# Proximal Causal Learning with Kernels: Two-Stage Estimation and Moment Restriction

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Arthur Gretton▲, Krikamol Muandet▲

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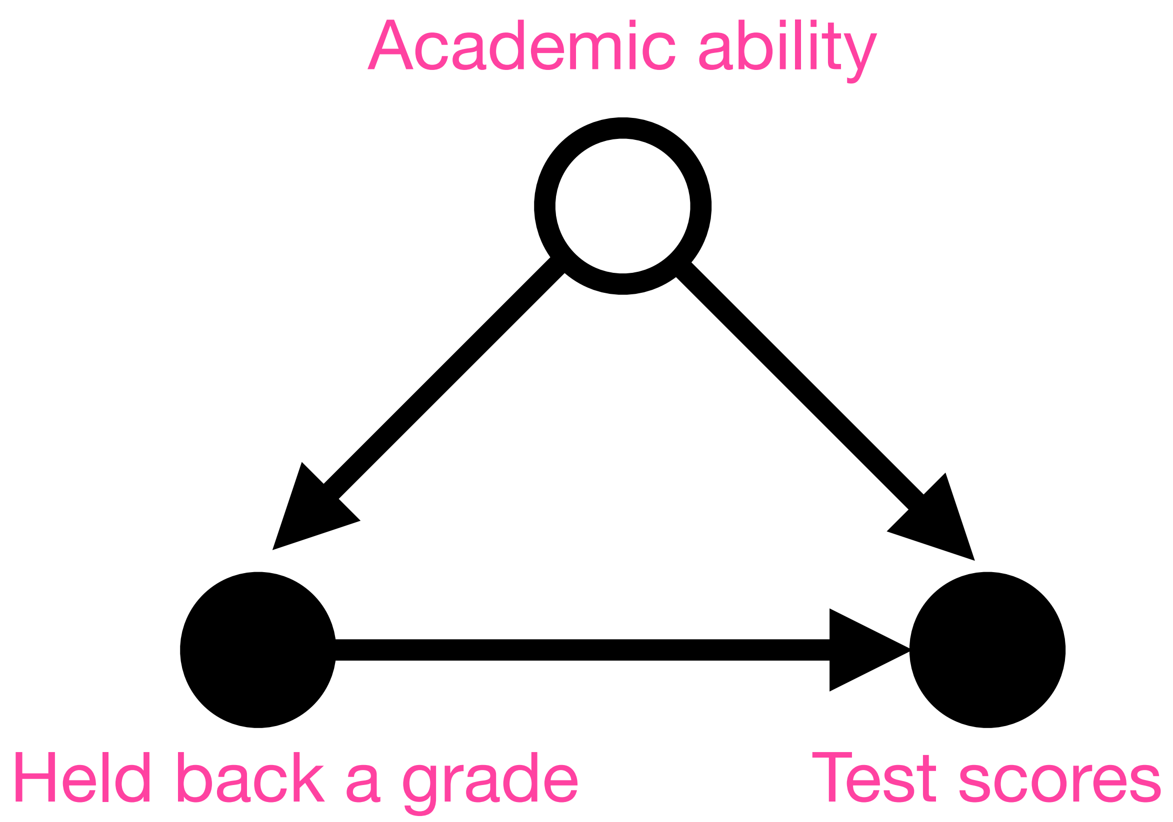
ENSAE ParisTech École Nationale de la Statistique et de l'Administration Économique
GATSBY
The Alan Turing Institute
University of Oxford

# Reliable Decision Making

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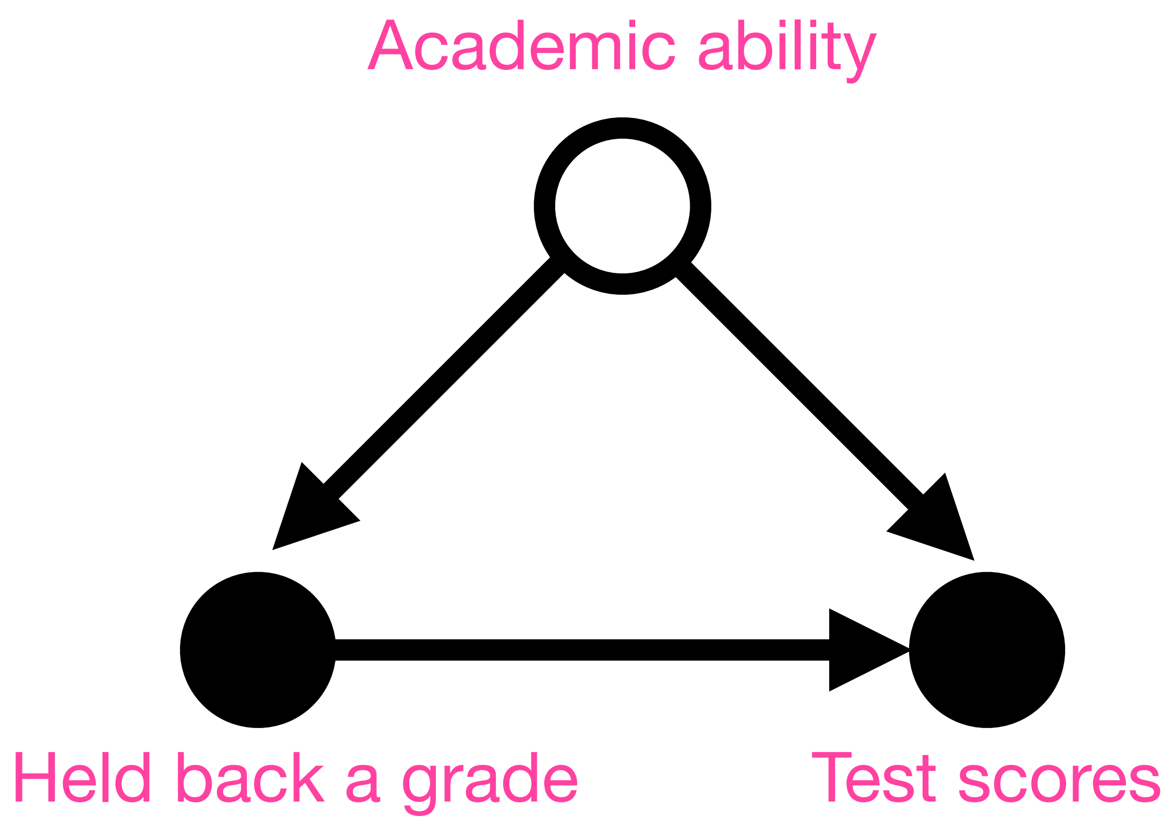


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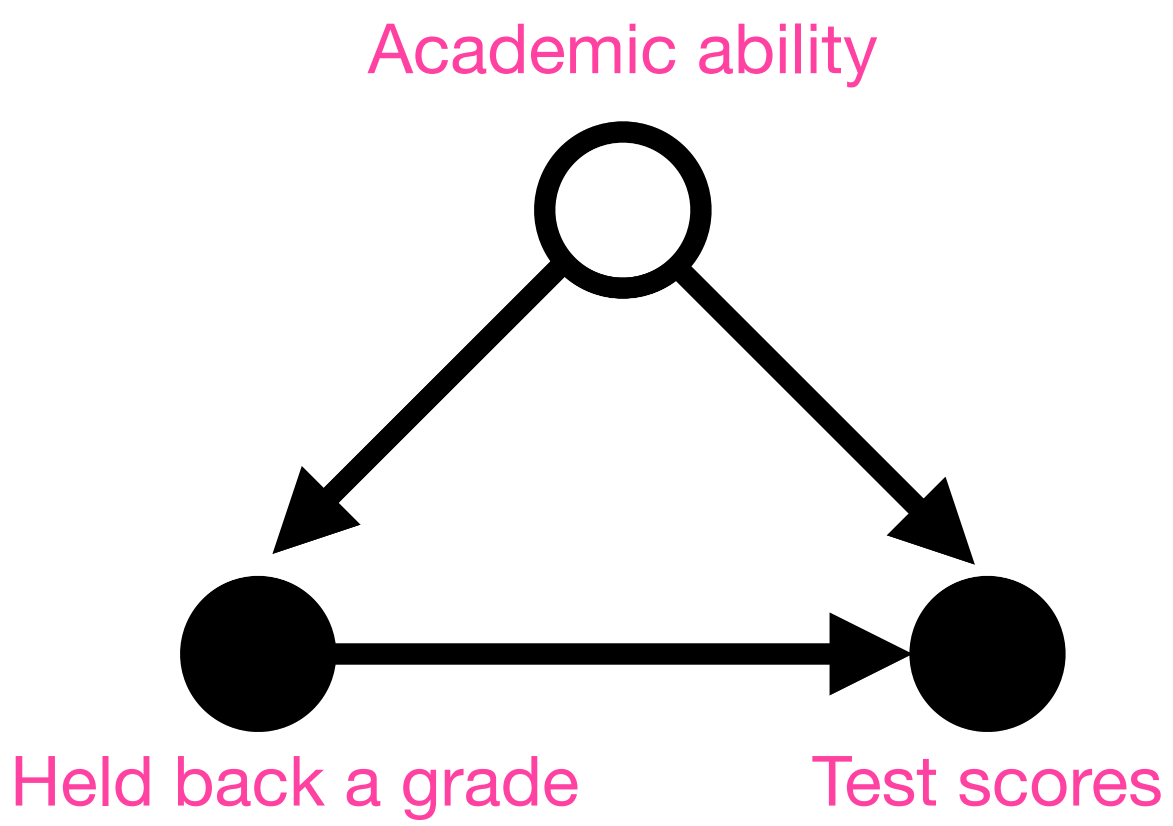


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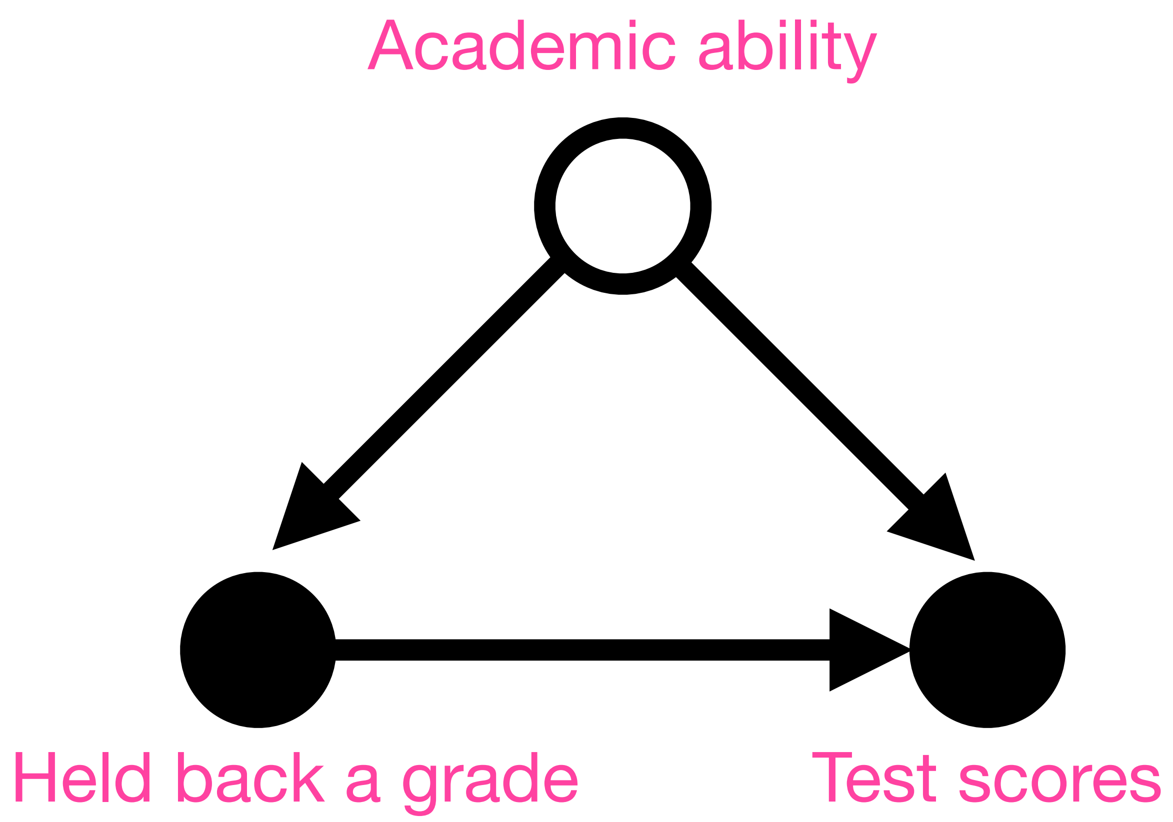


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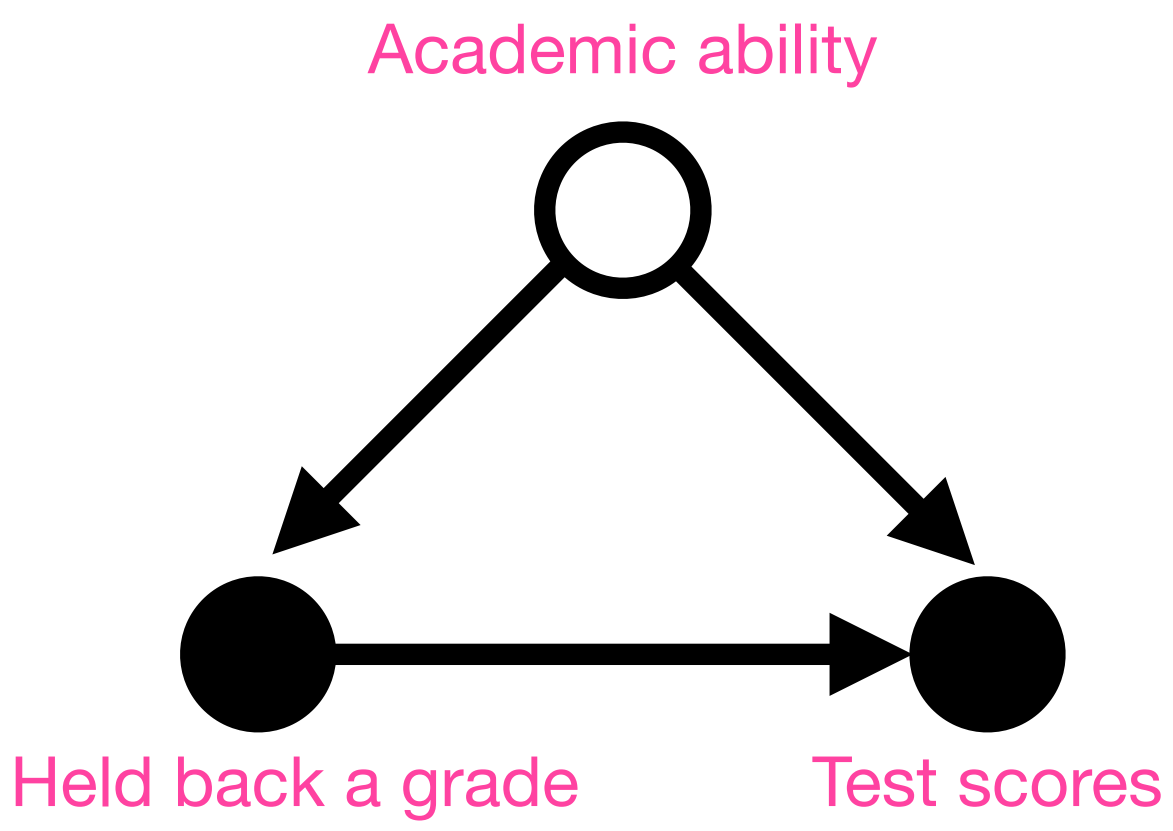


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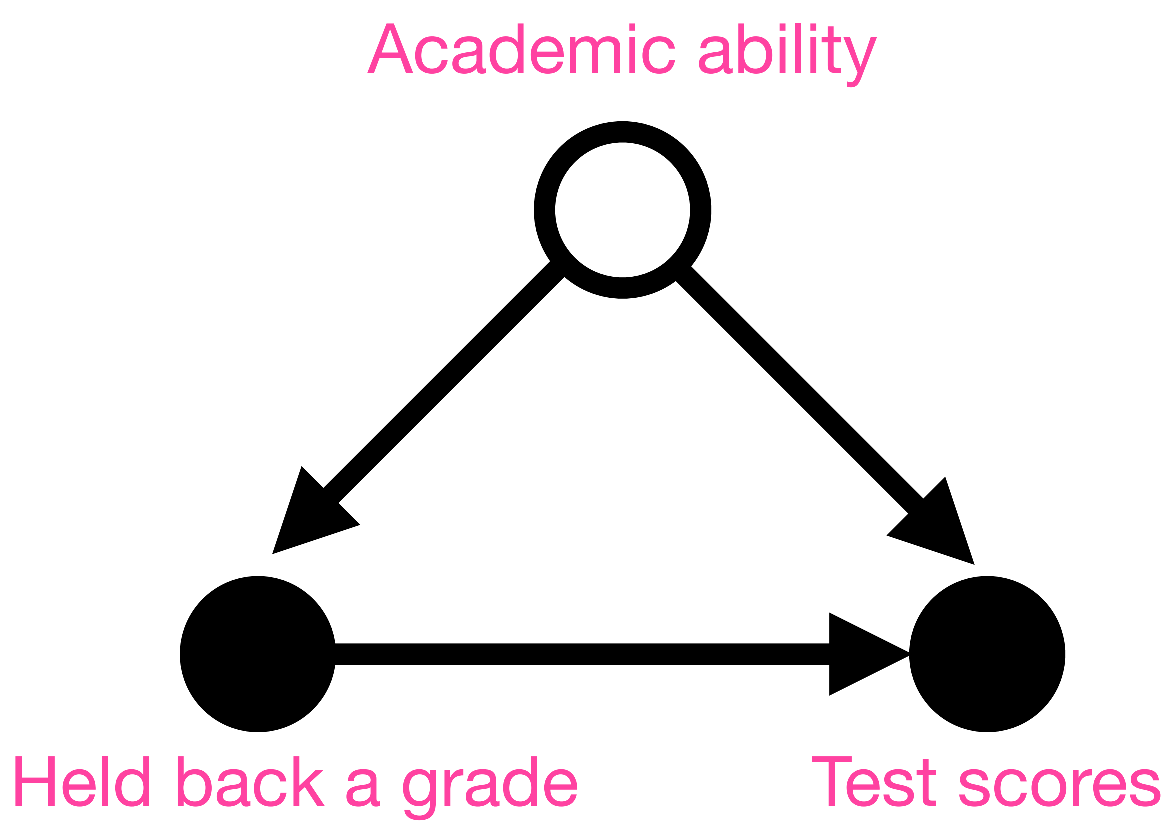


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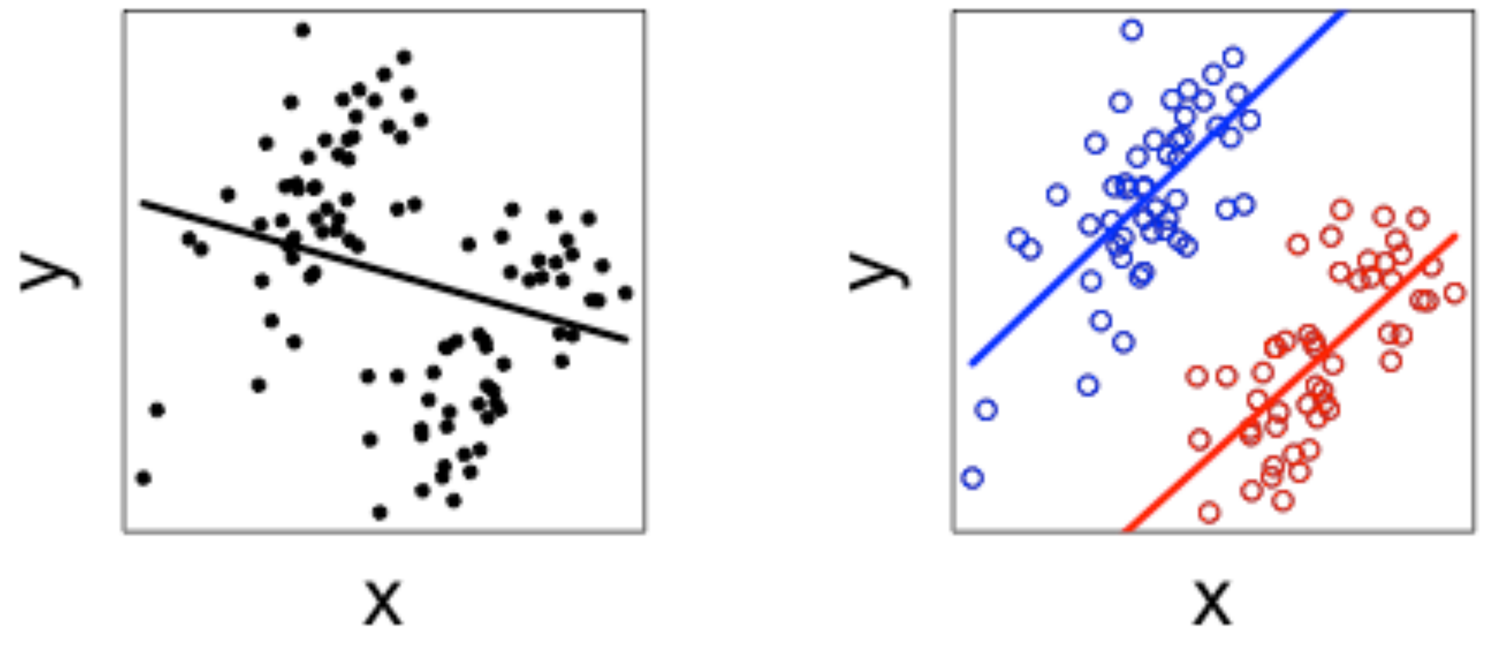


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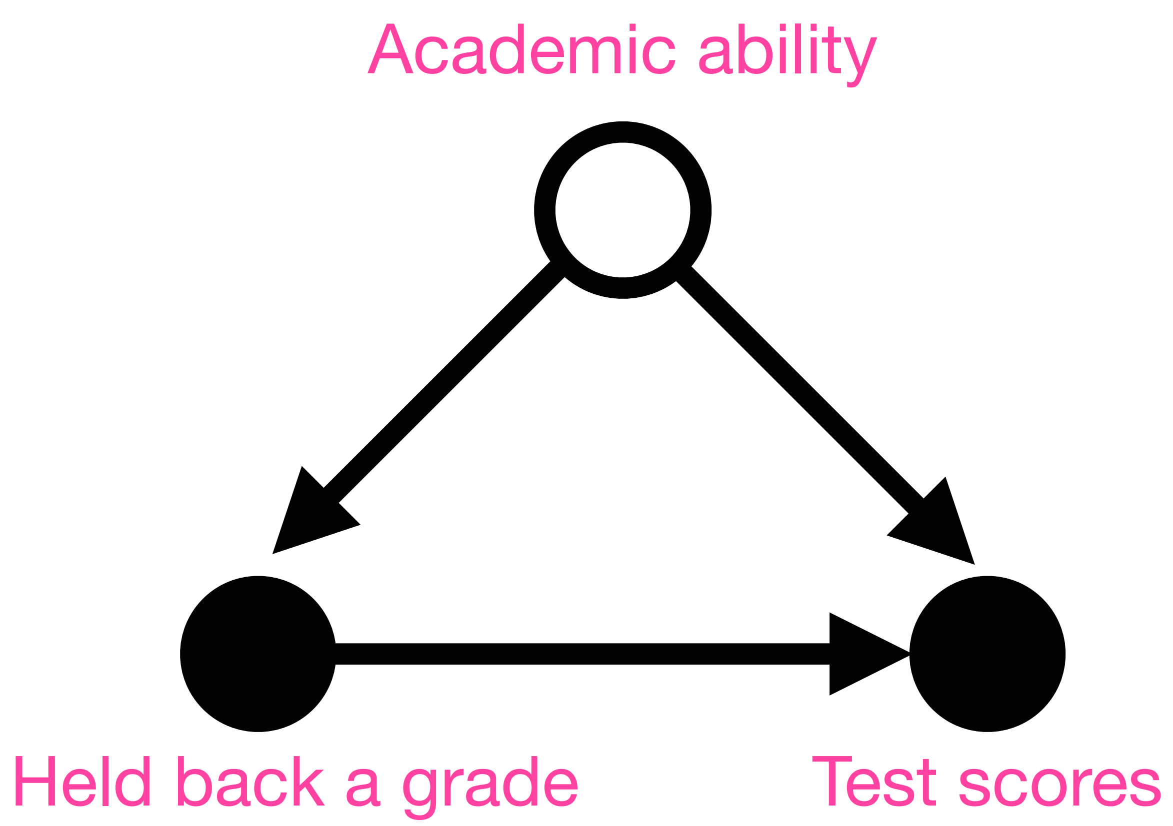


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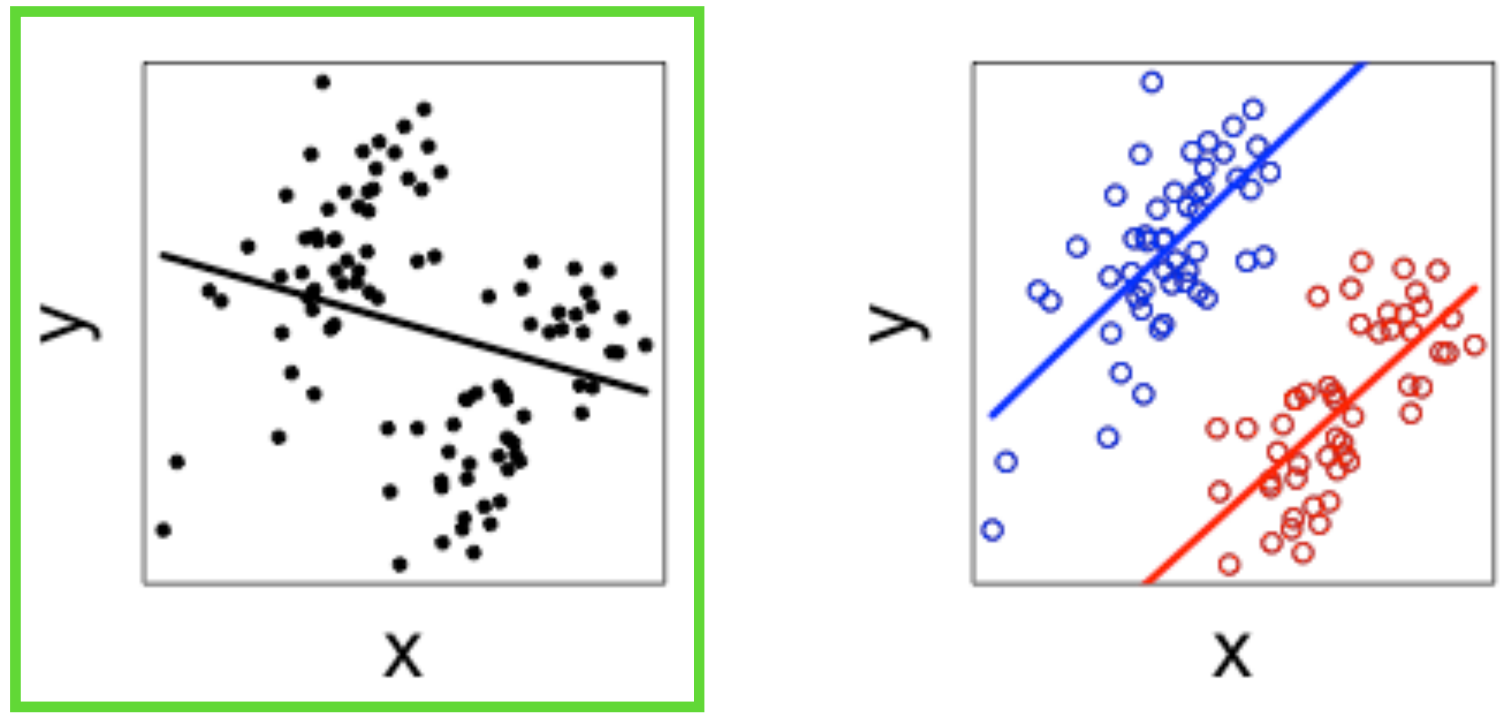

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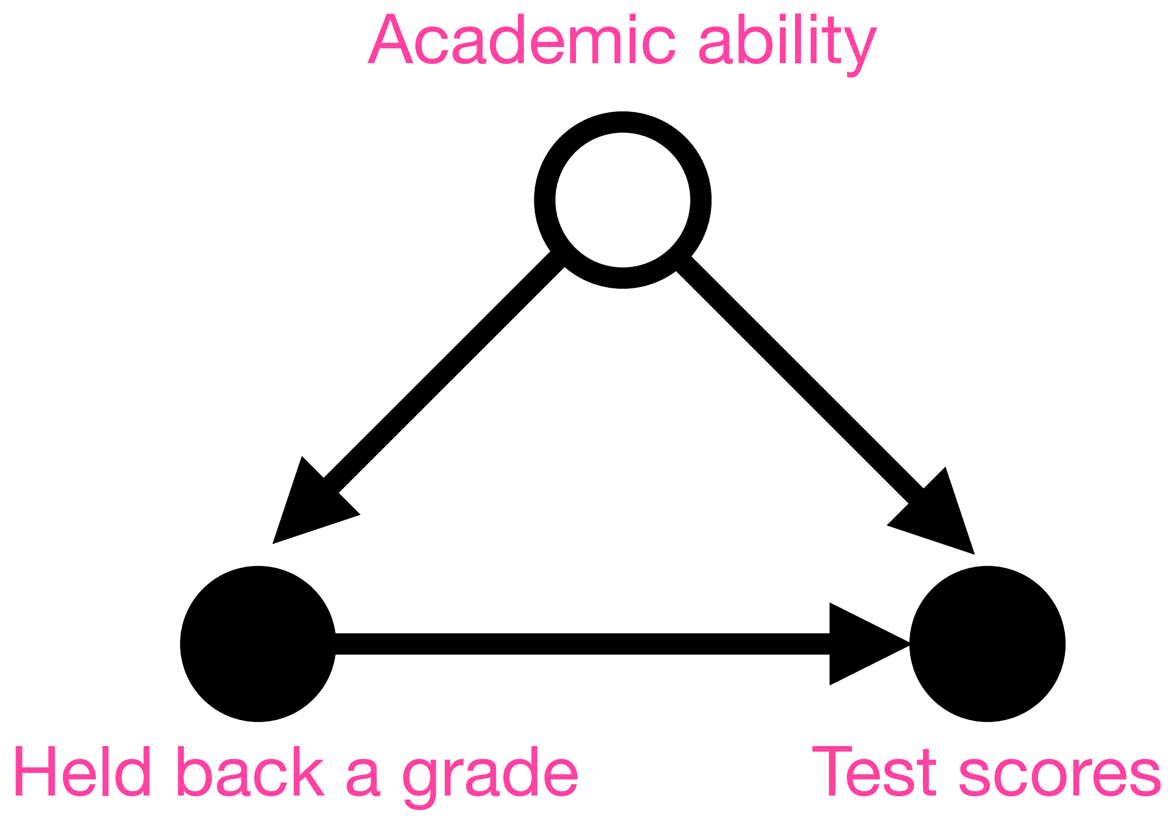


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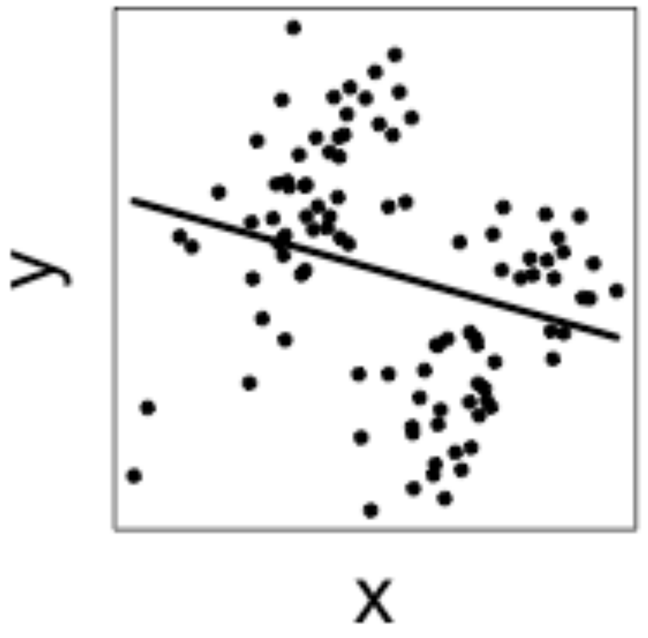


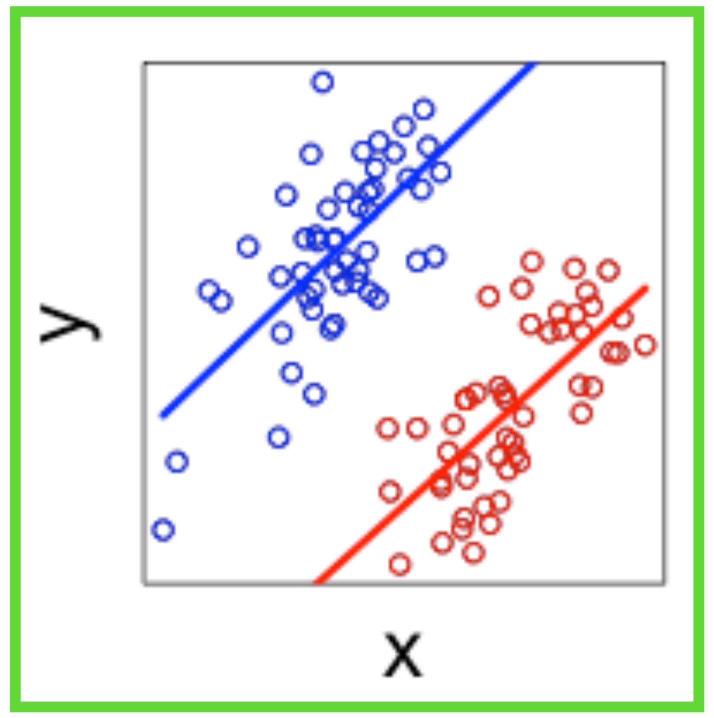

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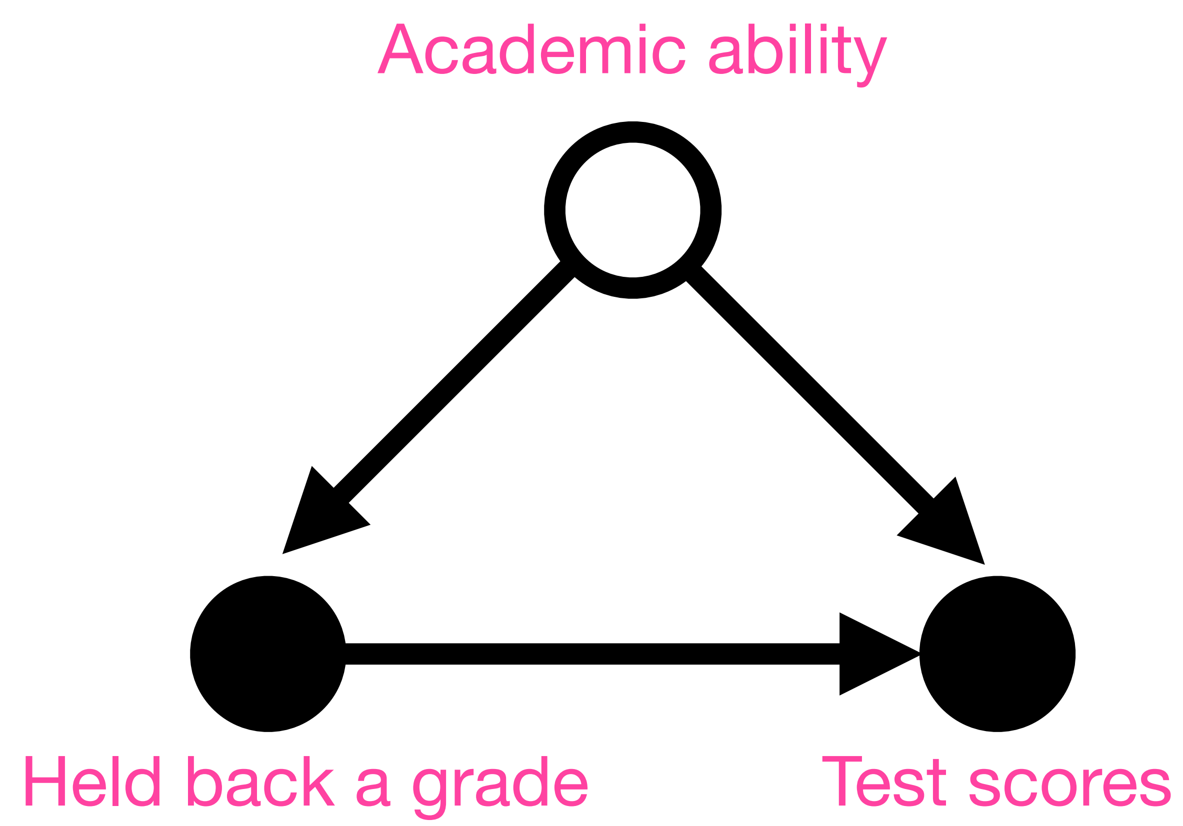


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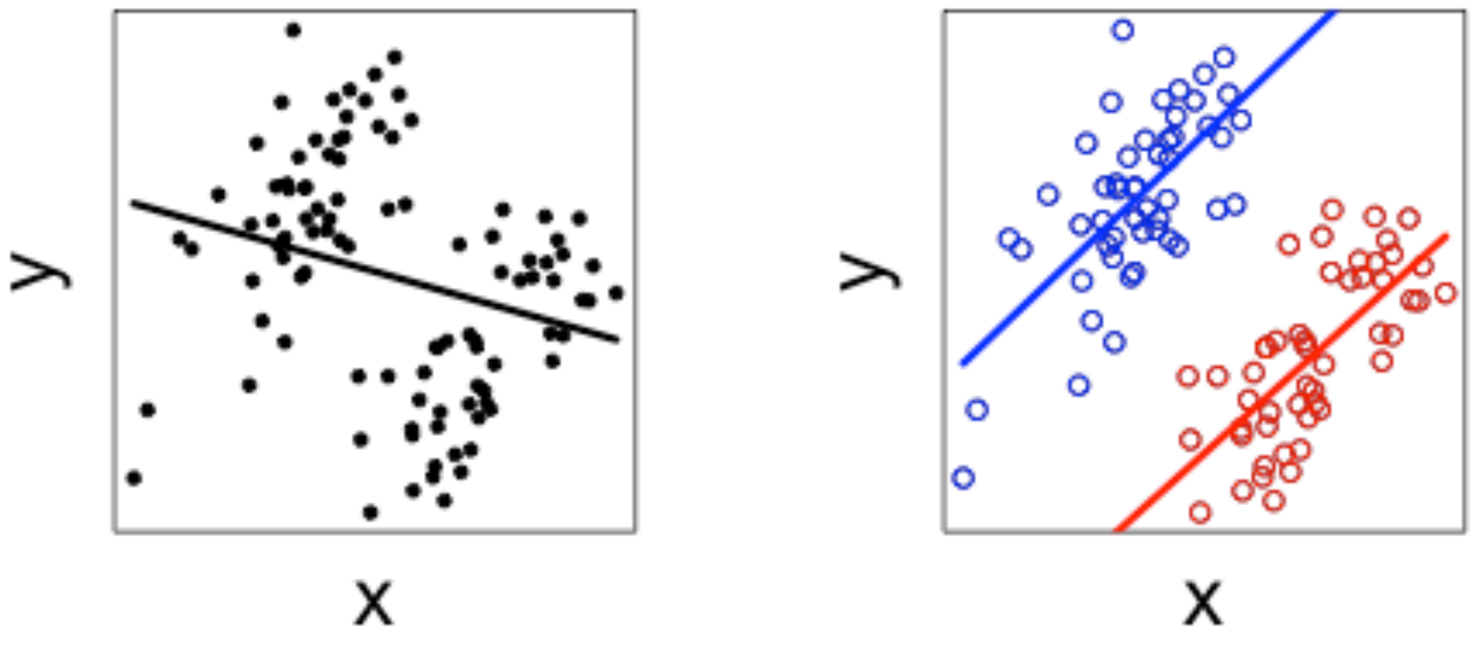

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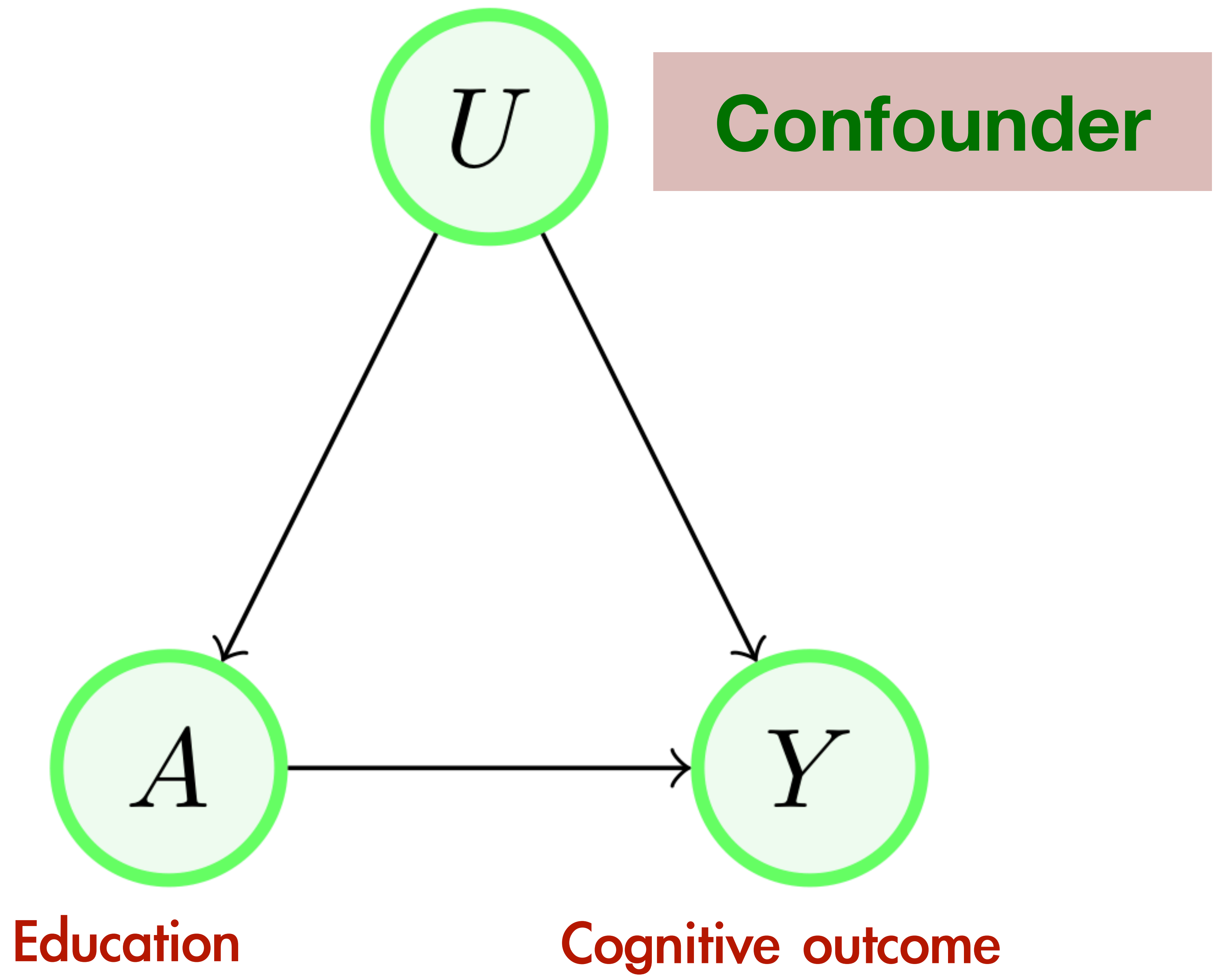

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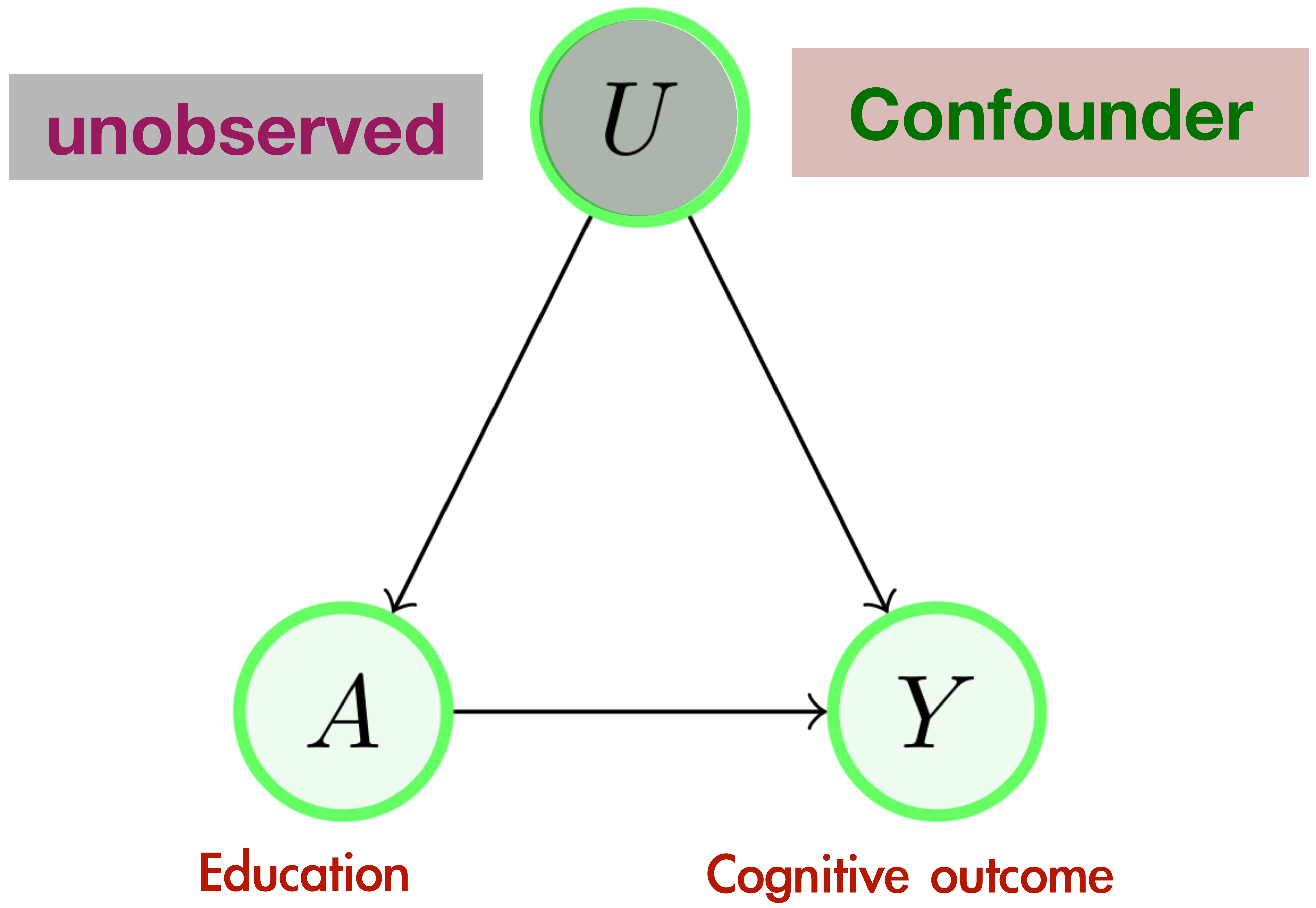

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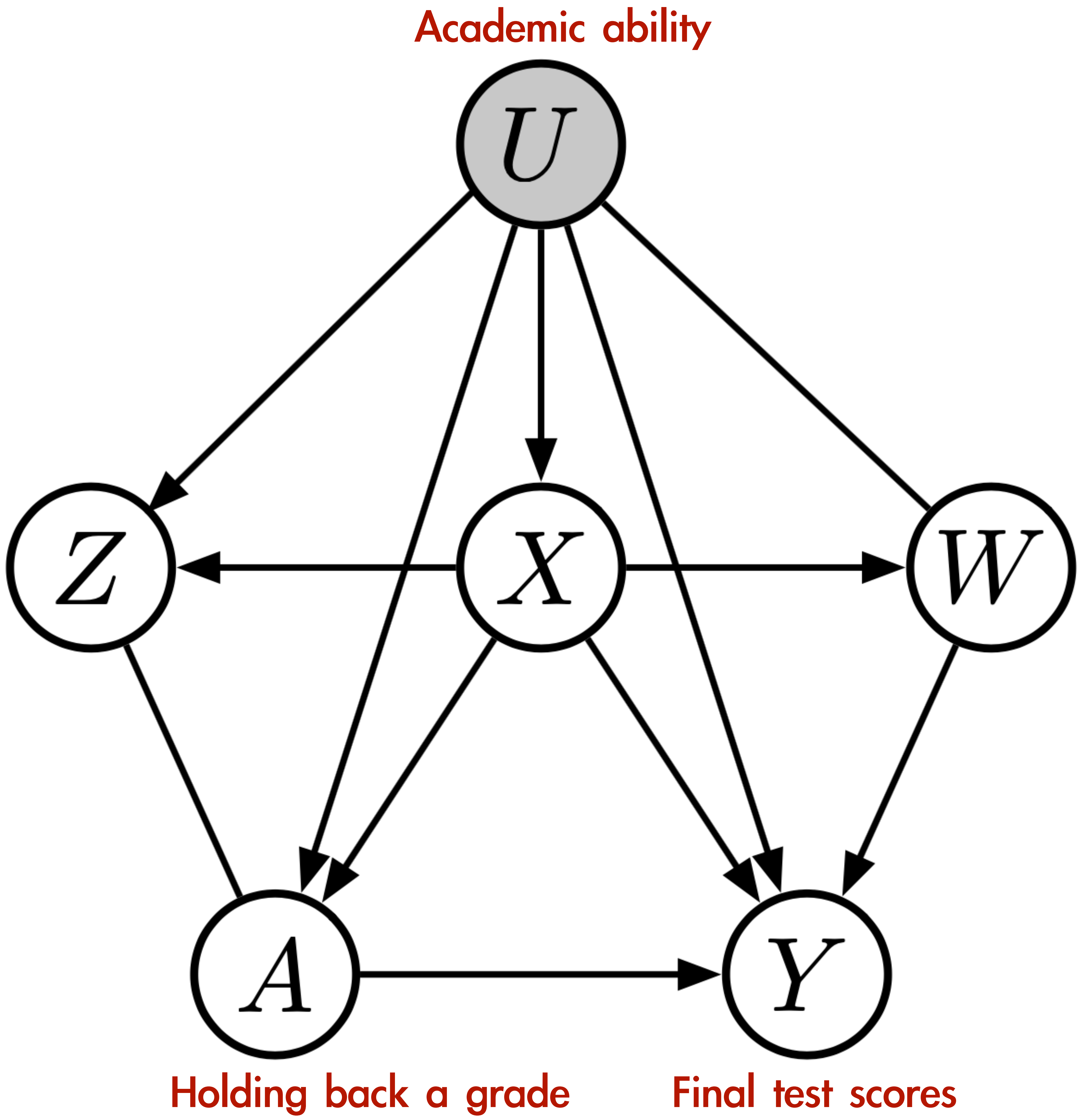

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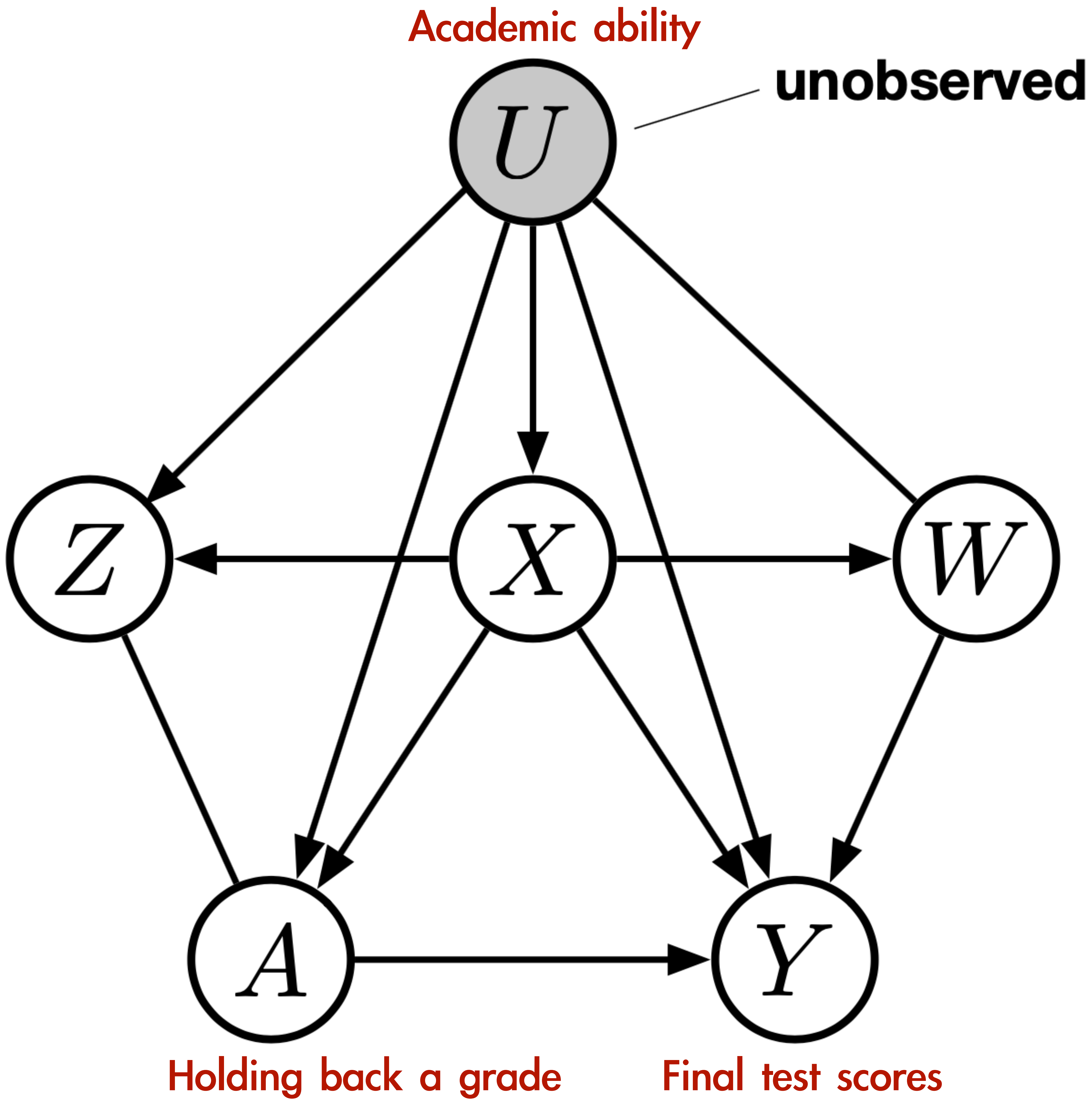

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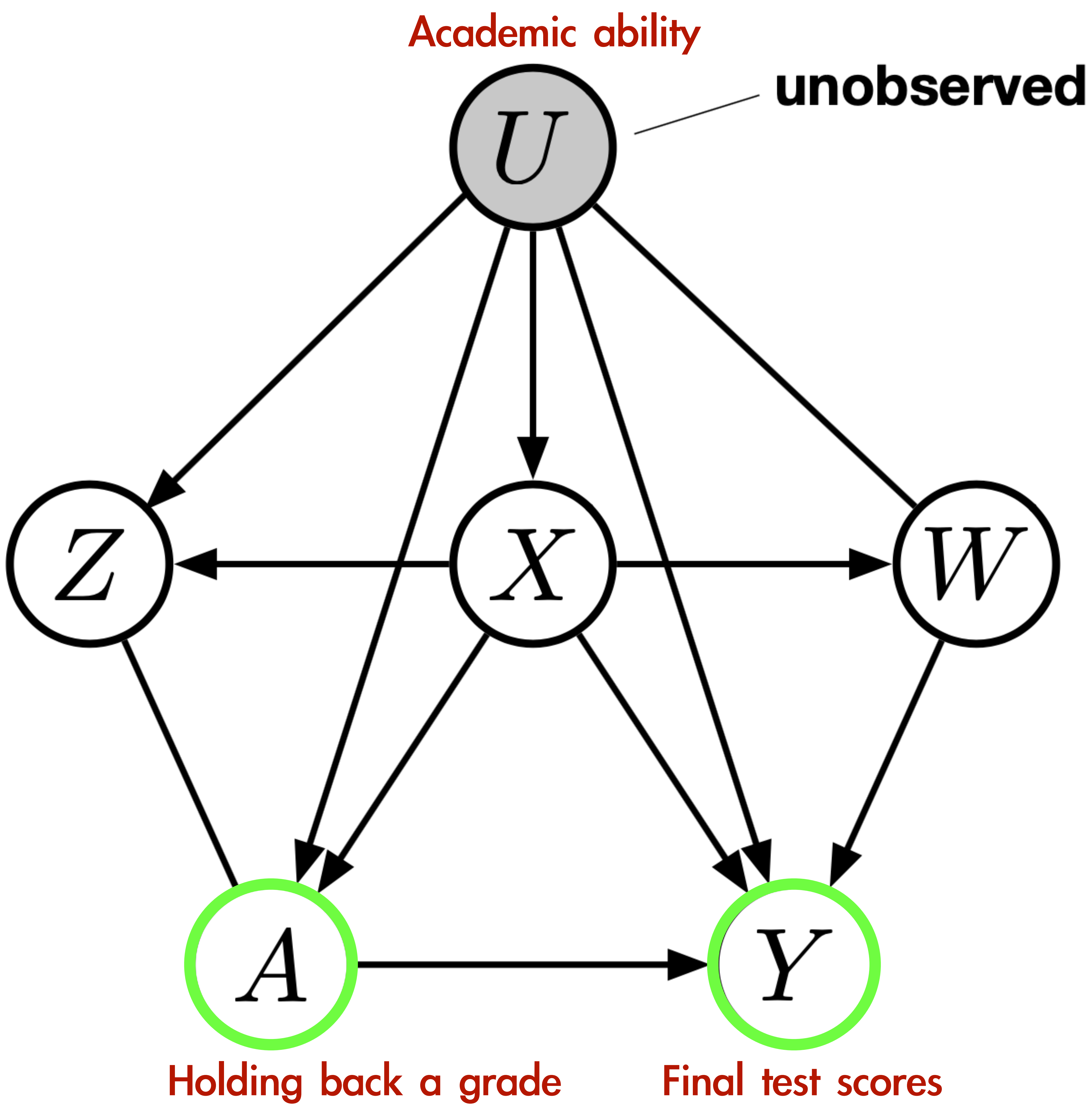

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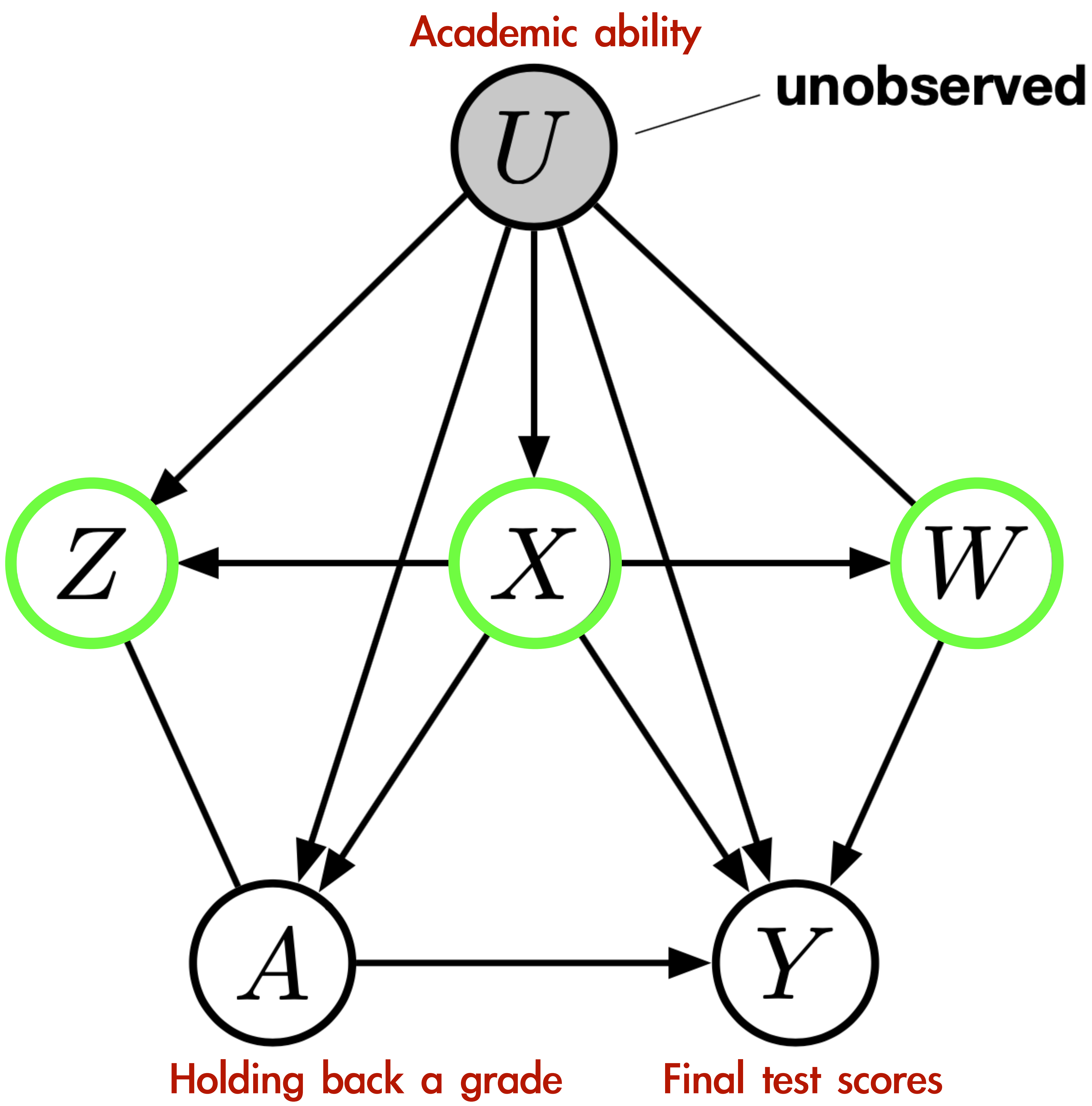

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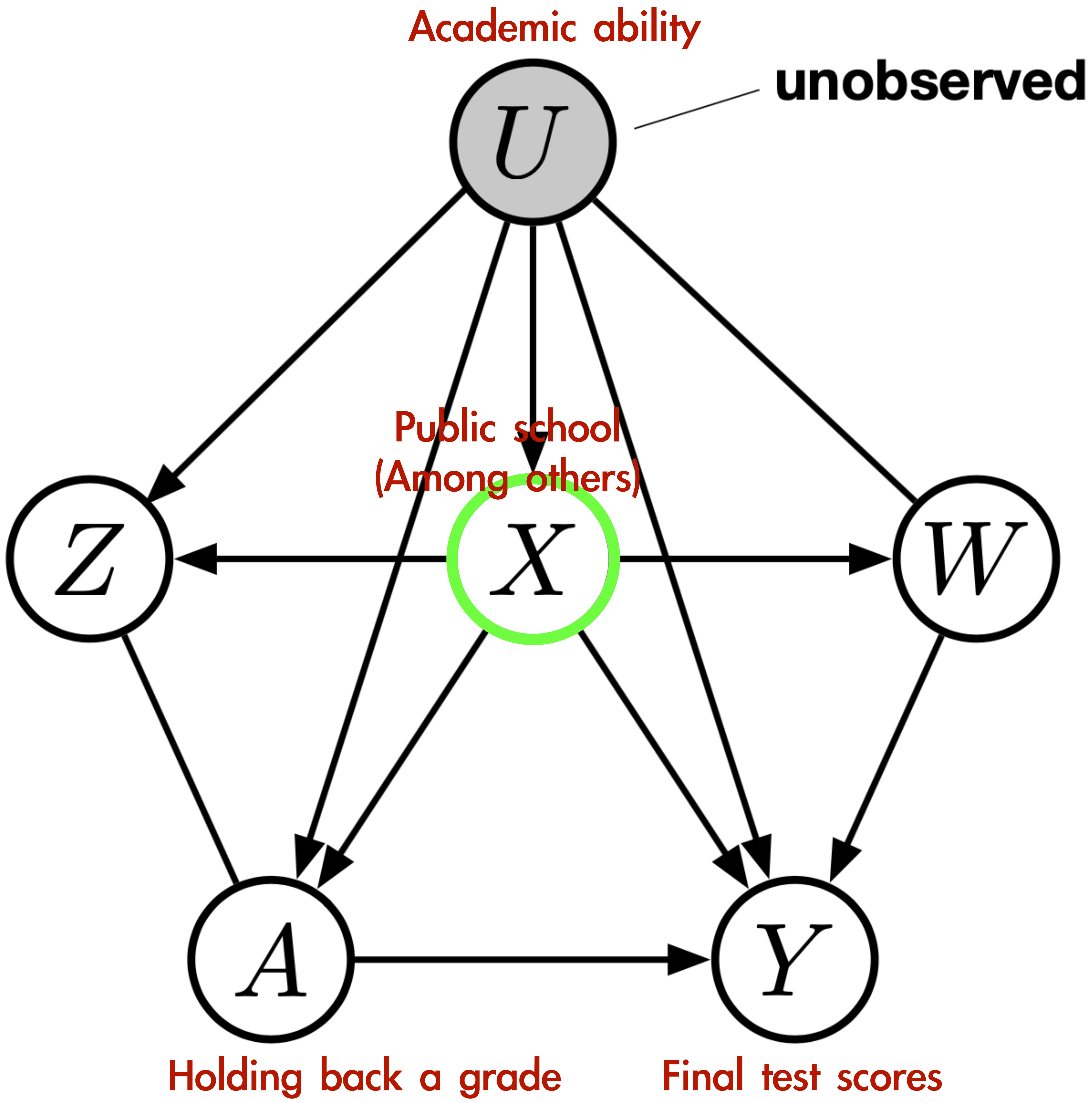

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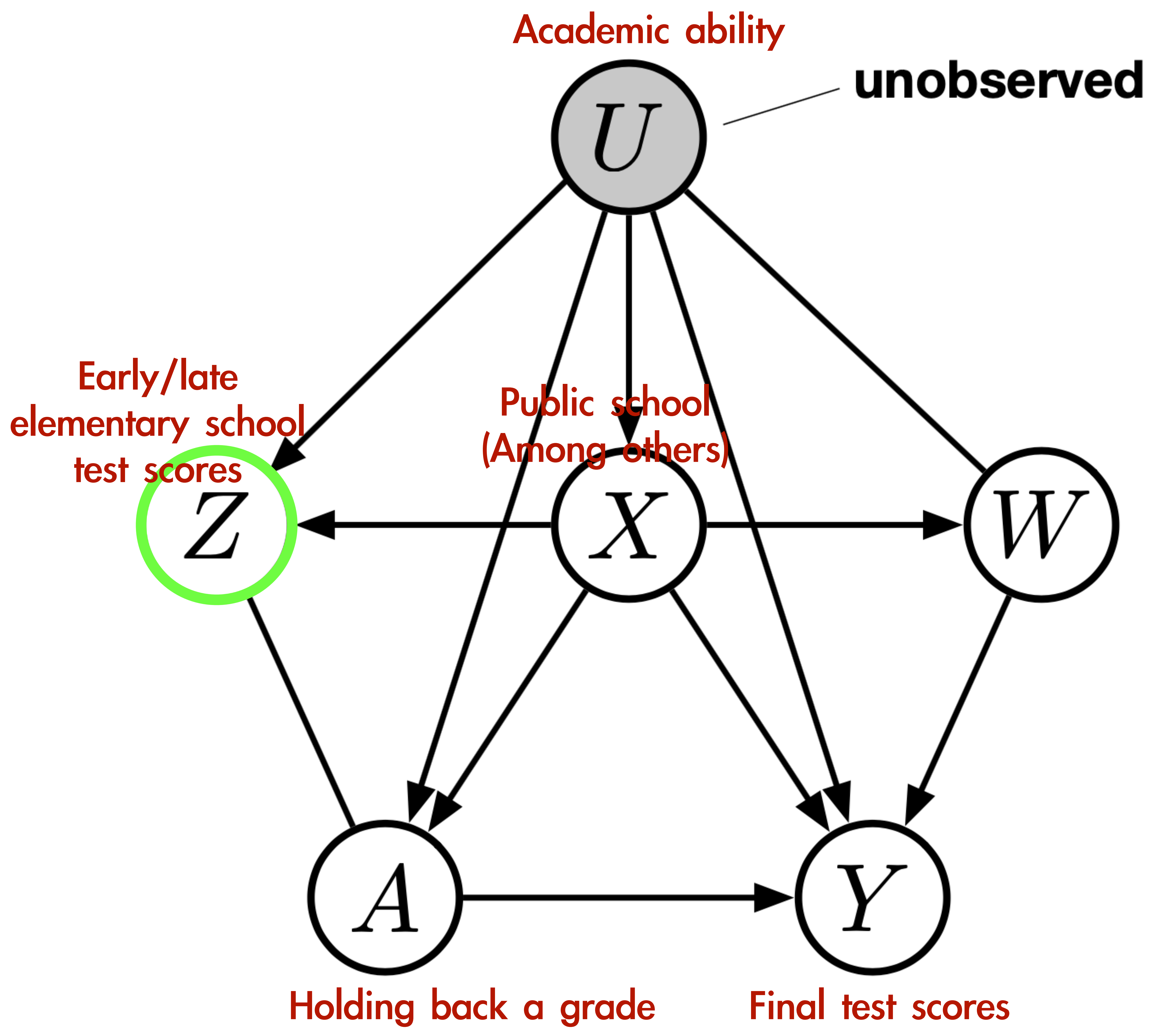

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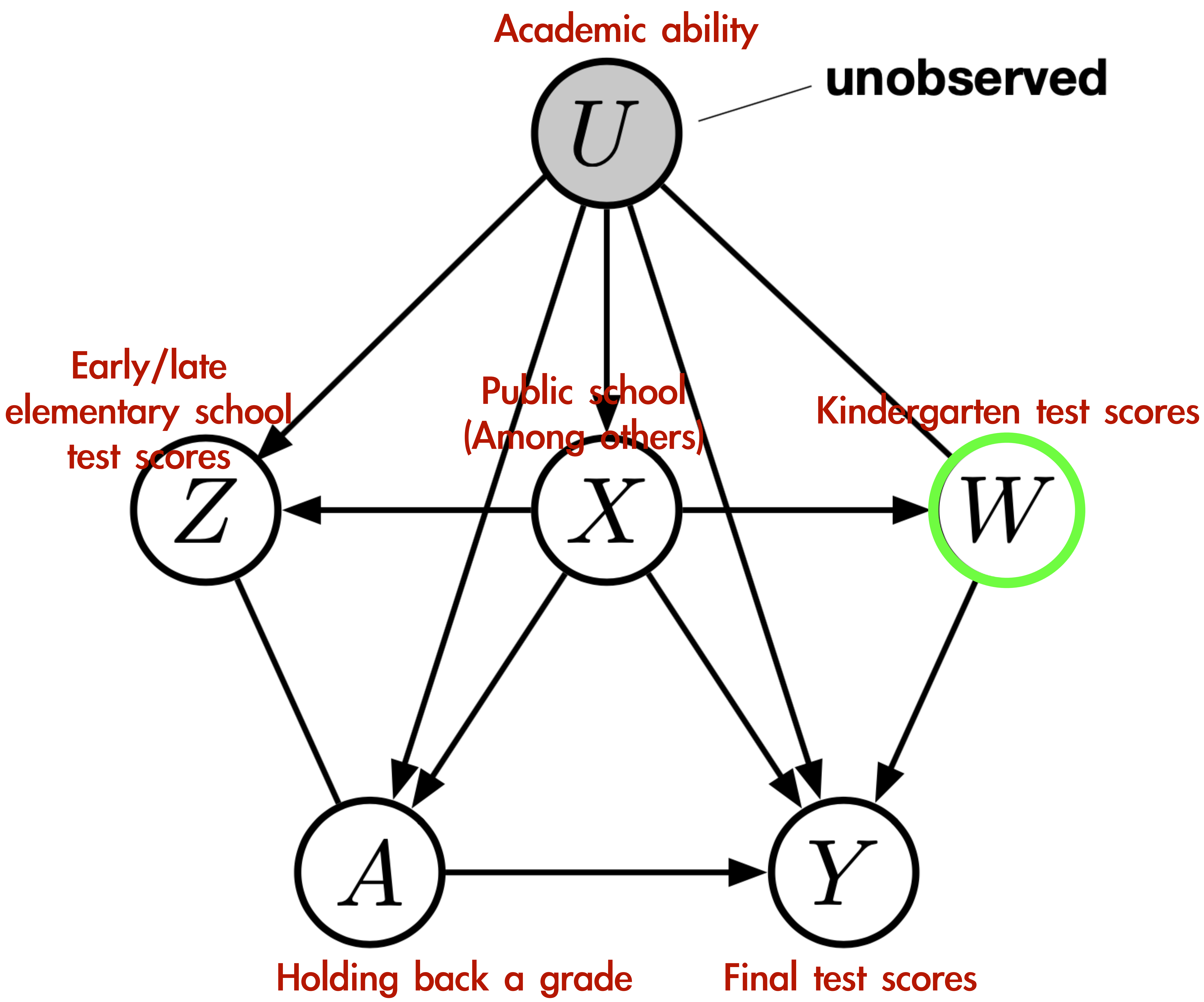

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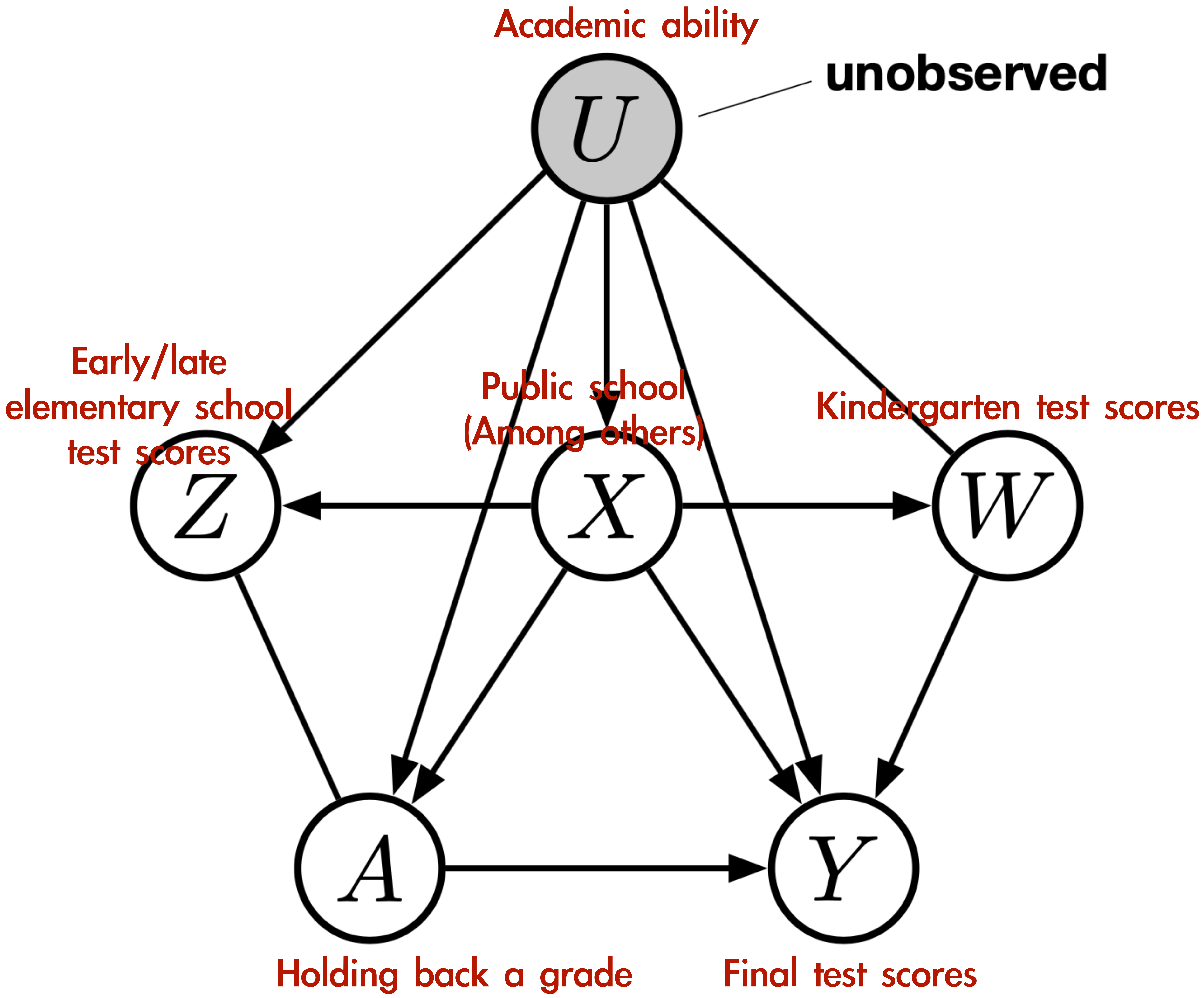

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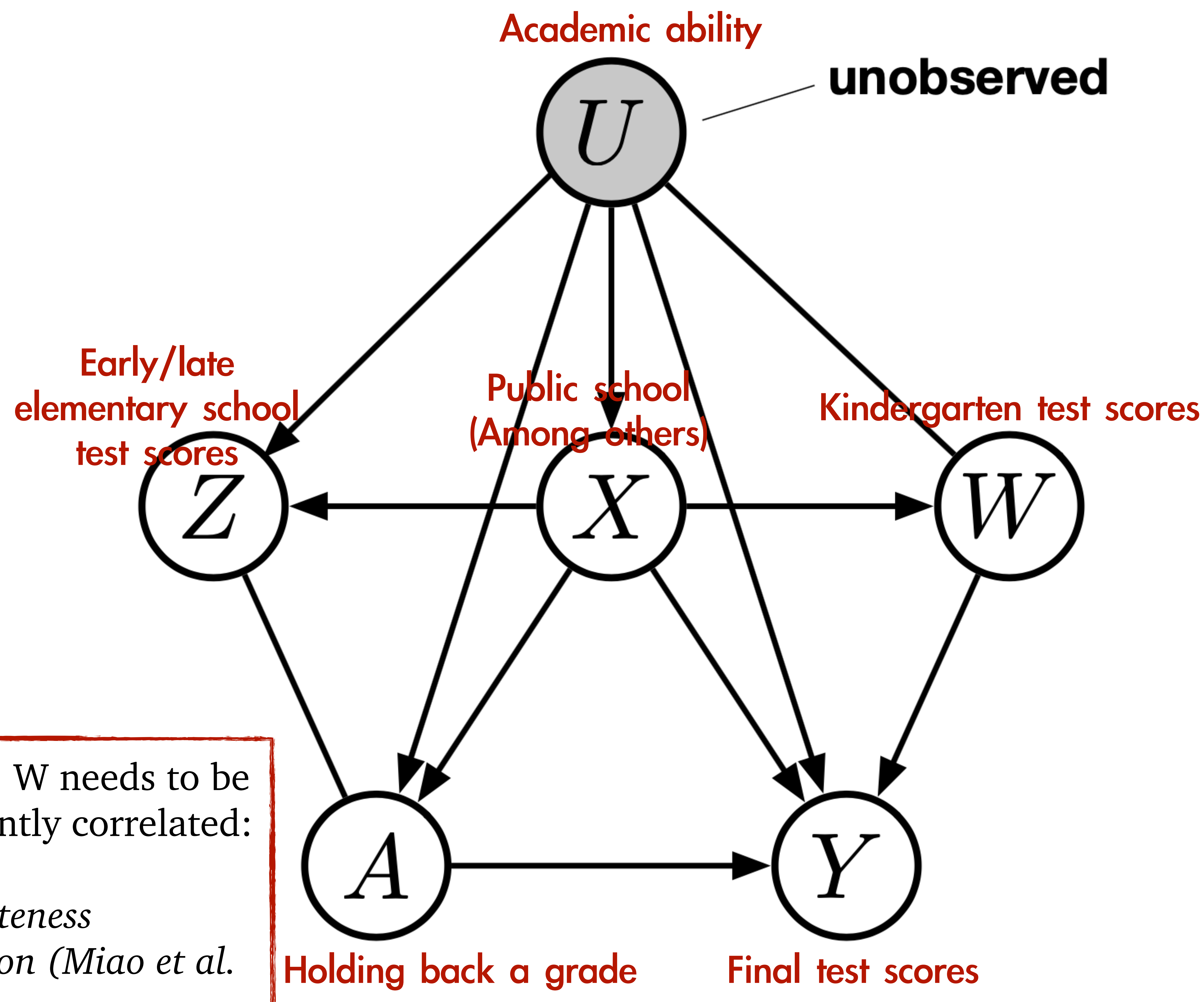


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*Completeness Condition (Miao et al. 2018)*

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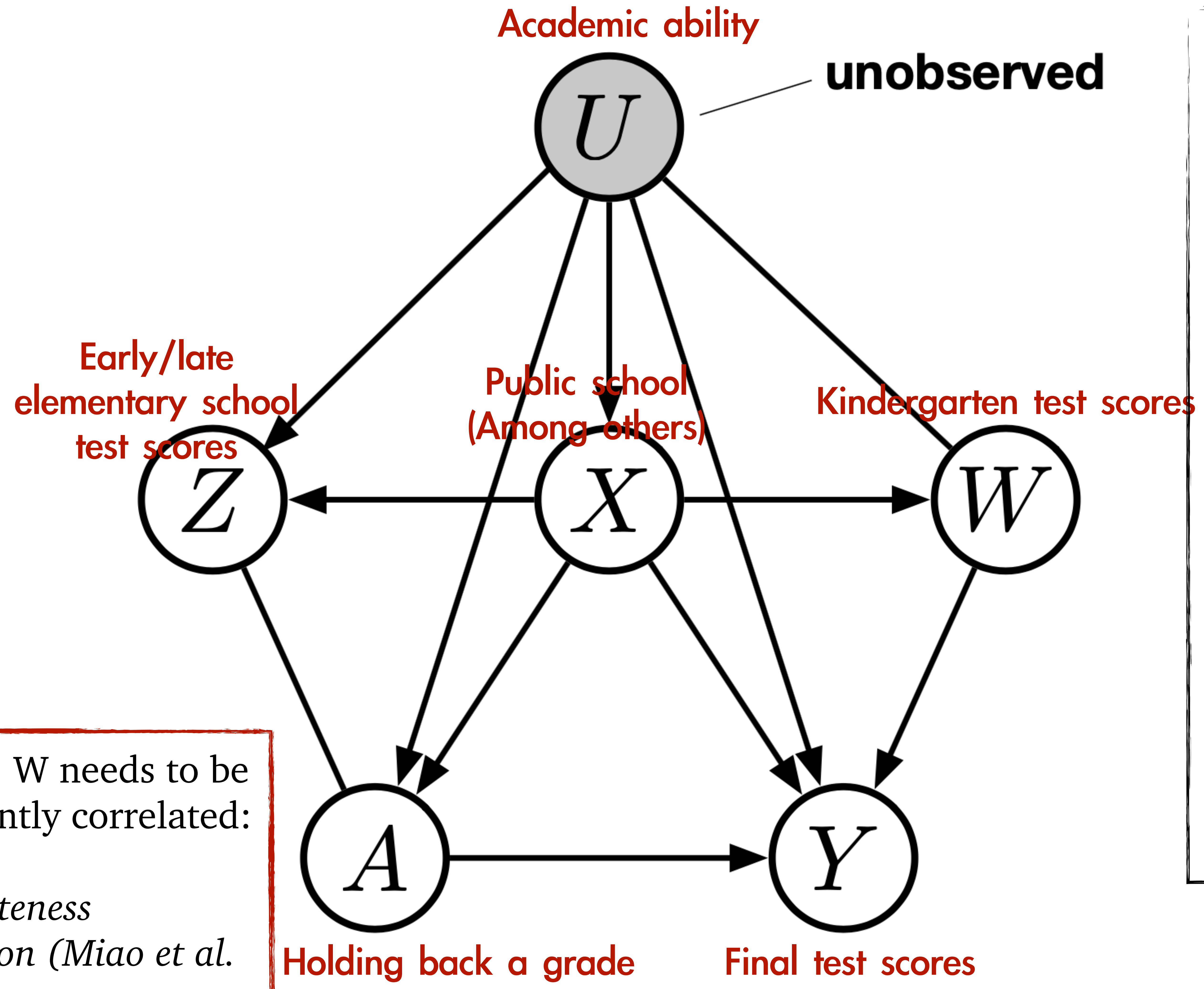


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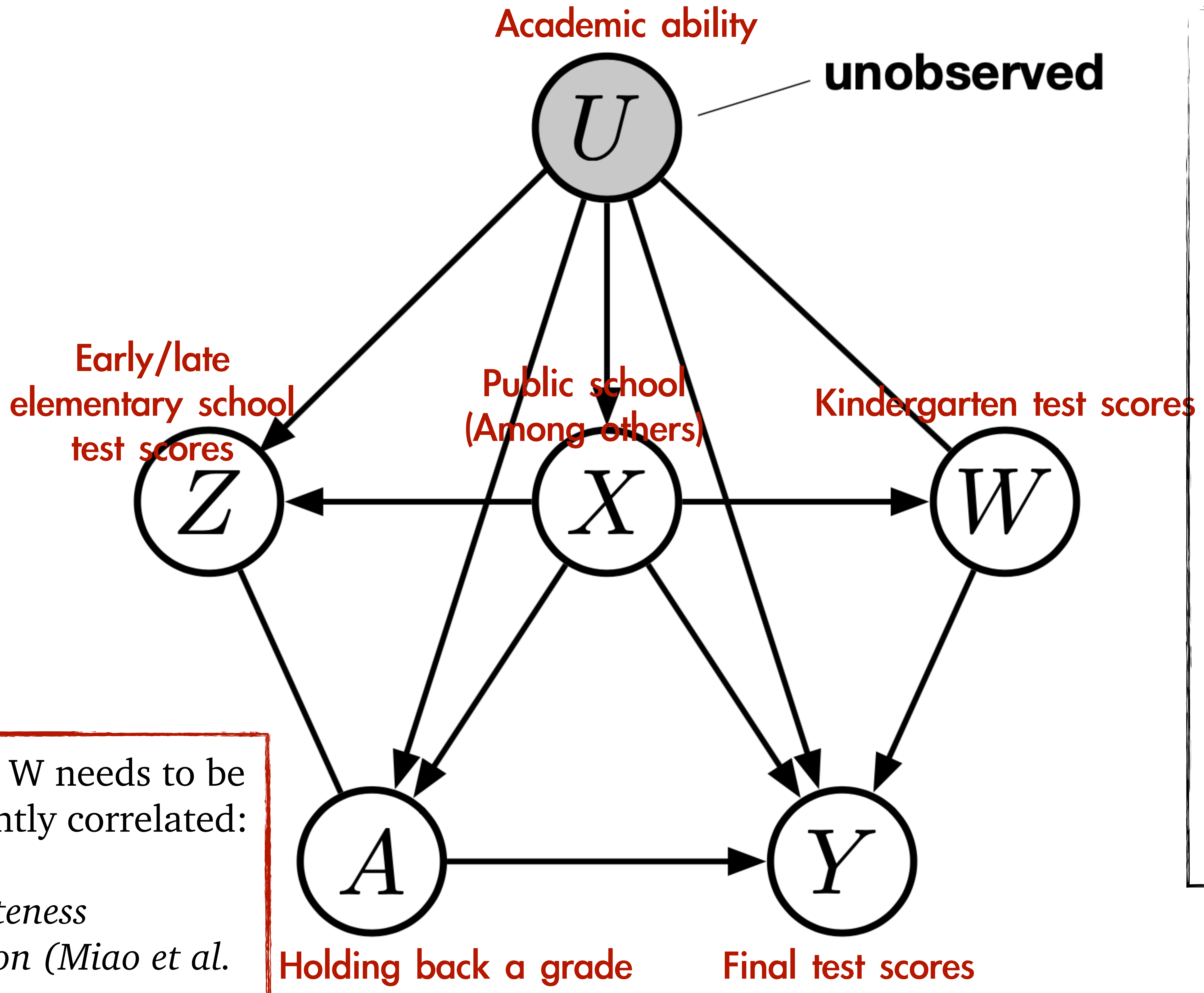


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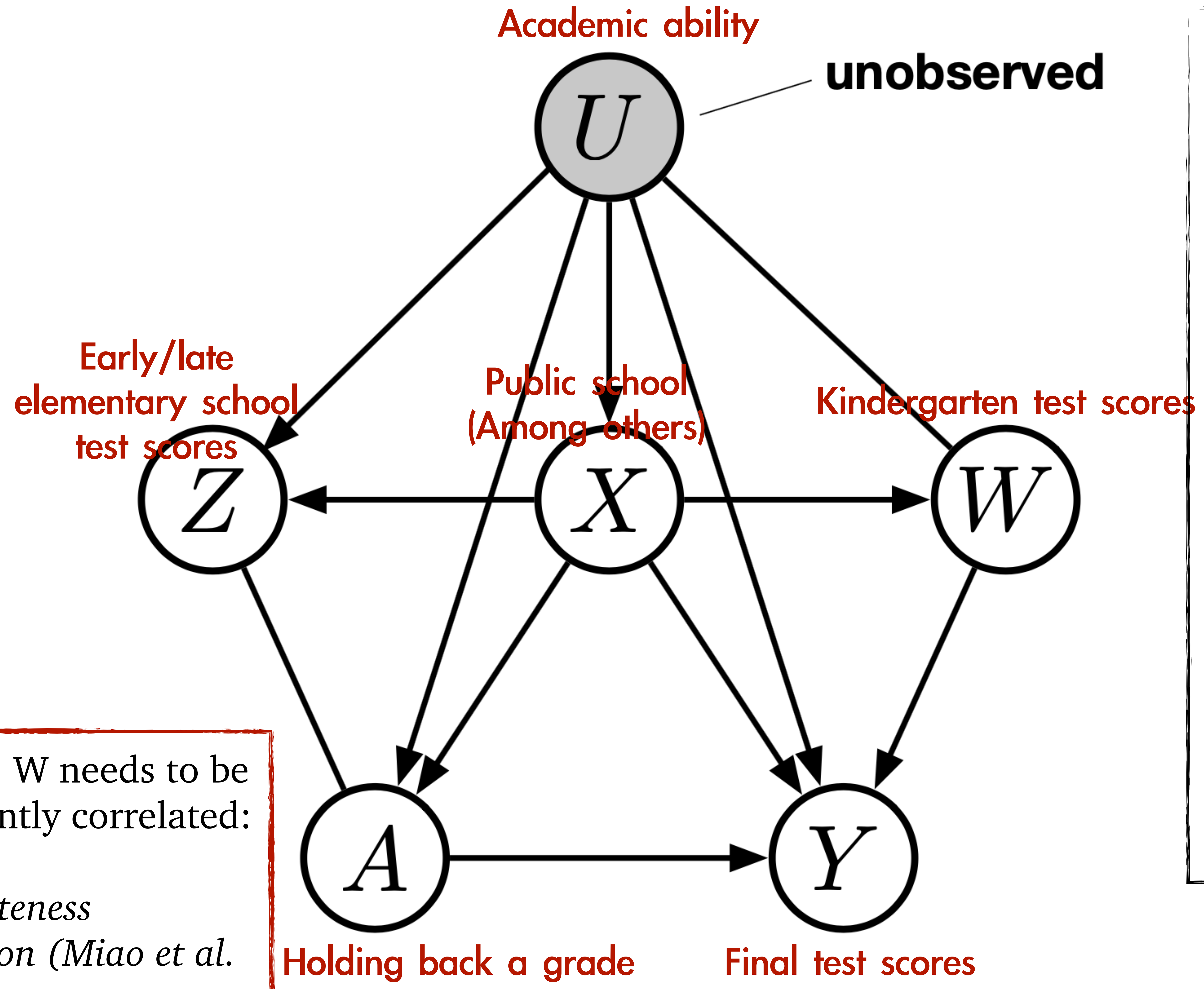


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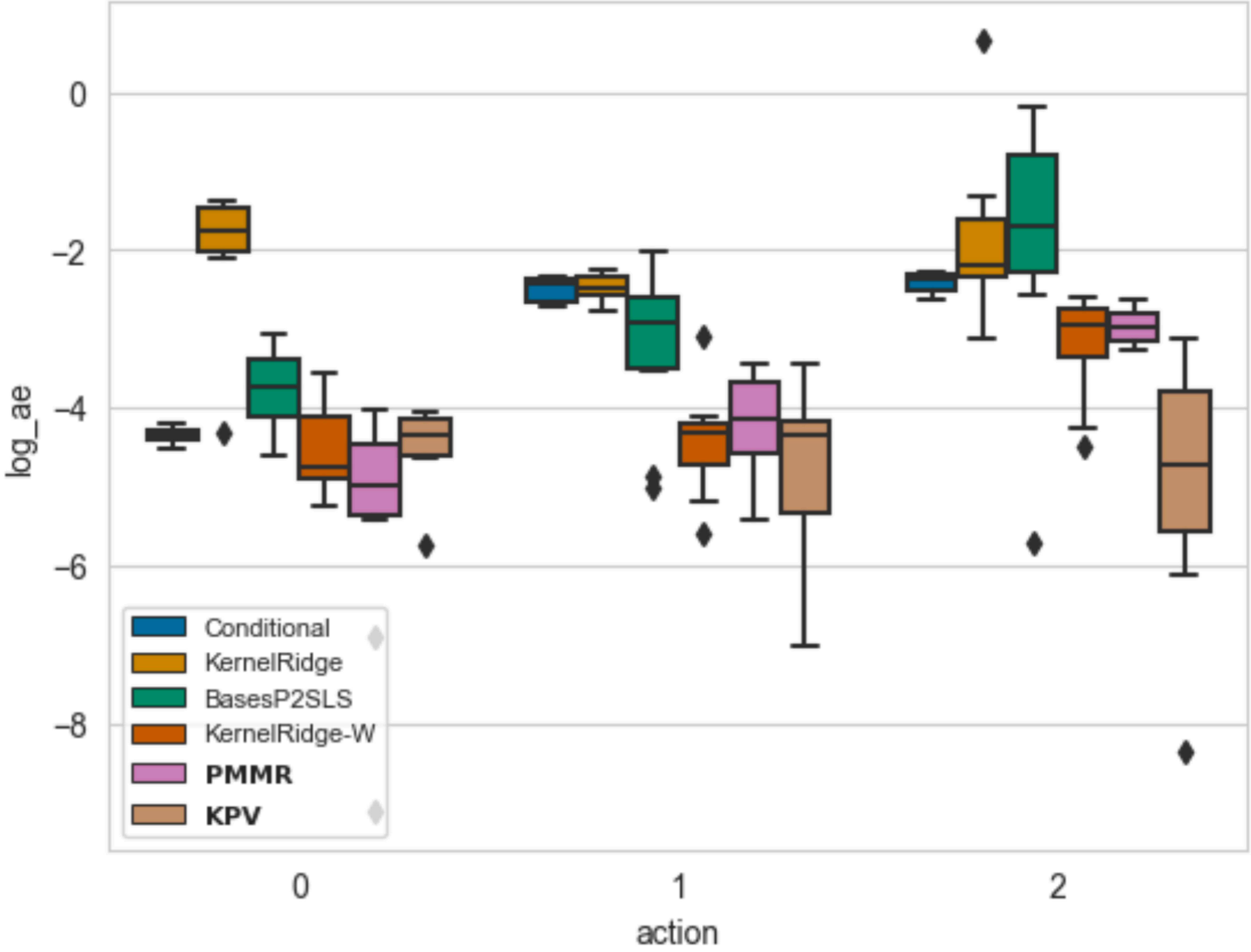
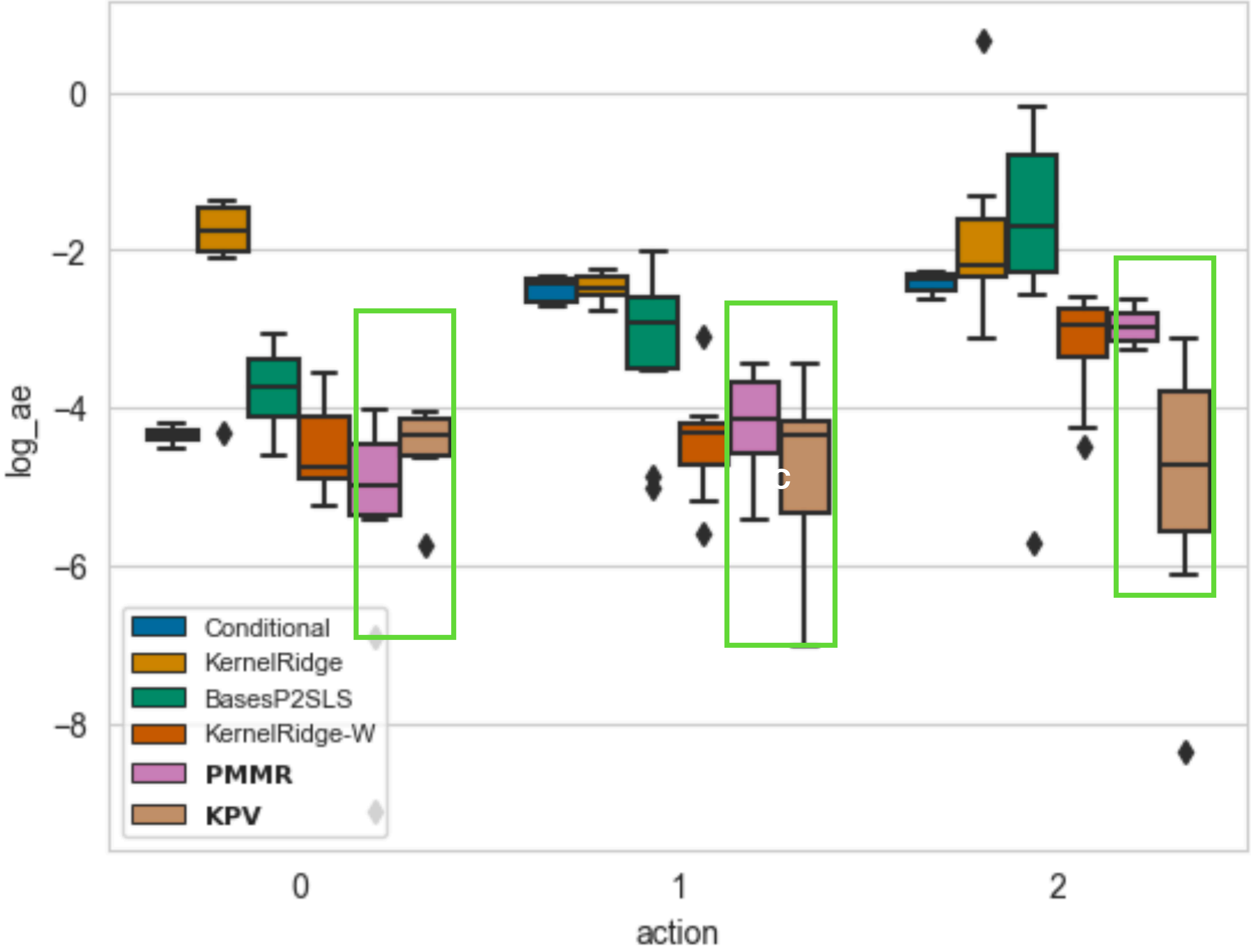
# Results




**Y: maths score**

# Results




Y: maths score