SNaRS - P5

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P5.1

Assuming the Poisson approximation with parameter $\lambda = \langle k \rangle = p(N-1)$.

1. Mean of the Poisson Distribution

$$\mathbb{E}[K] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}.$$

Let $j = k-1$:

$$\mathbb{E}[K] = e^{-\lambda}\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

2. Second Moment

Using the identity $k^2 = k(k-1) + k$, we get:

$$\mathbb{E}[K^2] = \mathbb{E}[K(K-1)] + \mathbb{E}[K].$$

Compute the first term:

$$\mathbb{E}[K(K-1)] = \sum_{k=0}^{\infty} k(k-1) \cdot \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=2}^{\infty} k(k-1) \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

Using:

$$k(k-1) \cdot \frac{\lambda^k}{k!} = \lambda^k \cdot \frac{k(k-1)}{k!} = \lambda^k \cdot \frac{1}{(k-2)!}$$

The sum becomes:

$$\mathbb{E}[K(K-1)] = e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!}$$

Let $j = k-2$:

$$\mathbb{E}[K(K-1)] = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+2}}{j!} = \lambda^2 e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda^2 e^{-\lambda} \cdot e^{\lambda} = \lambda^2$$

Thus:

$$\mathbb{E}[K^2] = \lambda^2 + \lambda.$$

3. Variance

$$\mathrm{Var}(K) = \mathbb{E}[K^2] - (\mathbb{E}[K])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

Variance = Mean = λ (Poisson property).

P5.5

In ER model of $G(N, E)$:

- The number of vertices N is fixed.
- The number of edges is exactly E.
- All graphs with exactly E edges are equally likely.

Counting the Number of Possible Graphs

There are $M = \binom{N}{2}$ possible edges. Selecting exactly E edges gives:

$$\Omega = \binom{M}{E}.$$

The partition function is simply the number of allowed combinations:

$$Z = \binom{M}{E}.$$

Degree Distribution in $G(N, E)$

The degree K_i of vertex i is distributed as a hypergeometric random variable:

- Total population (all possible edges): $M = \binom{N}{2}$
- Success states (edges connected to i): $N - 1$
- Total draws (the fixed number of edges in G): E

$$K_i \sim \mathrm{Hypergeometric}(M, N-1, E)$$

Mean degree:

$$\mathbb{E}[K_i] = \frac{(N-1)E}{M} = \frac{2E}{N}.$$

Variance:

$$\mathrm{Var}(K_i) = \frac{(N-1)E}{M}\left(1 - \frac{E}{M}\right) \cdot \frac{M-(N-1)}{M-1}.$$

Probability of a Graph

Since all graphs with E edges are have the same probability:

$$P(G) = \frac{1}{\binom{\binom{N}{2}}{E}}.$$

P5.7

Consider the Hamiltonian:

$$H(G) = \theta E(G).$$

When the number of edges is constrained to be exactly E_0, then:

$$E(G) = E_0 \Rightarrow H(G) = \theta E_0 = \text{constant}.$$

Thus, all allowed graphs (with E_0 edges) have identical energy.

Partition Function

The partition function sums over all graphs with exactly E_0 edges:

$$Z = \sum_{G:E(G)=E_0} e^{-\theta E(G)} = e^{-\theta E_0} \cdot \#\{G : E(G) = E_0\}.$$

Since the number of such graphs is $\binom{\binom{N}{2}}{E_0}$, we have:

$$Z = e^{-\theta E_0} \binom{\binom{N}{2}}{E_0}.$$

Probability of a Specific Graph

$$P(G) = \frac{e^{-\theta E_0}}{Z} = \frac{1}{\binom{\binom{N}{2}}{E_0}}.$$

All graphs with exactly E_0 edges are equally probable, regardless of θ.