Motion Control with Forces

EE366/CE366/CS380: Introduction to Robotics

Dr. Basit Memon

Electrical and Computer Engineering
Habib University

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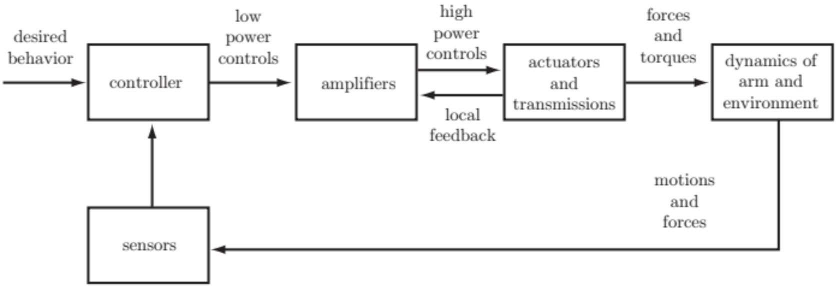
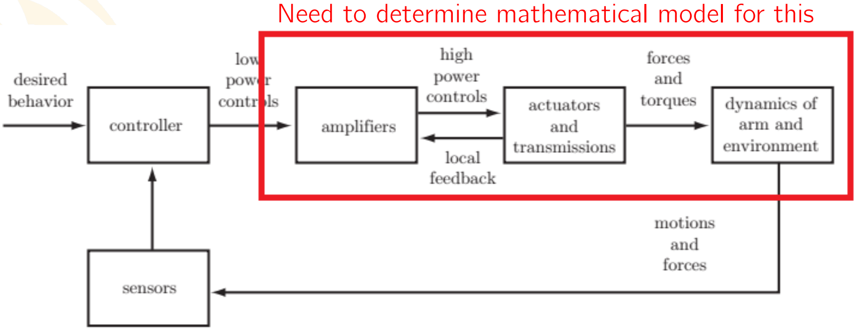


(a)

- **Find controller such that error between desired trajectory ($\theta_d, \dot{\theta}_d, \ddot{\theta}_d$) and actual trajectory is minimized, and effects of disturbance on trajectory are minimized.**

Determining the system model could yield a better controller.



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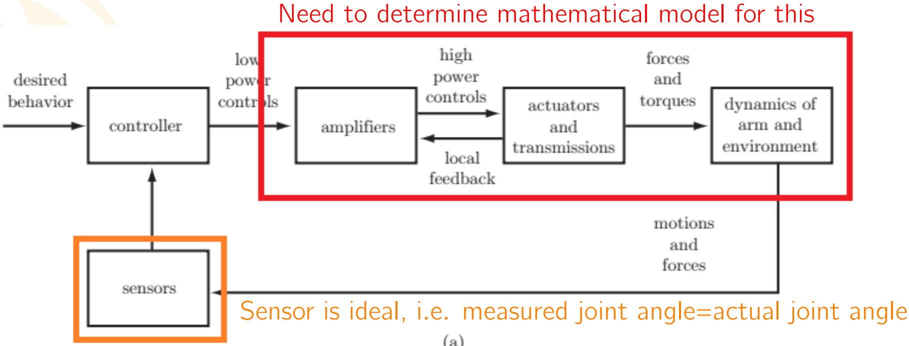

(a)

- Find controller such that error between desired trajectory $(\theta_d, \dot{\theta}_d, \ddot{\theta}_d)$ and actual trajectory is minimized, and effects of disturbance on trajectory are minimized.

What is the need for modeling?

- Position is the variable to be controlled and it has simple relationship with velocity commands.
  - Velocity is rate at which position changes in a direction.
  - Depending on the error, the rate can be adjusted.
- What is the relationship between position and voltage command?
  - Every motor physically accepts voltages that are then converted to position changes, speed changes, forces, torques at the link end via dynamics of intermediate components.

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A typical DC motor

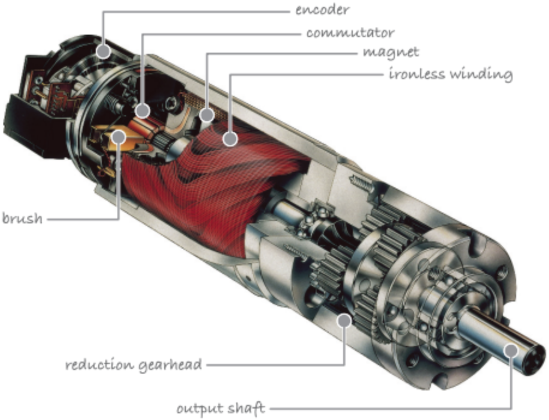
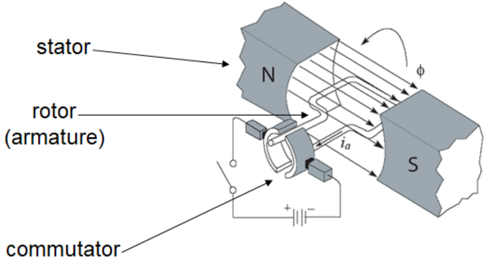


Fig. 9.6.
Schematic of an integrated motor-encoder-gearbox assembly (courtesy of maxon precision motors, inc.)

Figure: Courtesy of Corke's book

Rotor torque is proportional to current through armature coils.

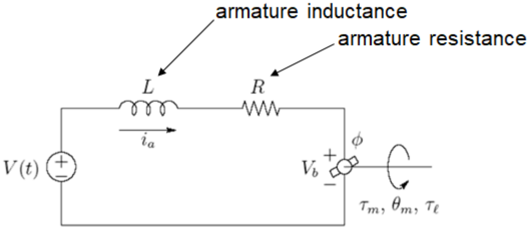


- Torque experienced by rotor is

$$\tau_m = K_m \, i_a,$$

assuming constant flux generated by permanent magnet.

- K_m is torque constant and i_a is armature current.



- Conductor moving in magnetic field generates voltage, called back EMF

$$V_b = K_b\,\dot{\theta}_m,$$

K_b is back EMF constant and ω_m is angular velocity of rotor.

$$L\frac{di_a}{dt} + Ri_a = V - V_b.$$

- K_m and K_b are always numerically equal.

What is relationship between position and torque of DC motor?

$$J_a\ddot{\theta}_m = \tau_m - \tau_f$$

Here J_a is inertia of the motor, τ_m is torque generated by rotor, and τ_f is friction torque. Typically,

$$\tau_f = B_m\dot{\theta}_m.$$

$$J_a\ddot{\theta}_m = \tau_m - B_m\dot{\theta}_m$$

$$J_a\ddot{\theta}_m = K_m i_a - B_m\dot{\theta}_m$$

$$J_a\dddot{\theta}_m = K_m\dot{i}_a - B_m\ddot{\theta}_m$$

$$J_a\dddot{\theta}_m = \frac{K_m}{L}\left(V - K_b\dot{\theta}_m - Ri_a\right) - B_m\ddot{\theta}_m$$

$$J_a\dddot{\theta}_m = \frac{K_m}{L}\left(V - K_b\dot{\theta}_m - R\frac{\tau_m}{K_m}\right) - B_m\ddot{\theta}_m$$

$$J_a\dddot{\theta}_m = \frac{K_m}{L}\left(V - K_b\dot{\theta}_m - R\frac{J_a\ddot{\theta}_m + B_m\dot{\theta}_m}{K_m}\right) - B_m\ddot{\theta}_m$$

We can approximate this third order ODE by second order since L/R is very small practically

$$J_a\ddot{\theta}_m = \frac{K_m}{R}V - \frac{K_mK_b}{R}\dot{\theta}_m - B_m\dot{\theta}_m$$

$$J_a\ddot{\theta}_m = \frac{K_m}{R}V - \frac{K_m^2}{R}\dot{\theta}_m - B_m\dot{\theta}_m$$

Another approximation in the model: Some frictions are ignored.

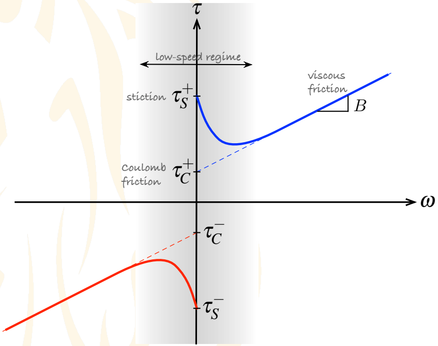


Figure: Typical friction vs speed curve

- Ignoring Coulomb friction, otherwise the model will be highly nonlinear.

$$
\begin{aligned}
J_a\ddot{\theta}_m &= \tau_m - \tau_f \\
&= \tau_m - B_m\dot{\theta}_m - \tau_C
\end{aligned}
$$

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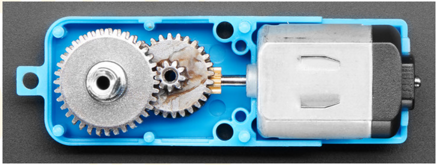
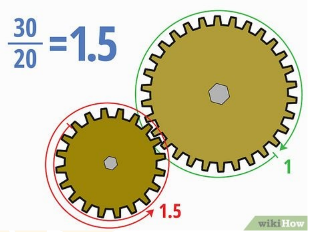




- **Gear Ratio (r)**

$$r = \frac{\text{Circumference of driven gear}}{\text{Circumference of driving gear}} = \frac{\text{Number of teeth of driven gear}}{\text{Number of teeth of driving gear}}$$

-

$$\dot{\theta}_l = \dot{\theta}_m / r$$

$$\tau_l = r\tau_m$$

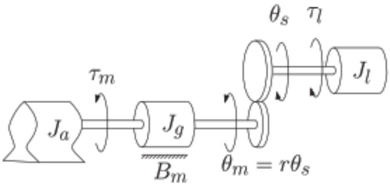


Figure: Lumped model of an actuator, gear train, and link. J_a, J_g, and J_l are actuator, gear, and load inertias respectively.

Model for the standalone motor is:

$$J_m\ddot{\theta}_m = \frac{K_m}{R}V - \frac{K_m^2}{R}\dot{\theta}_m - B_m\dot{\theta}_m$$

$J_m = J_a + J_g$ is lumped inertia of motor and gear assembly.

If motor is connected to the link, a portion of generated rotor torque supports the link

$$J_m\ddot{\theta}_m = \frac{K_m}{R}V - \frac{K_m^2}{R}\dot{\theta}_m - B_m\dot{\theta}_m - \frac{\tau_l}{r}$$

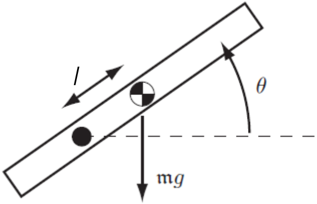


Figure: Single joint robot

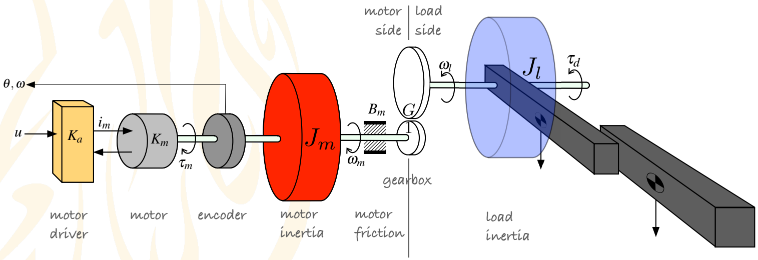
$$J_l\ddot{\theta}_l = \tau - mgl\cos\theta,$$

where m is the mass of the link, g is gravitational acceleration, and l is distance from axis of rotation to center of mass.

- τ is the external torque and contains the torque exerted by the motor and friction torque.

$$\tau = \tau_l - B_l\dot{\theta}_l$$

$$J_l\ddot{\theta}_l = \tau_l - B_l\dot{\theta}_l - mgl\cos\theta_l$$



- We can write a model for all three sub-systems combined, either on the motor side or on load side, i.e. in terms of θ_m or θ_l.

Model for motor + gear + link on motor side

$$
\begin{aligned}
J_m\ddot{\theta}_m &= \frac{K_m}{R}V - \frac{K_m^2}{R}\dot{\theta}_m - B_m\dot{\theta}_m - \frac{\tau_l}{r} \\
&= \frac{K_m}{R}V - \left(\frac{K_m^2}{R} + B_m\right)\dot{\theta}_m - \frac{J_l\ddot{\theta}_l + B_l\dot{\theta}_l + mgl\cos\theta_l}{r}
\end{aligned}
$$

Model for motor + gear + link on motor side

$$J_m\ddot{\theta}_m = \frac{K_m}{R}V - \left(\frac{K_m^2}{R} + B_m\right)\dot{\theta}_m - \frac{J_l\ddot{\theta}_l + B_l\dot{\theta}_l + mgl\cos\theta_l}{r}$$

Using the gear ratio: $\theta_l = \theta_m/r$,

$$\left(J_m + \frac{J_l}{r^2}\right)\ddot{\theta}_m = \frac{K_m}{R}V - \left(\frac{K_m^2}{R} + B_m + \frac{B_l}{r^2}\right)\dot{\theta}_m - \frac{mgl\cos(\theta_m/r)}{r}$$

Let $\left(J_m + \frac{J_l}{r^2}\right) = J_{eff}$, $\left(\frac{K_m^2}{R} + B_m + \frac{B_l}{r^2}\right) = B_{eff}$, $\frac{K_m}{R}V = u$

$$u = J_{eff}\ddot{\theta}_m + B_{eff}\dot{\theta}_m + \underbrace{\frac{mgl}{r}\cos(\theta_m/r)}_{\text{pesky nonlinear term}}$$

Model the nonlinear term as a disturbance to get a linear model.

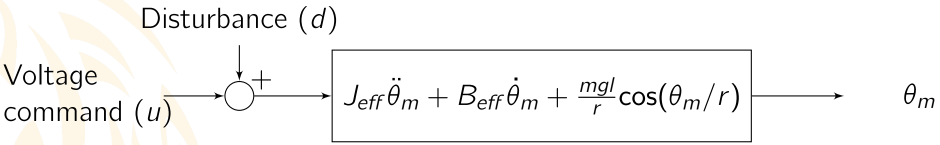
$$u = J_{eff}\ddot{\theta}_m + B_{eff}\dot{\theta}_m + \underbrace{\frac{mgl}{r}\cos(\theta_m/r)}_{\text{pesky nonlinear term}}$$

$$u = J_{eff}\ddot{\theta}_m + B_{eff}\dot{\theta}_m + d(t)$$

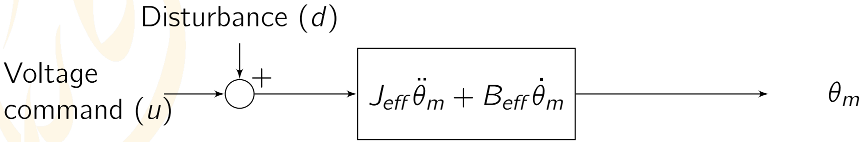
$$\boxed{u(t) - d(t) = J_{eff}\ddot{\theta}_m(t) + B_{eff}\dot{\theta}_m(t)}$$

- Linear ODE in θ. Control command is u.
- Notice that the disturbance term is reduced by r. Larger the gear ratio, smaller the disturbance term.

Summary of system model



approximated by:



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Objectives of single joint control

- **Stability:** The controller should not disturb the natural dynamics of the system, and the joint angle should always remain bounded.
- **Arbitrary trajectory tracking:** The controller should be able to make the joint angle follow any provided trajectory.
- **Good Disturbance Rejection:** The controller should null the effects of any disturbances (any torque that is not accounted in the model)

Setpoint tracking – P controller

- θ_d is constant. $\dot{\theta}_e = -\dot{\theta}$.
- P controller: $u = K_p(\theta_d - \theta)$
- Since the system is linear, we can determine the effect of u and d separately. So, set $d = 0$.
- Dynamics of the error are:

$$J_{eff}\ddot{\theta}_e + B_{eff}\dot{\theta}_e + K_p\theta_e = 0$$

Comparing to standard second order ODE

$$\ddot{\theta}_e + 2\zeta\omega_n\dot{\theta}_e + \omega_n^2\theta_e = 0$$

Setpoint tracking – P controller

$$J_{eff}\ddot{\theta}_e + B_{eff}\dot{\theta}_e + K_p\theta_e = 0$$
$$\ddot{\theta}_e + 2\zeta\omega_n\dot{\theta}_e + \omega_n^2\theta_e = 0$$

$$\omega_n = \sqrt{\frac{K_p}{J_{eff}}}$$
$$\Rightarrow \zeta = \frac{B_{eff}}{2J_{eff}} \times \sqrt{\frac{J_{eff}}{K_p}} = \frac{B_{eff}}{2\sqrt{K_pJ_{eff}}}$$

- For critical damping, i.e. $\theta_e \to 0$ at fastest rate, $\zeta = 1$ or
$$K_p = \frac{B_{eff}^2}{4J_{eff}}.$$

P controller: What happens because of disturbance?

$$u(t) - d(t) = J_{eff}\ddot{\theta}(t) + B_{eff}\dot{\theta}(t)$$

- When θ_d is achieved, there is a constant disturbance, i.e. nonlinear term $\frac{mgl}{r}\cos(\theta_m/r) = C$.
- Set $\theta_d = 0$. The error dynamics are:

$$J_{eff}\ddot{\theta}_e + B_{eff}\dot{\theta}_e + K_p\theta_e = -C$$

- If K_p is chosen for critical damping, then

$$\theta_e(t) = (a_1 + a_2 t)e^{-\omega_n t} - \boxed{\frac{C}{K_p}}$$

  - Error due to disturbance can be reduced by choosing large K_p.

$u - d = J_{eff}\ddot{\theta} + B_{eff}\dot{\theta}$

- $u(t) = K_p(\theta_d - \theta) + K_i \int_0^t (\theta_d - \theta)dt - K_d\dot{\theta}$
- θ_d is constant.

$$J_{eff}\ddot{\theta}_e + B_{eff}\dot{\theta}_e + K_p\theta_e + K_i \int_0^t \theta_e dt + K_d\dot{\theta}_e = d$$

- Because of the integral, we'll have to differentiate again and obtain a third order ODE.
- Since we're interested in final value of θ_e, i.e. $\lim_{t\to\infty} \theta_e$, only we can take advantage of final value theorem of Laplace transform

Steady state error with PID controller for setpoint tracking

$$J\ddot{\theta}_e + B\dot{\theta}_e + K_p\theta_e + K_i\int_0^t \theta_e dt + K_d\dot{\theta}_e = d$$

$$J_{eff} = J$$
$$B_{eff} = B$$

Taking Laplace transform,

$$Js^2E(s) + (B + K_d)sE(s) + K_pE(s) + \frac{K_i}{s}E(s) = D(s)$$

$$\left[Js^2 + (B + K_d)s + K_p + \frac{K_i}{s}\right]E(s) = D(s)$$

$$\frac{sD(s)}{Js^3 + (B + K_d)s^2 + K_ps + K_i} = E(s)$$

Steady-state error for PID controller is zero, with step disturbances

$$E(s) = \frac{sD(s)}{Js^3 + (B + K_d)s^2 + K_p s + K_i}$$

- For constant disturbance C, its Laplace transform is C/s.

According to final value theorem,

$$\lim_{t\to\infty} \theta_e(t) = \lim_{s\to 0} sE(s).$$

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sC}{Js^3 + (B + K_d)s^2 + K_p s + K_i} = 0$$

- So the error goes to zero for constant disturbance with PID controller.

How to choose K_p, K_d, and K_i?

- Set $K_i = 0$. Choose K_d and K_p to get the response at the right speed.
- Set K_i to satisfy the stability constraint.
- It can be shown that stability is assured if

$$\begin{aligned} K_p &> 0 \\ B + K_d &> 0 \\ K_i &> 0 \\ K_i &< \frac{(B + K_d)K_p}{J} \end{aligned}$$

Pseudocode for PID Control

```
time = 0                         // dt = servo cycle time
eint = 0                         // error integral
qprev = senseAngle               // initial joint angle q
loop
  [qd,qdotd] = trajectory(time) // from trajectory generator

  q = senseAngle                 // sense actual joint angle
  qdot = (q - qprev)/dt          // simple velocity calculation
  qprev = q

  e = qd - q
  edot = qdotd - qdot
  eint = eint + e*dt

  tau = Kp*e + Kd*edot + Ki*eint
  commandTorque(tau)

  time = time + dt
end loop
```

Figure 11.15: Pseudocode for PID control.

- $u(t) = K_p(\theta_d - \theta) + K_i \int_0^t (\theta_d - \theta)dt - K_d\dot{\theta}$
- $\ddot{\theta}_d = c$ is constant. It is quadratic trajectory.

$$u - d = J_{eff}\ddot{\theta} + B_{eff}\dot{\theta}$$

$$J_{eff}\ddot{\theta}_e + B_{eff}\dot{\theta}_e + K_p\theta_e + K_i\int_0^t \theta_e dt + K_d\dot{\theta}_e = d + J_{eff}c + (B_{eff} + K_d)ct$$

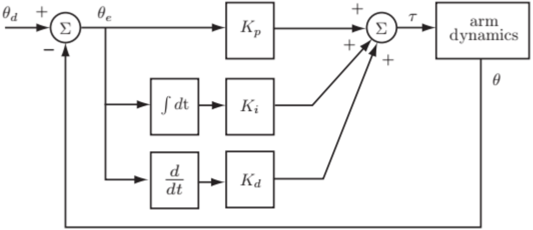
$$E(s) = \frac{sD(s) + J_{eff} + (B_{eff} + K_d)c/s}{Js^3 + (B + K_d)s^2 + K_p s + K_i}$$

$$E(s) = \frac{sD(s) + J_{eff} + (B_{eff} + K_d)c/s}{Js^3 + (B + K_d)s^2 + K_p s + K_i}$$

- For constant disturbance,

$$\lim_{s \to 0} sE(s) = \frac{(B_{eff} + K_d)c}{K_i}$$

- Error is zero for ramp trajectory, but finite for quadratic.



Will feedforward work for arbitrary trajectory tracking?

$$u - d = J_{eff}\ddot{\theta} + B_{eff}\dot{\theta}$$

- What should be feedforward signal?
- According to the model,
$$f(t) = J_{eff}\ddot{\theta}_d + B_{eff}\dot{\theta}_d.$$
- Control command for feedforward + feedback strategy is:
$$u = f + K_p\theta_e + K_i\int_0^t \theta_e dt + K_d\dot{\theta}_e.$$
- Substitute u in system equation to obtain error dynamics

FF + PID FB tracks arbitrary trajectory with step disturbance

$$u - d = J_{eff}\ddot{\theta} + B_{eff}\dot{\theta}$$

$$u = f + K_p\theta_e + K_i\int_0^t \theta_e dt + K_d\dot{\theta}_e$$

$$f = J_{eff}\ddot{\theta}_d + B_{eff}\dot{\theta}_d$$

$$J_{eff}\ddot{\theta}_e + (B_{eff} + K_d)\dot{\theta}_e + K_p\theta_e + K_i\int_0^t \theta_e dt = d$$

- Exact same equation for error as setpoint tracking
- Feedforward + PID feedback strategy makes error go to zero for any desired trajectory even in presence of step disturbance.

Can we do better?

- Recall that dynamics of system are:

$$u = J_{eff}\ddot{\theta}_m + B_{eff}\dot{\theta}_m + \underbrace{\frac{mgl}{r}\cos(\theta_m/r)}_{\text{disturbance prev.}}$$

- Can we use this information of disturbance somehow?

- What if

$$u = \underbrace{J_{eff}\ddot{\theta}_d + B_{eff}\dot{\theta}_d}_{\text{Feedforward}} + \underbrace{K_p\theta_e + K_i\int_0^t \theta_e dt + K_d\dot{\theta}_e}_{\text{PID feedback}} + \underbrace{\frac{mgl}{r}\cos(\theta_m/r)}_{\text{Gravity compensation}}$$

Gravity Compensation

- The gravity term can be perfectly canceled out and we don't have to use approximate model.
- Implementation: Measure θ_m and use it to compute the gravity compensation.

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