P6

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November 25, 2025

P6.1 (A model).
Attachment probability:

$$\Pi(k_i) = \frac{1}{t + m_0} \approx \frac{1}{t}.$$

Rate equation for the degree $k_i(t)$ of node i (born at time t_i):

$$\frac{dk_i}{dt} = m\,\Pi(k_i) = \frac{m}{t}.$$

Integrate from t_i (birth time) to t:

$$k_i(t) - k_i(t_i) = m \int_{t_i}^{t} \frac{1}{s}\, ds = m \ln \frac{t}{t_i}.$$

Typically $k_i(t_i) = m$ (node born with m initial links), so

$$k_i(t) = m \ln \frac{t}{t_i} + m.$$

Invert the relation to get t_i as function of k:

$$\ln \frac{t}{t_i} = \frac{k - m}{m} \quad \Rightarrow \quad t_i = t\, \exp\Big(- \frac{k - m}{m} \Big).$$

Nodes are born uniformly in time. The fraction of nodes with degree $\leq k$ equals the fraction with birth-time $\geq t_i$, hence

$$P(k_i \leq k) = \Pr(t_i \geq t\, e^{-(k-m)/m}) = 1 - \frac{t\, e^{-(k-m)/m}}{t} = 1 - e^{-(k-m)/m}.$$

Differentiate to obtain the degree probability density (or discrete version):

$$P(k) = \frac{d}{dk} P(k_i \leq k) = \frac{1}{m}\, e^{-(k-m)/m},$$

or (shifting normalization) for $k \geq m$:

$$P(k) \propto e^{-k/m}.$$

Thus the degree distribution is exponential with characteristic degree scale m.
P6.2 (B model).
Given (mean-field) rate equation from the handout:

$$\frac{dk_i}{dt} = \frac{N - 1}{N} \cdot \frac{k_i}{2t} + \frac{1}{N}.$$

(Explanation: factor $(N-1)/N$ appears because when we add an edge one endpoint is chosen among N vertices, and the preferential probability gives $k_i / \sum_j k_j$ with $\sum_j k_j \approx 2t$; the term $1/N$ accounts for the other endpoint chosen uniformly or equivalent source of increments.)

Rewrite in the standard linear form:

$$\frac{dk_i}{dt} - \frac{N-1}{2N} \cdot \frac{k_i}{t} = \frac{1}{N}.$$

Let $c = \frac{N-1}{2N}$. The integrating factor is t^{-c}:

$$\frac{d}{dt}\big(k_i(t)\, t^{-c}\big) \;=\; \frac{1}{N}\, t^{-c}.$$

Integrate from birth time t_i to t:

$$k_i(t)\, t^{-c} - k_i(t_i)\, t_i^{-c} = \frac{1}{N}\int_{t_i}^{t} s^{-c}\, ds = \frac{1}{N}\,\frac{t^{1-c} - t_i^{1-c}}{1-c}.$$

Thus

$$k_i(t) = k_i(t_i)\Big(\frac{t}{t_i}\Big)^{c} + \frac{1}{N(1-c)}\Big(t - t_i^{1-c} t^{c}\Big).$$

For long times $t \gg t_i$ the first term (and the mixed term) are subleading and the dominant asymptotic growth is linear in t:

$$k_i(t) \sim \frac{1}{N(1-c)}\, t.$$

Since $1 - c = 1 - \frac{N-1}{2N} = \frac{N+1}{2N}$, we obtain

$$\frac{1}{N(1-c)} = \frac{1}{N} \cdot \frac{2N}{N+1} = \frac{2}{N+1} \approx \frac{2}{N} \text{ for large } N.$$

So asymptotically

$$k_i(t) \approx \frac{2}{N+1}\, t \quad (\text{or } \approx \frac{2}{N}\, t \text{ for large } N),$$

which is consistent with the formula given in the handout (their simplified approximation $\approx 2t/N$).

P6.5 (derivation sketch).

Let $p_k(t)$ be the fraction of nodes with degree k at time t. When a new edge is added, the probability that a specific node of degree k receives an endpoint (and thus its degree increases to $k+1$) is approximately

$$\Pi_k = \frac{k}{\sum_j j\, n_j} \approx \frac{k}{2t},$$

since $\sum_j j n_j = 2t$ is the total degree after t added edges. Assume that per time step exactly one edge (two endpoints) is added; the rate equation (discrete-time) for $n_k(t) = N p_k(t)$ is

$$n_k(t+1) - n_k(t) \approx \frac{(k-1)}{2t} n_{k-1}(t) - \frac{k}{2t} n_k(t) + (\text{source term}).$$

Dividing by N and passing to large t, look for a stationary solution p_k:

$$0 \approx \frac{(k-1)}{2c} p_{k-1} - \frac{k}{2c} p_k + s_k,$$

where c is related to the growth constant and s_k summarizes the uniform contributions (endpoints chosen uniformly or from other processes). Neglecting small source corrections and rearranging yields a recurrence of the form

$$p_k \propto \frac{k-1}{k}\, p_{k-1}.$$

Iterating gives a solution that decays at least exponentially with k. A more careful balance including the source term leads to a geometric (exponential) stationary distribution:

$$p_k \simeq (1-q)\, q^{\,k-m} \quad (k \geq m),$$

for some $0 < q < 1$ determined by global conservation (normalization and mean degree). Thus model B yields an exponential (geometric) degree distribution in the stationary regime.