Kinematic Singularities

EE366/CE366/CS380: Introduction to Robotics

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- $\xi = J(q)\dot{q}$
- Jacobian relates joint velocities $\dot{q}$ to end-effector velocities, $\xi = (v^T, \omega^T)^T$
- $\xi = J_1 \dot{q}_1 + \cdots + J_n \dot{q}_n$
- End-effector velocities are linear combinations of columns of Jacobian
- For any arbitrary velocity, Jacobian should have 6 linearly independent columns, since $\xi \in \mathbb{R}^6$
- Rank of the Jacobian matrix depends on q

Definition

Configurations for which rank $J(q)$ is less than its maximum value are called singularities or singular configurations.

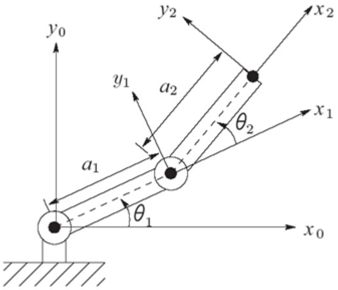
Why do we care?

- At singularities, motion in some directions will not be instantaneously possible.
- At/Near singularities, bounded end-effector velocities may require unbounded/large joint velocities.
- At singularities, bounded joint torques may produce unbounded end-effector forces and torques.
- At singularities, infinite solutions to the inverse kinematic problem may exist.
- Singularities often occur along workspace boundary.

How do we determine singularities?

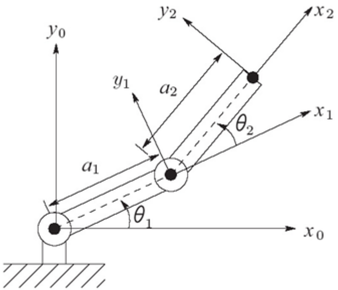
- Find out where the Jacobian loses its rank.
- Solve $\mathbf{det}\, J(q) = 0$ for q.
- Generally, complicated equations.
- Singularities of analytical Jacobian includes representational singularities as well.

Example: Two-links Planar Arm



- $J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$
- Maximal rank can be 2 here.
- If we consider the first 4 entries, then determinant of 4×4 matrix is $a_1 a_2 s_2$
- Assuming $a_1, a_2 \neq 0$, singular for $\theta_2 = 0, \pi$

At singularity, motion in some directions is impossible.

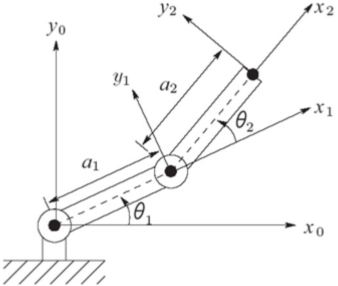


At singularity, we cannot move in direction of x_1.

$$
\begin{aligned}
{}^1J &= {}^1R_0 \, {}^0J \\
&= \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix} \\
&= \begin{bmatrix} -a_2 s_2 & -a_2 s_2 \\ a_1 + a_2 c_2 & a_2 c_2 \end{bmatrix}
\end{aligned}
$$

- At singularity,

$$
\begin{aligned}
{}^1J &= \begin{bmatrix} 0 & 0 \\ a_1 + a_2 & a_2 \end{bmatrix} \\
\Rightarrow {}^1\delta\dot{x} &= 0 \\
{}^1\delta\dot{y} &= (a_1 + a_2)\delta\dot{\theta}_1 + a_2\delta\dot{\theta}_2
\end{aligned}
$$



- $$^{1}J^{-1} = \frac{1}{a_1 a_2 s_2}\begin{bmatrix} a_2 c_2 & a_2 s_2 \\ -a_1 - a_2 c_2 & -a_2 s_2 \end{bmatrix}$$

- $$\delta\dot\theta_1 = \frac{c_2}{a_1 s_2}\,{}^{1}\delta\dot x + \frac{1}{a_1}\,{}^{1}\delta\dot y$$
$$\delta\dot\theta_2 = \frac{-(a_1 + a_2 c_2)}{a_1 a_2 s_2}\,{}^{1}\delta\dot x - \frac{1}{a_1}\,{}^{1}\delta\dot y$$

- When θ_2 is close to zero, $\sin\theta_2 \approx 0$
  - $\delta\dot\theta_1$ and $\delta\dot\theta_2$ are large even for small ${}^{1}\delta\dot x$ and ${}^{1}\delta\dot y$.

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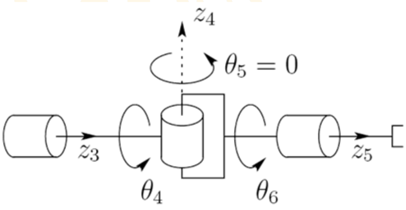
Do we really have to solve determinant equations?

- Singular configurations are independent of the choice of base or end frames, or the representation used for Jacobian.
- We can assign frames to make computation of determinant easier for us.
- For arms with spherical wrists, we can decouple the problem into arm singularities and wrist singularities.
- $J = \begin{bmatrix} J_P & J_O \end{bmatrix}$
- $J_O = \begin{bmatrix} z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ z_3 & z_4 & z_5 \end{bmatrix}$

- Wrist axes intersect at one point. We can assign frames such that $o_3 = o_4 = o_5 = o_6 = o$
- $J_O = \begin{bmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{bmatrix}$
- $J = \begin{bmatrix} J_{11} & O \\ J_{21} & J_{22} \end{bmatrix}$
- $\det J = \det J_{11} \, \det J_{22}$
- $\det J_{11} = 0$ gives arm singularities and $\det J_{22} = 0$ gives wrist singularities.
- J in this form will not give you the correct relation between velocities. Only used to determine singularities.

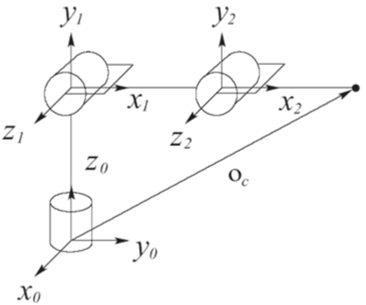
Spherical wrist is singular when $\theta_5 = 0, \pi$



- $J_{22} = \begin{bmatrix} z_3 & z_4 & z_5 \end{bmatrix}$
- Singularity occurs when z_3, z_4, and z_5 are linearly dependent
- This happens when z_3 and z_5 are colinear or $\theta_5 = 0, \pi$
- In general, whenever two revolute joint axes are collinear, a singularity results.

Arm Singularities

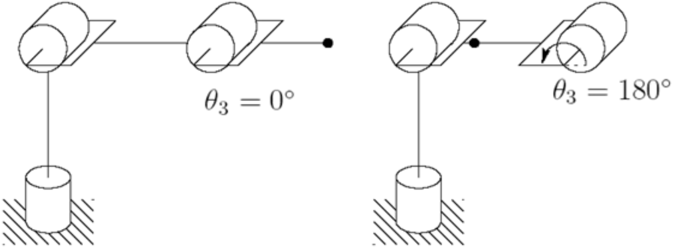
- Need to compute determinant of J_{11}, but with O_n at the wrist center O.
  - J_{11} has ith column $z_{i-1} \times (o - o_{i-1})$ if joint i is revolute and z_{i-1} if it is prismatic.
- Let's look at specific examples



- For indicated frame assignment, $\det J_{11} = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$
- Singular, if
  - $s_3 = 0$ or $\theta_3 = 0, \pi$
  - $a_2 c_2 + a_3 c_{23} = 0$

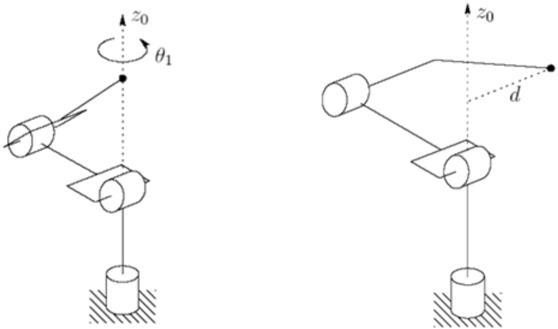
Arm Singularities – Articulated

- $s_3 = 0$
- Elbow singularity



Arm Singularities – Articulated

- $a_2 c_2 + a_3 c_{23} = 0$
- Shoulder singularity
- Not possible with an offset, but the points become unreachable



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