Project 3

① $X_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X_t + W_t,\ W_t \sim \mathcal{N}(0, \mathrm{diag}(1/10^2, 1))$

$Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} X_t + V_t,\ V_t \sim \mathcal{N}(0, 10^2)$

a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ Bu_t = 0,\ W = \begin{bmatrix} 1/10^2 & 0 \\ 0 & 1 \end{bmatrix},\ X_t \sim \mathcal{N}(u_{t|t}, \Sigma_{t|t})$

$H = \begin{bmatrix} 1 & 0 \end{bmatrix},\ V = 10^2$

Predict $u_{t+1|t} = Au_{t|t} + Bu_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u_{t|t}$

$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/10^2 & 0 \\ 0 & 1 \end{bmatrix}$

Kalman Gain $K_{t+1|t} = \Sigma_{t+1|t} H^T (H\Sigma_{t+1|t} H^T + V)^{-1}$

$= \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/10^2 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Big(\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot$

$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/10^2 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 10^2 \Big)^{-1}$

Update $u_{t+1|t+1} = u_{t+1|t} + K_{t+1|t}(Z_{t+1} - Hu_{t+1|t})$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u_{t|t} + K_{t+1|t}\left(\begin{bmatrix} 1 & 0 \end{bmatrix} X_{t+1} + V_{t+1} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u_{t|t}\right)$

$\Sigma_{t+1|t+1} = (I - K_{t+1|t} H)\Sigma_{t+1|t}$

$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - K_{t+1|t} \begin{bmatrix} 1 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/10^2 & 0 \\ 0 & 1 \end{bmatrix}\right)$

b) $P_h(Z_{t+1}|x): Z_{t+1} \sim \mathcal{N}([1\ 0]\, x_t,\ 10^2)$

$P_a(x|u_{t+1|t}, u_t): x \sim \mathcal{N}\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} u_{t+1|t},\ \begin{bmatrix}1/10^2 & 0\\ 0 & 1\end{bmatrix}\right)$

$$\frac{1}{\sqrt{2\pi}^d\sqrt{|\Sigma|}}\exp\left(-\frac{1}{2}(x-u)^T\Sigma^{-1}(x-u)\right)$$

$$P_a \rightarrow \frac{10}{\sqrt{2\pi}^d}\exp\left(-\frac{1}{2}(x-Au_{t+1|t})^T W^{-1}(x-Au_{t+1|t})\right)$$

$$P_h \rightarrow \frac{1/10}{\sqrt{2\pi}^d}\exp\left(-\frac{1}{2}(Z_{t+1}-Hx)^T V^{-1}(Z_{t+1}-Hx)\right)$$

$$\pi^k(x) = \frac{P_h(Z_{t+1}|x)\, P_a(x|u_{t+1|t}^{(k)}, u_t)}{\int P_h(Z_{t+1}|x)\, P_a(x|u_{t+1|t}^{(k)}, u_t)\,dx} \longrightarrow$$

$$= P\left(x;\ u_{t+1|t} + K_{t+1}(Z_{t+1} - Hu_{t+1|t}),\ (I - K_{t+1|t}H)\Sigma_{t+1|t}\right)$$

This is found via the Kalman Filter Update derivation, using this with Pa and Ph we can solve for $\pi^k(x)$

$$\pi^k(x) = \frac{1}{\eta_{t+1}}\exp\left(-\frac{1}{2}(Z_{t+1}-Hx)^T V^{-1}(Z_{t+1}-Hx)\right)\exp\left(-\frac{1}{2}(x-Au_{t+1|t})^T W^{-1}\cdot (x-Au_{t+1|t})\right)$$

$$= \frac{1}{\eta_{t+1}}\exp\left(-\frac{1}{2}\left(x-Au_{t+1|t}+k(Z_{t+1}-Hu_{t+1|t})\right)^T\cdot \left((I-kH)W\right)^{-1}\left(x-Au_{t+1|t}+k(Z_{t+1}-Hu_{t+1|t})\right)\right)$$

$$k = WH^T(HWH^T+V)^{-1} = \begin{bmatrix}1/10^2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\left([1\ 0]\begin{bmatrix}1/10^2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}+100\right)^{-1} = \begin{bmatrix}1/10^4\\ 0\end{bmatrix}$$

$$\Sigma = (I-kH)W = \left(I - \begin{bmatrix}1/10^4\\ 0\end{bmatrix}[1\ 0]\right)\begin{bmatrix}1/10^2 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}10^2/10^4 & 0\\ 0 & 1\end{bmatrix}$$

$$\pi^k(x) \sim G\left(x,\ Au_{t+1|t} + \begin{bmatrix}1/10^4\\ 0\end{bmatrix}(Z_{t+1}-Hu_{t+1|t}),\ \begin{bmatrix}10^2/10^4 & 0\\ 0 & 1\end{bmatrix}\right)$$

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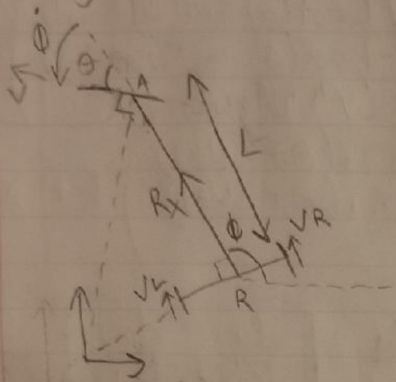
② 

a) $_R\dot{x} = v\cos\theta(t)$ and $_R\dot{y} = v\sin\theta(t)$ and $_R\dot{\theta} = \frac{v\sin\theta(t)}{L}$

These are in robot frame, in world frame -

$\dot{x}(t) = v\cos(\theta(t))\cos(\phi(t))$
$\dot{y}(t) = v\cos(\theta(t))\sin(\phi(t))$
$\dot{\phi}(t) = (V/L)\sin(\theta(t))$

b) $\phi(t) = \int \frac{V}{L}\sin(\theta)dt = A(t - t_0) \qquad A = \frac{V}{L}\sin(\theta)$

$\dot{y} = v\cos(\theta)\sin(A(t-t_0))$
$y = (-v\cos(\theta)\cos(A(t-t_0)))/A$

$x = (v\cos(\theta)\sin(A(t-t_0)))/A$

$$\left.\begin{aligned} \phi &= (V/L)\sin(\theta)(t-t_0) \\ x &= L\cot(\theta)\sin((V/L)\sin(\theta)(t-t_0)) \\ y &= -L\cot(\theta)\cos((V/L)\sin(\theta)(t-t_0)) \end{aligned}\right\} \text{Motion Model}$$

c)

$$X_t = \begin{pmatrix} x_t \\ y_t \\ \phi_t \end{pmatrix} \longrightarrow X_{t+1} = \begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \phi_{t+1} \end{pmatrix} = \begin{pmatrix} L\cot(\theta)\sin(\dot{\phi}(t-t_0)) \\ -L\cot(\theta)\cos(\dot{\phi}(t-t_0)) \\ \dot{\phi}(t-t_0) \end{pmatrix} \tau + \begin{pmatrix} x_t \\ y_t \\ \phi_t \end{pmatrix}$$

$$X_{t+1} = \begin{pmatrix} L\cot(\theta+w_\theta)\sin(((v+w_v)/L)\sin(\theta+w_\theta)) \\ -L\cot(\theta+w_\theta)\cos(((v+w_v)/L)\sin(\theta+w_\theta)) \\ ((v+w_v)/L)\sin(\theta+w_\theta) \end{pmatrix} \tau + X_t$$

$$A = \frac{da}{dx} = \left[\frac{da}{dx}, \frac{da}{dy}, \frac{da}{d\phi}\right], \quad Q = \frac{da}{dw} = \left[\frac{da}{dw_\theta}, \frac{da}{dw_v}\right]$$

$$w_t \sim \mathcal{N}(0, W) \longrightarrow w_\theta \sim \mathcal{N}(0, \sigma_\theta^2) \longrightarrow W = \sigma_\theta^2$$
$$v_t \sim \mathcal{N}(0, V) \longrightarrow w_v \sim \mathcal{N}(0, \sigma_v^2) \longrightarrow V = \sigma_v^2$$

Then plug into the EKF equations below -

Predict
$$u_{t+1|t} = a(u_{t|t}, u_t, 0)$$
$$\Sigma_{t+1|t} = A_t \Sigma_{t|t} A_t^T + Q_t W Q_t^T$$

Update
$$u_{t+1|t+1} = u_{t+1|t} + K_{t+1|t}(z_{t+1} - h(u_{t+1|t}, 0))$$
$$\Sigma_{t+1|t} = (I - K_{t+1|t} H_{t+1})\Sigma_{t+1|t}$$

Kalman Gain
$$K_{t+1|t} = \Sigma_{t+1|t} H_{t+1}^T (H_{t+1}\Sigma_{t+1|t} H_{t+1}^T + R_{t+1} V R_{t+1}^T)^{-1}$$

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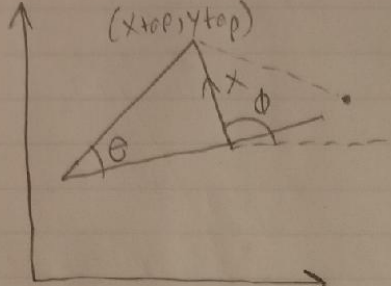
d)



$x_{top} = \cos(\phi)L + X$
$y_{top} = \sin(\phi)L + y$

$d = ||(X_{front} - y)$

Measurement Model:

$$d = \sqrt{(x_{top} - y_x)^2 + (y_{top} - y_y)^2}$$
$$= \sqrt{((\cos(\phi)L + X) - y_x)^2 + ((\sin(\phi)L + y) - y_y)^2}$$

$$H = \frac{dd}{dx} = \left[\frac{dd}{dx}, \frac{dd}{dy}, \frac{dd}{d\phi}\right]$$

$$J(x, y, \phi) = \begin{bmatrix} \partial x/\partial x & \partial x/\partial y & \partial x/\partial \phi \\ \partial y/\partial x & \partial y/\partial y & \partial y/\partial \phi \\ \partial z/\partial x & \partial z/\partial y & \partial z/\partial \phi \end{bmatrix}$$