Project 1

①

a) $R = \sqrt{-2\log(1-U)}$

$P(R \le r) = P(\sqrt{-2\log(1-U)} \le r) = P(U \le 1 - e^{-r/2})$

$= \int_0^{1-e^{-r/2}} \sqrt{-2\log(1-U)}\, dU$

$\theta = 2\pi V$

$P(\theta \le x) = P(2\pi V \le x) = P(V \le x/2\pi)$

$= \int_0^{x/2\pi} 2\pi V\, dV = \frac{x^2}{4\pi}$

b) $X = R\cos\theta$ and $Y = R\sin\theta$

$Y = f(\theta) \longrightarrow \theta = f^{-1}(Y) \longrightarrow \theta = 2\pi n - \sin^{-1}(Y/R) + \pi \quad n \in \mathbb{Z}$

$\frac{1}{\det(\frac{\partial f}{\partial x}(f^{-1}(Y))} P_X(f^{-1}(Y)) \longrightarrow P_X = P_{R\theta} = P(R)P(\theta) \longrightarrow$

$= P(-X)P(2\pi n + \pi)$

②

a) $D_{X,X'} = \frac{1}{8}(X-X')^T \sigma^{-1}(X-X') + \frac{1}{2}\ln\left(\frac{\sigma}{\sigma_1 \sigma_2}\right) \quad \sigma = \frac{\sigma_1+\sigma_2}{2}$

Since $X = X'$ and $\sigma_1 = \sigma_2 = \sigma^2 I_m$ here

$D_{X,X'} = \frac{1}{2}\ln(1) = 0$

2c) Since intra-class is 0, ratio will be 0, so as m increases the ratio will remain the same, implying that the accuracy is good.

b) In this case $\sigma_1 = \sigma_2 = \sigma^2 I_m$ but $X \neq X'$

$$D_{X,X'} = \frac{1}{8}(X-X')^T(\sigma^2 I_m)^{-1}(X-X') + \frac{1}{2}\ln(1)$$

③

a) $P(x_i,\lambda) = \frac{\lambda^x e^{-x}}{x!}$ show $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is MLE of λ

$$L(P(x_i,\lambda)) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-x}}{x!} \longrightarrow \log(L) = \log\left(\prod_{i=1}^{n} \frac{\lambda^x e^{-x}}{x!}\right) \longrightarrow$$

$$= \sum_{i=1}^{n} \ln\left(\frac{\lambda^x e^{-x}}{x!}\right) = \sum_{i=1}^{n}\left(\ln(e^{-\lambda}) + \ln(\lambda^x) - \ln(x!)\right) \longrightarrow$$

$$= -n\lambda + \sum \ln(\lambda^{x_i}) - \sum \ln(x!)$$

MLE: set $\frac{d}{d\lambda}(\log(L)) = 0 \longrightarrow$

$$-n + \frac{1}{\lambda}\sum x_i = 0 \longrightarrow \lambda = \frac{1}{n}\sum x_i$$

b) $P(\lambda|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$

$$\left(\frac{\beta}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)\underbrace{\left(\prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\right)}_{\text{Likelihood Poisson}} = \left(\frac{\beta}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\right)\left(\frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}\right)$$

$$= \frac{\beta^\alpha}{\prod x_i!\,\Gamma(\alpha)}\left(\lambda^{\alpha-1+\sum x_i}e^{-(\beta+n)\lambda}\right)$$

Project 1

Second part is Gamma distribution with

$\hat{\alpha} = \alpha + \sum_i x_i$ and $\hat{\beta} = \beta + n$

$P(\lambda | y) = \left(\lambda \middle| \alpha + \sum_{i=1}^{\hat{}} x_i, \beta + N\right) = \Gamma(\hat{\alpha}, \hat{\beta})$

c) MAP of a Gamma function where the distribution has the highest probability is

$\Gamma(\alpha, \beta) = \frac{\alpha - 1}{\beta} \longrightarrow MAP = \frac{\sum x_i + \alpha - 1}{N + \beta}$

d) $\hat{\eta} = e^{-2X}$

MLE $\longrightarrow X = \lambda(\eta) \rightarrow e^{-2\lambda} = \eta \rightarrow \lambda = -\frac{1}{2}\ln(\eta) = \lambda(\eta)$

$\lambda(\eta)$ into Poisson $P(X|\lambda) = \frac{\lambda^X e^{-X}}{X!}$ where $\lambda = \lambda(\eta) \longrightarrow$

$P(X|\lambda(\eta)) = \frac{\lambda(\eta)^X e^{-X}}{X!} = \frac{1}{X!}\left(-\frac{1}{2}\ln(\eta)\right)^X e^{\frac{1}{2}\ln(\eta)} \longrightarrow$

$\ln(P(X|\lambda(\eta))) = X\ln\left(-\frac{1}{2}\ln(\eta)\right) + \frac{1}{2}\ln(\eta) - \ln(X!) \longrightarrow$

$\frac{\partial \ln(P(X|\lambda(\hat{\eta})))}{\partial \hat{\eta}} = 0 \longrightarrow X\left(\frac{1}{-\frac{1}{2}\ln(\hat{\eta})}\right)\left(\frac{-1}{2\hat{\eta}}\right) + \frac{1}{2\hat{\eta}} = 0 \longrightarrow$

$\hat{\eta} = e^{-2X}$

e) Bias of $\hat{\eta} = e^{-2\lambda} - e^{\lambda(1/e^2 - 1)}$ using $e^t = \sum_{n=0}^{\infty} t^n/n!$

$$\text{Bias}(\hat{\eta}) = E[\hat{\eta}] - \eta \longrightarrow E[\hat{\eta}] = \sum \hat{\eta} P(X) = \sum e^{-2X} \frac{\lambda^X e^{-\lambda}}{X!}$$

$$= e^{-\lambda} \sum \frac{(\lambda e^{-2})^X}{X!} = e^{-\lambda} e^{\lambda e^{-2}} = e^{(e^{-2}-1)\lambda}$$

$$\text{Bias}(\hat{\eta}) = e^{(e^{-2}-1)\lambda} - e^{-2\lambda}$$

f) $(-1)^X$ is the only unbiased estimator of η

$$E[\hat{\eta}] = \sum \hat{\eta} P(X) = \sum (-1)^X \frac{\lambda^X e^{-\lambda}}{X!} = e^{-\lambda} \sum \frac{(-\lambda)^X}{X!}$$

$$= e^{-\lambda} e^{-\lambda} = e^{-2\lambda}$$

$$\text{Bias}(\hat{\eta}) = E[\hat{\eta}] - \eta = e^{-2\lambda} - e^{-2\lambda} = 0$$

Since Bias = 0 this is indeed unbiased

This Bias is bad because it will either be positive or negative based on X, it will never approach a true value for $\hat{\eta}$