Project 4

①

a) $P(Y_5 = S_3) = 1 - P(Y_5 = S_2) - P(Y_5 = S_1)$

$P(Y_5 = S_2) = 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 2 = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/4 = 1/32$

$1 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 2 = 1/2 \cdot 1/2 \cdot 1/4 \cdot 1/2 = 1/32$

$1 \rightarrow 1 \rightarrow 2 \rightarrow 2 \rightarrow 2 = 1/2 \cdot 1/4 \cdot 1/2 \cdot 1/2 = 1/32$

$1 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 2 = 1/4 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/32$

$P(Y_5 = S_1) = 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/16$

$P(Y_5 = S_3) = 1 - (1/32 + 1/32 + 1/32 + 1/32) - (1/16) = 13/16 = .8125$

b) We can solve this easier by checking $P(Y_5 = S_3 | X_5 = A)$

$B(A,3) = 0$, which means that at t_5 the probability that you observe an A if you are in S_3 is 0, so you cannot be in S_3 here, meaning

$P(Y_5 = S_3 | X_{1:5} = AABCABC) = 0$

c)

t	$\alpha_t(1)$	$\alpha_t(2)$	$\alpha_t(3)$
1	1/2	0	0
2	1/8	1/16	0
3	1/32	0	1/32
4	0	1/256	5/256
5	0	1/1024	0
6	0	0	1/4096
7	0	0	1/8192

$\alpha_{t+1}(i) = B(X_{t+1}, i) \sum_{j=1}^{N} T(i,j)\alpha_t(j)$

$\alpha_1(S_1) = B(A,1)\,\pi(1) = 1/2 \cdot 1 = 1/2$

$\alpha_1(S_2) = B(A,2)\,\pi(2) = 1/2 \cdot 0 = 0$

$\alpha_1(S_3) = B(A,3)\,\pi(3) = 0 \cdot 0 = 0$

$\alpha_2(S_1) = B(A,1)\left(T(1,1)\alpha_1(1) + T(1,2)\alpha_1(2) + T(1,3)\alpha_1(3)\right)$
$= (1/2)(1/2 \cdot 1/2 + 0 \cdot 0 + 0 \cdot 0) = 1/8$

$$\alpha_2(S_2) = B(A,2)\left(T(2,1)\alpha_1(1) + T(2,2)\alpha_1(2) + T(2,3)\alpha_1(3)\right)$$
$$= (1/2)(1/4 \cdot 1/2 + 1/2 \cdot 0 + 0 \cdot 0) = 1/16$$

$$\alpha_2(S_3) = B(A,3)\left(T(3,1)\alpha_1(1) + T(3,2)\alpha_1(2) + T(3,3)\alpha_1(3)\right)$$
$$= (0)(1/4 \cdot 1/2 + 1/2 \cdot 0 + 1 \cdot 0) = 0$$

Repeating this for each time step gives the table above

d) AABCABC means the states are in the order of
$S_1 \rightarrow S_1 \rightarrow S_1 \rightarrow S_2 \rightarrow S_2 \rightarrow S_3 \rightarrow S_3$

The only way to get AABCABC is by the sequence above $S_1 S_1 S_1 S_2 S_2 S_3 S_3$, so the posterior probability of AABCABC is 1

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(2)

a) $P(Z_1, Z_2, O_1, O_2) = \pi(Z_1)\prod_{i=1}^{2} B(O_i, Z_i)\, T(Z_1, Z_2)$

$\log\left((\theta_{Z_1 = z_1})(\theta_{Z_2 = z_2 | Z_1 = z_1})(\theta_{O_1 = o_1 | Z_1 = z_1} \cdot \theta_{O_2 = o_2 | Z_2 = z_2})\right)$

b) $\prod_{j=1}^{m} P(Z_1^j, Z_2^j, O_1^j, O_2^j) = \prod_{j=1}^{m}\left(\pi(Z_1^j)\prod_{i=1}^{2} B(O_i^j, Z_i^j)\, T(Z_1^j, Z_2^j)\right)$

$= \log\left(\prod_{j=1}^{m}\left((\theta_{Z_1^j = z_1})(\theta_{Z_2^j = z_2 | Z_1^j = z_1})(\theta_{O_1^j = o_1 | Z_1^j = z_1} \cdot \theta_{O_2^j = o_2 | Z_2^j = z_2})\right)\right)$

c) $\hat{\theta}_{Z_2 = z_2 | Z_1 = z_1} = T(Z_1, Z_2)$

$$T(i,j) = \frac{\sum_{k=1}^{K}\sum_{t=1}^{T} 1\{x_t^{(k)} = i,\ x_{t-1}^{(k)} = j\}}{\sum_{k=1}^{K}\sum_{t=1}^{T} 1\{x_{t-1}^{(k)} = j\}}$$

$$T(Z_1, Z_2) = \frac{\sum_{k=1}^{K}\sum_{t=1}^{T} 1\{x_t^{(k)} = Z_1,\ x_{t-1}^{(k)} = Z_2\}}{\sum_{k=1}^{K}\sum_{t=1}^{T} 1\{x_{t-1}^{(k)} = Z_2\}}$$

d) $\hat{\theta}_{O_i = o_i | Z_i \; z_i} = B(O_i, Z_i)$

$$B(i,j) = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T} 1\{z_t^{(k)} = i, x_t^{(k)} = j\}}{\sum_{k=1}^{K} \sum_{t=1}^{T} 1\{x_t^{(k)} = j\}}$$

$$B(O_i; Z_i) = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T} 1\{Z_t^{(k)} = O_i, x_t^{(k)} = Z_i\}}{\sum_{k=1}^{K} \sum_{t=1}^{T} 1\{x_t^{(k)} = Z_i\}}$$

Repeat for $i=1$ and $i=2$

e) $Q(Z_1 = z_1, Z_2 = z_2 | O_1^j, O_2^j)$

$$q^*(Z_{0:t}) = P_\theta(Z_{0:t} | O_{0:t}) = P_\theta(Z_1, Z_2 | O_1^j, O_2^j)$$

$$= \frac{P_t(Z_1, Z_2, O_1, O_2)}{P(O_1, O_2)}$$

$$= \frac{(\theta_{Z_1 = z_1})(\theta_{O_1 = o_1^j | Z_1 = z_1})(\theta_{Z_2 = z_2 | Z_1 = z_1})(\theta_{O_2 = o_2^j | Z_1 = z_2})}{\sum_{z_1, z_2} (\theta_{Z_2 = z_1})(\theta_{O_1 = o_1^j | Z_1 = z_1})(\theta_{Z_2 = z_2 | Z_1 = z_1})(\theta_{O_2 = o_2^j | Z_2 = z_2})}$$

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f) $\theta_{z_2=z_2|z_1=z_1} = P(Z_2=z_2 | Z_1=z_1)$

$$= \frac{\sum_j P(Z_2=z_2, Z_1=z_1 | o_1^j, o_2^j)}{\sum_j P(Z_1=z_1 | o_1^j, o_2^j)}$$

$$= \frac{\sum_j Q(Z_1=z_1, Z_2=z_2 | o_1^j, o_2^j)}{\sum_j \sum_{z_1 z_2} Q(Z_1=z_1, Z_2=z_2 | o_1^j, o_2^j)}$$

g) $\theta_{o_i=o_i | Z_{z_i}=z_i} = B(i,j) = \frac{\sum_{j=1}^{K} \sum_{t=1}^{T} \mathbb{1}(O_t^{(k)}=1)\, \gamma_t^{(k)}(j)}{\sum_{j=1}^{K} \sum_{t=1}^{T} \gamma_t^{(k)}(j)}$

$$\gamma_t^{(k)}(z) = \sum_{z_i} P(z_1, z_2 | o_1, o_2) = \sum_{z_i} Q$$

$$\theta_{o_i=o_i|z_i=z_i} = P(o_i=o_i | Z_i=z_i)$$

$$= \frac{\sum_j P(z_i=z_i, O_i=o_i | o_1^j, o_2^j)}{\sum_j P(z_i=z_1 | o_1^j, o_2^j)}$$

$$= \frac{\sum_j \sum_{z_k} \mathbb{1}(o_i=o_1^j, o_2^j)\, Q(Z_i=z_i, Z_k=z_k | o_1^j, o_2^j)}{\sum_{z_k} Q(Z_i=z_i, Z_k=z_k | o_1^j, o_2^j)}$$