Snars 7

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P7.1

The master equation:

$$N_k(t+1) - N_k(t) = \frac{k-1}{2t} N_{k-1}(t) - \frac{k}{2t} N_k(t) + \delta_{k,m}, \tag{1}$$

where m is the minimum degree and number of links each new node brings.

The stationary distribution is defined as:

$$P(k) = \lim_{t\to\infty} \frac{N_k(t)}{t}. \tag{2}$$

For large t, we can approximate $N_k(t+1) \approx N_k(t)$ and $\frac{N_k(t)}{t} \approx P(k)$ to obtain:

$$0 = \frac{k-1}{2} P(k-1) - \frac{k}{2} P(k) + \delta_{k,m}. \tag{3}$$

For $k > m$, this yields a recurrence:

$$P(k) = \frac{k-1}{k+2} P(k-1). \tag{4}$$

To determine $P(m)$, use the $k = m$ case:

$$P(m) = \frac{2}{m+2}. \tag{5}$$

Repeated application of the recurrence gives the closed form

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}. \tag{6}$$

As $k \to \infty$, $P(k) \sim k^{-3}$, recovering the BA exponent $\gamma = 3$.

P7.4

Definition of $Q(k)$

The distribution $Q(k)$ is defined as the probability that a node at the end of a randomly chosen edge has degree k. It therefore differs from $P(k)$, which is the degree distribution of a randomly selected vertex.

Counting Argument

The number of vertices with degree k is approximately $NP(k)$. Each vertex of degree k contributes exactly k edge ends. Hence, the total number of edge ends connected to vertices of degree k equals

$$kNP(k).$$

The total number of all edge ends in the network is equal to the sum of all degrees:

$$N\langle k\rangle.$$

Derivation of $Q(k)$

The probability that a randomly chosen edge end is attached to a vertex of degree k is given by

$$Q(k) = \frac{kNP(k)}{N\langle k\rangle}.$$

Canceling the factor N, we obtain

$$Q(k) = \frac{k}{\langle k\rangle}P(k).$$

Interpretation

Vertices with a larger degree are more likely to be reached by following a randomly selected edge. This explains the additional factor k in the relation between $Q(k)$ and $P(k)$.

P7.5

When traversing a network by following randomly chosen edges, the degree distribution of the reached vertices is not given by $P(k)$, but by the excess degree distribution

$$Q(k) = \frac{kP(k)}{\langle k\rangle},$$

since vertices with higher degree are more likely to be reached.

If a vertex of degree k is reached by following an edge, one of its edges has already been used. Therefore, the number of *new* outgoing edges available for further spreading is $k-1$. The average branching factor of the network is thus given by

$$B = \sum_k (k-1)Q(k).$$

The percolation threshold corresponds to the critical point at which the branching process becomes self-sustaining, i.e.

$$B = 1.$$

This condition can be rewritten as

$$\sum_k (k-1)Q(k) = 1 \quad \Longleftrightarrow \quad \sum_k kQ(k) = 2.$$

Using the definition of $Q(k)$, we obtain

$$\sum_k kQ(k) = \sum_k k\frac{kP(k)}{\langle k \rangle} = \frac{1}{\langle k \rangle}\sum_k k^2 P(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}.$$

Thus, the condition for the existence of a giant connected component (or, equivalently, for percolation) is

$$\frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2.$$

This criterion shows that the emergence of large-scale connectivity is governed by the ratio of the second to the first moment of the degree distribution.

P7.6

Degree Distribution in ER Graphs

For Erdős–Rényi graphs, the degree distribution follows a Poisson distribution with mean

$$\lambda = \langle k \rangle.$$

The first two moments are given by

$$\langle k \rangle = \lambda, \qquad \langle k^2 \rangle = \lambda^2 + \lambda.$$

Applying the Percolation Condition

Substituting these moments into the percolation criterion derived in Task 7.5 yields

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\lambda^2 + \lambda}{\lambda} = \lambda + 1.$$

Critical Threshold

At the percolation threshold the equality holds:

$$\lambda + 1 = 2,$$

which implies

$$\lambda = 1.$$

Hence, the critical average degree is

$$\langle k \rangle_c = 1.$$

Since for ER graphs $\langle k \rangle \approx pN$, the critical connection probability is

$$p_c \approx \frac{1}{N}.$$

P7.8

Let $G_0(x) = \sum_{k=0}^{\infty} P(k)x^k$.

(a) $G_0(1) = 1$

This follows directly from normalization $\sum_k P(k) = 1$ as $x = 1$.

(b) $G_0'(1) = \langle k \rangle$

Differentiating, we obtain:

$$G_0'(x) = \sum_k kP(k)x^{k-1}, \tag{7}$$

then evaluating at $x = 1$:

$$G_0'(1) = \sum_k kP(k), \tag{8}$$

$P(k)$ can be seen as a fraction of all nodes that has a degree k meaning, that the sum corresponds to the average degree:

$$G_0'(1) = \langle k \rangle, \tag{9}$$

(c) $\frac{d}{dx}[xG_0'(x)]\big|_{x=1} = \langle k^2 \rangle$

Calculate:

$$\frac{d}{dx}[xG_0'(x)] = G_0'(x) + xG_0''(x),$$

$$G_0''(x) = \sum_k k(k-1)P(k)x^{k-2}$$

$$\frac{d}{dx}[xG_0'(x)]_{x=1} = \sum_k kP(k) + \sum_k k(k-1)P(k) = \sum_k kP(k)[1+(k-1)] = \sum_k k^2 P(k) = \langle k^2 \rangle.$$

(d) $G_1(x) = \frac{G_0'(x)}{G_0'(1)}$

Let's recall:

$$F(k) = \frac{k+1}{\langle k \rangle}P(k+1). \tag{10}$$

Then:

$$G_1(x) = \sum_{k=0}^{\infty} F(k)x^k = \frac{1}{\langle k \rangle}\sum_{j=1}^{\infty} jP(j)x^{j-1} = \frac{G_0'(x)}{G_0'(1)}. \tag{11}$$

(e) $G_1'(1) = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1$

Differentiating $G_1(x)$:

$$G_1'(x) = \frac{G_0''(x)}{G_0'(1)} = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}. \tag{12}$$

For x=1:

$$G_1'(1) = \frac{G_0''(1)}{G_0'(1)} = \frac{\sum_k k(k-1)P(k)}{\langle k \rangle} = \frac{\sum_k k^2 P(k) - \sum_k kP(k)}{\langle k \rangle} = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}. \tag{13}$$

P7.9

For a Poisson degree distribution:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}, \tag{14}$$

calculate:

$$G_0(x) = e^{-\langle k \rangle} \sum_{k=0}^{\infty} \frac{(\langle k \rangle x)^k}{k!} = e^{-\langle k \rangle} e^{\langle k \rangle x} = e^{\langle k \rangle (x-1)}. \tag{15}$$

P7.10

Using $G_1(x) = G_0'(x)/G_0'(1)$ and $G_0(x) = e^{\langle k \rangle (x-1)}$, we have:

$$G_1(x) = e^{\langle k \rangle (x-1)} = G_0(x). \tag{16}$$

P7.11

$h_1(s)$ is the probability that starting from the end of a randomly selected edge, we can explore s vertices in finite number of steps.

Arriving at a vertex via an edge yields the excess-degree distribution $F(k)$, meaning the vertex has k additional edges. If these k branches collectively contribute $s - 1$ nodes (probability of which is described as $h_k(s-1)$), then, to get a distribution over any randomly chosen edge, you need to sum over all values $k = 0...s - 1$, yielding:

$$h_1(s) = \sum_{k=0}^{s-1} F(k)h_k(s - 1). \tag{17}$$

P7.12

(a) $H_1(x) = x \sum_{k=0} F(k)H_k(x)$

$$H_1(x) = \sum_{s=0}^{\infty} h_1(s)x^s = \sum_{s=0}^{\infty}[\sum_{k=0}^{s-1} F(k)h_k(s - 1)]x^s = x \sum_{s=1}^{\infty}\sum_{k=0}^{s-1} F(k)h_k(s - 1)x^{s-1} =$$

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$$= x \sum_k F(k) \sum_{s=1}^{\infty} h_k(s-1)x^{s-1} = x \sum_k F(k) \sum_{s=0}^{\infty} h_k(s)x^s = x \sum_k F(k)H_k(x)$$

(b) $H_k(x) = [H_1(x)]^k$

If there us k independent branches, each one has distribution generated by $H_1(x)$. The sum of independent random variables corresponds to multiplication of generating functions. Thus, the distribution of sum of all the independent branches has a generating function of:

$$H_k(x) = [H_1(x)]^k$$

(c) $H_1(x) = xG_1(H_1(x))$

Using P7.11:

$$H_1(x) = \sum_s x^s \sum_k F(k)h_k(s-1) \quad = x \sum_k F(k)H_k(x) = x \sum_k F(k)[H_1(x)]^k = xG_1(H_1(x)).$$

$$(18)$$

(d) $H_0(x) = xG_0(H_1(x))$

$$H_0(x) = \sum_s \sum_k P(k)h_k(s)x^s = x \sum_k P(k) \sum_s h_k(s)x^s = x \sum_k P(k)H_k(x) = x \sum_k P(k)[H_1(x)]^k = xG_0(H_1($$

$$(19)$$

P7.13

$$\langle s \rangle = H_0'(1) = 1 + G_0'(1)H_1'(1). \tag{20}$$

Differentiate $H_1(x) = xG_1(H_1(x))$:

$$H_1'(x) = G_1(H_1(x)) + xG_1'(H_1(x))H_1'(x). \tag{21}$$

Set $x = 1$, use $G_1(1) = 1$ and $H_1(1) = 1$:

$$H_1'(1) = 1 + G_1'(1)H_1'(1) \quad \Rightarrow \quad H_1'(1) = \frac{1}{1 - G_1'(1)}. \tag{22}$$

Use $G_1'(1) = \frac{\langle k^2 \rangle}{\langle k \rangle} - 1$ to obtain

$$H_1'(1) = \frac{\langle k \rangle}{2\langle k \rangle - \langle k^2 \rangle}. \tag{23}$$

Thus

$$\langle s \rangle = 1 + \frac{\langle k \rangle^2}{2\langle k \rangle - \langle k^2 \rangle}. \tag{24}$$

P7.14

Let $v = H_1(1)$. The fixed-point condition is

$$v = G_1(v). \qquad (25)$$

The probability is

$$P_\infty = 1 - H_0(1) = 1 - G_0(v). \qquad (26)$$

For ER, using $G_0 = G_1 = e^{\langle k \rangle (x-1)}$, we compute

$$v = e^{\langle k \rangle (v-1)}, \qquad (27)$$

so with $v = 1 - P_\infty$ we obtain the equation:

$$P_\infty = 1 - e^{-\langle k \rangle P_\infty}. \qquad (28)$$

P7.16

$$P(k) = (\alpha - 1)m^{\alpha-1}k^{-\alpha}, \quad k = m, \dots, k_{\max}, \qquad (29)$$

with $k_{\max} = mN^{1/(\alpha-1)}$.

$$\langle k \rangle = \int_m^{k_{max}} k \frac{(\alpha-1)m^{\alpha-1}}{k^\alpha} = \frac{\alpha-1}{2-\alpha}m^{\alpha-1}[k_{max}^{2-\alpha} - m^{2-\alpha}]$$

$$\langle k^2 \rangle = \int_m^{k_{max}} k^2 \frac{(\alpha-1)m^{\alpha-1}}{k^\alpha} = \frac{\alpha-1}{3-\alpha}m^{\alpha-1}[k_{max}^{3-\alpha} - m^{3-\alpha}]$$

Let's set $C = (\alpha-1)m^{\alpha-1}$ and consider cases:

- $3 < \alpha$ - elements $(2-\alpha), (3-\alpha)$ are negative so those parts vanish when $k_{max} \to \infty$

$$\langle k \rangle = C \frac{-m^{2-\alpha}}{2-\alpha} = \frac{1-\alpha}{2-\alpha}m$$

$$\langle k^2 \rangle = C \frac{-m^{3-\alpha}}{3-\alpha} = \frac{1-\alpha}{3-\alpha}m^2$$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = |\frac{2-\alpha}{3-\alpha}|m$$

- $2 < \alpha < 3$ - then $(2-\alpha) < 0$, but $(3-\alpha) > 0$. When $k_{max} \to \infty$

$$\langle k \rangle = C \frac{-m^{2-\alpha}}{2-\alpha} = \frac{1-\alpha}{2-\alpha}m$$

$$\langle k^2 \rangle = C \frac{k_{max}^{3-\alpha} - m^{3-\alpha}}{3-\alpha} = \frac{1-\alpha}{3-\alpha}(m^2 - m^{\alpha-1}k_{max}^{3-\alpha})$$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{2-\alpha}{3-\alpha}(m - m^{\alpha-2}k_{max}^{3-\alpha}) \approx |\frac{2-\alpha}{3-\alpha}|m^{\alpha-2}k_{max}^{3-\alpha} \quad \text{for} \quad k_{max} >> m$$

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- $\alpha < 2$ - then both $(2 - \alpha)$ and $(3 - \alpha)$ are positive. When $k_{max} \to \infty$

$$\langle k \rangle = C\frac{k_{max}^{2-\alpha} - m^{2-\alpha}}{2 - \alpha} = \frac{1-\alpha}{2-\alpha}(m - m^{\alpha-1}k_{max}^{2-\alpha}) \approx \frac{1-\alpha}{2-\alpha}(-m^{\alpha-1}k_{max}^{2-\alpha}) \quad \text{for} \quad k_{max} >> m$$

$$\langle k^2 \rangle = C\frac{k_{max}^{3-\alpha} - m^{3-\alpha}}{3 - \alpha} = \frac{1-\alpha}{3-\alpha}(m^2 - m^{\alpha-1}k_{max}^{3-\alpha}) \approx \frac{1-\alpha}{3-\alpha}(-m^{\alpha-1}k_{max}^{3-\alpha}) \quad \text{for} \quad k_{max} >> m$$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} \approx |\frac{2-\alpha}{3-\alpha}|k_{max} \quad \text{for} \quad k_{max} >> m$$

P7.17

Having degree distribution of scale-free network:

$$P(k) = (\alpha - 1)k_{min}^{\alpha-1}k^{-\alpha}$$

And the polylogarithm function:

$$Li_\alpha(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^\alpha}$$

From definition of G_0:

$$G_0(x) = \sum_{k=0}^{\infty} P(k)x^k = \sum_{k=0}^{\infty}(\alpha - 1)k_{min}^{\alpha-1}k^{-\alpha}x^k = (\alpha - 1)k_{min}^{\alpha-1}\sum_{k=1}^{\infty}k^{-\alpha}x^k = (\alpha - 1)k_{min}^{\alpha-1}Li_\alpha(x)$$

For $k_{min} = 1$:

$$G_0(x) = (\alpha - 1)Li_\alpha(x)$$

Diffrentiating:

$$G_0'(x) = (\alpha - 1)\sum_{k=1}^{\infty}k\frac{x^{k-1}}{k^\alpha} = (\alpha - 1)\frac{1}{x}\sum_{k=1}^{\infty}\frac{x^k}{k^{\alpha-1}} = \frac{(\alpha - 1)}{x}Li_{\alpha-1}(x)$$

As for G_1:

$$G_1(x) = \frac{G_0'(x)}{G_o'(1)} = \frac{\frac{(\alpha-1)}{x}Li_{\alpha-1}(x)}{\langle k \rangle}$$

From P.16, for $\alpha > 2$ we have: $\frac{1-\alpha}{2-\alpha}k_{min}$ using this for $k_{min} = 1$ we get:

$$G_1(x) = \frac{\frac{(\alpha-1)}{x}Li_{\alpha-1}(x)}{\frac{1-\alpha}{2-\alpha}} = \frac{\alpha - 2}{x}Li_{\alpha-1}(x)$$