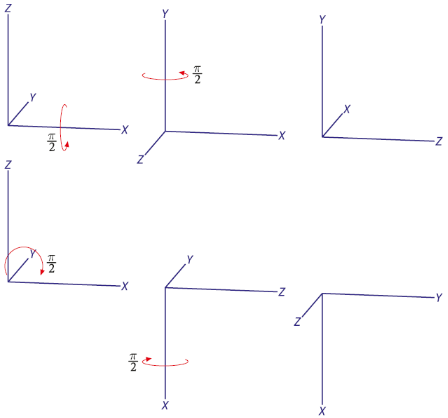
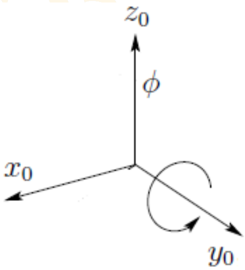


Figure: Rotation in 3D is not commutative

Physical Interpretation of order of rotation [2]

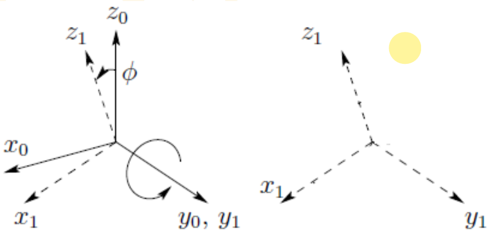
- Rotation is not commutative.
- What is the interpretation of $R_1 R_2$ versus $R_2 R_1$?

Right multiplication is rotation about current axis.

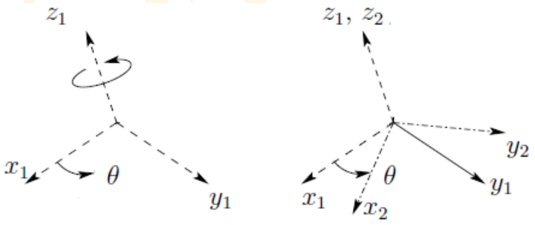


- $R = R_y(\phi)R_z(\theta)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about z by an angle θ.
- Imagine we have two coincident frames 0,1. We rotate frame 1 about y_0 by ϕ to get ${}^0R_1$.

Right multiplication is rotation about current axis.

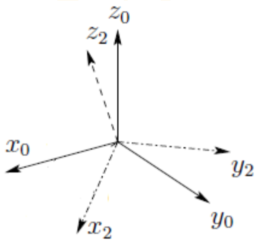


- $R = R_y(\phi)R_z(\theta)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about z by an angle θ.
- Imagine we have two coincident frames 0,1. We rotate frame 1 about y_0 by ϕ to get ${}^0R_1$.



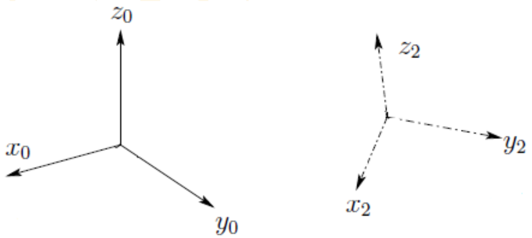
- $R = R_y(\phi)R_z(\theta)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about z by an angle θ.
- Another frame 2 is coincident with 1, and we rotate frame 2 about z_1 by θ to get ${}^1R_2$.

Right multiplication is rotation about current axis.

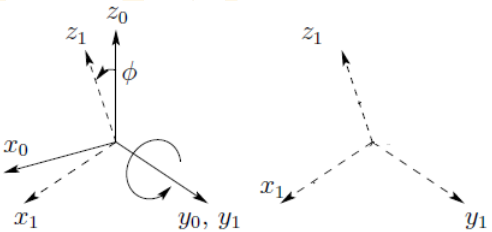


- $R = R_y(\phi)R_z(\theta)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about z by an angle θ.
- So, it makes sense that R represents rotation from 0 to 2 as ${}^0R_1\,{}^1R_2 = {}^0R_2$.

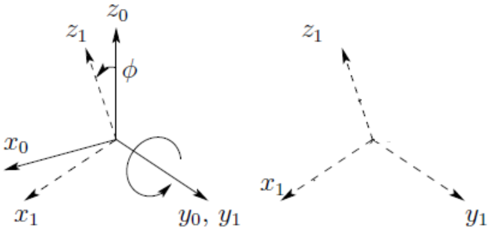
Left multiplication is rotation about fixed axis.



- $R_z(\theta)R_y(\phi)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about original fixed frame z by an angle θ.



- $R_z(\theta)R_y(\phi)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about original fixed frame z by an angle θ.
- Imagine we have two coincident frames 0,1. We rotate frame 1 about y_0 by ϕ to get ${}^0R_1$.



- $R = R_z(\theta)R_y(\phi)$ corresponds to rotation about y-axis by angle ϕ first followed by a rotation about original fixed frame z by an angle θ.
- Assume $R_z(\theta) = {}^0R_3$ is a rotation with respect to 0 frame, expressed in coordinates of 0 frame.

$$R = R_z(\theta)R_y(\phi) = {}^0R_3\,{}^0R_1$$

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Euler's Rotation Theorem

> Any displacement of a rigid body in 3D space where one point of the body remains fixed is equivalent to a single rotation about some axis that runs through the fixed point.

- Euler represented every rotation as a sequence of rotations about orthogonal axes, where the angles of rotation are now called Euler Angles.
- Davenport generalized result to non-orthogonal axes. These sequences of angles are called Davenport sequences.

Parameterization of Rotation

1. Z-Y-Z Euler Angles
2. X-Y-Z Fixed Angles
3. Angle-Axis Representation
4. Euler Parameters/ Unit Quaternions
5. Exponential

Rotate $\{0\}$ first about $\hat{z}_0$ by an angle ϕ to obtain $\{a\}$, then about current $\hat{y}_a$ by an angle θ to obtain $\{b\}$, and about current $\hat{z}_b$ by an angle ψ to obtain $\{1\}$.

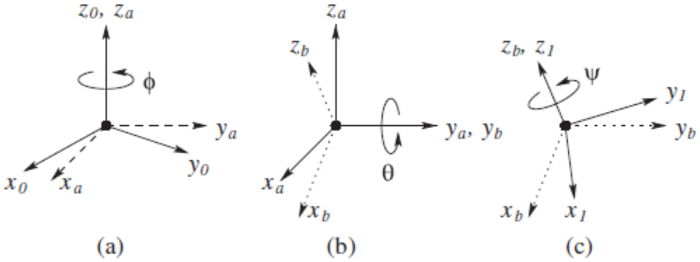

Figure: Source: Intro to Robotics, Mechanics and Control

$$
\begin{aligned}
{}^{0}_{1}R_{ZYZ}(\phi,\theta,\psi) &= R_z(\phi)\,R_y(\theta)\,R_z(\psi) \\
{}^{0}R_1 &= {}^{0}R_a\;{}^{a}R_b\;{}^{b}R_1 \\
&= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix}
\end{aligned}
$$

01. Rotation Matrix to Z-Y-Z Euler Angles

$$\begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_{13}^2 + r_{23}^2 = (c\phi s\theta)^2 + (s\phi s\theta)^2$$

$$\sin\theta = \pm\sqrt{r_{13}^2 + r_{23}^2}$$

$$\theta = \arctan 2(\pm\sqrt{r_{13}^2 + r_{23}^2}, r_{33})$$

$$\psi = \arctan 2\left(\pm r_{32}, \mp r_{31}\right)$$

$$\phi = \arctan 2\left(\pm r_{23}, \pm r_{13}\right)$$

- Singularity if $\theta = 0°$ or $\theta = 180°$
- $s\theta = 0$ and $c\theta = \pm 1$
- $\begin{bmatrix} c_{\phi+\psi} & -s_(\phi+\psi) & 0 \\ s_(\phi+\psi) & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -c_{\phi-\psi} & -s_(\phi-\psi) & 0 \\ s_(\phi-\psi) & c_{\phi-\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Can only determine $(\phi+\psi)$ or $(\phi-\psi)$, and there are infinitely many solutions.

Rotate $\{B\}$ first about $\hat{X}_A$ by an angle γ (roll), then about $\hat{Y}_A$ by an angle β (pitch), and about $\hat{Z}_A$ by an angle α (yaw).

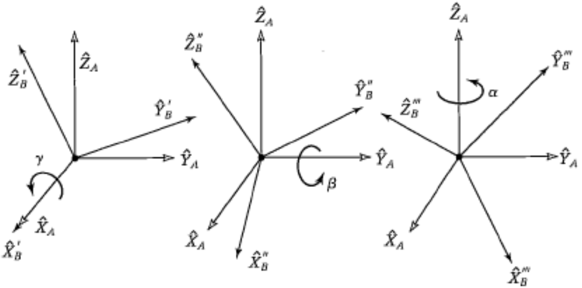

Figure: Source: Intro to Robotics, Mechanics and Control

$$
\begin{aligned}
{}^{A}R_{B} = {}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \\
&= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
\end{aligned}
$$

02. Rotation Matrix to Z-Y-X Fixed Angles

$$\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma - c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$(c\alpha c\beta)^2 + (s\alpha c\beta)^2 = r_{11}^2 + r_{21}^2$$

$$\cos\beta = \sqrt{r_{11}^2 + r_{21}^2}$$

$$\beta = \arctan 2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

$$\alpha = \arctan 2\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)$$

$$\gamma = \arctan 2\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)$$

- Singularity if $\beta = \pm 90°$
- $c\beta = 0$ and $s\beta = \pm 1$
- $\begin{bmatrix} 0 & -s(\alpha-\gamma) & c(\alpha-\gamma) \\ 0 & c(\alpha-\gamma) & s(\alpha-\gamma) \\ -1 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -s(\alpha+\gamma) & -c(\alpha+\gamma) \\ 0 & c(\alpha+\gamma) & -s(\alpha+\gamma) \\ 1 & 0 & 0 \end{bmatrix}$
- Can only determine $(\alpha+\gamma)$ or $(\alpha-\gamma)$

03. Angle-Axis Representation [2]

Rotate $\{B\}$ about ${}^{A}\hat{K}$ by an angle θ according to right-hand rule.

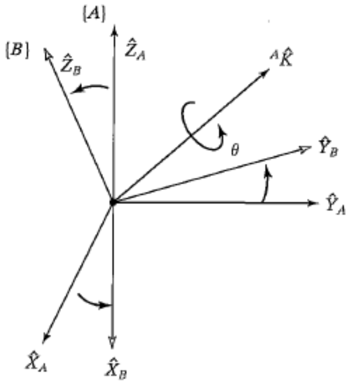

Figure: Source: Intro to Robotics, Mechanics and Control

- Rotation $R = R_z(\alpha)\, R_y(\beta)$ aligns world z-axis to k-vector.
-

$$
\begin{aligned}
R_{\hat{K}}(\theta) &= R\, R_z(\theta)\, R^{-1} \\
&= R_z(\alpha)\, R_y(\beta)\, R_z(\theta)\, R_y(-\beta)\, R_z(-\alpha)
\end{aligned}
$$

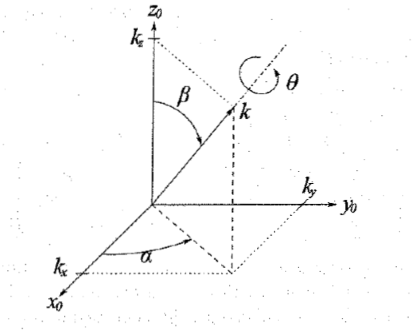

Figure 2.12: Rotation about an arbitrary axis.

Figure: Source: Robot Modeling and Control

$$R_{\hat{K}}(\theta) = \begin{bmatrix} k_x^2\, v\theta + c\theta & k_x k_y\, v\theta - k_z\, s\theta & k_x k_z\, v\theta + k_y\, s\theta \\ k_x k_y\, v\theta + k_z\, s\theta & k_y^2\, v\theta + c\theta & k_y k_z\, v\theta - k_x\, s\theta \\ k_x k_z\, v\theta - k_y\, s\theta & k_y k_z\, v\theta + k_x\, s\theta & k_z^2\, v\theta + c\theta \end{bmatrix}$$

where $v\theta = 1 - c\theta$. This can be obtained by realizing that

$$\cos\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \qquad \sin\beta = \frac{\sqrt{k_x^2 + k_y^2}}{1}$$

$$\sin\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \qquad \cos\beta = \frac{k_z}{1}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- Solution is not unique – $(\hat{K}, \theta)$ or $(-\hat{K}, -\theta)$
- $\hat{K}$ is undefined for $\theta = 0°$ or $\theta = 180°$.

- Four parameters – $\left({}^{A}\hat{K}_x, {}^{A}\hat{K}_y, {}^{A}\hat{K}_z, \theta\right)$
- Unit vector has only two independent parameters because of unit length constraint.
- Angle and unit direction can be combined into three parameters –

$$\left(\theta\ {}^{A}\hat{K}_x, \theta\ {}^{A}\hat{K}_y, \theta\ {}^{A}\hat{K}_z\right)$$

- If $\hat{K} = (k_x, k_y, k_z)^T$ then

$$\epsilon_1 = k_x \sin\frac{\theta}{2}$$
$$\epsilon_2 = k_y \sin\frac{\theta}{2}$$
$$\epsilon_3 = k_z \sin\frac{\theta}{2}$$
$$\epsilon_4 = \cos\frac{\theta}{2}$$

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

- Quaternion: Scalar + Vector – hypercomplex number

$$q = s + v_1 i + v_2 j + v_3 k$$

Euler Parameters/ Unit Quaternions

$$R_\epsilon = \begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix}$$

$$\epsilon_1 = \frac{r_{32}-r_{23}}{4\epsilon_4}$$

$$\epsilon_2 = \frac{r_{13}-r_{31}}{4\epsilon_4}$$

$$\epsilon_3 = \frac{r_{21}-r_{12}}{4\epsilon_4}$$

$$\epsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

[1] **Richard M Murray, Zexiang Li, and S Shankar Sastry.**
A mathematical introduction to robotic manipulation.
CRC press, 2017.

[2] **Mark W Spong, Seth Hutchinson, and Mathukumalli Vidyasagar.**
Robot modeling and control.
John Wiley & Sons, 2020.