# Divide-and-Conquer: Polynomial Multiplication

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Algorithmic Design and Techniques
Algorithms and Data Structures at edX

### Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

### Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

$$A(x) = 3x^2 + 2x + 5$$

$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Input: Two n-1 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$  $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ 

#### Output:

Input: Two 
$$n-1$$
 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$   
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ 

Output: The product polynomial: 
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

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$$n-1$$
 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$   
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Output: The product polynomial: 
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where:  $c_{2n-2} = a_{n-1}b_{n-1}$ 

Input: Two 
$$n-1$$
 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$   
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ 

Output: The product polynomial: 
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$
  
 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$ 

Input: Two 
$$n-1$$
 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$   $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ 

Output: The product polynomial:  $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$  where:  $c_{2n-2} = a_{n-1}b_{n-1}$   $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$  ...  $c_2 = a_2b_0 + a_1b_1 + a_0b_2$ 

Input: Two 
$$n-1$$
 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$   $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ 

Output: The product polynomial:  $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ 

where: 
$$c_{2n-2}=a_{n-1}b_{n-1}\ c_{2n-3}=a_{n-1}b_{n-2}+a_{n-2}b_{n-1}$$

 $c_2 = a_2b_0 + a_1b_1 + a_0b_2$  $c_1 = a_1b_0 + a_0b_1$ 

Input: Two n-1 degree polynomials:  $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$  $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ Output: The product polynomial:  $c_{2n-2}x^{2n-2}+c_{2n-3}x^{2n-3}+\cdots+c_1x+c_0$ where:  $c_{2n-2} = a_{n-1}b_{n-1}$  $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$ 

$$c_2 = a_2b_0 + a_1b_1 + a_0b_2$$
 $c_1 = a_1b_0 + a_0b_1$ 
 $c_0 = a_0b_0$ 

#### Example

Input: n = 3, A = (3, 2, 5), B = (5, 1, 2)

Input: 
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^2 + 2x + 5$$

Input: 
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$

Input: 
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

### Example

Input: 
$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Output: C = (15, 13, 33, 9, 10)

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```
product \leftarrow Array[2n-1] for i from 0 to 2n-2:
```

 $product[i] \leftarrow 0$ 

```
product \leftarrow Array[2n-1]
for i from 0 to 2n-2:
```

 $product[i] \leftarrow 0$ for i from 0 to n-1:

for i from 0 to n-1:

 $product[i+j] \leftarrow product[i+j] + A[i] \times B[j]$ 

```
product \leftarrow Array[2n-1]
for i from 0 to 2n-2:
```

 $product[i] \leftarrow 0$ 

for i from 0 to n-1:

return *product* 

for i from 0 to n-1:

 $product[i+j] \leftarrow product[i+j] + A[i] \times B[j]$ 

```
\begin{array}{l} \textit{product} \leftarrow \textit{Array}[2n-1] \\ \textit{for } \textit{i} \;\; \textit{from 0 to } 2n-2: \\ \textit{product}[\textit{i}] \leftarrow 0 \\ \textit{for } \textit{i} \;\; \textit{from 0 to } n-1: \\ \textit{for } \textit{j} \;\; \textit{from 0 to } n-1: \\ \textit{product}[\textit{i}+\textit{j}] \leftarrow \textit{product}[\textit{i}+\textit{j}] + \textit{A}[\textit{i}] \times \textit{B}[\textit{j}] \\ \textit{return } \;\; \textit{product} \end{array}
```

Runtime:  $O(n^2)$ 

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Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_n$ 

$$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \dots + a_{\frac{n}{2}}$$

$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$

Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$   $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$ 

Let 
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where  $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$   $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_0$ 

Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$   $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$ 

Let 
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where  $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$   $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$ 

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$   $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$ 

Let 
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where  $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$   $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$ 

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

lacktriangle Calculate  $D_1E_1, D_1E_0, D_0E_1$ , and  $D_0E_0$ 

Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$   $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$ 

Let 
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where  $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$   $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$ 

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

lacktriangle Calculate  $D_1E_1, D_1E_0, D_0E_1$ , and  $D_0E_0$ 

Let 
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$   $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$ 

$$D_1(x) = a_{n-1}x^2 + a_{n-2}x^2 + \dots + a_{\frac{n}{2}}$$

$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$

$$E_1(x) = E_1(x)x^{\frac{n}{2}} + E_0(x) \text{ where}$$

$$E_1(x) = b_{-1}x^{\frac{n}{2}-1} + b_{-2}x^{\frac{n}{2}-2} + \dots + b_n$$

$$E_{1}(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$$

$$E_{0}(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_{0}$$

$$AB = (D_{1}x^{\frac{n}{2}} + D_{0})(E_{1}x^{\frac{n}{2}} + E_{0})$$

= 
$$(D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$$
  
• Calculate  $D_1E_1$ ,  $D_1E_0$ ,  $D_0E_1$ , and  $D_0E_0$ 

Recurrence:  $T(n) = 4T(\frac{n}{2}) + kn$ .

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
  

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{B(x)} = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ 

$$A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{B(x)} = x^3 + 2x^2 + 3x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
  

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ 

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
  

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_0(x) = 2x + 1$ 

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
  

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $E_1(x) = x + 2$ 

$$A(x) = 4x + 3x + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

# Polynomial Mult: Divide & Conquer

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ 

 $E_1(x) = x + 2$ 

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
  

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$A(x) = 4x^3 + 3x^2 + 2x + 1$	
$B(x) = x^3 + 2x^2 + 3x + 4$	

 $D_0(x) = 2x + 1$  $E_0(x) = 3x + 4$ 

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

 $D_1 E_1 = 4x^2 + 11x + 6$ 

 $E_1(x) = x + 2$ 

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_0(x) = 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $E_0(x) = 3x + 4$ 

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$a_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3 E_1(x) = x + 2$$

$$=x+2$$

$$= x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 



$$E_0(x) = 3x + 4$$
  
 $D_1E_0 = 12x^2 + 25x + 12$ 

$$3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_0E_1 = 2x^2 + 5x + 2$ 

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 

$$E_0(x) = 3x + 4$$
$$D_1 E_0 = 12x^2 + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$_{1}E_{0}=1$$

$$12x^2 + 2$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

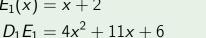
$$B(x) = x^{3} + 2x^{2} + 3x +$$

$$D_{1}(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $D_0E_1 = 2x^2 + 5x + 2$ 

$$D_1(x) = 4x + 3$$
  
$$E_1(x) = x + 2$$



$$E_0(x) = \frac{3x + 4}{2x + 6}$$
  
 $D_1E_0 = 12x^2 + 2x^2$ 

$$D_1$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$D_2 E_0 = 6x^2 + 11x + 4$$

$$= 6.$$

$$= 6x^2$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$+ 11x$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D(x) = x^{2} + 2x^{2} + 3x + 6$$

$$D_{1}(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_0E_1 = 2x^2 + 5x + 2$ 

AB =

$$x + 3$$
  
 $x + 2$ 

$$E_1(x) = x + 2$$
  
 $D_1 E_1 = 4x^2 + 11x + 6$ 

$$E_0(x) = 3x + 4$$

$$D_1 E_0 = 12x^2 + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$D_2 E_1 = 6x^2 + 11x + 4$$

$$E_0=6$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

 $D_0E_1 = 2x^2 + 5x + 2$ 

 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $E_1(x) = x + 2$ 

$$E_1(x) = x + 2$$
  
 $D_1 E_1 = 4x^2 + 11x + 6$ 

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$P_1 E_0 =$$

 $D_0(x) = 2x + 1$  $E_0(x) = 3x + 4$ 

 $D_0 E_0 = 6x^2 + 11x + 4$ 

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_0E_1 = 2x^2 + 5x + 2$ 

 $AB = (4x^2 + 11x + 6)x^4 +$ 

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

$$= 6$$

 $)x^{2} +$ 

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D(x) = x + 2x + 3x + 3$$
  
 $D_1(x) = 4x + 3$   
 $E_1(x) = x + 2$ 

 $(12x^2 + 25x + 12)$ 

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$
  
 $AB = (4x^2 + 11x + 6)x^4 + 6x^4$ 

$$D_0 E_0 =$$

 $D_0(x) = 2x + 1$  $E_0(x) = 3x + 4$ 

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $)x^{2} +$ 

$$25x + 1$$

$$D_1 E_0 = 12x^2 + 25x + 12$$



$$B(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$
  
$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1 E_1 = 4x^2 + 11x + 6$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$
 $AB = (4x^2 + 11x + 6)x^4 +$ 

$$(4x^{2} + 11x + 6)x^{4} + (12x^{2} + 25x + 12 + 2x^{2} + 5x + 2)x^{2} +$$

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

$$= 6x^2 +$$

$$D_1 E_0 = 12x^2 + 23x + 1$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = 1x + 3$$

$$E_1(x) = x + 2$$

 $6x^2 + 11x + 4$ 

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 

$$D_1 E_1 = 4x^2 + 11x + 6$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$
 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$ 

$$D_0E_0=$$

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$+11x$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 2x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
 $D_1(x) = 4x + 3$   $D_0(x) = 2x + 1$   
 $E_1(x) = x + 2$   $E_0(x) = 3x + 4$ 

$$D_1(x) = 4x + 3$$
  
 $E_1(x) = x + 2$ 

 $6x^2 + 11x + 4$ 

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 

$$D_1 E_1 = 4x^2 + 11x + 0$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$
  
 $AB = (4x^2 + 11x + 6)x^4 + 2x^2$ 

$$+6)x^4+$$

$$(4x^2 + 11x + 6)x^4 +$$
  
 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$ 

 $=4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$ 

$$5x + 2$$

$$D_1 E_0 = 12x^2 + 25x +$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$E_0(x) = 3x + 4$$
  
 $D_1 E_0 = 12x^2 + 25x + 12$ 

# Function Mult2( $A, B, n, a_l, b_l$ )

if 
$$n = 1$$
:
$$R[0] = A[a_l] * B[b_l] ; return R$$

### Function Mult2 $(A, B, n, a_l, b_l)$

$$R = array[0..2n - 1]$$
  
if  $n = 1$ :  
 $R[0] = A[a_I] * B[b_I]$ ; return  $R$   
 $R[0..n - 2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$ 

if 
$$n = 1$$
:  
 $R[0] = A[a_I] * B[b_I]$ 

 $\frac{n}{2}$ 

$$R[0] = A[a_I] * B[b_I]$$
; return  $R$   
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$   
 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_I + \frac{n}{2}, b_I + \frac{n}{2})$ 

if 
$$n = 1$$
:
$$R[0] = A[a_l] * B[b_l] ; return R$$

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$$
  
 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_I + \frac{n}{2}, b_I + \frac{n}{2}, a_I + \frac{n}{2},$ 

$$R[n..2n-2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2})$$

$$A[n..2n-2] - \text{Multi2}(A, B, \frac{\pi}{2}, a_l + \frac{\pi}{2})$$

 $D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$ 

if 
$$n = 1$$
:  
 $R[0] = A[a_l] * B[b_l]$ ; return  $R$   
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$ 

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$$
  
 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$ 

$$D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$$
  
 $D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$   
 $R[\frac{n}{2} \dots n + \frac{n}{2} - 2] + D_1 E_0 + D_0 E_1$ 

$$R[0] = A[a_I] * B[b_I]$$
; return  $R$   
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$   
 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_I + \frac{n}{2})$ 

if n = 1:

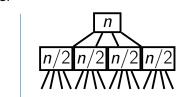
return R

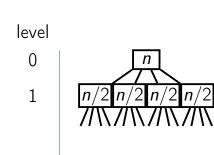
$$R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$$
  
 $D_0E_1 = Mult2(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$   
 $D_1E_0 = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$ 

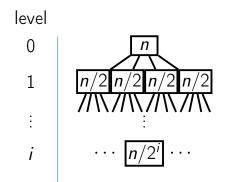
 $D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$  $R[\frac{n}{2} \dots n + \frac{n}{2} - 2] + D_1 E_0 + D_0 E_1$ 

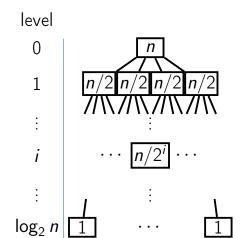


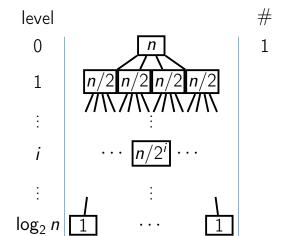
level

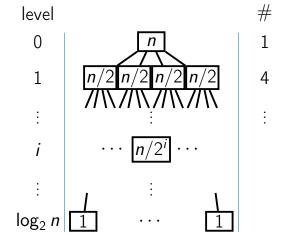


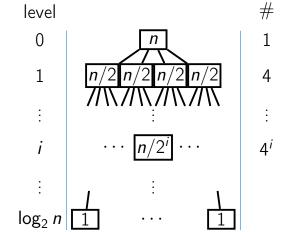


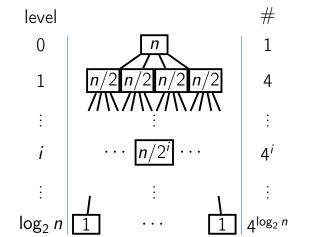


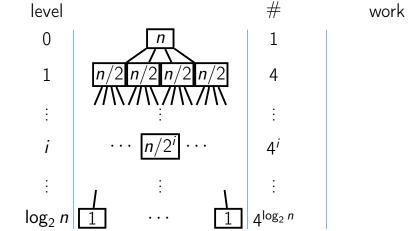


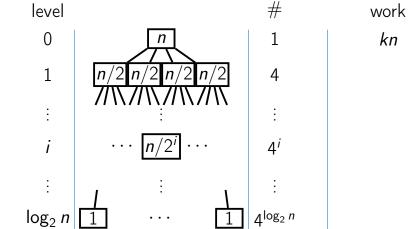


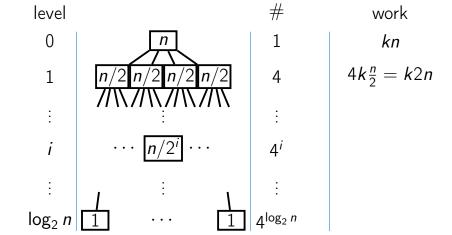


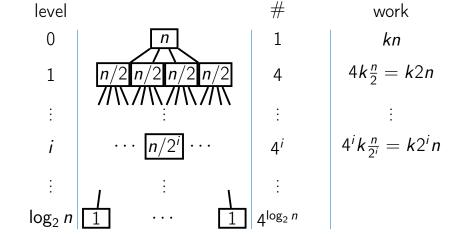


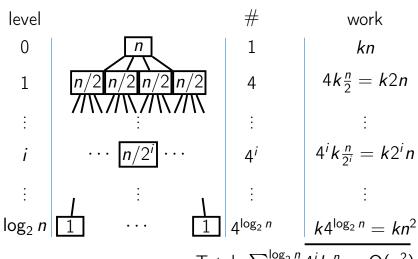








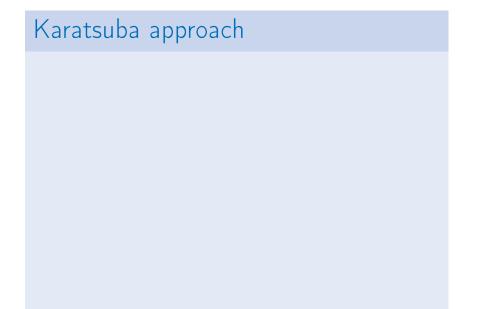




Total:  $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$ 

#### Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer



$$A(x) = a_1x + a_0$$

$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$ 

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$x + 1$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$ 

$$A(x) = a_1x + a_0$$

Rewrite as:

$$B(x) = b_1 x + b_0$$

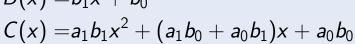
Needs 4 multiplications



















$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$
  
Needs 4 multiplications

Rewrite as: 
$$C(x) = a_1b_1x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x + a_0b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$
  
 $C(x) = a_1 b_1 x^2 + b$ 

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$
  
Needs 4 multiplications

Rewrite as:

Rewrite as: 
$$C(x) = a_1b_1x^2 +$$

 $a_0b_0$ 

Needs 3 multiplications

$$x^{2}$$

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$ 

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 +$ 

 $a_0b_0$ 

Needs 3 multiplications

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$ 

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

 $C(x) = a_1b_1x^2 +$ 

 $a_0b_0$ 

Needs 3 multiplications

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$ 

# Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 3$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$

Karatsuba Example
$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

Karatsuba Example
$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

 $D_0(x) = 2x + 1$ 

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = \frac{x}{2} + \frac{2}{3}$$

$$D_1(x) = 4x + 3$$
  $D_0(x) = 2x + 1$   
 $E_1(x) = x + 2$ 

Karatsuba Example
$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

 $E_0(x) = 3x + 4$ 

 $E_1(x) = x + 2$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$
  
 $D_{1}(x) = 4x + 3$ 

$$D_1(x) = 4x + 3$$
  
$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 

 $D_0(x) = 2x + 1$  $E_0(x) = 3x + 4$ 

Karatsuba Example
$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

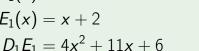
$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

$$D_1(x) = 4x + 3$$
  
 $E_1(x) = x + 2$ 

$$(x) = 4x + 3$$
  
 $(x) = x + 2$   
 $(x) = 4x^2 + 11x + 6$ 



 $E_0(x) = 3x + 4$ 

 $D_0 E_0 = 6x^2 + 11x + 4$ 

## Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_x(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$
  
$$E_1(x) = x + 2$$

 $(D_1 + D_0)(E_1 + E_0) =$ 

$$(x^2 + 2)$$
  $(x^2 + 2)$   $(x^2 + 2)$ 

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0E_0=$$

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$+11X$$

Karatsuba Example
$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_1(x) = 4x + 3$$
  
 $E_2(x) = x + 2$ 

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1E_1 = 4x^2 + 11x + 6$ 

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)$ 

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

 $D_0 E_0 = 6x^2 + 11x + 4$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$D_1(x) = x + 2x + 3x + 4$$
  
 $D_1(x) = 4x + 3$ 

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$(x) = x + 2$$

$$(x) = 4x^2 + 11x + 6$$

$$E_1(x) = x + 2$$
  
 $D_1 E_1 = 4x^2 + 11x + 6$ 

$$0 = x + 2$$

$$1 = 4x^2 + 11x + 6$$

$$\frac{1 - x + 2}{1 = 4x^2 + 11x + 6}$$

$$11x + 6$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6$$
  $D_0E_0 = 6x^2$   
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

 $D_0(x) = 2x + 1$  $E_0(x) = 3x + 4$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$0) = 4x + 3$$

$$0) = x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1 E_1 = 4x^2 + 11x + 6$ 

 $= 24x^2 + 52x + 24$ 

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 3$$

$$D_1E_1 = 4x^2 + 11x + 6$$
  $D_0E_0 = 6x^2 + 11x + 4$   
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

 $D_0(x) = 2x + 1$ 

$$x^2 + 11x +$$

# Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
  
$$D_1(x) = 4x + 3$$

 $AB = (4x^2 + 11x + 6)x^4 +$ 

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$
  
 $D_1 E_1 = 4x^2 + 11x + 6$ 

$$1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
  $D_0E_0 = 6$   
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

$$E_0(x) = 3x + 4$$
$$D_0 E_0 = 6x^2 + 1$$

 $= 24x^2 + 52x + 24$ 

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$(4x + 6)$$

 $D_0(x) = 2x + 1$ 

# Karatsuba Example $A(x) = 4x^{3} + 3x^{2} + 2x + 1$ $B(x) = x^{3} + 2x^{2} + 3x + 4$ $D_{1}(x) = 4x + 3$ $D_{0}(x) = 2x + 1$ $E_{1}(x) = x + 2$ $D_{1}E_{1} = 4x^{2} + 11x + 6$ $D_{0}E_{0} = 6x^{2} + 11x + 4$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $= 24x^2 + 52x + 24$ 

 $)x^{2} +$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$D(x) = x + 2x + 3x + 4$$
  
 $D_1(x) = 4x + 3$ 

 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $(24x^2 + 52x + 24)$ 

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$
  

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
  $D_0E_0 = 6$   
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

$$(6x + 6) = (6x + 6)$$

$$E_0 = (6x + 4)$$

$$D_0 E_0 = 6x^2 + 11x + 4$$
4)(4x + 6)

$$52x + 24$$

 $D_0(x) = 2x + 1$ 

 $E_0(x) = 3x + 4$ 

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $)x^{2} +$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$ $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$ $E_1(x) = x + 2$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

 $D_1 E_1 = 4x^2 + 11x + 6$ 

 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $E_0(x) = 3x + 4$ 

 $= 24x^2 + 52x + 24$ 

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$ 

 $D_0 E_0 = 6x^2 + 11x + 4$ 

 $)x^{2} +$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$ $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$ $E_1(x) = x + 2$ $D_1 E_1 = 4x^2 + 11x + 6$

 $AB = (4x^2 + 11x + 6)x^4 +$ 

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ 

 $E_0(x) = 3x + 4$ 

 $-(6x^2+11x+4))x^2+$ 

 $= 24x^2 + 52x + 24$ 

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$ 

 $D_0 E_0 = 6x^2 + 11x + 4$ 

### Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$ $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$ $E_1(x) = x + 2$ $E_0(x) = 3x + 4$ $D_1 E_1 = 4x^2 + 11x + 6$ $D_0 E_0 = 6x^2 + 11x + 4$ $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$ $= 24x^2 + 52x + 24$ $AB = (4x^2 + 11x + 6)x^4 +$ $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

 $6x^2 + 11x + 4$ 

 $-(6x^2+11x+4))x^2+$ 

```
Karatsuba Example
A(x) = 4x^3 + 3x^2 + 2x + 1
B(x) = x^3 + 2x^2 + 3x + 4
D_1(x) = 4x + 3
E_1(x) = x + 2
D_1 E_1 = 4x^2 + 11x + 6
```

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

$$E_{1}(x) = x + 2$$

$$E_{0}(x) = 3x + 4$$

$$D_{1}E_{1} = 4x^{2} + 11x + 6$$

$$D_{0}E_{0} = 6x^{2} + 11x + 4$$

$$(D_{1} + D_{0})(E_{1} + E_{0}) = (6x + 4)(4x + 6)$$

$$= 24x^{2} + 52x + 24$$

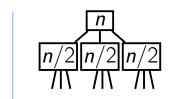
$$AB = (4x^{2} + 11x + 6)x^{4} + (24x^{2} + 52x + 24 - (4x^{2} + 11x + 6))$$

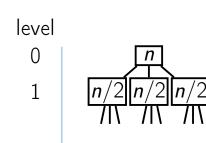
$$- (6x^{2} + 11x + 4)x^{2} + (6x^{2} + 11x + 4)$$

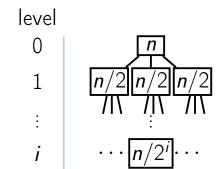
$$= 4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$$

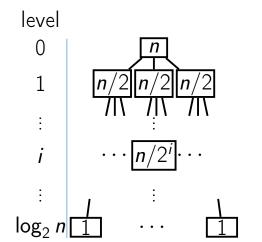


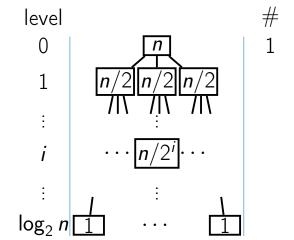
level

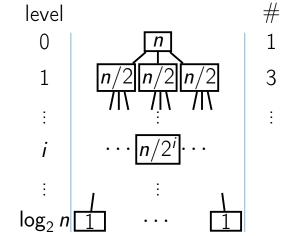


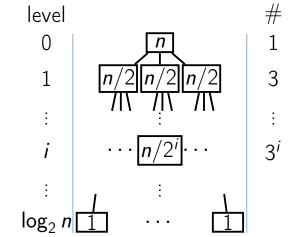


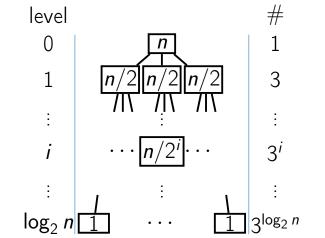


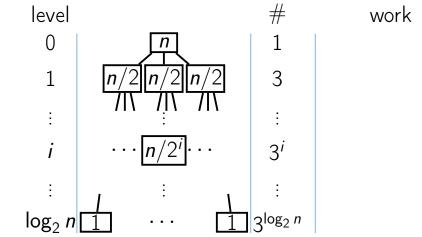


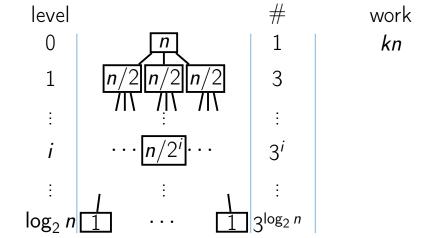


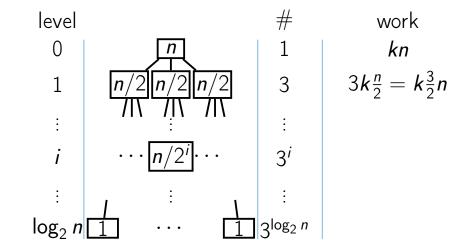












level

 $\log_2 n$ 

#

work

 $3^{\log_2 n} k 3^{\log_2 n} = k n^{\log_2 3}$ 

level # work

0 | 
$$n/2$$
 |  $n/2$  |  $n/$ 

 $=\Theta(n^{1.58})$