Inverse Kinematics

EE366/CE366/CS380: Introduction to Robotics

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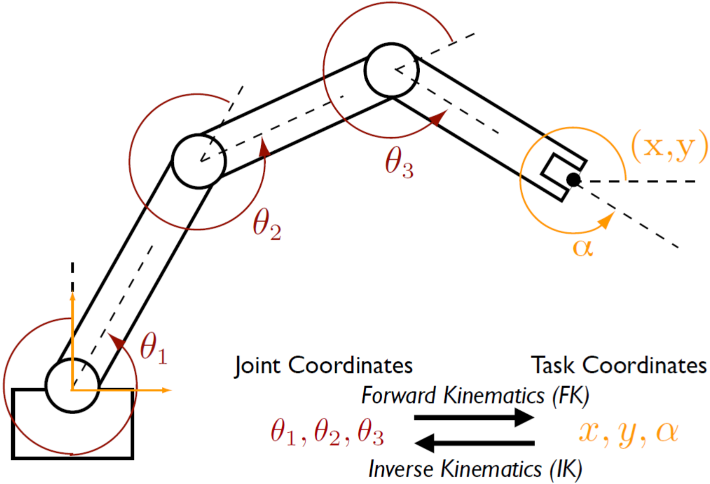
Kinematics: Study of motion (position, velocity, acceleration, etc.) without regard to the forces, torques that cause it. Geometric description of motion.

Forward Kinematics: Calculation of position and orientation of end-effector from its joint coordinates.

Inverse Kinematics: Determine the values of the joint coordinates, given the end-effector's position and orientation.

Kinematics establishes link between joint and task coordinates.



General Problem

Given desired position and orientation of end-effector,

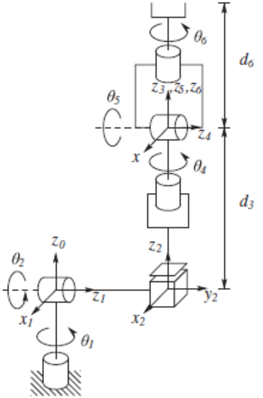
$$T = \begin{bmatrix} R & p \\ \mathbf{0} & 1 \end{bmatrix},$$

find all possible joint variables, $(q_1, \cdots, q_n)$, that satisfy the equation

$$^0T_n(q_1, \cdots, q_n) = T.$$

- 12 nonlinear transcendental equations in n unknowns.

Example: IK of Stanford manipulator



$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}
$$

- Will there always be a solution?
- Will solution be unique?
- Is any solution admissible/realizable?

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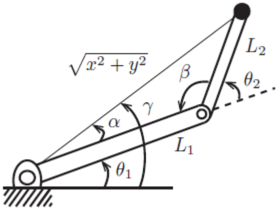


Figure: Source: Modern Robotics

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

IK of 2R-planar chain: Finding θ_2

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

- $x^2 + y^2 = L_1^2 + L_2^2 + 2L_1 L_2 \cos\theta_2$
- $\cos\theta_2 = \frac{x^2+y^2-L_1^2-L_2^2}{2L_1 L_2}$
  - For solution to exist, RHS should be between -1 and 1.
- $\sin\theta_2 = \pm\sqrt{1-\cos^2\theta_2}$
- $\theta_2 = \arctan 2(\sin\theta_2, \cos\theta_2)$
  - Two possible solutions.

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\
&= L_1 \cos\theta_1 + L_2 \cos\theta_1 \cos\theta_2 - L_2 \sin\theta_1 \sin\theta_2 \\
&= (\underbrace{L_1 + L_2 \cos\theta_2}_{k_1}) \cos\theta_1 - \underbrace{L_2 \sin\theta_2}_{k_2} \sin\theta_1 \\
&= k_1 \cos\theta_1 - k_2 \sin\theta_1
\end{aligned}
$$

$$
\begin{aligned}
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \\
&= L_1 \sin\theta_1 + L_2 \sin\theta_1 \cos\theta_2 + L_2 \cos\theta_1 \sin\theta_2 \\
&= (\underbrace{L_1 + L_2 \cos\theta_2}_{k_1}) \sin\theta_1 + \underbrace{L_2 \sin\theta_2}_{k_2} \cos\theta_1 \\
&= k_1 \sin\theta_1 + k_2 \cos\theta_1
\end{aligned}
$$

$$x = k_1 \cos\theta_1 - k_2 \sin\theta_1$$
$$y = k_1 \sin\theta_1 + k_2 \cos\theta_1$$

where,

$$k_1 = L_1 + L_2 \cos\theta_2$$
$$k_2 = L_2 \sin\theta_2$$

Let,

$$r = +\sqrt{{k_1}^2 + {k_2}^2}$$
$$\gamma = \arctan 2(k_2, k_1)$$

So that,

$$k_1 = r\cos\gamma$$
$$k_2 = r\sin\gamma$$

and,

$$x = r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1$$
$$y = r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1$$

$$
\begin{aligned}
x &= r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1 \\
&= r\cos(\gamma+\theta_1) \\
y &= r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1 \\
&= r\sin(\gamma+\theta_1)
\end{aligned}
$$

where,

$$
\begin{aligned}
r &= +\sqrt{k_1{}^2 + k_2{}^2} \\
\gamma &= \arctan 2(k_2, k_1) \\
k_1 &= L_1 + L_2\cos\theta_2 \\
k_2 &= L_2\sin\theta_2
\end{aligned}
$$

$$
\Rightarrow \frac{x}{r} = \cos(\gamma+\theta_1)
$$
$$
\frac{y}{r} = \sin(\gamma+\theta_1)
$$

$$
\begin{aligned}
\Rightarrow \gamma + \theta_1 &= \arctan 2\left(\frac{y}{r}, \frac{x}{r}\right) \\
&= \arctan 2(y, x)
\end{aligned}
$$

$$
\Rightarrow \theta_1 = \arctan 2(y, x) - \gamma
$$

IK of 2R-planar chain: Finding θ_1

$$
\begin{aligned}
r &= +\sqrt{{k_1}^2 + {k_2}^2} \\
\gamma &= \arctan 2(k_2, k_1) \\
k_1 &= L_1 + L_2 \cos\theta_2 \\
k_2 &= L_2 \sin\theta_2
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= \arctan 2(y, x) - \gamma \\
&= \arctan 2(y, x) - \arctan 2(k_2, k_1) \\
&= \arctan 2(y, x) - \arctan 2(L_2 \sin\theta_2, L_1 + L_2 \cos\theta_2)
\end{aligned}
$$

Inverse Kinematics of 2R-planar chain using **geometric method**

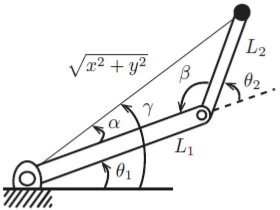


Figure: Source: Modern Robotics

- From the law of cosines,

$$x^2 + y^2 = L_1^2 + L_2^2 - 2L_1L_2\cos\beta$$

- $\beta = \arccos\left(\frac{L_1^2+L_2^2-x^2-y^2}{2L_1L_2}\right)$
- $\alpha = \arccos\left(\frac{L_1^2+x^2+y^2-L_2^2}{2L_1\sqrt{x^2+y^2}}\right)$
-

$$\begin{aligned}\theta_2 &= 180^\circ - \beta \\ \theta_1 &= \gamma - \alpha \\ &= \arctan 2(y, x) - \alpha\end{aligned}$$

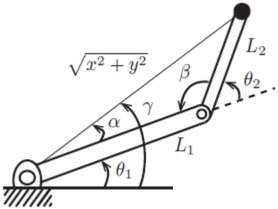


Figure: Source: Modern Robotics

- Other solution is:

$$\theta_2 = -(180^\circ - \beta)$$
$$\theta_1 = \gamma + \alpha$$

Existence of Solution

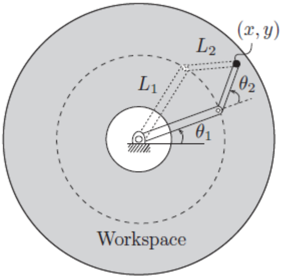


Figure: Source: Modern Robotics

- What does it mean for RHS of $\cos\theta_2$ equation to not be in $[-1, 1]$?
- The goal position and orientation of end-effector are not included in the workspace.
  - If point is included in **reachable workspace** then we can reach the point in at least one possible orientation.
  - If point is included in **dexterous workspace** then we can reach the point in any orientation.

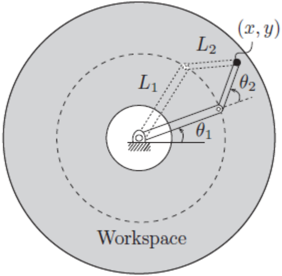
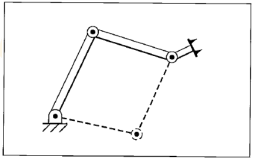


Figure: Source: Modern Robotics

- Workspace for $L_1 \neq L_2$
  - Outer radius: $L_1 + L_2$; Inner radius: $|L_1 - L_2|$
- Two possible orientation. Only one possible at the boundary.
- Assumption: All joints can rotate 360°
  - Seldom true.
  - Otherwise, workspace is further constrained.
  - Same workspace with $\theta_1 \in [0^\circ, 360^\circ)$ and $\theta_2 \in [0^\circ, 180^\circ]$.

Given a manipulator, how many IK solutions does it have?



a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

FIGURE 4.5: Number of solutions vs. nonzero a_i.

Figure: Source: Introduction to Robotics: Mechanics and Control

- Need to know all solutions, as the system has to be able to choose one.
- The more link parameters are non-zero, bigger the maximum number of possible solutions.
  - Upper bound on IK solutions for general 6 dof manipulator is 16 [a].
- Infinite solutions at singularities or for kinematically redundant manipulators.

[a] Manseur, Rachid, and Keith L. Doty. "A robot manipulator with 16 real inverse kinematic solution sets." The International Journal of Robotics Research 8.5 (1989): 75-79.

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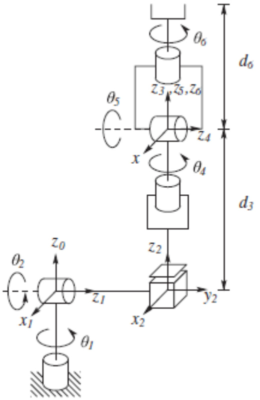

$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}
$$

Closed-form solutions of 6 DOF manipulators

It is possible to find all solutions (not necessarily closed-form) of any single series chain of revolute and prismatic joints having a total of six degrees of freedom.

Sufficient conditions for existence of closed-form solution

A 6-dof manipulator admits a closed-form inverse kinematics solution if [a]

1. Three consecutive revolute joint axes intersect at a common point
2. Any three joints are prismatic

[a] Kinematics of Manipulators under Computer Control, Pieper, 1968

We can kinematically decouple position from orientation [1]

- Under these conditions, we can decouple inverse kinematics problem into two problems:
  - Inverse position kinematics
  - Inverse orientation kinematics
- This is why manipulators with spherical wrist are popular $\rightarrow$ Three rotational axes intersect

A spherical wrist satisfies conditions for kinematic decoupling.

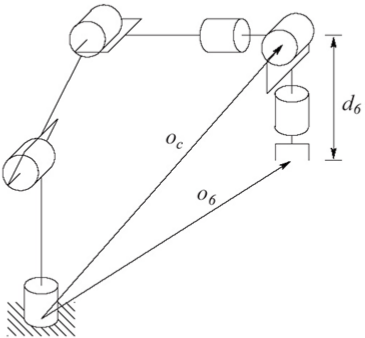

Figure: Source: Robot Modeling and Control

- z_3, z_4, and z_5 intersect. Say at o_c.
- Motion of these joints will not affect o_c
- **o_c is function of only first three joint variables.**
- **Plan:** $o_6 \rightarrow o_c \rightarrow (q_1, q_2, q_3)$

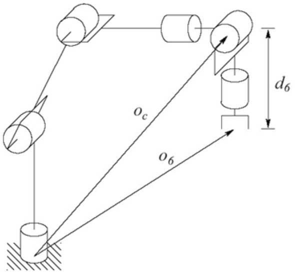

Figure: Source: Robot Modeling and Control

$$o = {}^0o_6 = {}^0o_c + d_6\ {}^0\hat{z}_5$$

z_5 and z_6 are in same direction and z_6 in $\{0\}$ frame is third column of R. So,

$$o = {}^0o_c + d_6\ R\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}x_c\\y_c\\z_c\end{bmatrix} = \begin{bmatrix}o_x - d_6 r_{13}\\o_y - d_6 r_{23}\\o_z - d_6 r_{33}\end{bmatrix}$$

- Solve equations to find values of first 3 joints.

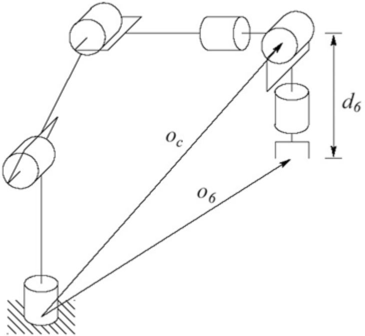

Figure: Source: Robot Modeling and Control

$$R = {}^0R_3\,{}^3R_6$$

where R is desired orientation.

$$\Rightarrow {}^3R_6 = {}^0R_3^{-1}\,R = {}^0R_3^T\,R$$

- ${}^0R_3$ only depends on first three joint angles.
- Last three joint angles determined from the above equation.
- A set of Euler angles can be used to solve for them.

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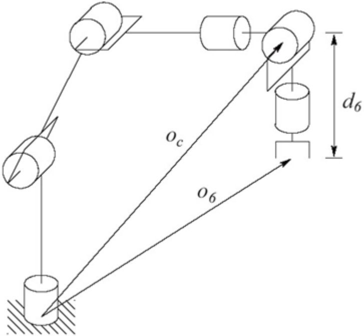


Figure: Source: Robot Modeling and Control

- Mostly encounter common robot manipulator configurations.
- Simple configurations can be studied geometrically as most DH parameters are zero.
- **Geometric approach:** Solve for q_i by projecting manipulator onto $x_{i-1} - y_{i-1}$ plane according to DH convention, and then solving a trigonometry problem.

Problem: IK of Articulated (RRR) arm with spherical wrist

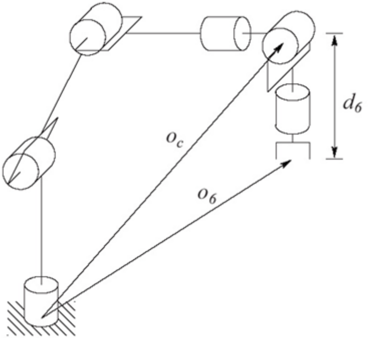


Figure: Source: Robot Modeling and Control

- Find the wrist center, o_c, location of intersection point of last three joint axes, from the end-effector position o_6.
- Find θ_1, θ_2, and θ_3 using o_c and either the geometrical approach or analytically.
- Say $o_c = (x_c, y_c, z_c)$.

- The DH parameters for the arm are:

Link	a_i	α_i	d_i	θ_i
1	0	$90°$	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	d_3	θ_3

- Multiplying A_i,

$$
{}^0T_3 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & d_3 s_1 + a_2 c_1 c_2 + a_3 c_1 c_{23} \\ s_1 c_2 3 & -s_1 s_{23} & -c_1 & -d_3 c_1 + a_2 s_1 c_2 + a_3 s_1 c_{23} \\ s_{23} & c_{23} & 0 & d_1 + a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Solve equations for θ_1, θ_2, and θ_3:

$$
\begin{aligned}
d_3 s_1 + a_2 c_1 c_2 + a_3 c_1 c_{23} &= x_c \\
-d_3 c_1 + a_2 s_1 c_2 + a_3 s_1 c_{23} &= y_c \\
d_1 + a_2 s_2 + a_3 s_{23} &= z_c
\end{aligned}
$$

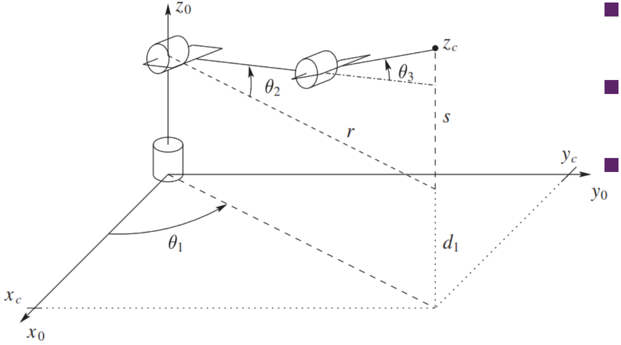


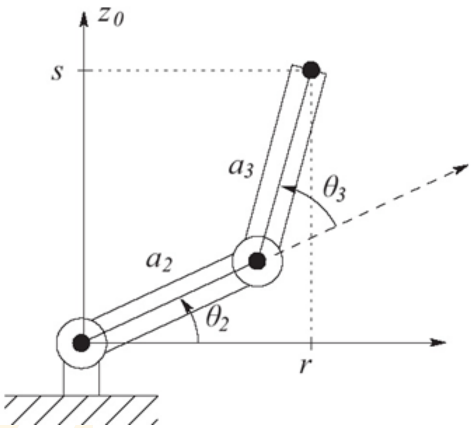
Figure: Source: Robot Modeling and Control

- Project o_c onto $x_0 - y_0$ plane.
- $\theta_1 = \arctan 2(y_c, x_c)$
- Another solution is:

$$\theta_1 = 180^\circ + \arctan 2(y_c, x_c)$$

- Corresponding θ_2 and θ_3 will be different.

Inverse Position Kinematics of RRR arm: Geometric Method



- To find θ_2, θ_3 project onto plane formed by $x_1 - y_1$ $(x_1 - z_0)$.
- This is 2 link planar case.
- Using law of cosines:

$$\cos(180^\circ - \theta_3) = -\frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$D := \cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$r^2 = x_c^2 + y_c^2$$

$$s^2 = (z_c - d_1)^2$$

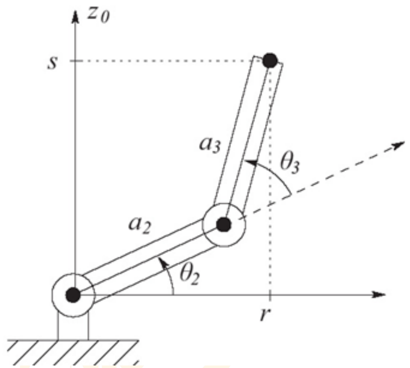


Figure: Source: Robot Modeling and Control

- $\theta_3 = \arctan 2\left(\pm\sqrt{1-D^2}, D\right)$
- Two solutions for θ_3 – Elbow down and Elbow up

$$\theta_2 = \arctan 2\left(s, r\right) - \arctan 2(a_3 \sin\theta_3, a_2 + a_3\cos\theta_3)$$

- Two solution pairs for (θ_2, θ_3)

- In general, maximum of 4 possible solutions.

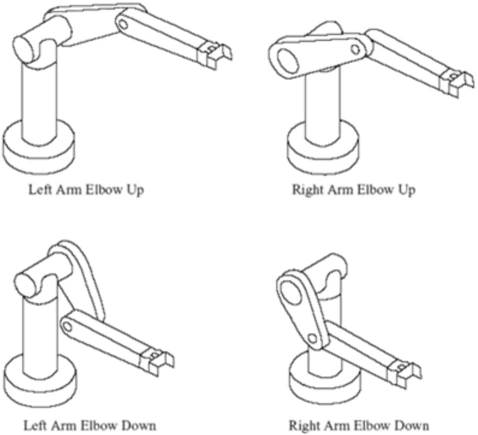


Figure: Example: PUMA

- Singular configuration – $x_c = y_c = 0$
- Infinite solutions for θ_1

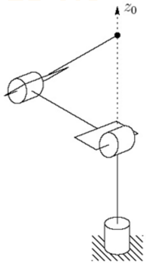


Figure: Source: Robot Modeling and Control

- With offset, wrist center cannot intersect z_0 configurations.

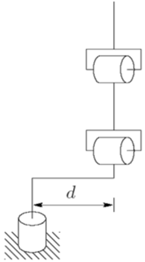
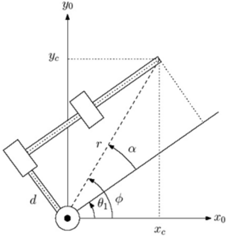


Figure: Source: Robot Modeling and Control

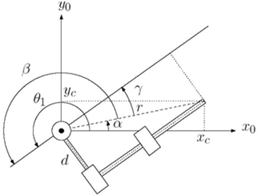
- Left Arm

$$\theta_1 = \phi - \alpha$$
$$\phi = \arctan 2(y_c, x_c)$$
$$\alpha = \arctan 2\left(d, \sqrt{x_c^2 + y_c^2 - d^2}\right)$$



- Right Arm

$$\theta_1 = \alpha + \beta$$
$$\alpha = \arctan 2(y_c, x_c)$$
$$\beta = \gamma + 180^\circ$$
$$\gamma = \arctan 2\left(d, \sqrt{x_c^2 + y_c^2 - d^2}\right)$$



See book for solutions of θ_2, θ_3.

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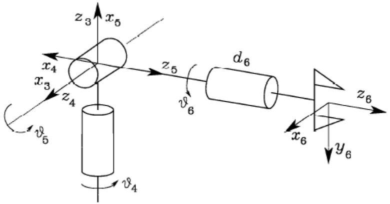

Figure: Source: Robotics–Modeling, Planning and Control

$$
{}^3T_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Solve the equation ${}^3R_6 = {}^0R_3^T\, R$ for θ_4, θ_5, and θ_6, where R is desired orientation.

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}^T \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Notice the RHS is completely known as we have determined θ_1, θ_2, and θ_3 from inverse position kinematics.

${}^3R_6$ is equivalent to ZYZ Euler rotation sequence.

$$
\begin{aligned}
R_{ZYZ} &= R_z(\phi)\, R_y(\theta)\, R_z(\psi) \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

Compare with

$$
{}^3R_6 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 \\ -s_5 c_6 & s_5 s_6 & c_5 \end{bmatrix}
$$

Use Euler inverse formulas to find orientation joint angles.

- $\theta_4 = \phi$; $\theta_5 = \theta$; $\theta_6 = \psi$
- $$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$
- Two cases:
  - Singular case: Middle angle, $\theta = 0$
    - Occurs when both t_{13} and t_{23} are zero.
  - Nonsingular case: $\theta \neq 0$

- **Non-singular Case**
- Two possible solutions

$$\theta = \arctan 2\left(\sqrt{1 - t_{33}^2}, t_{33}\right)$$
$$\phi = \arctan 2\left(t_{23}, t_{13}\right)$$
$$\psi = \arctan 2\left(t_{32}, -t_{31}\right)$$

OR

$$\theta = \arctan 2\left(-\sqrt{1 - t_{33}^2}, t_{33}\right)$$
$$\phi = \arctan 2\left(-t_{23}, -t_{13}\right)$$
$$\psi = \arctan 2\left(-t_{32}, t_{31}\right)$$

- **Singular Case**
- Infinite possible solutions in each singular case

$$\theta = 0$$
$$\phi = 0$$
$$\psi = \arctan 2\left(t_{21}, t_{11}\right)$$

OR

$$\theta = 180^\circ$$
$$\phi = 0$$
$$\psi = -\arctan 2\left(t_{21}, t_{22}\right)$$

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}^T \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Non-singular case - Solution 1:

$$\theta_5 = \text{arctan}\,2\left(\sqrt{1-(r_{13}s_1 - r_{23}c_1)^2}, r_{13}s_1 - r_{23}c_1\right)$$

$$\theta_4 = \text{arctan}\,2\left(-r_{13}c_1s_{23} - r_{23}s_1s_{23} + r_{33}c_{23}, r_{13}c_1c_{23} + r_{23}s_1c_{23} + r_{33}s_{23}\right)$$

$$\theta_6 = \text{arctan}\,2\left(r_{12}s_1 - r_{22}c_1, -r_{11}s_1 + r_{21}c_1\right)$$

Derivation of ZYZ Euler Inverse Formulas: Non-singular case

$$\begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

- Both of t_{13} and t_{23} are not zero. $\Rightarrow t_{13}^2 + t_{23}^2 = \sin^2\theta \neq 0 \Rightarrow \theta \neq 0$
- So,

$$\theta = \arctan 2\left(\sqrt{1 - t_{33}^2}, t_{33}\right)$$

or

$$\theta = \arctan 2\left(-\sqrt{1 - t_{33}^2}, t_{33}\right)$$

$$\begin{bmatrix} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

- If θ is given by the first equation, then $\sin\theta > 0$, and

$$\phi = \arctan 2\,(t_{23}, t_{13})$$
$$\psi = \arctan 2\,(t_{32}, -t_{31})$$

- If θ is given by the second equation, then $\sin\theta < 0$, and

$$\phi = \arctan 2\,(-t_{23}, -t_{13})$$
$$\psi = \arctan 2\,(-t_{32}, t_{31})$$

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

- If $\theta = n\pi$, then

$$R_{ZYZ} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

- If $t_{33} = 1$, then $\theta = 0$ and

$$R_{ZYZ} = \begin{bmatrix} c_{\phi\psi} & -s_{\phi\psi} & 0 \\ s_{\phi\psi} & c_{\phi\psi} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

- $\phi + \psi = \arctan 2(t_{21}, t_{11})$
- Sum $\phi + \psi$ is unique
- But, infinitely many solutions for (ϕ, ψ).
- As convention, might as well let $\phi = 0$

Derivation of ZYZ Euler Inverse Formulas: Singular case

- If $t_{33} = -1$, then $\theta = 180^\circ$ and

$$R_{ZYZ} = \begin{bmatrix} -\cos(\phi - \psi) & -\sin(\phi - \psi) & 0 \\ \sin(\phi - \psi) & \cos(\phi - \psi) & 0 \\ 0 & 0 & \pm - 1 \end{bmatrix}$$

- $\phi - \psi = \arctan 2(t_{21}, t_{22})$
- Infinitely many solutions for (ϕ, ψ).

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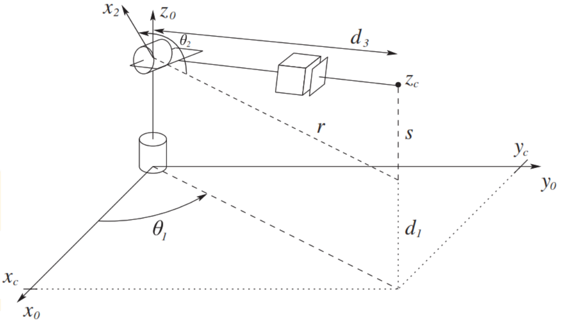

Figure: Source: Robot Modeling and Control

- θ_1=?
  - $\theta_1 = \arctan 2(y_c, x_c)$
- Another solution?
  - $\theta_1 = \arctan 2(x_c, y_c) + 180°$
- Singularity?
  - Yes

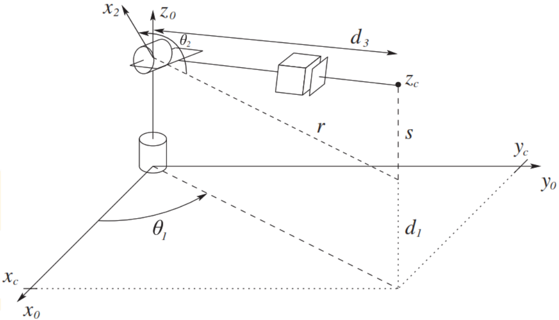

Figure: Source: Robot Modeling and Control

- θ_2=?
  - $\theta_2 = \arctan 2(s, r) + 90^\circ$
- d_3 =?
  - $d_3 = \sqrt{r^2 + s^2}$
- We have two solutions.
- Can revise solution for the offset case.

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[1] **Mark W Spong, Seth Hutchinson, and Mathukumalli Vidyasagar.**
Robot modeling and control.
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