ENS DE LYON

Afsin Ozdemir

Internship Report

From Brunn-Minkowski to Sobolev Type Inequalities via Mass Transportation

Supervisor

Ivan Gentil

Institute Camille Jordan

Department of Mathematics

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Chapter 1

Introduction

The mass transportation approach comes up in all known proofs of Brunn-Minkowski inequality,

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n} \tag{1.1}$$

where A and B are compact Borel sets in $\mathbb{R}^n$ and $|\cdot|$ is the n-dimensional Lebesgue measure. Also, it is well known that one can derive Isoperimetric inequalities from (1.1), which can be found in the famous book *Topics in Optimal Transportation* [6] by C.Villani. These links raise questions such as could it possible to derive any other type of well-known inequalities? The first example can be taken as the sharp Sobolev inequalities. The classical statement of that is, for $n \geq 2$, $p \in [1,n)$, $p^* = \frac{np}{n-p}$, and any smooth enough function f on $\mathbb{R}^n$, following inequality holds,

$$\|f\|_{L^{p^*}} \leq \frac{\|h_p\|_*}{(\int_{\mathbb{R}^n} |\nabla h_p|^{1/p})} \left(\int_{\mathbb{R}^n} |\nabla f|^p \right)^{1/p}, \tag{1.2}$$

here

$$h_p(x) := (1 + |x|^{\frac{p}{p-1}})^{\frac{p-n}{p}},$$

where $\|\cdot\|_{L^p}$ denotes the L^p-norms with respect to the Lebesgue measure, and unless it stated otherwise we will take integrals with respect to the Lebesgue measure. The classical statement of the Sobolev inequalities were proven by S.Sobolev for the cases $p = 2$ and $p \neq 2$, however the statement (1.2) were taken from the work [7] of D. Cordero, B. Nazaret and C. Villani for the optimal constant h_p. It should be noted that it is not the first exhibit of optimal constants, first examples were given in [1, 3], but their approach is not our concern in this work. We will focus on how it is possible to obtain the sharp Sobolev inequality and more generally, how we can obtain larger family of Gagliardo-Nirenberg inequalities via different mass transportation approaches. Gagliardo-Nirenberg inequalities can be stated in the following form, for $p > 1$, $\alpha = ap/(a-p)$ and $\beta = p(a-1)/(a-p)$

$$\|f\|_{L^\alpha} \leq C\|\nabla f\|_{L^p}^{\theta}\|f\|_{L^\beta}^{1-\theta},$$

where θ is fixed as scaling in variance. In the work [7], it has proven that mass transportation can be applied to general type Sobolev inequalities (and the family contains it, Gagliardo-Nirenberg types inequalities) with an arbitrary norm on $\mathbb{R}^n$. It was a simple and novel approach, which uses the Monge-Ampère equation. Before going through the explanations of it, mass transportation has to be mentioned. We say that a measurable map $T : X \to Y$ transports a measure μ on X onto ν on Y, if

$$\int_X H(T)\mathrm{d}\mu = \int_Y H\mathrm{d}\nu,$$

for all $H \in L^1(\mathrm{d}\nu)$ and denoted as $T\#\mu = \nu$. Along with this definition, we need to mention the work of Brenier, which was mainly developed to solve the Monge-Kantorovich transportation problem, but we will not mention it here. The paper [7] mainly developed the way one can use the following Theorem, to use the Monge-Ampère equation, which will be discussed shortly.

Theorem 1.1 (Brenier's Theorem)**.** *If μ and ν are two probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to Lebesgue measure, then there exist convex function φ such that $\nabla\varphi\#\mu = \nu$. Furthermore, $\nabla\varphi$ is uniquely determined by μ almost everywhere.*

Now we are ready to present the notion of Monge-Ampère equation. Let $d\mu(x) = f(x)dx$ and $d\nu(y) = g(y)dy$ be two probability measures, absolutely continuous with respect to the Lebesgue measure. Take an arbitrary H from $\mathcal{C}_c^\infty(\mathbb{R}^n)$. By definition of the transport, we have

$$\int H(\nabla\varphi(x))f(x)dx = \int H(y)g(y)dy.$$

If we do the change of variable $y = \nabla\varphi$, we obtain,

$$\int H(\nabla\varphi(x))f(x)dx = \int H(\nabla\varphi(x))g(\nabla\varphi(x))\det(\nabla^2\varphi(x))dx.$$

Therefore in the weak sense we obtained,

$$f(x) = g(\nabla\varphi(x)) \det(\nabla^2\varphi(x)), \tag{1.3}$$

where $\nabla^2\varphi$ is the Alexandrov Hessian of φ, which is the absolutely continuous part of the distributional Hessian of the convex function φ. If g is positive we can write it as

$$\det \nabla^2(\varphi(x)) = \frac{f(x)}{g(\nabla\varphi)}. \tag{1.4}$$

This is a particular case of the Monge-Ampère equation. Solutions of the Monge-Ampère type equations is an old topic, however, it is not the major concern of this work. For further details, one can check the dedicated chapter of the book [6]. We are going to focus on 1.3, which is helping us to prove functional inequalities.

Now if we turn into our main topic, we want to concentrate on the cases involves any norm rather than just using the Euclidean norm on $\mathbb{R}^n$. So, the inequality 1.2, turns into the following form:

$$\|f\|_{L^{p^*}} \leq \frac{\|h_p\|_{L^{p^*}}}{(\int \|\nabla h\|_*^p)^{1/p}} \left(\int_{\mathbb{R}^n} \|\nabla f\|_*^p\right)^{1/p}, \tag{1.5}$$

where $\|y\|_* := \sup_{\|x\|\leq 1} x.y$ and $h_p(x) := (1 + \|x\|^{\frac{p}{p-1}})^{\frac{p-n}{p}}$. In this direction first theorem appears in our hands as follows,

Theorem 1.2 (A convex Sobolev Inequality). *Let $n \geq 2$ and $W : \mathbb{R}^n \to (0, +\infty)$ such that $\liminf_{+\infty} \frac{W(x)}{|x|^\gamma} > 0$ for some $\gamma > \frac{n}{n-1}$. For any $g : \mathbb{R}^n \to (0, +\infty)$ with $g^{-n}|\nabla g|^{\gamma/\gamma-1} \in L^1$ and*

$$\int g^{-n} = \int W^{-n},$$

one has

$$\int W^*(\nabla g) g^{-n} \geq \frac{1}{n-1} \int W^{1-n}, \tag{1.6}$$

where W^* is the Legendre transform explained in the Chapter 2. □

In Chapter 3, we shall see how it is possible to obtain the sharp Sobolev and Gagliardo-Nirenberg inequalities from a generalized version of this theorem.

In Chapter 4 we proceed with another idea, would it be possible to achieve these results using the Brunn-Minkowski inequality? In the work [5], S.Bobkov and M.Ledoux showed that it is possible to obtain different functional inequalities, such as the logarithmic Sobolev, etc. from the functional form of Brunn-Minkowski inequality, which is known as Prékopa-Leindler inequality. We can also obtain them by using another form which is known as Borell-Brascamp-Lieb (BBL) inequality, which can be stated as follows; for $g, W, H : \mathbb{R}^n \to [0, +\infty]$ with

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx + ty) \leq sg(x) + tW(y)$$

and $\int W^{-n} = \int g^{-n} = 1$,then,

$$\int H^{-n} \geq 1. \tag{1.7}$$

Instead of using the method of S.Bobkov and M.Ledoux, here we are explaining how monotone mass transform can be used to reach the sharp functional inequalities. For this purpose, we are using a different version of BBL inequality,

Theorem 1.3 (An extended Borell-Brascamp-Lieb inequality)**.** *Let $n \geq 2$. Let $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0, 1]$, $t = 1 - s$ be such that*

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx + ty) \leq sg(x) + tW(y)$$

and $\int W^{-n} = \int g^{-n} = 1$. Then

$$\int H^{1-n} \geq s \int g^{1-n} + t \int W^{1-n}. \tag{1.8}$$

After giving the relations between convex inequalities and different types of BBL for a specific class of functions, we will explain how to recover the sharp Sobolev inequalities and Gagliardo-Nirenberg inequalities. In the second part of chapter 4, recovering the new trace Sobolev inequalities explained briefly which was inspired from the work [8] of B.Nazaret. In addition, we stated the new trace Gagliardo-Nirenberg inequalities, which was the one the main results of the paper [13].

In Chapter 5, we will present a new results for the Sobolev inequalities with a weighted measure, which will be represented as $d\mu = e^{-V(x)}dx$. We proved that we can obtain the Sobolev inequalities for this measure with a new functional approach from the work of F.Bolley, et al. [13].

Finally, in the Appendix A, inf-convolution approach for the cases $\mathbb{R}^n_+$, $\mathbb{R}^n$ and μ (on a convex cone) explained ,and given without going through all the details.

Chapter 2

Definitions and Preliminaries

2.1 Definitions, Preliminaries, and Notations

Before starting giving the main objective of this report, we are starting with preliminary tools that we are going to use in the following parts. First to avoid confusion, $\|f\|_p$ stand for $L^p(\mathbb{R}^n)$ norm of a function f. We are considering Euclidean norm $|\cdot|$ on $\mathbb{R}^n$ and $x.y$ denotes the Euclidean scalar product for $x, y \in \mathbb{R}^n$. Let us begin by recalling the familiar notion of Legendre transform,

Definition 2.1. *Let $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Its Legendre transform W^* defined by*

$$W^*(y) = \sup_{x\in\mathbb{R}^n} \{y.x - W(x)\},$$

for all $y \in \mathbb{R}^n$ and the function W is differentiable almost everywhere in its domain with

$$W^*(\nabla W(x)) + W(x) = x.\nabla W(x).$$

For further details, readers can check the first chapter of the book of H. Brezis [9].

Until this point, necessary stuff was given related to optimal transport. However, since we are dealing with convex (and concave) functions we need to have more tools, geometrically, to obtain inequalities. First, we are dealing with $n \times n$ positive symmetric matrices, which are going to be denoted as $S^n_{++}(\mathbb{R})$. One example of this type of matrix $\nabla^2\varphi$ leads up to the way of proving functional inequalities. Before presenting the first lemma, the following definitions are needed.

Definition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^+$ and θ be in $[0, 1]$.*

- *We call f concave if,*

$$f(\theta x + (1-\theta)y) \geq f(\theta x) + f((1-\theta)y), \quad \forall x, y \in \mathbb{R}^n.$$

- *We call f log-concave if,*

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{(1-\theta)}, \quad \forall x, y \in \mathbb{R}^n.$$

We shall not directly be using the log-concavity notion, however, it is important to have complete work. The lemma that we are going to use more directly about concavity is the following.

Lemma 2.1. *Let M be in $S^n_{++}(\mathbb{R})$.*

1. *The map $\phi : S^n_{++}(\mathbb{R}) \to \mathbb{R}$ that maps $M \longmapsto det(M)$ is log-concave.*
2. *For every $k \in (0, 1/n]$, the map $M \longmapsto \det^k M$ is concave over the set of positive symmetric matrices. Concavity inequality around the identity implies*

$$det^k M \leq 1 - nk + k\mathrm{tr}(M) \tag{2.1}$$

for all $M \in S^n_{++}(\mathbb{R})$ and where $\mathrm{tr}(M)$ is the trace of the matrix M.

and we need some properties of positive concave functions as well,

Lemma 2.2. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function. Then Φ is non-decreasing.*

Proof. Which is very easy to see by definition, let $0 \leq x < z$ and $\lambda \in (0, 1)$, then $\Phi(\lambda x + (1-\lambda)z) \leq \lambda\Phi(x) + \lambda\Phi(z)$. Take $1 - \lambda := \frac{y-x}{z-x}$, so that $\lambda x + (1-\lambda)z = y$. So we obtained

$$\Phi(z) \leq \frac{\Phi(y) - \Phi(x)}{y - x}(z - y) + \Phi(y).$$

Assume that $\Phi(y) < \Phi(x)$ for some $x < y$, so the term $\frac{\Phi(y)-\Phi(x)}{y-x}(z - y)$ is negative. If we take

$\limsup_{z\to\infty}$, we get

$$\limsup_{z\to\infty} \Phi(z) \leq \limsup_{z\to\infty} \left(\frac{\Phi(y) - \Phi(x)}{y - x}(z - y) + \Phi(y) \right) = -\infty.$$

This is a contradiction so that Φ is non-decreasing. $\square$

Chapter 3

Functional Inequalities

3.1 Warm up

Brunn-Minkowski is a well-known inequality that can be proven in different ways. As a warm-up, we want to prove its functional equivalent which is known as Prékopa-Leindler inequality by using optimal transport. In our context, it is a cornerstone approach to the relation between functional inequalities and mass transport.

Theorem 3.1 (Prékopa-Leindler Inequality)**.** *Let f, g and h be positive functionals from $\mathbb{R}^n$. Suppose following inequality holds for any $x, y \in \mathbb{R}^n$ and for arbitrary $\lambda \in (0,1)$:*

$$h((1-\lambda)x + \lambda y) \geq (f)^{1-\lambda}(g)^{\lambda}$$

Then we have

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \tag{3.1}$$

Sketch of the Proof. Without loss of generality assume that $\int f = \int g = 1$. Our aim is to show $\int h \geq 1$. By Brenier's theorem 1.1, there exist a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\varphi \# f(x)dx = g(y)dy$. After changing variables as $z = (1-\lambda)x + \lambda\nabla\varphi(x)$ and using 2.1, we obtain

$$\begin{aligned}\int h(z)dz &= \int h((1-\lambda)x + \lambda\nabla\varphi(x)) \det((1-\lambda)I_n + \lambda\nabla^2\varphi(x))dx \\ &\geq \int f(x)^{1-\lambda} g(\varphi(x))^{\lambda} \det((1-\lambda)I_n + \lambda\nabla^2\varphi(x))dx \\ &\geq \int f(x)^{1-\lambda} g(\varphi(x))^{\lambda} \det(\nabla^2\varphi(x))dx \\ &= \int f(x)^{1-\lambda} f^{\lambda}(x)dx = 1,\end{aligned}$$

where I_n is $n \times n$ identity matrix. □

Unfortunately, this proof cannot be counted as a complete one since we are doing change of variables, by assuming extra regularisation of Brenier's map, where it is also possible to prove it by McCann's idea from his PhD thesis [4], which was given in [6]. We presented this proof since the idea of it is going to have an important role in the following chapter.

3.2 Deriving convex Gagliardo-Nirenberg inequalities

Our first aim is to derive Sobolev and Gagliardo Nirenberg inequalities from more general Functional inequality which is the generalization of theorem 1.2.

Theorem 3.2 (Convex Inequalities). *Let $n \geq 1$. Let $a \geq n$ (and $a > 1$ in case $n = 1$) and let $W : \mathbb{R}^n \to (0, +\infty)$ be such that*

$$\int W^{-a} = 1$$

and

$$\exists \gamma > \max\{\frac{n}{a-1, 1}\}, \qquad \liminf_{x \to +\infty} \frac{W(x)}{|x|^{\gamma}} > 0.$$

For any $g \in W^{1,1}_{loc}$ positive such that $g^{-a}|\nabla g|^{\gamma/(\gamma-1)}$ is integrable and

$$\int g^{-a} = 1,$$

one has

$$(a-1)\int W^*(\nabla g)g^{-a} + (a-n)\int g^{1-a} \geq \int W^{1-a}. \tag{3.2}$$

Furthermore, equality holds in the case of $g = W$ and is convex.

Choice of γ and integrability of $g^{-a}|\nabla g|^{\gamma/(\gamma-1)}$ are based on technical reasons which rise because of the regularity problem of the Brenier's map.

Before proceeding with the proof, as a warm-up, let us look at how we can derive sharp Sobolev inequality out of it. First, let define $W : \mathbb{R}^n \to (0, +\infty)$ as following:

$$W(x) = \frac{||x||^q}{q} + C, \qquad x \in \mathbb{R}^n,$$

where positive constant C is chosen so that $\int W^{-a} = 1$. Then also we have Legendre transform of W,

$$W^*(y) = \frac{||y||_*^p}{p} - C, \qquad y \in \mathbb{R}^n,$$

where $1/p + 1/q = 1$.

We shall apply Theorem 3.2 with fixed function W. It is easy to notice that $C > 0$ is well defined and $\int W^{1-a}$ is finite for

$$\begin{cases} \text{If } a \geq n+1, \text{ then } p > 1, \\ \text{If } a \in [n, n+1), \text{ then } 1 < p < \frac{n}{n+1-a} = \tilde{p} \ \ (\ \tilde{p} = n \text{ when } a = n). \end{cases} \tag{3.3}$$

It is easy to see that, first one and for the second one, it should be seen, for $a \in [n, n+1)$, then to satisfy the condition $\int \|\nabla g\|_*^p g^{-a} \leq \int \|\nabla g\|_*^{\gamma/\gamma-1} g^{-a}$, p should be in $(1, \tilde{p}]$. Note that assumption on W in Theorem 3 is satisfied, with the observation of (3.3), if $\gamma = q$. Assume (a, p) is an admissible tuple (satisfies previous conditions) and thus for any function g with $\int g^{-a} = 1$ and $|\nabla g|^{1-\frac{a}{p}} \in L^p$, we get

$$D \leq \frac{a-1}{p} \int \frac{\|\nabla g\|_*^p}{g^a} + (a-n) \int g^{1-a} \tag{3.4}$$

where $D = (a-1)C + \int W^{1-a}$ and $a > 1$ is fixed. This is an essential inequality to derive Sobolev and Gagliardo-Nirenberg Inequalities.

Sobolev Inequalities: A computationally basic example is deriving Sobolev inequalities. Let assume that $a = n$ and $n \geq 2$ and $p \in (1, n)$. Then (3.4) turns into

$$\frac{Dp}{n-1} \leq \int \frac{\|\nabla g\|_*^p}{g^{-n}}$$

where $\int g^{-n} = 1$ and $\nabla g^{1-\frac{n}{p}} \in L^p$. Let's put, $g = f^{\frac{p}{p-n}}$, we have

$$\frac{Dp}{n-1} \leq \left|\frac{p}{n-p}\right|^p \int \|\nabla f\|_*^p f^{\frac{np}{p-n}} f^{\frac{-np}{p-n}}$$

By assumption we know, $\int f^{\frac{np}{p-n}} = 1$, so we can add it as factor without any trouble. Hence we get,

$$\frac{Dp}{n-1}\left|\frac{n-p}{p}\right|^p \left(\int f^{\frac{np}{p-n}}\right)^{\frac{n-p}{n}} \leq \int \|\nabla f\|_*^p$$

And it is clear that this inequality is optimal since holds for the equality hold for $g = W$ and convex or equivalently when $f(x) = \left(C + \frac{\|x\|^q}{q}\right)^{\frac{p-n}{p}}$. So we recover the classical result after removing the sign condition;

Theorem 3.3 (Sobolev Inequalities)**.** *Let $n \geq 2$ and $p \in (1,p)$ and $p^* = \frac{np}{n-p}$ (Sobolev conjugate of p). The inequality*

$$\left(\int |f|^{p^*}\right)^{1/p^*} \leq C_{n,p}\left(\int \|\nabla f\|_*^p\right)^{1/p}$$

holds for every f with $f \in L^{p^*}$ and $\nabla f^p \in L^p$, where $C_{n,p}$ is optimal constant reach by the function $x \mapsto (1+\|x\|^q)^{\frac{p-n}{p}}$.

3.2.1 Gagliardo-Nirenberg Inequalities

Right now, on the contrary of Sobolev cases we are considering about $a > n$ and $p \neq a$. Let's $h = g^{1-\frac{a}{p}}$ this time, inequality (3.4) turns into

$$D \leq \frac{a-1}{p}\left(\frac{p}{p-a}\right)^p \int \|\nabla h\|_*^p + (a-n)\int h^{p\frac{a-1}{p-a}}.$$

For simplicity write it as the following form for positive constant D_2

$$1 \leq D_2 \int \|\nabla h\|_*^p + (a-n)\int h^{p\frac{a-1}{p-a}}.$$

This is true for all h with $\nabla h \in L^p$ and $\int h^{p\frac{a-1}{p-a}} = 1$. If we remove normalization process, we get

$$\left(\int h^{\frac{ap}{a-p}}\right)^{\frac{a-p}{a}} \leq D_2 \int \|\nabla h\|_*^p + (a-n)\int h^{p\frac{a-1}{a-p}}\left(\int h^{\frac{ap}{a-p}}\right)^{\frac{1-p}{a}}$$

We are one step closer to have Gagliardo-Nirenberg inequalities, after doing new arrangement such as putting $h(x) = f(\lambda x)$ and optimize over $\lambda > 0$, we have

$$\left(\int f^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}(1-\frac{1-p}{a-p}\omega)} \leq D_3 \left(\int \|f\|_*^p\right)^{\frac{1-\omega}{p}} \left(\int f^{p\frac{a-1}{a-p}}\right)^{\frac{1-p}{p(a-1)}\frac{a-1}{a-p}\omega}$$

where $\omega = \frac{p(a-n)}{p(a-n)+n} \in (0,1)$. Now its splitting into two cases where the sign of $\left(1 - \frac{1-p}{a-p}\omega\right)$ and $\frac{a-1}{a-p}\omega$. If $p < a$ then both coefficients are positive, by 3.3 it can be checked. This gives us the first case of the theorem we are going to state in a second. If $p > a$, then again by 3.3 both coefficients are negative, we stated it in the second case of the theorem.

Theorem 3.4 (Gagliardo-Nirenberg Inequalities.)**.** *Let $n \geq 1$ and $a > n$.*

- *For any $1 < p < a$, the inequality*

$$\left(\int |f|^{\frac{ap}{p-a}}\right)^{\frac{a-p}{ap}} \leq C^+_{n,p,a} \left(\int \|\nabla f\|_*^p\right)^{\frac{\theta}{p}} \left(\int |f|^{p\frac{a-1}{p-a}}\right)^{\frac{a-p}{p(a-1)}(1-\theta)} \tag{3.5}$$

  holds for any function f which has finite integrals and $\theta \in (0,1)$ is given by

$$\frac{a-p}{a} = \theta\frac{n-p}{n} + (1-\theta)\frac{a-p}{a-1}$$

  and $C^+_{n,p,a}$ is positive optimal constant given by the extremal function $x \mapsto (a + \|x\|^q)^{\frac{p-a}{p}}$.

- *If $p > a$ when $a \leq n+1$, or if $p \in \left(a, \frac{n}{n+1-a}\right)$ when $a \in [n.n+1)$, then the inequality*

$$\left(\int |f|^{p\frac{ap}{p-a}}\right)^{\frac{a-p}{p(a-1)}} \leq C^-_{n,p,a} \left(\int \|\nabla f\|_*^p\right)^{\frac{\theta'}{p}} \left(\int |f|^{\frac{ap}{a-p}}\right)^{\frac{a-p}{ap}(1-\theta')} \tag{3.6}$$

  holds for any function f with finite integral, where $\theta' \in (0,1)$ is given by

$$\frac{p-a}{a-1} = \theta'\frac{p-n}{n} + (1-\theta')\frac{p-a}{a},$$

  and $C^-_{n,p,a}$ is the optimal constant given by the extremal function $x \mapsto (1 + \|x\|^q)^{\frac{p-a}{p}}$.

Proof of the Theorem 3.2. Let φ be the Brenier's map between the measure $g^{-a}dx$ and $W^{-a}dx$, where we can write the transport as $\nabla\varphi\#g^{-a} = W^{-a}$, so by 1.3 we have

$$W(\nabla\varphi) = g\left(\det(\nabla^2\varphi)\right)^{1/a}$$

and by assumption $a \geq n$, we can deduce almost everywhere,

$$\left(\det(\nabla^2\varphi)\right)^{-a} \leq 1 - \frac{n}{a} + \frac{1}{a}\Delta\varphi$$

by 2.1 for $k = 1/a$. Notes that here and below $\Delta\varphi = \mathrm{tr}(\nabla^2\varphi)$. Integrating both sides with respect to $g^{-a}dx$ we get

$$\int W(\nabla\varphi)g^{-a} \leq \left(1 - \frac{n}{a}\right)\int g^{1-a} + \frac{1}{a}\int \Delta\varphi g^{1-a}.$$

Now assume for $a > \gamma/(\gamma - 1)$, we have almost everywhere

$$\int \Delta\varphi g^{1-a} \leq (a-1)\int \nabla\varphi.\nabla g\, g^{-a}. \tag{3.7}$$

With this assumption, we obtained

$$a\int W(\nabla\varphi)g^{-a} \leq (a-n)\int g^{1-a} + (a-1)\int \nabla g.\nabla\varphi g^{-a}.$$

But by definition of Legendre Transform 2.1,

$$\nabla g.\nabla\varphi \leq W(\nabla\varphi) + W^*(\nabla g),$$

after collecting terms on one side we get

$$\int W(\nabla\varphi)g^{-a} \leq (a-1)\int W^*(\nabla g)g^{-a} + (a-n)\int g^{1-a}.$$

Finally, $\int W(\nabla\varphi)g^{-a} = \int W^{1-a}$ by transport $\nabla\varphi\#g^{-a} = W^{-a}$, so have desired inequality which is,

$$(a-1)\int W^*(\nabla g)g^{-a} + (a-n)\int g^{1-a} \geq \int W^{1-a}. \tag{3.8}$$

This finish the proof for $a > \gamma/(\gamma - 1)$. However we need to verify (3.7), for this, the idea is coming from [11] where is an extension of the argument of Lemma 7 in [7]. For this, define $g_\varepsilon^{1-\frac{a}{\gamma'}} := \min\{g^{1-\frac{a}{\gamma'}}(x/(1-\varepsilon)), g^{1-\frac{a}{\gamma'}}\chi(\varepsilon x)\}$ for a cut-off function χ, for instance such that $0 \geq \chi \geq 1$, $\chi(x) = 1$ when $|x| \leq 1/2$ and $\chi(x) = 0$, whenever $|x| \geq 1$. We want to show the argument (3.7) for g_ε instead of g, then we let $\varepsilon \to 0$. For this we want to show the sequence $\nabla g_\varepsilon^{1-\frac{a}{\gamma'}}$ is bounded in $L^{\gamma'}$. So, we need to show, both $g^{1-\frac{a}{\gamma'}}(x/(1-\varepsilon)$ and $g^{1-\frac{a}{\gamma'}}\chi(\varepsilon x)$ are in

$L^{\gamma'}$. It is easy to observe for $\nabla\left(g^{1-\frac{a}{\gamma'}}(x/(1-\varepsilon))\right)$ by change of variable $y = x/(1-\varepsilon)$, because we get $(1-\frac{a}{\gamma'})|\nabla g|g^{-\frac{a}{\gamma'}}$ which is bounded in $L^{\gamma'}$ by assumption. For the sequence $g^{1-\frac{a}{\gamma'}}\chi(\varepsilon x)$ also bounded, since

$$2^{1-\gamma'}\int\left|\nabla\left(g^{1-\frac{a}{\gamma'}}\chi(\varepsilon x)\right)\right|^{\gamma'}dx \leq \int\left|\nabla g^{1-\frac{a}{\gamma'}(x)}\right|^{\gamma'}|\chi(\varepsilon x)|^{\gamma'} + \varepsilon^{\gamma'}\int g^{\gamma'-a}(x)\left|\nabla\chi\right|^{\gamma'}(\varepsilon x)dx.$$

First term is bounded since $|\chi| \leq 1$ and $\nabla g^{1-\frac{a}{\gamma'}} \in L^{\gamma'}$ by assumption $|\nabla g|g^{1-\frac{a}{\gamma'}}$ is integrable. For the second term using Hölder's inequality for $a/(a-\gamma')$ with $a > \gamma'$ and change of variables gives us,

$$\varepsilon^{\gamma'(1-\frac{a}{n})}\left(\int g^{-a}\right)^{1-\frac{a}{\gamma'}}\left(\int|\nabla\chi|^{a}\right)^{\frac{\gamma'}{a}}$$

and hence it is bounded(uniformly) in ε for $a \geq n$.

For the case of $\gamma\prime \leq a$, we want to use the same argument as before by approximation. For this purpose, fix some $s > a/(a-1)$, so that the s' is in $(1, a)$. Now define, $W_\varepsilon := Z_\varepsilon(W(x)+\varepsilon|x|^s)$ with $Z_\varepsilon \in \mathbb{R}$ such that $\int_{\mathbb{R}_n} W_\varepsilon^{-a} = 1$. Observe that we have $s' \leq \gamma'$, so by using Hölder's inequality and the assumption of the statement, $g^{-a}|\nabla g|^{s'}$ is integrable. Everything in the last hypothesis is satisfied here, so (3.8) turns into,

$$(a-1)\int(W_\varepsilon)^*(\nabla g)g^{-a} + (a-n)\int g^{1-a} \geq \int W_\varepsilon^{1-a},$$

where we have $Z_\varepsilon \to 1$ and $W_\varepsilon \to W$. Here is the time to use dominated converges theorem, right hand side converges to $\int W^{1-a}$ by it. For the left hand side consider $W_\varepsilon \geq Z_\varepsilon W$, so we got $\int(W_\varepsilon)^*(\nabla g)g^{-a} \leq Z_\varepsilon\int W^*(\frac{\nabla g}{Z_\varepsilon})g^{-a}$, by dominated converges theorem, it converges to $\int W^*(\nabla g)(g^{-a})$. Therefore, we obtained (3.8) for g and W. $\square$

Chapter 4

Dynamical formulations and trace inequalities

We started our journey with the generalization of 1.2, in this chapter we hereby focus on the generalization of 1.3 and applications on dynamical formulations and Sharp Trace Inequalities on a half-space. Let's begin with that generalization which concerns convex case,

Theorem 4.1. *(Φ-Borell-Brascamp-Lieb inequality). Let $a \geq n \geq 1$ (if $n = 1$, then $a > 1$) and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function. Let also $g, W, H : \mathbb{R}^n \to [0, +\infty]$ be Borel functions and $s \in [0, 1]$ and $t = 1 - s$, be such that*

$$\forall x, y \in \mathbb{R}^n, \qquad H(sx + ty) \leq sg(x) + tW(y), \tag{4.1}$$

and $\int W^{-a} = \int g^{-a} = 1$. Then we have

$$\int \Phi(H)H^{-a} \geq s \int \Phi(g)g^{-a} + t \int \Phi(W)W^{-a} \tag{4.2}$$

First Proof. By Brenier's Theorem, there is a convex map $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ such that its unique gradient gives us $\nabla\varphi \# g^{-a} = W^{-a}$. So the corresponding Monge-Ampère equation will be

$$W(\nabla\varphi) = g \det(\nabla^2\varphi)^{1/a}.$$

Moreover, by 2.2, we have that Φ is non-decreasing and also $\frac{\Phi(x)-\Phi(0)}{x}$ is non-increasing, so that

$x \mapsto \Phi(x)x^{-a}$ is non-increasing. We compute that

$$\begin{aligned}
\int \Phi(H)H^{-a} &= \int \Phi(H(z))H^{-a}(z)dz \\
&= \int \Phi(H(sx+t\nabla\varphi))H^{-a}(sx+t\nabla\varphi)\det(sI_n+t\nabla^2\varphi)dx \\
&\geq \int \Phi(sg(x)+tW(\nabla\varphi))(sg(x)+tW(\nabla\varphi))^{-a}\det(sI_n+t\nabla^2\varphi)dx \\
&\geq \int (s\Phi(g(x)+t\Phi(W(\nabla\varphi)))(sg(x)+tg\det(\nabla^2\varphi)^{1/a})^{-a}det(sI_n+t\nabla^2\varphi)dx \\
&\geq \int (s\Phi(g(x))+t\Phi(W(\nabla\varphi))g^{-a}(s+t\det(\nabla^2\varphi)^{1/a})^{-a}det(sI_n+t\nabla^2\varphi)dx,
\end{aligned}$$

where we used change of variables with $z = sx + \nabla\varphi(x)$. Since $1/a \leq 1/n$, by lemma 2.1 we know that

$$(\det(sI_n+t\nabla^2\varphi)^{1/a})^a \geq (s\det(I_n)^{1/a}+t\det(\nabla^2\varphi)^{1/a})^a, \tag{4.3}$$

and with $\int W^{-a} = \int g^{-a} = 1$ and $\nabla\varphi\# g^{-a} = W^{-a}$, we get

$$\int \Phi(H)H^{-a} \geq \int (s\Phi(g(x))+t\Phi(W(\nabla\varphi)))g^{-a} = s\int \Phi(g)g^{-a} + \int \Phi(W)W^{-a}.$$

□

Second Proof. We are going to use McCann's idea. Let $(\rho_t)_{t\in[0,1]}$ be density path between $\rho_0 = g^{-a}$ and $\rho_1 = W^{-a}$. It is defined as following in [13]; for each t in $[0,1]$, ρ_t is defined as $\rho_t = \nabla\varphi_t\#\rho_0$, where $\varphi_t(x) = s\frac{|x|^2}{2} + t\varphi(x)$. And if we use Monge-Ampère equation two times, on $\rho_1 = \nabla\varphi\#\rho_0$, and $\rho_t = \nabla\varphi_t\#\rho_0$ and the determinant inequality (4.3), we get

$$\begin{aligned}
\rho_t(\nabla\varphi_t) &= g^{-a}\det(\nabla^2\varphi_t)^{-1} \\
&\leq g^{-a}(s+tdet(\nabla^2\varphi))^{-a} \\
&= (sg+tgdet(\nabla^2\varphi))^{-a} \\
&= (sg+tW(\nabla\varphi))^{-a}
\end{aligned}$$

Therefore we have

$$\rho_t(\nabla\varphi_t) \leq (sg+tW(\nabla\varphi))^{-a}.$$

Now, multiply each side with $\Phi(sg + tW(\nabla\varphi))$, we get

$$\Phi(sg + tW(\nabla\varphi))\rho_t(\nabla\varphi_t) \leq \Phi(sg + tW(\nabla\varphi))(sg + W(\nabla\varphi))^{-a}.$$

The function Φ is concave and $x \mapsto \Phi(x)x^{-a}$ is non-increasing, so we have,

$$\begin{aligned}[s\Phi(g(x)) + t\Phi(W(\nabla\varphi(x)))]\,\rho_t(\nabla\varphi_t(x)) &\leq \Phi(H(sx + t\nabla\varphi(x)))H(sx + t\nabla\varphi(x))^{-a}\\ &= \Phi(H(\nabla\varphi_t(x)))H(\nabla\varphi_t(x))^{-a}.\end{aligned}$$

We can see the equation as following, base on the fact that the $\rho_t(\nabla\varphi_t) > 0$, following quantity obtained,

$$s\Phi(g(x)) + t\Phi(W(\nabla\varphi(x))) \leq \frac{\Phi(H(\nabla\varphi_t))H(\nabla\varphi_t)^{-a}}{\rho_t(\nabla\varphi_t)}\mathcal{X}_{\{\rho_t(\nabla\varphi_t)>0)\}},$$

and we integrate both sides with respect to ρ_0, from the left side we are obtaining,

$$\int [s\Phi(g) + t\Phi(W(\nabla\varphi))]\,\rho_0 = s\int \Phi(g)g^{-a} + t\int \Phi(W)W^{-a}.$$

And from the right hand side, by using $\nabla\varphi_t\#\rho_0 = \rho_t$, we get,

$$\int \frac{\Phi(H(\nabla\varphi_t))H(\nabla\varphi_t)^{-a}}{\rho_t(\nabla\varphi_t)}\mathcal{X}_{\{\rho_t(\nabla\varphi_t)>0)} = \int_{\{\rho_t>0\}} \Phi(H)H^{-a} \leq \int \Phi(H)H^{-a},$$

hence, we get the desired inequality 4.2. $\square$

4.1 Dynamical Formulations

Until this point, we have been doing our calculations on time-independent factors. Right now our plan is to add one more variable which can be seen as a time variable,

Define

$$Q_h^W(g)(x) = \begin{cases} \inf_{y\in\mathbb{R}^n}\{g(y) + hW(\frac{x-y}{h})\}, & \text{if } h > 0,\\ g(x) & \text{if } , h = 0\end{cases} \tag{4.4}$$

Or equivalently,

$$Q_h^W(g)(x) = \inf_{y\in\mathbb{R}^n}\{g(x - hz) + hW(z)\}.$$

If we consider the assumption (4.1), we can deduce

$$H(x) = sQ^W_{t/s}(g)(x/s)$$

if H is the biggest function that satisfies it. Now let us combine these formulas with the previous theorem.

Theorem 4.2 (Dynamical Reformulation of Φ-Borell-Brascamp-Lieb inequalities)**.** *Let $a \geq n \geq 1$ ($a > 1$ in case of $n = 1$), and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function. Let also assume $g, W : \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ be Borel functions with $\int W^{-a} = \int g^{-a} = 1$. Then for any $h \geq 0$, the inequality (4.2) is equivalent to*

$$(1+h)^{a-n} \int \Phi\left(\frac{1}{1+h} Q^W_h(g)\right) (Q^W_h(g))^{-a} \geq \frac{1}{1+h} \int \Phi(g) g^{-a} + \frac{h}{1+h} \int \Phi(W) W^{-a}. \quad (4.5)$$

And if we consider the case when $a = n$ and $\Phi(x) = x$, we get

$$\forall h \geq 0, \qquad \int Q^W_h(g)^{1-n} \geq \int g^{1-n} + h \int W^{1-n}. \quad (4.6)$$

And for (4.4) and (4.5) equalities are hold if $g = W$ and convex.

Proof of this theorem doesn't require any effort, it is enough to put 4.4 into Theorem 4.1, then we get desired inequality. The equation 4.6 admits a linearization as a convex inequality when $h \to 0$, which is

$$Q^W_h g = g - hW^*(\nabla g) + o(h).$$

It was detailedly explained in the [13] in Appendix A. We are considering a special case of Theorem 4.2, when $\Phi(x) = x$ and for an admissible (It is going to be explained in below) functions g and W. Which we obtained,

$$(a-1) \int W^*(\nabla g) g^{-a} + (a-n) \int g^{1-a} \geq \int W^{1-a}.$$

So, for restricted class $\mathcal{F}^a$ of functions (g, W), we obtained the Theorem 3.2 and therefore we can recover Sobolev and Gagliardo-Nirenberg inequalities from it.

Remark: The classical Borell-Brascamp-Lieb inequality admits the following dynamical formulation: if $g, W : \mathbb{R}^n \to (0, +\infty)$ are such that $\int g^{-n} = \int W^{-n} = 1$, then

$$\int Q_h^W(g)^{1-n} \geq 1, \qquad h \geq 0.$$

For h tending to 0 we recover the convexity inequality 3.2 with $a = n+1$, which is

$$\int_{\mathbb{R}^n} \frac{W^*(\nabla g)}{g^{n+1}} \geq 0, \tag{4.7}$$

which derived in [11]. But this inequality only implies Gagliardo-Nirenberg inequalities for $a \geq n+1$ and does not imply Sobolev inequality which was pointed out in [5]. This was one of the main focuses of the work [13].

4.2 Trace Sobolev Inequalities

In this part, by respecting the method we mentioned earlier in the previous chapter, we are going to derive Sobolev inequalities from the convex inequality. First, for notational convenience, let us give the following definitions.

Definition 4.1.

- *For any $n \geq 2$, define*

$$\mathbb{R}^n_+ = \left\{ z = (u, x), \;\; u \geq 0, x \in \mathbb{R}^{n-1} \right\}.$$

Then clearly, $\partial \mathbb{R}^n_+ = \left\{ z = (0, x), \;\; u \geq 0, x \in \mathbb{R}^{n-1} \right\} = \mathbb{R}^{n-1}$

- *For $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $h \in \mathbb{R}$, we denote,*

$$\mathbb{R}^n_{+he} = \mathbb{R}^n_+ + he = \left\{ (u, x), \;\; u \geq h, \;\; x \in \mathbb{R}^{n-1} \right\}.$$

Our objective is to prove

Theorem 4.3. *(Trace Sobolev inequality from [8]). For any $n \geq p > 1$ and for $\tilde{p} = p(n-1)/(n-p)$ the Sobolev inequality*

$$\left(\int_{\partial \mathbb{R}^n_+} f^{\tilde{p}} \right)^{1/\tilde{p}} \leq D_{n,p} \left(\int_{\mathbb{R}^n_+} \|f\|_*^p \right)^{1/p},$$

holds for any $\mathcal{C}^1$ and compactly supported functions f on $\mathbb{R}^n_+$. Here $D_{n,p}$ is the optimal constant *given by the extremal function*

$$h_p(z) = \|z+e\|^{-\frac{n-p}{p-1}}, \;\; z \in \mathbb{R}^n_+.$$

□

We have the result for $\mathcal{C}^1$ and compactly supported functions, nevertheless, it can be extended to any other space by using approximation. B.Nazaret gave the idea of adding a vector e (in his paper it was $e = (-1, 0)$) to the map W in [8], which we obtained his main statement. However, in this work, we are not going to prove optimality of the $D_{n,p}$, which can also be found in the papers [8] and [13]. Now we can start with the equivalent of the Theorem 4.1 for half-spaces. Let us take $\Phi(x) = x$, then we have the following Theorem.

Theorem 4.4. *(Trace Borell-Brascamp-Lieb inequality). Let $a \geq n$, $g : \mathbb{R}^n_+ \to (0, +\infty)$ and $W : \mathbb{R}^n_{+e} \to (0, +\infty)$ such that $\int_{\mathbb{R}^n_+} g^{-a} = \int_{\mathbb{R}^n_{+e}} W^{-a} = 1$. Then $\forall h > 0$,*

$$(1+h)^{a-n} \int_{\mathbb{R}^n_{+he}} Q_h^W(g)^{1-a} \geq \int_{\mathbb{R}^n_+} g^{1-a} + h \int_{\mathbb{R}^n_{+e}} W^{1-a}, \tag{4.8}$$

where for $(u, x) \in \mathbb{R}^n_{+he}$,

$$Q_h^W(g)(u,x) = \inf_{(v,y)\in\mathbb{R}^n_+,\ 0\leq v\leq u-h} \left\{ g(v,y) + hW(\frac{u-v}{h}, \frac{x-y}{h}) \right\}.$$

And the equality happens if $g(z) = W(z+e)$ and convex.

Proof. First we are going to extend our functions to $\mathbb{R}^n$ as follows, define $\tilde{g} : \mathbb{R}^n \to (0, +\infty]$ and $\tilde{W} : \mathbb{R}^n \to (0, +\infty]$ by

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if} \quad x \in \mathbb{R}^n_+ \\ +\infty, & \text{if} \quad x \notin \mathbb{R}^n_+ \end{cases} \quad \text{and} \quad \tilde{W} = \begin{cases} W(x), & \text{if} \quad x \in \mathbb{R}^n_{+e} \\ +\infty, & \text{if} \quad x \notin \mathbb{R}^n_{+e} \end{cases}. \tag{4.9}$$

Then, we get $\int_{\mathbb{R}^n} \tilde{g}^{-a} = \int_{\mathbb{R}^n} \tilde{W}^{-a} = 1$. Now we can apply the Theorem 4.2 for the functions $\tilde{g}$ and $\tilde{W}$, and for any $h > 0$, we have,

$$(1+h)^{a-n} \int_{\mathbb{R}^n} Q_h^{\tilde{W}}(\tilde{g})^{1-a} \geq \int_{\mathbb{R}^n_+} \tilde{g}^{1-a} + \int_{\mathbb{R}^n_{+e}} \tilde{W}^{1-a}$$

where,

$$Q_h^{\tilde{W}}(\tilde{g})(u,x) = \inf_{(u,y)\in\mathbb{R}^n} \left\{ \tilde{g}(u,y) + h\tilde{W}\left(\frac{u-v}{h}, \frac{x-y}{h}\right) \right\}.$$

We can restrict infimum into the case of $0 \leq v \leq u-h$, since otherwise $Q_h^{\tilde{W}}(\tilde{g})$ would be infinity. So, we can deduce

$$\int_{\mathbb{R}^n} Q_h^{\tilde{W}}(\tilde{g})(u,x) = \int_{\mathbb{R}^n_{+he}} Q_h^{\tilde{W}}(\tilde{g})(u,x) = \int_{\mathbb{R}^n_{+he}} Q_h^{W}(g)(u,x).$$

Hence, we obtained (4.7). For the equality, if $g(z) = W(z+e)$ and convex, by convexity, we have,

$$Q_h^W(g)(u,x) = (1+h)W\left(\frac{u+1}{h+1}, \frac{x}{h+1}\right)$$

for any $(u,x) \in \mathbb{R}^n_{+he}$. □

As mentioned in the previous chapter, we could obtain convex inequality from Borell-Brascamp-Lieb type inequality. It is also the case for $\mathbb{R}^n_+$, so by A.4 and using the following identity,

$$\int_{\mathbb{R}^n_{+he}} Q_h^W(g)^{1-a} = \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx.$$

So for admissible couple (g,W) (Definition A.1) we have,

$$\left.\frac{d}{dh}\right|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx = -\int_{\partial\mathbb{R}^n_+} g^{1-a}dx + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}dz, \tag{4.10}$$

where W^* is the Legendre transform of W which is

$$W^*(y) = \inf_{x\in\mathbb{R}^n_{+e}} \{x.y - W(x)\}, \qquad y \in \mathbb{R}^n. \tag{4.11}$$

So now before proceeding to obtain the new trace Sobolev inequalities. We have the following proposition as a result of the above.

Proposition 4.1 (Trace convex inequality)**.** . *Let $a \geq n$. Let $g : \mathbb{R}^n_+ \to (0,+\infty)$ and W : $\mathbb{R}^n_{+e} \to (0,+\infty)$ belong to $\mathcal{F}^a_+$ with W convex and $\int_{\mathbb{R}^n_+} g^{-a} = \int_{\mathbb{R}^n_{+e}} W^{-a} = 1$. Then,*

$$(a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a} + (a-n)\int_{\mathbb{R}^n_+} g^{1-a} \geq \int_{\mathbb{R}^n_{+e}} W^{1-a} + \int_{\partial\mathbb{R}^n_+} g^{1-a} \tag{4.12}$$

Moreover (4.11) is an equality when $g(z) = W(z+e)$ for $z \in \mathbb{R}^n_+$ and convex.

Therefore here, by using the method we used in the previous chapter we can obtain Sobolev and Gagliardo-Nirenberg inequalities. Let $a \geq n$, $p \in (0,n)$ and $q = \frac{p}{p-1}$. Let be $W(z) = C\frac{\|x\|^q}{q}$ for $z \in \mathbb{R}^n_{+e}$ with the C such that $\int_{\mathbb{R}^n_{+e}} W^{1-a} = 1$. To apply the method first we need to show it satisfies the conditions $(C1) \sim (C4)$ in the definition A.1 with a proper g. Fortunately, for $\gamma = q$ we have $(C1)$ and $(C2)$, since $q > n/(a-1)$ for $a \geq n$. Also, we have

$$W^*(y) = \sup_{z \in \mathbb{R}^n_+} \left\{ y.z - C\frac{\|z\|^q}{q} \right\} \leq \sup_{z \in \mathbb{R}^n} \left\{ y.z - C\frac{\|z\|^q}{q} \right\} = C^{1-p}\frac{\|y\|^p_*}{p}. \tag{4.13}$$

After putting everything into Proposition 4.1, we get

$$C^{1-p}\frac{(a-1)}{p}\int_{\mathbb{R}^n_+} \frac{\|\nabla g\|^p_*}{g^a} + (a-n)\int_{\mathbb{R}^n} g^{1-a} \geq \int_{\mathbb{R}^n_{+e}} W^{1-a} + \int_{\partial\mathbb{R}^n_+} g^{1-a}, \tag{4.14}$$

for a function g, f which satisfies $(C3)$ and $(C4)$ with $\int_{\mathbb{R}^n_+} g^{-a} = 1$, so that $(g, W) \in \mathcal{F}^a_+$.

Note: How can we be sure about optimality of (4.13), even though we use the inequality (4.12)? It is easy to see, if we consider about the equality case when $g(z) = W(z+e)$ for $x \in \mathbb{R}^n_+$, then the point in (4.10) is reached at the point $\nabla g(x)$ is reached in $\mathbb{R}^n_+$ and it is an equality.

Deriving Trace Sobolev inequalities: Like in chapter 3, let's assume $a = n$. Equation (4.13) turns into;

$$C^{1-p}\frac{(n-1)}{p}\int_{\mathbb{R}^n_+} \frac{\|\nabla g\|^p_*}{g^n} - \int_{\mathbb{R}^n_{+e}} W^{1-n} \geq \int_{\partial\mathbb{R}^n_+} g^{1-n}$$

Like in the $\mathbb{R}^n$ case, put $f = g^{\frac{p-n}{p}}$, so that $\int_{\mathbb{R}^n_+} f^{\frac{pn}{p-n}} = 1$, so inequality transforms to.

$$\int_{\partial\mathbb{R}^n_+} f^{\frac{p(n-1)}{p-n}} \leq C^{1-p}\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n_+} \|f\|^p_* - \int_{\mathbb{R}^n_{+e}} W^{1-n} \tag{4.15}$$

We want to obtain a result for all compactly supported $\mathcal{C}^1$ functions in $\mathbb{R}^n_+$, which does not necessarily means it vanishes in $\partial\mathbb{R}^n_+$. Define

$$f_\varepsilon(x) = \varepsilon|x+e|^{-\frac{n-p}{p-1}} + c_\varepsilon f(x),$$

for compactly supported $f \in \mathcal{C}^1$ in $\partial\mathbb{R}^n_+$, where c_ε is a quantity such that $\int_{\mathbb{R}^n_+} f_\varepsilon^{\frac{np}{n-p}} = 1$. Then, $g_\varepsilon = f_\varepsilon^{-\frac{p}{n-p}}$ satisfies $(C3)$ and $(C4)$. Moreover, when $\varepsilon \to 0$, $c_\varepsilon \to 1$ and then inequality (4.14) holds for the function f. To keep notation simple, let us define the following:

$$\tilde{p} = \frac{p(n-1)}{n-p},\ \beta = \left(\int_{\mathbb{R}n_+} f^{\frac{pn}{n-p}} dz\right)^{\frac{n-p}{np}},\ A = C^{1-p}\frac{n-1}{p}\left(\frac{p}{n-p}\right)^p \text{ and } B = \int_{\mathbb{R}^n_{+e}} W^{1-n} dz$$

So after removing normalization in (4.14) we get, for any f,

$$\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}} \leq A\int_{\mathbb{R}^n_+} \|f\|_*^p dz \beta^{\tilde{p}-p} - B\beta_{\tilde{p}},$$

We want to apply Young's inequality $xy \leq x^u/u + y^v/v$ here. So first to get it put $u = \frac{\tilde{p}}{p} = \frac{n-1}{n-p}$ and $v = \frac{\tilde{p}}{\tilde{p}-p}$(which satisfy $u, v > 1$ and $1/u + 1/v = 1$). Now put them into the equation:

$$\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}} \leq Bv\left[\frac{A}{Bv}\int_{\mathbb{R}^n_+} \|f\|_*^p \beta^{\tilde{p}-p} - \frac{1}{v}\beta^{\tilde{p}}\right].$$

For,

$$x = \frac{A}{Bv}\int_{\mathbb{R}^n_+} \|f\|_*^p \text{ and } y = \beta^{\tilde{p}-p}$$

yields

$$\left(\int_{\partial\mathbb{R}^n_+} f^{\tilde{p}}\right)^{1/\tilde{p}} \leq \frac{A^{1/p}}{(Bv)^{\frac{p-1}{p(n-1)}}}\left(\frac{n-p}{n-1}\right)^{\frac{n-p}{p(n-1)}}\left(\int_{\mathbb{R}^n_+} \|f\|_*^p\right).$$

After removing sign conditions we obtain Theorem 4.3.

Remark: After playing with coefficients and using scaling argument like in the Chapter 3, it is possible to obtain trace Gagliardo-Nirenberg inequalities as in the following form,

Theorem 4.5 (Trace Gagliardo-Nirenberg inequalities)**.** *For any $a \geq n > p > 1$, the Gagliardo-Nirenberg inequality*

$$\left(\int_{\partial\mathbb{R}^n_+} |f|^{p\frac{a-1}{a-p}} dx\right)^{\frac{a-p}{p(a-1)}} \leq D_{n,p,a}\left(\int_{\mathbb{R}^n_+} \|\nabla f\|_*^p dz\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^n_+} f^{p\frac{a-1}{a-p}}\right)^{1-\theta\frac{a-p}{p(a-1)}} \tag{4.16}$$

holds for any $\mathcal{C}^1$ and compactly supported function on $\mathbb{R}^n_+$. here $\theta \in [0,1]$ and

$$\frac{n-1}{n}\frac{a-p}{a-1} = \theta\frac{n-p}{n} + (1-\theta\frac{a-p}{a-1})$$

and $D_{n,p,a}$ is the optimal constant, reached when

$$f(z) = h_p(z) = \|z+e\|^{-\frac{a-p}{p-1}}, \quad z \in \mathbb{R}^n_+.$$

When $a = n$, then we recover the trace Sobolev inequality of the Theorem 4.3

Chapter 5

New Model

At the moment we are shifting our focus, but this is not too far away from original purpose. From this point, we want to derive the sharp Sobolev inequalities in a measure $d\mu(x) = e^{-V(x)}dx$. Even though we are not going to mention details about how it is useful in the field of partial differential equations, one example can be found in the paper of H.Brascamp and E.Lieb [2], where they were using $d\mu$ to derive Prékopa-Leindler inequality and using obtained results to derive new inequalities for the fundamental solution of diffusion equation with a convex potential. And on the other hand, as we mentioned before, in the paper [5], S.Bobkov and M.Ledoux inspired from the their work, and showed how it is possible to derive well-known inequalities by using $d\mu$ from the Brunn-Minkowski inequality, such as entropic version of the Brascamp-Leindler inequality. For further details, we suggest reader to consult their works. In this chapter, our concern is to derive the Sobolev inequalities in the following form: let $p \geq 2$ and $p^* = \frac{np}{n-p}$, where $n \geq d \geq 2$ and $\Omega \subset \mathbb{R}^d$, we have

$$\left(\int_\Omega |f|^{p^*} d\mu \right)^{1/p^*} \leq \left(\int_\Omega \|\nabla f\|_p d\mu \right)^{1/p},$$

with the method we presented in chapter 4. Before starting, it has to be mention that, when we say convex cone, we mean a subset of a linear space which is closed under linear combinations with positive coefficient. Also, in this chapter, $d \geq 2$ will be a fixed integer.

5.1 First look

Now assume that we have probability measure μ with probability density $e^{-V(x)}$, which means $\frac{d\mu}{dx} = e^{-V(x)}$. Let $n > d$ be any real number, and $V : \Omega \to \mathbb{R}^+$ be a convex function, where Ω is a convex cone in $\mathbb{R}^d$. V satisfies the following statements;

- $\nabla^2 V \geq \frac{1}{n-d} \nabla V \otimes \nabla V$,
- For $\lambda \in \mathbb{R}$, $e^{-V(\lambda x)} = \lambda^{n-d} e^{-V(x)}$.
- $\int_\Omega e^{-V(x)} dx < +\infty$

One example would be important for us, which is $d\mu = x^A dx$, where

$$x^A := \prod_{i=1}^{d} |x|^{A_i}, \tag{5.1}$$

where $A_i \in \mathbb{R}^+$ and $A = (A_1, \ldots, A_d)$. Essentially, we want to show, how to derive the Sobolev inequalities, with this weighted measure. Note that, it was done by C.Xavier and R.Xavier in [10] with a different form, for whole $\mathbb{R}^n$ and focused on the case of half-spaces in the paper [12] by V.H.Nguyen with optimal constant.

As before, first step is to prove BBL inequality in our new context,

Theorem 5.1 (Borell-Brascamp-Lieb inequality for $d\mu$)**.** *Let $g, H, W : \Omega \to \mathbb{R}^+$,where $\Omega \subseteq \mathbb{R}^d$ so that for all $x, y \in \Omega$ and for $t, s \in (0, 1)$ with $t = 1 - s$,*

$$H(sx + ty) \leq sg(x) + tW(y).$$

We have

$$\int_\Omega H^{1-n} d\mu \geq s \int_\Omega g^{1-n} d\mu + t \int_\Omega W^{1-n} d\mu, \tag{5.2}$$

where $d\mu = e^{-V(x)} dx$. And equality holds if $g = W$ and convex.

Even though it was smooth at the case of 4.1, we need to consider about convexity problem that appears during the proof. However, simple technical lemma solves the problem during the proof.

Proof: Like in the previous sections, let us start with Brenier's map between the Lebesgue measures $\nabla\varphi \# g^{-n} d\mu = W^{-n} d\mu$. Now we get different result for Monge-Ampère Equation which is,

$$W(\nabla\varphi) = ge^{\frac{V-V(\nabla\varphi)}{n}} det^{1/n}(\nabla^2\varphi). \tag{5.3}$$

It is necessary to give the two proof after this part, first one is crucial since the idea behind it, unfortunately there is an assumption of change of variables without a proof. Luckily, it is solvable by using McCann's idea [4], which is the second proof,

First Proof. Now, start with

$$\begin{aligned}
\int_\Omega H^{1-n} d\mu &= \int_\Omega H^{1-n}(z)e^{-V(z)}dz \\
&= \int_\Omega H^{1-n}(sx + t\nabla\varphi(x))P(s,x,\varphi)dx \\
&\geq \int_\Omega (sg(x) + tW(\nabla\varphi(x)))^{1-n} P(s,x,\varphi)dx \\
&= \int_\Omega (sg(x) + tW(\nabla\varphi(x)))(sg(x) + tW(\nabla\varphi(x)))^{-n} P(s,x,\varphi)dx
\end{aligned}$$

where $P(s,x,\varphi) = e^{-V(sx+t\nabla\varphi(x))}\det(sI_n + t\nabla^2\varphi(x))$. Now putting (5.2) to the equation we get

$$\begin{aligned}
&= \int_\Omega (sg + tW(\nabla\varphi))(sg + tge^{\frac{V-V(\nabla\varphi)}{n}} det^{1/n}(\nabla^2\varphi))^{-n} P(s,x,\varphi)dx \\
&= \int_\Omega (sg + tW(\nabla\varphi))g^{-n}e^{-V}(se^{\frac{-V}{n}} + te^{\frac{-V(\nabla\varphi)}{n}} det^{1/n}(\nabla^2\varphi))^{-n} P(s,x,\varphi)dx
\end{aligned}$$

Right now, if we can prove the following statement, it is going to be enough to conclude;

$$(se^{\frac{-V}{n}} + te^{\frac{-V(\nabla\varphi)}{n}} det^{1/n}(\nabla^2\varphi))^{-n} e^{-V(sx+t\nabla\varphi(x))}\det(sI_n + t\nabla^2\varphi(x)) \geq 1,$$

which can be seen as

$$e^{\frac{-V(sx+t\nabla\varphi(x))}{n}}\det(sI_n + t\nabla^2\varphi(x))^{1/n} \geq (se^{\frac{-V}{n}} + te^{\frac{-V(\nabla\varphi)}{n}} det^{1/n}(\nabla^2\varphi)) \tag{5.4}$$

This statement clearly true for $V = 0$, since $1/d \geq 1/n$, and the function is $H \mapsto det^{1/k}(H)$ is concave for $k \in (0, 1/d]$.

Lemma 5.2. *The inequality 5.4 holds.*

Proof of the lemma. At the first glance, it does not seems easy. However, let's check what do we know until this point. Let's define $\Gamma : S^n_{++} \to \mathbb{R}^+$ such that $\Gamma(A) = \det^{1/d}(A)$, which is an concave function. To show 5.4, we are seeking for a concave connection of functions on the left side. However, if we just consider the function, $e^{\frac{-V(x)}{n}}$ we cannot conclude anything since it's not concave. Moreover, this statement would be obviously wrong in general. Fortunately, we have a different option: if we define $\psi(x) = e^{\frac{-V(x)}{n-d}}$, then it is concave, which can be seen as following,

$$(\psi(x))' = \frac{1}{n-d}(-\nabla V(x) e^{\frac{-V(x)}{n-d}})$$

and the second derivative is,

$$(\psi(x))'' = \frac{1}{(n-d)^2}(\nabla V(x) \otimes \nabla V(x) e^{\frac{-V(x)}{n-d}}) + \frac{1}{n-d}(-\nabla^2 V(x) e^{\frac{-V(x)}{n-d}}) \leq 0$$

by the assumption 5.1. Therefore, the functions Γ and ψ are concave. Now define

$$\Theta(u, v) = u^{1-\alpha} v^{\alpha},$$

where $\alpha = d/n$.

Claim: Θ is concave,

Proof of the Claim: Let's calculate first and second derivative of Θ,

$$\begin{aligned}
\Theta_u &= (1-\alpha) u^{-\alpha} v^{\alpha}, \\
\Theta_v &= (\alpha) u^{1-\alpha} v^{\alpha-1}, \\
\Theta_{uu} &= (1-\alpha)(-\alpha) u^{-\alpha-1} v^{\alpha}, \\
\Theta_{vv} &= (\alpha)(\alpha-1) u^{1-\alpha} v^{\alpha-2}, \\
\Theta_{uv} &= (1-\alpha)(\alpha) u^{-\alpha} v^{\alpha-1}.
\end{aligned}$$

So the Hessian of Θ is

$$\begin{vmatrix} \Theta_{uu} & \Theta_{uv} \\ \Theta_{vu} & \Theta_{vv} \end{vmatrix} = (1-\alpha)^2(\alpha)^2 u^{-1} v^{2\alpha-2} - (1-\alpha)^2(\alpha)^2 u^{-1} v^{2\alpha-2} = 0. \tag{5.5}$$

So, Θ is concave. Now we have everything needed. Let $x, y \in \mathbb{R}^d$and $A, B \in S^n_{++}$ and $s \in (0, 1)$

whereas $t = 1 - s$

$$\begin{aligned}\Theta(\psi(sx+ty)),\Gamma(sA+tB)) &\geq \Theta(s\psi(x)+t\psi(y), s\Gamma(A)+t\Gamma(B)) \\ &= \Theta(s(\psi(x),\Gamma(A))+t(\psi(x),\Gamma(y))) \\ &\geq s\Theta(\psi(x),\Gamma(A))+t\Theta(\psi(y),\Gamma(B)).\end{aligned}$$

If we put $y = \nabla\varphi$, $A = I_n$ and $B = \nabla^2\varphi$, then we get,

$$e^{\frac{-V(sx+t\nabla\varphi(x))}{n}} \det(sI_n + t\nabla^2\varphi(x))^{1/n} \geq (se^{\frac{-V(x)}{n}} + te^{\frac{-V(\nabla\varphi(x))}{n}} det^{1/n}(\nabla^2\varphi)(x)).$$

□

Hence we have

$$\int_\Omega H^{1-n} d\mu \geq s\int_\Omega g^{1-n} d\mu + t\int_\Omega W^{1-n} d\mu.$$

□

Second Proof. Let $(\rho_t)_{t\in[0,1]}$, be the density map between $\rho_0 = g^{-n}d\mu$ and $\rho_1 = W^{-n}d\mu$. Define it as following, $\nabla\varphi_t\#\rho_0 = \rho_t$, where $\varphi_t(x) = s\frac{|x|^2}{2} + t\varphi(x)$. If we apply Monge-Ampère two times for $\nabla\varphi_t\#\rho_0 = \rho_t$ and $\nabla\varphi\#\rho_0 = \rho_1$,

$$\begin{aligned}\int_\Omega H(\nabla\varphi_t)\rho_0 &= \int_\Omega H(\nabla\varphi_t(x))g^{-n}(x)e^{-V(x)}dx \\ &= \int_\Omega H(\nabla\varphi_t(x))\rho_t(\nabla\varphi_t(x))e^{-V(\nabla\varphi_t(x))}\det(\nabla^2\varphi_t(x))dx,\end{aligned}$$

so we get (in the weak sense)

$$\begin{aligned}\rho_t(\nabla\varphi_t(x)) &= g^{-n}e^{-V(x)}e^{+V(\nabla\varphi_t(x))}\det(\nabla^2\varphi_t(x))^{-1} \\ &= g^{-n}e^{-V(x)}[e^{\frac{-V(\nabla\varphi_t(x))}{n}}\det(\nabla^2\varphi_t(x))^{1/n}]^{-n}.\end{aligned}$$

By using previous lemma and (5.3),

$$\begin{aligned}\rho_t(\nabla\varphi_t) &\leq g^{-n}e^{-V}[se^{\frac{-V}{n}} + te^{\frac{-V(\nabla\varphi)}{n}}\det(\nabla^2\varphi)]^{-n} \\ &= [sg + tge^{\frac{V-V(\nabla\varphi)}{n}}\det(\nabla^2\varphi)]^{-n} \\ &= [sg + tW(\nabla\varphi)]^{-n}.\end{aligned}$$

Like in the chapter 4, multipling both side by $sg + tW(\nabla\varphi)$,

$$\rho_t(\nabla\varphi_t)(sg + W(\nabla\varphi)) \leq (sg + W(\nabla\varphi))^{1-n} \leq H(\nabla\varphi_t)^{1-n},$$

and considering the fact $\rho_t(\nabla\varphi_t) > 0$ for ρ_0 almost every x, we have

$$[sg + W(\nabla\varphi)] \leq \frac{H(\nabla\varphi_t)^{1-n}}{\rho_t(\nabla\varphi_t)}\chi_{\{\rho_t(\nabla\varphi_t)>0\}}.$$

Take integral with respect to $\rho_0 d\mu$ for both sides. For the left side we obtain

$$\int_\Omega [sg + W(\nabla\varphi)]\rho_0 d\mu = s\int_\Omega g^{1-n}d\mu + t\int_\Omega W^{1-n}d\mu,$$

and for the other side,

$$\int_\Omega \left[\frac{H(\nabla\varphi_t)^{1-n}}{\rho_t(\nabla\varphi_t)}\chi_{\{\rho_t(\nabla\varphi_t)>0\}}\right]\rho_0 d\mu = \int_{\{\rho_t>0\}} H^{1-n}d\mu \leq \int_\Omega H^{1-n}d\mu$$

The proof is complete. □

5.2 Dynamical Formulations

As we show in the previous chapters, to convex inequality from Borell-Brascamp-Lieb inequality, we need to obtain dynamical formulation of the theorem. First we need to give a new definition:

$$Q_h^W(g)(x) := \inf_{y\in\Omega}\{g(y) + hW(\frac{x-y}{h})\},$$

and therefore with the constraint of the Theorem 5.1,

$$H(x) = sQ_{t/s}^W(g)(x/s),$$

if the H is the largest function satisfy the constraint.

Theorem 5.3. *Let $n \geq d > 1$. Let also $g, W : \Omega \to [0, +\infty]$. For any $h \geq 0$, Theorem 5.1 is equivalent to*

$$\int_\Omega Q_h^W(g)^{1-n}d\mu \geq \int_\Omega g^{1-n}d\mu + h\int_\Omega W^{1-n}d\mu \tag{5.6}$$

and equality holds for $g = W$ and convex.

Proof. Let $h = \frac{t}{s}$, for $s, t \in (0,1)$ and $t = 1 - s$. We compute

$$\begin{aligned}
\int_\Omega H^{1-n} d\mu &= \int_\Omega s^{1-n} [Q^W_{t/s}(g)(x/s)]^{1-n} e^{-V(x)} dx \\
&= s^{d-n+1} \int_\Omega [Q^W_{t/s}(g)(x)]^{1-n} e^{-V(sx)} dx \\
&= s^{d-n+1} \int_\Omega [Q^W_{t/s}(g)(x)]^{1-n} e^{-V(x)} s^{n-d} dx \\
&= s \int_\Omega [Q^W_{t/s}(g)(x)]^{1-n} e^{V(x)} dx,
\end{aligned}$$

Divide both sides with s, and put h, so we get

$$\int_\Omega Q^W_h(g)^{1-n} d\mu \geq \int_\Omega g^{1-n} d\mu + h \int_\Omega W^{1-n} d\mu.$$

□

With the same approach, we obtain the following inequality by the theorem A.6

Theorem 5.4 (Convex inequality for $d\mu$)**.** *Let $n \geq d > 1$, and the couple $(g, W) \in \mathcal{F}^n$. We have*

$$(n-1) \int_\Omega \frac{W^*(\nabla g)}{g^n} d\mu \geq \int_\Omega W^{1-n} d\mu \tag{5.7}$$

Which is apparently almost the same inequality we have in chapter 4 to derive the Sobolev inequalities. So, if we put $W(x) = C\frac{\|x\|^p}{p}$, with C satisfies that $\int W^{-n} d\mu = 1$ like in chapter 4, we obtain the following inequality,

$$\left(\int_\Omega \|\nabla f\|_p d\mu \right)^{1/p} \geq \left(\int_\Omega |f|^{p^*} d\mu \right)^{1/p^*} \tag{5.8}$$

where $p^* = \frac{np}{p-n}$.

Remark. *Note that our coefficients are different right now. We can also recover the case for if we take $d = n$, then $p^* = \frac{nd}{n-d}$, which is the more known version stated in [10].*

Appendix A

Inf-Convolution

A.1 Time Derivative for Half-Space

In this section we are going to explain which assumptions are needed to obtain linearisation for the case $x \in \mathbb{R}^n_+$. Let $a \geq n$ and let $g : \mathbb{R}^n_+ \to (0, +\infty)$, $W : \mathbb{R}^n_{+e} \to (0, +\infty)$ such that $\int_{\mathbb{R}^n_+} g^{-a}$, $\int_{\mathbb{R}^n_{+e}} W^{-a}$ are finite. Also, assume that g and W are assumed to be $\mathcal{C}^1$ in the interior of the respective domain of definition. Moreover, we assume,

$$\lim_{x \in \mathbb{R}^n_{+e}, |x| \to \infty} \frac{W(x)}{|x|} = +\infty \tag{A.1}$$

We will explain and give sufficient conditions to get desired inequality,

$$\left.\frac{d}{dh}\right|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u,x)dudx = -\int_{\partial\mathbb{R}^n_+} g^{1-a}dx + (a-1)\int_{\mathbb{R}^n_+} \frac{W^*(\nabla g)}{g^a}dz. \tag{A.2}$$

First, we need to have the following the lemma to obtain pointwise convergence,

Lemma A.1. *For all $x \in \operatorname{int}(\mathbb{R}^n_+)$,*

$$\left.\frac{\partial}{\partial h}\right|_{h=0} Q_h^W g(x) = -W^*(\nabla g(x)).$$

Proof of the lemma is detailedly explained in the Appendix part of the paper [13]. One side is coming directly from the definitions and on the other side, authors use convergence trick by using the domain. In the next definition, we collect all the assumptions made to prove the following lemmas.

Definition A.1 (The set $\mathcal{F}_+^a$ of admissible couples in $\mathbb{R}_+^n$)**.** *Let $n \geq 2$, $g : \mathbb{R}_+^n \to (0, +\infty)$ and $W : \mathbb{R}_{+e}^n \to (0, +\infty)$. We say couple belongs to $\mathcal{F}_+^a$ with $a \geq n$ if the following four conditions are satisfied for some γ:*

(C1) $\gamma > \max\{\frac{n}{a-1}, 1\}$*;*

(C2) there exists a constant $A > 0$ such that $W(x) \geq A|x|^\gamma$for all $x \in \mathbb{R}_{+e}^n$;

(C3) there exists a constant $B > 0$ such that $|\nabla g(x)| \leq B(|x|^{\gamma-1} + 1)$ for all $x \in \mathbb{R}_+^n$;

(C4) there exists a constant $C > 0$ such that $C(|x|^\gamma + 1) \leq g(x)$ for all $x \in \mathbb{R}_+^n$.

Lemma A.2. *Assume $(C1) \sim (C2)$. Then, we find constant $h_1 > 0$ such that , for all $h \in (0, h_1)$ and $x \in \mathbb{R}_{+he}^n$*

$$-C_1 h(|x|^\gamma + 1) \leq W_h^W g(x) - g(x) \leq C_2 h(|x|^{\gamma-1} + 1),$$

where $C1$ and $C2$ are independent from $h > 0$.

We need this lemma to obtain the following statement,

Lemma A.3. *Assume $(C1) \sim (C4)$. Then we find constants C_0 and $h_2 > 0$ such that for all $h \in (0, h_2)$ and $x \in \mathbb{R}_{+he}^n$*

$$\left| \frac{Q_h^W g(x)^{1-a} - g(x)^{1-a}}{h} \right| \leq \frac{C_0}{1 + |x|^{\gamma(a-1)}}.$$

We have the theorem for derivation as following;

Theorem A.4. *In above notation, assume that the couple (g, W) is in $\mathcal{F}_+^a$. Then we have,*

$$\frac{d}{dh}\bigg|_{h=0} \int_h^\infty \int_{\mathbb{R}^{n-1}} Q_h^W(g)^{1-a}(u, x) du dx = -\int_{\partial \mathbb{R}_+^n} g^{1-a} dx + (a-1) \int_{\mathbb{R}_+^n} \frac{W^*(\nabla g)}{g^a} dz.$$

A.2 Time Derivative for whole space

We only give results and conditions for the $\mathbb{R}^n$ case. Let $g, W : \mathbb{R}^n \to (0, +\infty)$ such that g is $\mathcal{C}^1$ and $\int g^{-n} = \int W^{-n} = 1$.

Definition A.2. *(The set $\mathcal{F}^a$ of admissible couples in $\mathbb{R}^n$). Let $n \geq 2$, $g, W : \mathbb{R}_+^n \to (0, +\infty)$. We say couple belongs to $\mathcal{F}^a$ with $a \geq n(a > 1$ if $n = 1)$ if the following four conditions are satisfied for some γ:*

(C1) $\gamma > \max\{\frac{n}{a-1}, 1\}$;

(C2) there exists a constant $A > 0$ *such that* $W(x) \geq A|x|^\gamma$ *for all* $x \in \mathbb{R}^n$;

(C3) there exists a constant $B > 0$ *such that* $|\nabla g(x)| \leq B(|x|^{\gamma-1} + 1)$ *for all* $x \in \mathbb{R}^n$;

(C4) there exists a constant $C > 0$ *such that* $C(|x|^\gamma + 1) \leq g(x)$ *for all* $x \in \mathbb{R}^n$.

Theorem A.5. *Assume that the couple* (g, W) *is in* $\mathcal{F}^a$. *Then the derivative at* $h = 0$ *of the map*

$$h \mapsto \int Q_h^W(g)^{1-a},$$

where $h \in (0, +\infty)$, *is equal to*

$$(1-a) \int \frac{W^*(\nabla g)}{g^a}.$$

A.3 Time derivation for weighted measure

Now we present what we have for the Ω case. Let $g, W : \Omega \to (0, +\infty)$ such that g is $\mathcal{C}^1$ and $\int_\Omega g^{-n} = \int_\Omega W^{-n} = 1$.

Theorem A.6. *If* $(g, W) \in \mathcal{F}^d$, *then the derivative of the following map*

$$h \mapsto \int_\Omega Q_h^W(g)^{1-n} d\mu,$$

at $h = 0$, *is*

$$(n-1) \int_\Omega \frac{W^*(\nabla g)}{g^{-n}} d\mu.$$

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