Solutions to Project P9: Random Walks and PageRank

1 P9.1

Let $G = (V, E)$ be a finite, connected, undirected graph with $|V| = N$. Its adjacency matrix A is symmetric, $A_{ij} = A_{ji}$. The degree of vertex i is

$$k_i = \sum_{j=1}^{N} A_{ij}.$$

We consider the simple random walk on G with transition probabilities

$$P_{ji} = \mathbb{P}(X_{t+1} = j \mid X_t = i) = \frac{A_{ij}}{k_i}.$$

The transition matrix P is column-stochastic:

$$\sum_{j=1}^{N} P_{ji} = 1.$$

Stationary distribution

A probability vector $\pi = (\pi_1, \dots, \pi_N)$ is called stationary if

$$P\pi = \pi, \qquad \sum_{i=1}^{N} \pi_i = 1.$$

Detailed balance condition

A sufficient condition for stationarity is the *detailed balance* condition:

$$\pi_i P_{ji} = \pi_j P_{ij} \quad \text{for all } i, j.$$

If this condition holds, then

$$\sum_i \pi_i P_{ji} = \sum_i \pi_j P_{ij} = \pi_j \sum_i P_{ij} = \pi_j,$$

so π is stationary.

Verification

We propose

$$\pi_i = \frac{k_i}{\sum_{\ell=1}^{N} k_\ell}.$$

Then

$$\pi_i P_{ji} = \frac{k_i}{\sum_\ell k_\ell} \cdot \frac{A_{ij}}{k_i} = \frac{A_{ij}}{\sum_\ell k_\ell}.$$

Since the graph is undirected, $A_{ij} = A_{ji}$, and thus

$$\pi_i P_{ji} = \pi_j P_{ij}.$$

Hence the detailed balance condition holds.

Uniqueness

Because the graph is finite and connected, the Markov chain is irreducible. If the graph is also non-bipartite, the chain is aperiodic. By the Perron–Frobenius theorem, the stationary distribution is unique and equals

$$\boxed{\pi_i = \frac{k_i}{\sum_j k_j}.}$$

Interpretation

The stationary probability of a node is proportional to its degree. Therefore, high-degree nodes are visited more frequently in the long-time limit. This result is fundamental in the theory of random walks on graphs