L-Shaped Drawings of Series-Parallel Graphs

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ABSTRACT
An orthogonal drawing of a planar graph G is a drawing of G such that each vertex is drawn as a point, each edge is drawn as a sequence of alternate horizontal and vertical line segment, and any two edges do not cross except at their common end. An L-shaped drawing of a graph G is an orthogonal drawing of G where each edge has exactly one bend. Clearly the maximum degree of a graph G is at most 4 if G has an L-shaped drawing. It is known that every planar graph of maximum degree 3 does not have L-shaped drawing. In this paper we study L-shaped drawings of series-parallel graphs in variable embedding settings. We show that every series-parallel graph of maximum degree 3 and every maximal outerplane graph of maximum degree 4 admit L-shaped drawings.

KEYWORDS: Graph Drawing, L-shaped drawing, Series Parallel Graphs.

1. INTRODUCTION

An orthogonal drawing of a planar graph *G* is a drawing of *G* such that each vertex is drawn as a point, each edge is drawn as a sequence of alternate horizontal and vertical line segment, and any two edges do not cross except at their common end. Orthogonal drawings of planar graphs have been extensively studied in literature [1, 10, 8, 7, 11]. A bend is a point where an edge changes its direction in a drawing. If an edge change its direction once then it turns as L-shaped. An L-shaped drawing of a graph *G* is an orthogonal drawing of *G* where each edge has exactly one bend. An edge with straight-line is not allowed in an L-shaped drawing. Figure 1(c) illustrates an example of an L-shaped drawing of the graph shown in Fig. 1(b).

If *G* has a vertex of degree five or more, then *G* has no orthogonal drawing. On the other hand, if *G* has no vertex of degree five or more, that is, the maximum degree $\triangle$ of *G* is at most four, then *G* has an orthogonal drawing, but may need bends. It is known that every tree with n vertices and with degree four admits a planar L-shaped point-set embedding on every diagonal point set with n point [3]. It is known that every planar graph does not have L-shaped drawing. Figure 1(a) shows an example of graph that does not admit an L-shaped drawing. Rahman *et. al* showed that not all series-parallel graphs admit no bend orthogonal drawings [9]. Orthogonal drawings of planar graphs have practical applications in circuit schematics, VLSI layouts, aesthetic layout of diagrams and computational geometry [10, 8, 7, 11]. The problem of obtaining an L-shaped drawing of a given graph has important practical applications in the fields of VLSI layout where connectors are L-shaped for some specified functions.

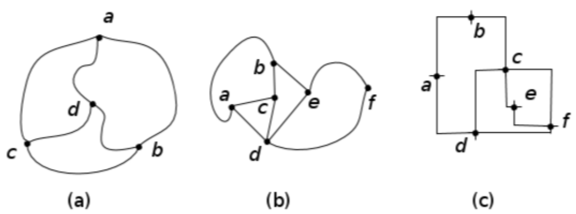

Fig. 1. (a) A graph *G* that has no orthogonal drawing and (c) an L-shaped drawing of the graph as shown in b).

2. MAIN RESULTS

In this paper we investigate L-shaped drawings of two non-trivial subclasses of planar graphs called series-parallel graphs and maximal outerplane graphs. We show that every series-parallel graphs of degree 3 whose source and sink degree 2 admits an L-shaped drawing. We also show that every maximal outerplane graph of maximum degree 4 admits an L-shaped drawing. Furthermore these drawings can be found in linear-time.

2.1 Series-Parallel Graphs

A path $u = u_1, \dots, u_k = v$ between vertices u and v in G is denoted by $P(u, v)$. A graph $G = (V, E)$ is called a series-parallel graph (with source s and sink t) if either G consists of a pair of vertices connected by a single edge or there exist two series-parallel graphs $G_i = (V_i, E_i)$, $i = 1, 2$, with source s_i and sink t_i such that $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and either $s = s_1$, $t_1 = s_2$ and $t = t_2$ or $s = s_1 = s_2$ and $t = t_1 = t_2$ [6]. A biconnected component of a series-parallel graph is also a series-parallel graph. By definition, a series-parallel graph G is a connected planar graph and G has exactly one source s and exactly one sink t.

We show that every series-parallel graph of maximum degree 3 admits an L-shaped drawing. Let G be a graph and x be a vertex of G. If we draw x on the plane we get four horizontal vertical half lines starting form x. We call these are ports of x. We denote four ports of a vertex x by α, β, γ and δ where α, β, γ and δ are the left, top, right and down half line starting from x, respectively. We call a port p of x is closed when an edge is already drawn using the port p in x and p is open if it is open for drawing an edge of x.

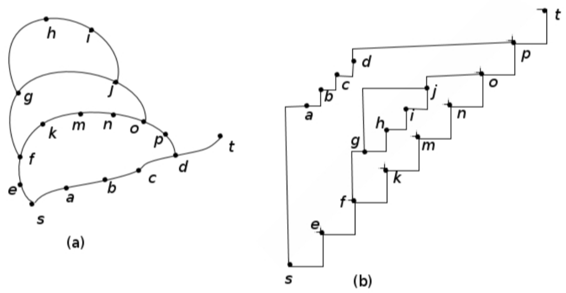

Fig. 2. (a) A series-parallel graph G (b) an L-shaped drawing of G.

We now have the following lemma.

Lemma 1. *Let l be a line with slope 1. Let p and q be the two points on l with open ports β and γ, respectively. Let* $P(u, v) = (u = u_1), u_2, \dots, (u_k = v)$ *be a path. Then P (u, v) admits an L-shaped drawing where u and v are drawn on the points p and q, respectively, and the vertices* $u_2, u_3, \dots, u_{k-1}$ *lie on a line with slope 1. Furthermore, the port β is open for* $u_i (2 \le i < k - 2)$ *and γ is open for* $u_j (2 < j \le k - 2)$.

Proves of lemma and theorem are omitted. Then we have the following theorem.

Theorem 1. *Let G be a connected series-parallel graph of maximum degree 3, and source and sink of G with at most degree 2. Then G admits an L-shaped drawing.*

2.2 Maximal Outerplanar Graph

A plane graph with all its vertices on the outer face is an outerplanar graph. An outerplanar graph is maximal outerplanar graph if no more edges can be inserted without violating its outerplanar property. Let $G = (V, E)$ be a simple maximal outerplanar graph with vertex set V and edge set E, where $|V| = n$. Throughout this paper we use deg(v) to denote the degree of the vertex v. The highest possible degree of a maximal outerplanar graph G can be 4 for an orthogonal drawing of G. The following property of a maximal outerplanar graph is known.

Lemma 2. [4] *Let G be a maximal outerplanar graph of* $n \ge 3$ *vertices. Then the following (i)- (iv) hold. (i) G has* $2n-3$ *edges; (ii) G has* $n-3$ *inner edges; (iii) G has* $n - 2$ *inner faces and each inner face is a triangle and, (iv) G has at least two vertices of degree 2.*

Although a maximal outerplanar graph is a subclass of series-parallel graph, we give an algorithm to compute an L-shaped drawing of a maximal outerplane graph of maximum degree 4. In this section we get an L-shaped drawing of a maximal outerplane graph whose embedding is fixed. Figure 3(a) illustrates a maximal outerplane graph G of maximum degree 4 and Fig. 3(b) shows an L-shaped drawing of G.

In the L-shaped drawing of a maximal outerplane graph G, every vertex is placed in different horizontal line and vertical line for n vertices and there is no gap between two consecutive horizontal lines as well as two consecutive vertical lines, So G admits an L-shaped drawing on a grid size of $O(n) \times O(n)$.

We now have the following theorem. Proof of theorem is omitted.

Theorem 2. *Let G be a maximal outerplane graph of n vertices of maximum degree 4. Then G admits an L-shaped drawing on a grid size of O(n) × O(n) and such drawing can be computed in linear time.*

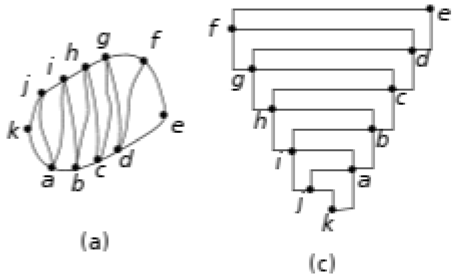

Fig.3. (a) A maximal outerplane graph *G* of maximum degree 4 and (b) an L-shaped drawing of *G*.

3. CONCLUSION

In this paper we have studied L-shaped drawings of series-parallel graphs. We have shown that a series-parallel graph of n vertices of maximum degree 3 has an L-shaped drawing and such a drawing can be found in *O(n)* time. We also have given an algorithm to compute an L-shaped drawing of a maximal outerplane graph of maximum degree 4 on a grid size of *O(n)* × *O(n)* and such drawing can be computed in linear-time.

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