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Video 2.1
Sampath Kannan

Optimization

- **Objective Function:** A function that assigns a value to each feasible solution

Optimization

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- **Optimization Problem:** Find the solution with the maximum(or minimum) objective function value

Optimization: Examples

Optimization problems appear everywhere!

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Optimization problems appear everywhere!

- **Shortest** path from location A to location B
- **Maximum** value of goods you can buy on a budget
- **Smallest** number of changes you need to make to transform one string into another
- Locating k hospitals in acommunity to **minimize** the maximum time anyone has to travel
- Compute the value of a function in the **fewest** steps

Optimization: Solutions

- Brute Force approach: look at objective function value of each possible solution and take the best
- There can be exponentially many solutions. Brute-force approach can take too long
- **Dynamic Programming** - efficient way to find the optimal solution for some problems
  - When can we use dynamic programming?
  - How can we apply it?

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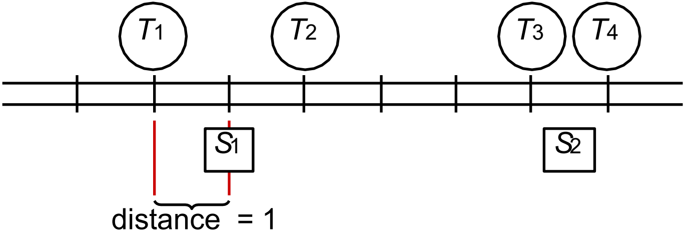
Video 2.2

Sampath Kannan

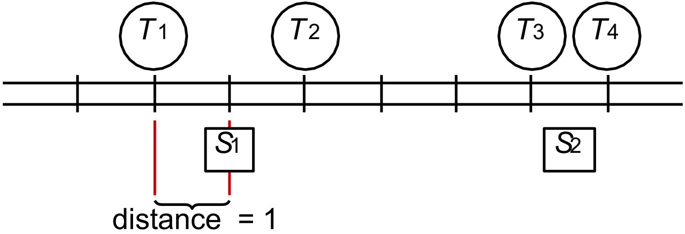
Station Placement

We want to place *k* stations along a train line so that the maximum distance between a town and its nearest station is minimized.



This diagram shows a cost 1 solution for *k* = 2 where the towns are located at positions 1, 3, 6 and 7.

Notation



- Notation: The towns area sequence of integers $t_1, t_2, \cdots, t_n$ and the stations area sequence of rational numbers $s_1, s_2, \cdots, s_k$

Top Level Decisions

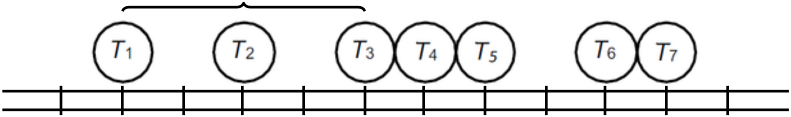
- Once we have the stations, we know which towns will use which.

Top Level Decisions

- Once we have the stations, we know which towns will use which.
- We can make a **top-level decision** about how many towns will use the left most station. Don’t know the answer so we have to try all possibilities!
- This is the idea of dynamic programming, we explore all choices and take the best one

Top Level Decisions

If the first *i* towns use the left most station.. .



Top Level Decisions

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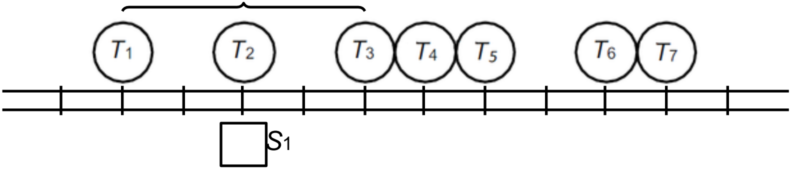

station should be at

$(t_i + t_1) / 2$

Top Level Decisions

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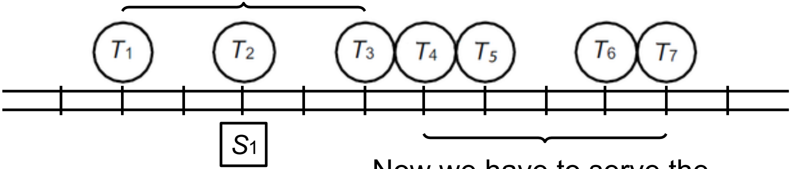

station should be at ($t_i + t_1$) / 2

Now we have to serve the remaining $n - i$ towns with $k - 1$ stations

Recurrence

- Locate([*i* , *j*], *l*) finds the best location for *A* stations to serve towns t_i through t_j and returns the maximum distance from any town to its nearest station.

Recurrence

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- Recursive idea:

$$\text{Locate}([i, j], l) = \min_{x \in [i, j-1]} \max {}^{\cdot} \frac{t_x - t_i}{2}, \text{Locate}([x + 1, j], l - 1)$$

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Video 2.3
Sampath Kannan

Computing Locate

- Locate (*t*[*i* , *j*], *k*) finds the best location for *k* stations to serve towns t_i through t_j and returns the maximum distance from any town to its nearest station.
- Recursive idea: For *k* > 1

Locate(*t*[*i*, *j*], *k*) =

$$\min_{x \in \{i \ldots j-1\}} \max\left(\frac{t_x - t_i}{2}, \text{Locate}(t[x+1, j], k-1) \right)$$

- What does this mean? Why is it correct?

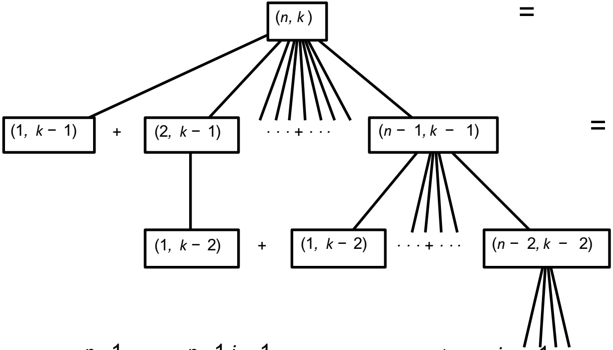
Computing Locate

- We want to compute Locate([1, n], k), the placement of k stations for n towns
- $T(n, k)$ = the time it takes to solve this problem

$$T(n, k) = \sum_{j=1}^{n-1} T(j, k-1) + n$$

$$T(1, k) = 1$$

$$T(n, 1) = 1$$



$$= n$$

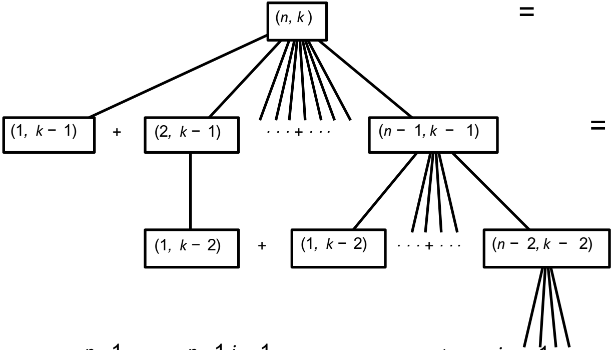
$$= \sum_{i_1=1}^{n-1} i_1$$

$$= \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{i_1-1} i_2$$

$$= n + \sum_{i_1=1}^{n-1} i_1 + \sum_{i_1=1}^{n-1} \sum_{i_2=1}^{i_1-1} i_2 + \cdots + \sum_{i_1=1}^{n-1} \cdots \sum_{i_k-1=1}^{i_{k-2}-1} i_{k-1}$$

$$= n + \sum_{i=1}^{k-1} \sum_{m=1}^{n-i} m\binom{n-1-m}{i-1}$$

- This grows exponentially fast!



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- This grows exponentially fast!
- Subproblem (1, *k* − 2) gets caled by (2, *k* − 1), (3, *k* − 1) *. ..*
- If we don’t recompute we can save time

Types of Subproblems

- What are the subproblems we see?

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- What are the subproblems we see?
- The new list of towns is always a suffice of the original list ($n - 1$ suffixes)
- The number of stations is always between 1 and k
- The total number of subproblems is at most k ($n - 1$)
- Dynamic programming works when there aren't too many subproblems

Dynamic programming Locate

- Instead of thinking about a recursion tree, think about an array of subproblems *C*
- *C* [*i* , *j*] = the minimum cost of placing *j* station to serve towns *i* through *n*

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```
Initialize C[i,j] = null
for all i let C[n,i] = 1
Locate(t[start,end],k):
  if t[start,k] = null
    c = Inf
    for x in start...end-1 do
      c = min(c, max((t[x]-t[end])/2,
                           Locate(t[x-1,end],k-1)))
    C[start,k] = c
    return c
  else
    return C[start,k]
```

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- Each subproblem takes at most n operations to solve (remember there are at most $(n - 1)\ k$ subproblems)
- So the new running time is $O(n^2k)$
- We can also avoid the recursion entirely:

```
Locate(t[start,end],k):
  for all i let C[i,1] = (t[end]-t[i])/2
  for all i let C[end,i] = 0
  for j from 2 to k
    for i from end-1 to 1
      c = Inf
      for x from i to end-1
        c = min(c, max((t[x]-t[i])/2,C[x+1.j-1))
      C[i,j] = c
    return C[1,k]
```

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Video 2.4
Sampath Kannan

Dynamic Programming

What are the general properties of problems where dynamic programming is applicable?

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- **Optimal Substructure:** In order to solve the whole problem optimally, we need to solve certain subproblems optimally
- **Not-too-many subproblems:** The same few subproblems keep recurring, so we do not need to solve too many subproblems.

Dynamic Programming - Optimal Substructure

”Optimal Substructure”- when is it present?

- Another Example: Shortest path between two cities A and B.

Dynamic Programming - Optimal Substructure

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  - Shortest path from A to B:
    - Take the shortest path from A to C
    - Then take the shortest path from C to B.
  - This problem has optimal substructure!
    - Why? Optimal solutions is composed of the optimal solution to subproblems
    - Problem: It might be difficult to find this city C

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Video 2.5

Sampath Kannan

Dynamic Programming: LCS

Longest Common Subsequence(LCS)

- Given: two strings $s = s_1 s_2 \ldots s_m$ and $t = t_1\ t_2 \ldots t_n$
- Strings are over some alphabet (may be english, may be something else)

Dynamic Programming: LCS

Longest Common
Subsequence(LCS)

T	U	C	S	O	N

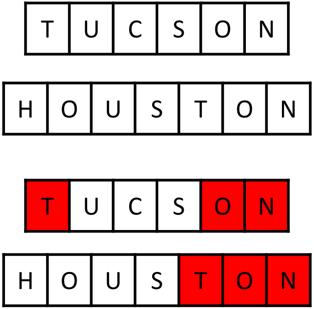
H	O	U	S	T	O	N

- Example: *s* = *TUCSON*, *t* = *HOUSTON*

Dynamic Programming: LCS

Longest Common Subsequence(LCS)

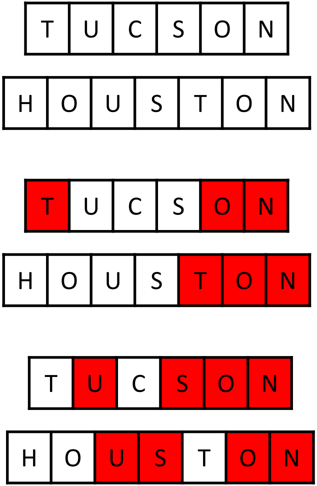
- Example: *s* = *TUCSON*, *t* = *HOUSTON*
- One common subsequence: TON (length 3)



Dynamic Programming: LCS

Longest Common Subsequence(LCS)

- Example: *s* = *TUCSON*, *t* = *HOUSTON*
- One common subsequence: TON (length 3)
- Longer one: USON (length 4, best)



T U C S O N

H O U S T O N

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LCS: Top-Level Question

Can LCS be solved using dynamic programming?

- Given two subsequences, what might be a question for a top level decision?
- Note: the question must be such that its answers cover all possible solutions
- Idea 1: Does the longest common subsequence include the last letter of both strings?

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Other ideas?

LCS: Top-Level Question

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Can we come up with a better top-level decision?

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- First, a lemma:
- *Lemma:* If the last symbols of two strings match, there is a longest common subsequence that uses these symbols

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- First, a lemma:
- *Lemma:* If the last symbols of two strings match, there is a longest common subsequence that uses these symbols
  - Simple proof by contradiction!
  - Suppose both strings have *x* as their last symbol, but we find a LCS that does not end in *x* .
  - We could add *x* to the end of our current LCS, creating a new, better LCS.
  - If some other occurrence of *x* in either string is used in LCS, it doesn’t hurt to replace it with the last occurrence of *x* .

LCS: Top-Level Question

T	U	C	S	O	N

H	O	U	S	T	O	N

Can we come up with a better top-level decision?

- Good top-level question: If the last symbols don’t match, which one do we want to drop from consideration?
  - Note that one of them must be dropped!
  - We are not ignoring any possible solutions
  - How is this question better than our first approach?

LCS: Dynamic Programming

- Let LCS(i,j) representt he length of the longest common subsequence using the first i symbols of s and the first j symbols of t.
- Note that we could store the actual sequences themselves by carefully keeping track of indices

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if s_i = t_j:
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- We can write a simple recursive algorithm to implement this
- How slow would it be? Would it be resolving any of the same problems?

LCS: Dynamic Programming

LCS: Dynamic
Programming

LCS: Dynamic Programming

LCS: Dynamic Programming

- Instead,we’ll use a bottom-up solution:
  - Solve the smalest subproblems first, build up to bigger solutions
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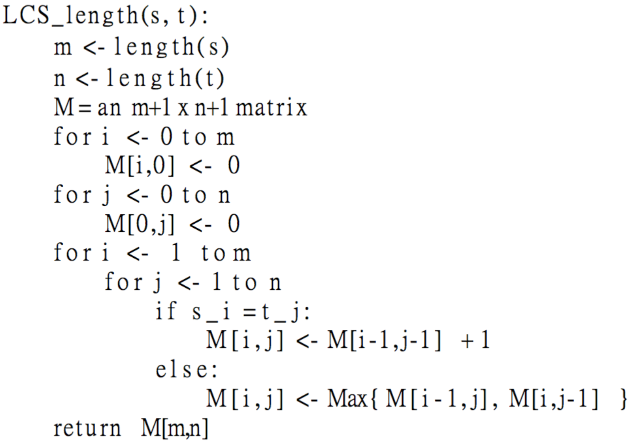
  - *LCS*(*m* , *n*)

LCS: Dynamic Programming

LCS: Dynamic Programming

- Over all algorithm:
  - Create an (m + 1)x (n + 1) grid, where slot (i , j) represents the solution to $LCS(i,j)$.
  - Fill in the trivial subproblem solutions (LCS (i , 0) and LCS (0,j))
    Fill the grid row by row (or column by column) using the previous recurrence.

LCS: Algorithm And Runtime



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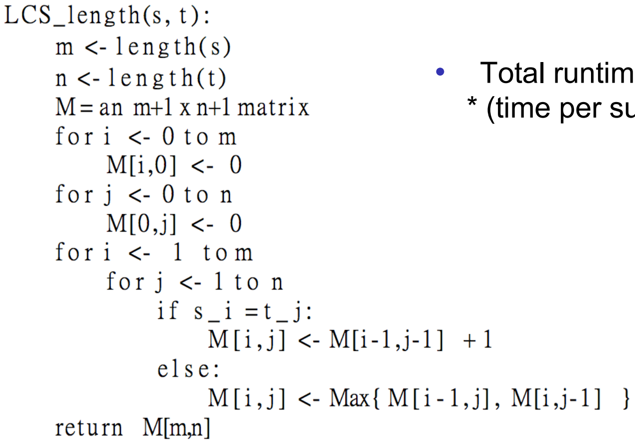

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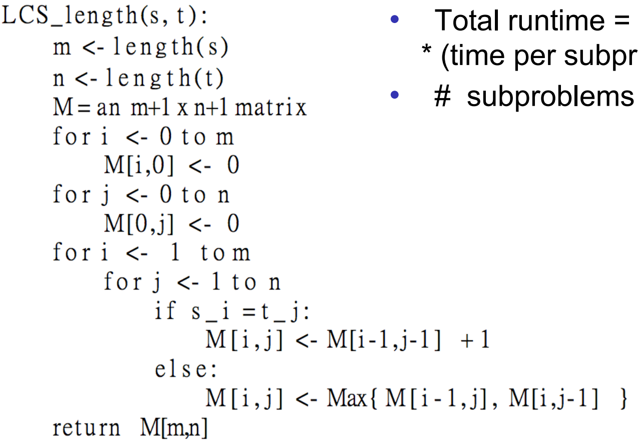

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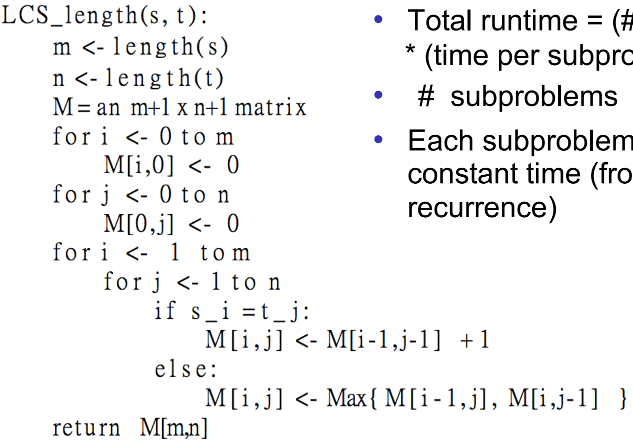

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- Total runtime = (# subproblems) * (time per subproblem)
- # subproblems = *mn*
- Each subproblem is calculated in constant time (from the recurrence)

LCS: Algorithm And Runtime

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    return M[m,n]
```

- Total runtime = (# subproblems) * (time per subproblem)
- # subproblems = *mn*
- Each subproblem is calculated in constant time (from the recurrence)

- **Total runtime of LCS:** *O*(*mn*)

LCS: Original Example

T	U	C	S	O	N

H	O	U	S	T	O	N

LCStable

		T	U	C	S	O	N
		0	0	0	0	0	0
H	0						
O	0						
U	0						
S	0						
T	0						
O	0						
N	0						

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LCStable

		T	U	C	S	O	N
		0	0	0	0	0	0
H	0	0					
O	0	0					
U	0	0					
S	0	0					
T	0	1					
O	0	1					
N	0	1					

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LCStable

		T	U	C	S	O	N
		0	0	0	0	0	0
H	0	0	0				
O	0	0	0				
U	0	0	1				
S	0	0	1				
T	0	1	1				
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		0	0	0	0	0	0
H	0	0	0	0	0		
O	0	0	0	0	0		
U	0	0	1	1	1		
S	0	0	1	1	2		
T	0	1	1	1	2		
O	0	1	1	1	2		
N	0	1	1	1	2		

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		0	0	0	0	0	0
H	0	0	0	0	0	0	
O	0	0	0	0	0	1	
U	0	0	1	1	1	1	
S	0	0	1	1	2	2	
T	0	1	1	1	2	2	
O	0	1	1	1	2	3	
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		0	0	0	0	0	0
H	0	0	0	0	0	0	0
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U	0	0	1	1	1	1	1
S	0	0	1	1	2	2	2
T	0	1	1	1	2	2	2
O	0	1	1	1	2	3	3
N	0	1	1	1	2	3	4

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Optimal Static Binary Search Trees

- Input: n keys (in sorted order), $k_1 < k_2 < \ldots < k_n$, along with probability of each key being accessed, $p_1, p_2, \ldots, p_n$

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P (access) access time

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access time

key	a	b	c	d	e	f	g	h	i	j
P(access)	0.15	0.05	0.08	0.1	0.2	0.05	0.1	0.07	0.15	0.05

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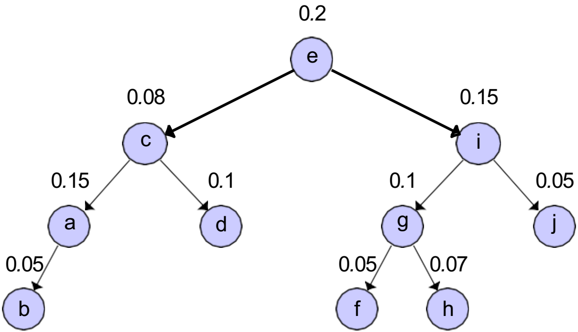

Optimal Static BSTs

- Question for top-level decision

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  - Optimal Sub structure Property:
    - The left and right subtrees should also be optimal BSTs for their elements

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k_1 k_n

Goal: create optimal BST for keys $k_1 \ldots k_n$

Optimal Static BSTs

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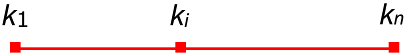

Choose k_i as root

Optimal Static BSTs

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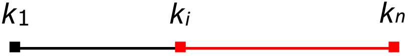

Compute right subtree (Using keys $k_{i+1} \ldots k_n$)

Optimal Static BSTs

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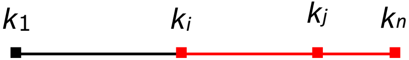

Choose k_j as root (of the right subtree)

Optimal Static BSTs

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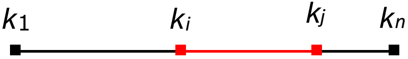

Compute its left subtree
(Using keys $k_{i+1} \ldots k_{j-1}$)

Optimal Static BSTs

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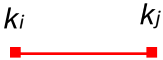

Optimal Static BSTs

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- What subproblems might we have to solve?
  - Potentially one for each contiguous set of keys $[k_i \ldots k_j]$
- How many are there?
  - Each contiguous set is defined by choosing two keys from our set (smallest and largest key in the interval)
  - $\binom{n}{2} = \frac{n(n-1)}{2} \quad O(n^2)$

Optimal Static BSTs: Recurrence

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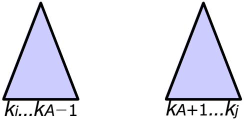

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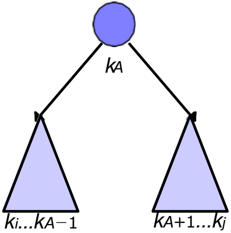

Search times for every element in both subtrees increases by 1

P(access element in left subtree)

$= \Sigma_{m=i}^{A-1} p_m$

Optimal Static BSTs: Recurrence

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$$Min_{i \leq \ell \leq j} \{ \overbrace{T(i, \ell - 1) + 1 * \sum_{m=i}^{\ell-1} p_m}^{\text{left subtree access time}} + \overbrace{T(\ell + 1, j) + 1 * \sum_{m=\ell+1}^{j} p_m}^{\text{right subtree access time}} + \overbrace{1 * p_\ell}^{\text{root access time}} \}$$

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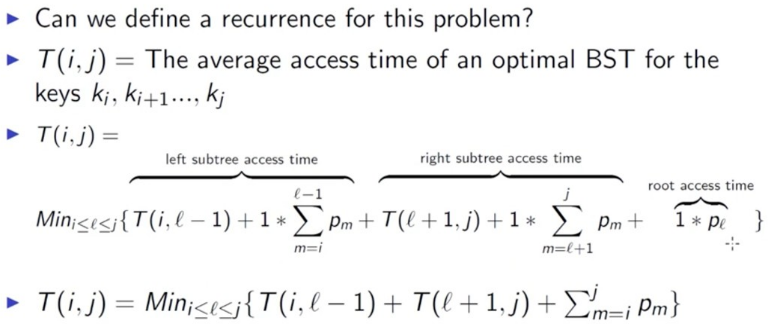

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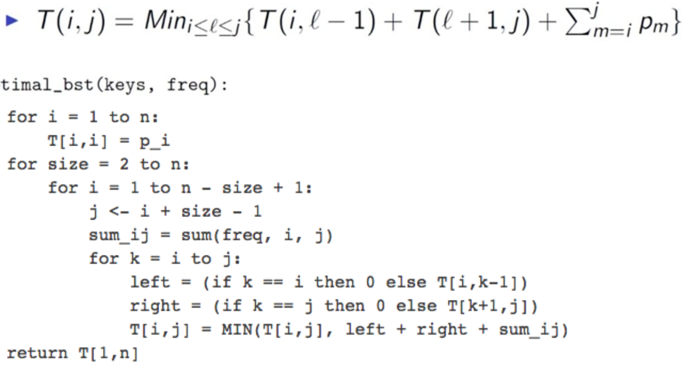

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```
optimal_bst(keys, freq):
  for i = 1 to n:
      T[i,i] = p_i
  for size = 2 to n:
      for i = 1 to n - size + 1:
          j <- i + size - 1
          sum_ij = sum(freq, i, j)
          for k = i to j:
              left = (if k == i then 0 else T[i,k-1])
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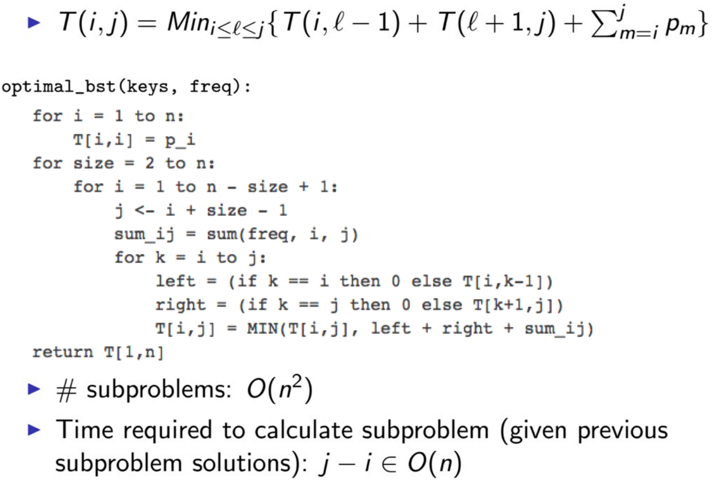
              T[i,j] = MIN(T[i,j], left + right + sum_ij)
  return T[1,n]
```

Optimal Static BSTs: Recurrence



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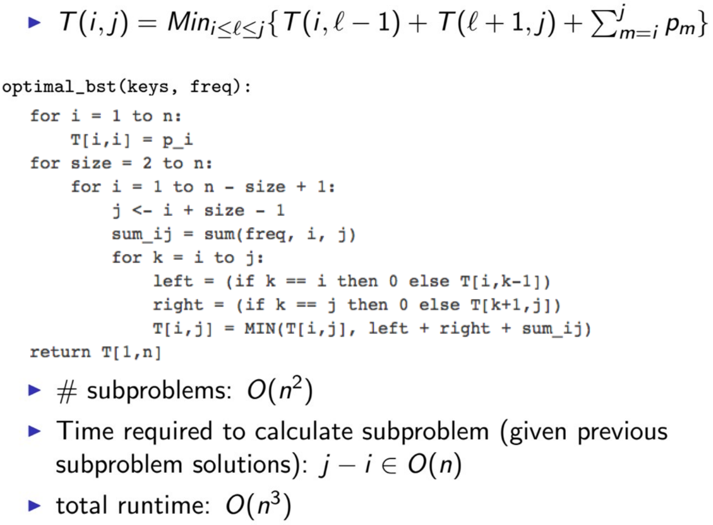

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- Time required to calculate subproblem (given previous subproblem solutions): $j - i \in O(n)$
- total runtime: $O(n^3)$

Dynamic Programming - Design

- How do we design dynamic programming algorithms?
- Understand if your problem has the two properties
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Dynamic Programming - Design

- How do we design dynamic programming algorithms?
- Understand if your problem has the two properties
- If so, ask what the top-level decision question is
    - There could be many possible questions
    - Picking the right one is an art
- Next, design a recursive algorithm that solves the whole problem by considering each answer to the top-level question, and recursively solving the resulting subproblems

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- Basic algorithm **only computes the cost of the solution.** Keeping track of the actual solution requires some extra book keeping

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Greedy Algorithms

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- For some problems, we can immediately decide what the best answer to the top-level decision question is.
- If so, we can make that choice and solve the resulting smaller problems.
- Algorithms that work like this are called **greedy algorithms:**

  We greedily choose the best option at the moment because it will also lead to the best solution overall.

- In general, greedy algorithms are easy to design, but hard to prove correct.

Greedy: Example

- **Input:** n jobs, $j_1, j_2, ... j_n$, and at ime T
- Job j_i takes t_i units of time to complete
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- **Input:** n jobs, $j_1, j_2, \ldots j_n$, and a time T
- Job j_i takes t_i units of time to complete
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- **Goal:** Order the jobs in away that maximizes total # of jobs completed

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  - Ø Why? Maximizes amount of time remaining after completion of 1 job

Greedy: Example

- Solution:
  - Sort the jobs in increasing order of t_i (time for each job)
  - Complete as many jobs as possible in this order, until T units of time have passed

Greedy: Example

- Solution:
  - Sort the jobs in increasing order of t_i (time for each job)
  - Complete as many jobs as possible in this order, until T units of time have passed
- Correctness of the greedy solution may seem in clear in this case.
- In more complicated problems, this is less obvious!

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Greedy Algorithms

Another (more complicated) scheduling problem:

- **Input:** nevents, $e_1, e_2, \ldots, e_n$
- Each event e_i starts at time s_i and finishes at time f_i
- Only one event can be running at any given time
- **Goal:** Find the maximum number of events we can schedule

Scheduling: Some Ideas

Top level decision: Which event to schedule first? What are some greedy choices that we could make?

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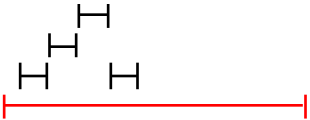

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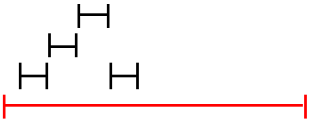

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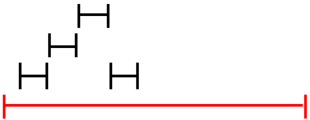

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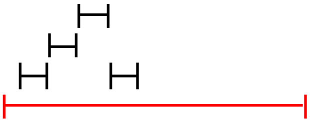

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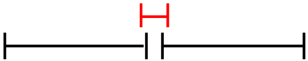

Scheduling: Correct Greedy Algorithm

- Idea: Sort events by finish time!
- Let us renumber the events in order of increasing finish time,
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- Pick e_1; eliminate events that overlap with e_1; Recurse.

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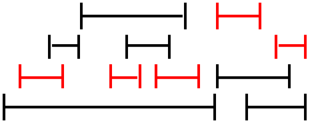

Scheduling: Correct Greedy Algorithm

- Idea: Sort events by finish time!
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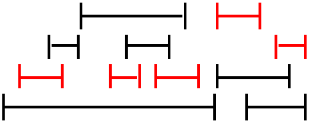

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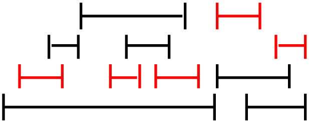

- Intuition: Our greedy first choice leaves the maximum time for remaining events.
- Optimum couldn't do better.

Greedy Algorithm: Correctness

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- After the greedy choice is made, solving the rest of the problem optimally will solve the entire problem optimally (optimal substructure)

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Optimal substructure:

- Consider an optimal solution$O = e_{o_1} e_{o_2} \ldots e_{o_k}$
- The solution $e_{o_2} \ldots e_{o_k}$ must be the optimal sequence of events for the time interval $[s_{o_2} \ldots f_{o_k}]$
- Otherwise, take the better sequence and replace$e_{o_2} \ldots e_{o_k}$ with it!

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Video 2.9

Sampath Kannan

File Compression

- Suppose we are sending English text over the internet, we need a way to encode the symbols as a sequence of ones and zeros.
- We'd like to minimize the total amount of information sent (the length of the sequence).
- Idea: use shorter encoding for more frequent symbols (e.g., 'e', 'r', 't').

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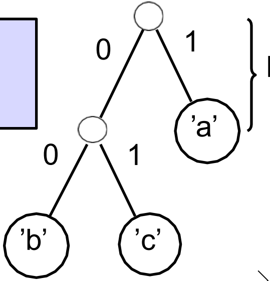
- Bad Example:
    - *a*›→010*, f* ›→0101*, g*›→1110*, s* ›→110*, . . .*
- Problem: 0101110 can be $\boxed{010}\boxed{1110}$ = 'ag', or $\boxed{0101}\boxed{110}$ = 'fs'
- Solution: No code can be a prefix of another

Prefix Codes

Let the alphabet be$\Sigma = \{a, b, c\}$ and for each symbol $x \in \Sigma$ we let $f(x)$ be the frequency of x and $d(x)$ be the length of the encoding of x.

$a \mapsto 1$
$b \mapsto 00$
$c \mapsto 01$



Note: depth(x) = $d(x)$

Goal: minimize $\sum_{x \in Sigma} d(x)f(x)$

Attempt 1 (Greedy):

Most frequent symbol›→0, recurse on remaining symbols

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$\Sigma = \{a, b, c, d\}$
$f(a) = 0.26, f(b) = 0.255$
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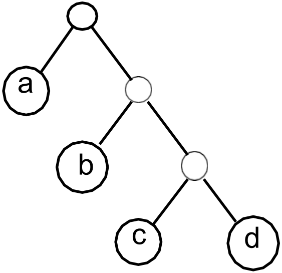

Attempt 1 (Greedy):
Most frequent symbol›→0,
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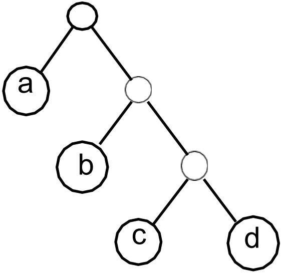

The cost of this tree is ≈ 2.2 but if we gave each symbol a code of length 2 the cost would be 2

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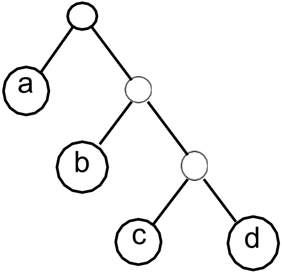

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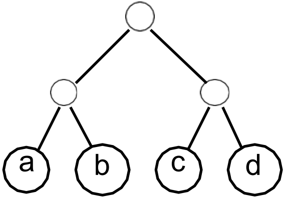

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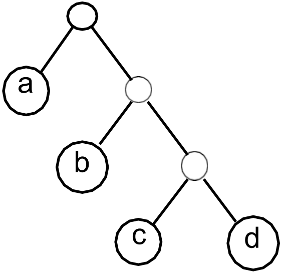

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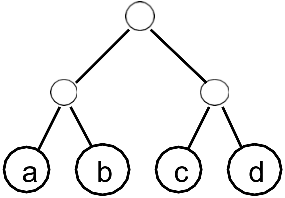
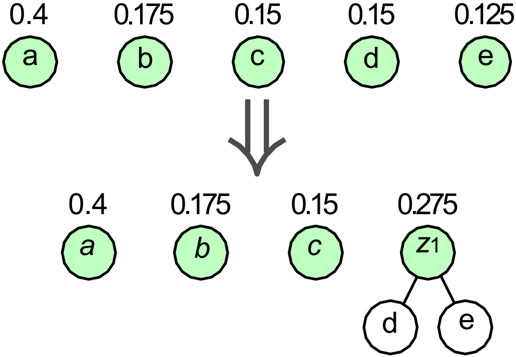


Consider: $f(a) = 1/2$, $f(b) = 1/4$, $f(c) = 1/8$, $f(d) = 1/8$.

Cost = 2 but attempt 1 is better (1.75)

Correct Algorithm

Idea: The two lowest frequency symbols *x* , *y* will be siblings so replace them with a new node *z* of frequency *f* (*x*) + *f* (*y*) and recursively solve the problem



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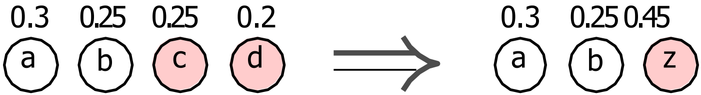
Huffman Coding

Replace two lowest frequency symbols x, y with a new symbol z of frequency $f(x) + f(y)$ and recurse on $n - 1$ symbols.



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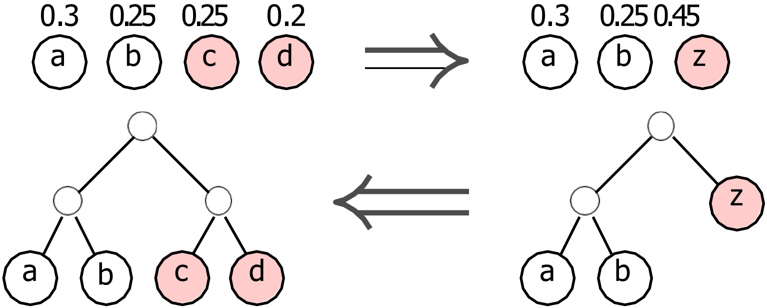
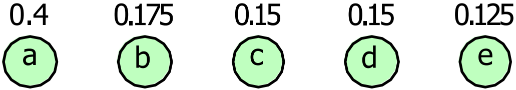


Take the resulting tree on $n - 1$ symbols and replace z with x and y

Example Huffman



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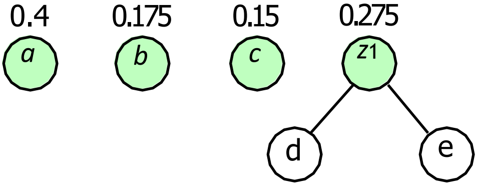

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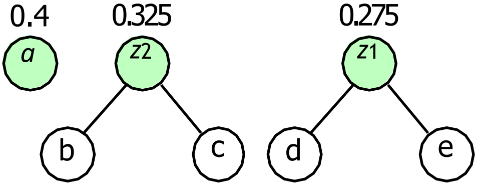

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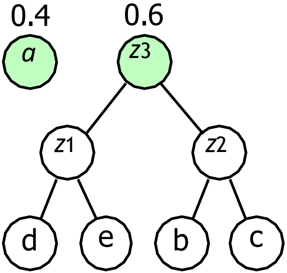

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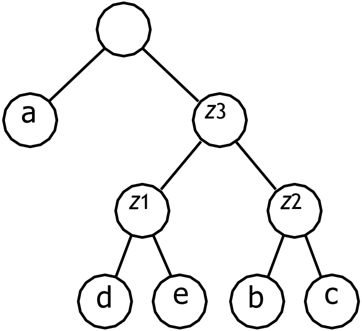

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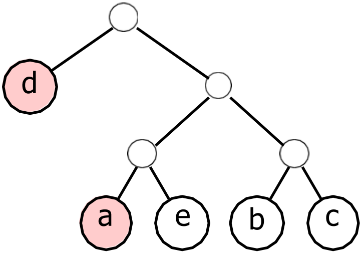
- If an optimal tree existed with higher frequency symbols at the deepest level we could swap the lowest frequency symbol with one of the deepest ones and get a tree with a lower cost. This is contradiction since we started by saying the tree was optimal.

Example Swap

Recall the previous example where $f(a) = 0.4$ and $f(d) = 0.15$. Here is an encoding tree where one of the lowest frequency symbol (d) is swapped with a higher frequency one (a).

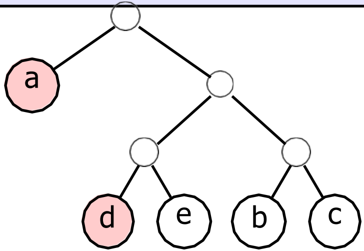


The string“. . . aadaadaadaa . . . ” gets encoded to:

100	100	0	100	100	0	100	100	0	100	100

Example Swap

See how the encoding length shrinks when we swap ’a’ and ’d’



“. .. aadaadaadaa .. .” is now encoded to:

0	0	100	0	0	100	0	0	100	0	0