Lecture 6:
Theoretical Fundamentals of MDPs and Dynamic Programming

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This Lecture

- Last lectures: MDP, DP, Model-free Prediction, Model-free Control (Sample-versions of VI/PI)
- This lecture:
  - Mathematical formalism behind the MDP framework.
  - Revisit the Bellman equations and introduce their corresponding operators.
  - Re-visit the paradigm of dynamic programming: VI and PI.
- Next lectures: approximate versions of these paradigms, mainly in the absence of perfect knowledge of the environment.

Preliminaries
(Quick Recap of Functional Analysis)

Normed Vector Spaces

- Normed Vector Spaces: vector space $\mathcal{X}$ + a norm $\|.\|$ on the elements of $\mathcal{X}$.
- Norms are defined a mapping $\mathcal{X} \to \mathbb{R}$ s.t:
  1. $\|x\| \geq 0, \forall x \in \mathcal{X}$ and if $\|x\| = 0$ then $x = \mathbf{0}$.
  2. $\|\alpha x\| = |\alpha| \|x\|$ (homogeneity)
  3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ (triangle inequality)

- For this lecture:
  - Vector spaces: $\mathcal{X} = \mathbb{R}^d$
  - Norms:
    - max-norm/L_∞ norm $\|.\|_\infty$
    - (weighted) L_2 norms $\|.\|_{2,\rho}$

Contraction Mapping

Definition

Let $\mathcal{X}$ be a vector space, equipped with a norm $||.||$. An mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a α-contraction mapping if for any $x_1, x_2 \in \mathcal{X}$, $\exists \alpha \in [0, 1)$ s.t.

$$\|\mathcal{T}x_1 - \mathcal{T}x_2\| \leq \alpha \|x_1 - x_2\|$$

- If $\alpha \in [0, 1]$, then we call $\mathcal{T}$ non-expanding
- Every contraction is also (by definition) Lipschitz, thus it is also continuous. In particular this means:

$$\text{If } x_n \to_{\|.\|} x \text{ then } \mathcal{T}x_n \to_{\|.\|} \mathcal{T}x$$

Fixed point

Definition

A point/vector $x \in \mathcal{X}$ is a fixed point of an operator $\mathcal{T}$ if $\mathcal{T}x = x$.

Banach Fixed Point Theorem

Theorem (Banach Fixed Point Theorem)

Let $\mathcal{X}$ a complete normed vector space, equipped with a norm $||.||$ and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ a γ-contraction mapping, then:

1. *$\mathcal{T}$ has a unique fixed point $x \in \mathcal{X}$: $\exists! x^* \in \mathcal{X}$ s.t. $\mathcal{T}x^* = x^*$*
2. *$\forall x_0 \in \mathcal{X}$, the sequence $x_{n+1} = \mathcal{T}x_n$ converges to x^* in a geometric fashion:*

$$\|x_n - x^*\| \leq \gamma^n \|x_0 - x^*\|$$

Thus $\lim_{n\to\infty} \|x_n - x^*\| \leq \lim_{n\to\infty} (\gamma^n \|x_0 - x^*\|) = 0.$

MDPs and DP

(Recap) MDPs

- Markov Decision Processes (MDPs) formally describe an environment:

$$\mathcal{M} = (\mathcal{S}, \mathcal{A}, p, r, \gamma)$$

- Almost all RL problems can be formalized as MDPs, e.g.
  - Optimal control primarily deals with continuous MDPs
  - Partially observable problems can be converted into MDPs
  - Bandits are MDPs with one state

(Recap) Value functions

- State value function, for a policy π :

$$v_\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1} | s_0 = s; \pi\right]$$

- Action value function, for a policy π:

$$q_\pi(s, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1} | s_0 = s; a_0 = a, \pi\right]$$

- Optimal value functions: $q^* = \max_\pi q_\pi$ $(v^* = \max_\pi v_\pi)$

(Recap) Bellman Equations

Theorem (Bellman Expectation Equations)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, for any policy π, the value functions obey the following expectation equations:

$$v_\pi(s) = \sum_a \pi(s,a) \left[r(s,a) + \gamma \sum_{s'} p(s'|a,s) v_\pi(s') \right] \quad (1)$$

$$q_\pi(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|a,s) \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a') \quad (2)$$

(Recap) The Bellman Optimality Equation

Theorem (Bellman Optimality Equations)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, the optimal value functions *obey the following expectation equations:*

$$
\begin{aligned}
v^*(s) &= \max_a \left[r(s,a) + \gamma \sum_{s'} p(s'|a,s) v^*(s') \right] && (3) \\
q^*(s,a) &= r(s,a) + \gamma \sum_{s'} p(s'|a,s) \max_{a' \in \mathcal{A}} q^*(s',a') && (4)
\end{aligned}
$$

Bellman Operators

The Bellman Optimality Operator

Definition (Bellman Optimality Operator $T^*_{\mathcal{V}}$)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, let $\mathcal{V} \equiv \mathcal{V}_{\mathcal{S}}$ be the space of bounded real-valued functions over $\mathcal{S}$. We define, point-wise, the Bellman Optimality operator $T^*_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ as:

$$(T^*_{\mathcal{V}} f)(s) = \max_a \left[r(s,a) + \gamma \sum_{s'} p(s'|a,s) f(s') \right], \forall f \in \mathcal{V} \tag{5}$$

As a common convention we drop the index $\mathcal{V}$ and simply use $T^* = T^*_{\mathcal{V}}$

Properties of the Bellman Operator T^*

1. It has one unique fixed point v^*.

$$T^* v^* = v^*$$

2. T^* is a γ-contraction wrt. to $\|.\|_\infty$

$$\|T^* v - T^* u\|_\infty \leq \gamma \|v - u\|_\infty, \forall u, v \in \mathcal{V}$$

3. T^* is monotonic:

$$\forall u, v \in \mathcal{V} \text{ s.t. } u \leq v, \text{ component-wise, then } T^* u \leq T^* v$$

Properties of the Bellman Operator T^* (Proofs)

Prop. (2): T^* is a γ-contraction wrt. to $\|.\|_\infty$

Proof.

$$
\begin{aligned}
|T^* v(s) - T^* u(s)| &= \left|\max_a \left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} v(s')\right] - \max_b \left[r(s,b) + \gamma \mathbb{E}_{s''|s,b} u(s'')\right]\right| && (6)\\
&\leq \max_a \left|\left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} v(s')\right] - \left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} u(s')\right]\right| && (7)\\
&= \gamma \max_a |\mathbb{E}_{s'|s,a} [v(s') - u(s')]| && (8)\\
&\leq \gamma \max_{s'} |[v(s') - u(s')]| && (9)
\end{aligned}
$$

Thus we get:

$$\|T^* v - T^* u\|_\infty \leq \gamma \|v - u\|_\infty, \forall u, v \in \mathcal{V}$$

$\square$

Note: Step (6)-(7) uses: $|\max_a f(a) - \max_b g(b)| \leq max_a |f(a) - g(a)|$

Properties of the Bellman Operator T^* (Proofs)

Prop. (3): T^* is monotonic

Proof.

Given $v(s) \le u(s), \forall s \Rightarrow r(s,a) + \mathbb{E}_{s'|s,a} u(s') \le r(s,a) + \mathbb{E}_{s'|s,a} v(s')$.

$$
\begin{aligned}
T^* v(s) - T^* u(s) &= \max_a \left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} v(s') \right] - \max_b \left[r(s,b) + \gamma \mathbb{E}_{s''|s,b} u(s'') \right] && (10)\\
&\le \max_a \left(\left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} v(s') \right] - \left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} u(s') \right] \right) && (11)\\
&\le 0, \forall s. && (12)
\end{aligned}
$$

Thus $T^* v(s) \le T^* u(s), \forall s \in \mathcal{S}$. $\square$

Value Iteration through the lens of the Bellman Operator

Value Iteration

- Start with v_0.
- Update values: $v_{k+1} = T^* v_k$.

As $k \to \infty$, $v_k \to_{\|.\|_\infty} v^*$.

Proof: Direct application of the *Banach Fixed Point Theorem*.

$$
\begin{aligned}
\|v_k - v^*\|_\infty &= \quad \|T^* v_{k-1} - v^*\|_\infty \\
&= \quad \|T^* v_{k-1} - T^* v^*\|_\infty \quad &\text{(fixed point prop.)} \\
&\leq \quad \gamma \|v_{k-1} - v^*\|_\infty \quad &\text{(contraction prop.)} \\
&\leq \quad \gamma^k \|v_0 - v^*\|_\infty \quad &\text{(iterative application)}
\end{aligned}
$$

The Bellman Expectation Operator

Definition (Bellman Expectation Operator)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, let $\mathcal{V} \equiv \mathcal{V}_{\mathcal{S}}$ be the space of bounded real-valued functions over $\mathcal{S}$. For any policy $\pi : \mathcal{S} \times \mathcal{A} \to [0, 1]$, we define, point-wise, the Bellman Expectation operator $T^{\pi}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ as:

$$(T^{\pi}_{\mathcal{V}} f)(s) = \sum_{a} \pi(s, a) \left[r(s, a) + \gamma \sum_{s'} p(s'|a, s) f(s') \right], \forall f \in \mathcal{V} \tag{13}$$

Properties of the Bellman Operator T^π

1. It has one unique fixed point v_π.

$$T^\pi v_\pi = v_\pi$$

2. T^π is a γ-contraction wrt. to $\|.\|_\infty$

$$\|T^\pi v - T^\pi u\|_\infty \leq \gamma \|v - u\|_\infty, \forall u, v \in \mathcal{V}$$

3. T^π is monotonic:

$$\forall u, v \in \mathcal{V} \text{ s.t. } u \leq v, \text{ component-wise, then } T^\pi u \leq T^\pi v$$

Properties of the Bellman Operator T^π (Proofs)

Prop. (2): T^π is a γ-contraction wrt. to $\|.\|_\infty$

Proof.

$$\begin{aligned} T^\pi v(s) - T^\pi u(s) &= \sum_a \pi(a|s)\left[r(s,a) + \gamma \mathbb{E}_{s'|s,a} v(s') - r(s,a) - \gamma \mathbb{E}_{s'|s,a} u(s')\right] \\ &= \gamma \sum_a \pi(a|s) \mathbb{E}_{s'|s,a}\left[v(s') - u(s')\right] \qquad (14) \\ \Rightarrow |T^\pi v(s) - T^\pi u(s)| &\leq \gamma \max_{s'} |\left[v(s') - u(s')\right]| \end{aligned}$$

Thus we get:

$$\|T^\pi v - T^\pi u\|_\infty \leq \gamma \|v - u\|_\infty, \forall u, v \in \mathcal{V}$$

$\square$

Note: (14) gives us also Prop. (3), monotonicity of T^π.

Policy Evaluation

Policy Evaluation

- Start with v_0.
- Update values: $v_{k+1} = T^{\pi} v_k$.

As $k \to \infty$, $v_k \to_{\|.\|_\infty} v_\pi$.

Proof: Direct application of the *Banach Fixed Point Theorem*.

(Summary) Dynamic Programming with Bellman Operators

Value Iteration

- Start with v_0.
- Update values: $v_{k+1} = T^* v_k$.

Policy Iteration

- Start with π_0.
- Iterate:
  - Policy Evaluation: v_{π_i}
    - (E.g. For instance, by iterating T^{π}: $v_k = T^{\pi_i} v_{k-1} \Rightarrow v_k \to v^{\pi_i}$ as $k \to \infty$)
  - Greedy Improvement: $\pi_{i+1} = \arg\max_a q_{\pi_i}(s, a)$

Similarly for $q^\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ functions

Definition (Bellman Expectation Operator)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, let $\mathcal{Q} \equiv \mathcal{Q}_{\mathcal{S},\mathcal{A}}$ be the space of bounded real-valued functions over $\mathcal{S} \times \mathcal{A}$. For any policy $\pi : \mathcal{S} \times \mathcal{A} \to [0, 1]$, we define, point-wise, the Bellman Expectation operator $T^\pi_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ as:

$$(T^\pi_{\mathcal{Q}} f)(s, a) = r(s, a) + \gamma \sum_{s'} p(s'|a, s) \sum_{a' \in \mathcal{A}} \pi(a'|s') f(s', a') \ , \forall f \in \mathcal{Q}$$

- This operator has unique fixed point which corresponds to the *action-value function* q_π in our MDP $\mathcal{M}$.
- Same properties as T^π: γ-contraction and monotonicity.

Similarly for $q^* : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ functions

Definition (Bellman Optimality Operator)

Given an MDP, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, let $\mathcal{Q} \equiv \mathcal{Q}_{\mathcal{S},\mathcal{A}}$ be the space of bounded real-valued functions over $\mathcal{S} \times \mathcal{A}$. We define the Bellman Optimality operator $T^*_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}$ as:

$$(T^*_{\mathcal{Q}} f)(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|a,s) \max_{a' \in \mathcal{A}} f(s',a') \ , \forall f \in \mathcal{Q}$$

- This operator has unique fixed point which corresponds to the *action-value function* q^* in our MDP $\mathcal{M}$.
- Same properties as T^*: γ-contraction and monotonicity.

Approximate Dynamic Programming

Approximate DP

- So far, we have assume perfect knowledge of the MDP and perfect/exact representation of the value functions.
- Realistically, more often than not:
  - We won't know the underlying MDP (like in the last two lectures)
  - We won't be able to represent the value function exactly after each update (lectures to come)

Approximate DP

- Realistically, more often than not:
  - We won't know the underlying MDP.
    $\Rightarrow$ sampling/estimation error, as we don't have access to the true operators T^{π} (T^{*})
  - We won't be able to represent the value function exactly after each update.
    $\Rightarrow$ approximation error, as we approximate the true value functions within a (parametric) class (e.g. linear functions, neural nets, etc).
- Objective: Under the above conditions, come up with a policy π that is (close to) optimal.

(Reminder) Value Iteration

Value Iteration

- Start with v_0.
- Update values: $v_{k+1} = T^* v_k$.

As $k \to \infty$, $v_k \to_{\|.\|_\infty} v^*$.

Approximate Value Iteration

Approximate Value Iteration

- Start with v_0.
- Update values: $v_{k+1} = \mathcal{A} T^* v_k$. $(v_{k+1} \approx T^* v_k)$

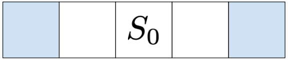
Question: As $k \to \infty$, $v_k \to_{\|.\|_\infty} v^*$? **X**
Answer: In general, no.

Hopeless?

Example

Recall policy evaluation example (Lecture 3):

Simple MDP

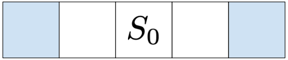


$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

Example

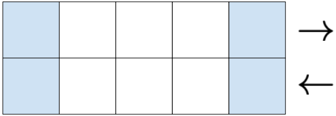
Recall policy evaluation example (Lecture 3):

Simple MDP



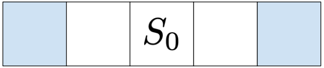
$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$q(s, a)$



Example

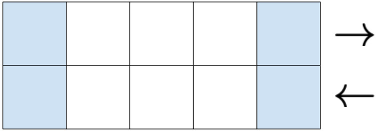
Simple MDP



$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$$\boxed{\pi = \text{uniform}}$$

$q(s, a)$



POLICY EVALUATION

$$q(s, a) \rightarrow r(s, a) + \gamma \mathbb{E}_{\pi}[q(s', a')]$$

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$\pi = \text{uniform}$

$q(s, a)$

	-1.0	-1.0	-1.0		$\rightarrow$
	-1.0	-1.0	-1.0		$\leftarrow$

POLICY EVALUATION

$$q(s, a) \rightarrow r(s, a) + \gamma \mathbb{E}_\pi[q(s', a')]$$

K=1

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$$\pi = \text{uniform}$$

$q(s, a)$

	-1.0	-1.0	-1.0		$\rightarrow$
	-1.0	-1.0	-1.0		$\leftarrow$

	-1.9	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-1.9		$\leftarrow$

POLICY EVALUATION

$$q(s, a) \rightarrow r(s, a) + \gamma \mathbb{E}_\pi [q(s', a')]$$

K=1

K=2

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$$\pi = \text{uniform}$$

$q(s,a)$

	-1.0	-1.0	-1.0		$\rightarrow$
	-1.0	-1.0	-1.0		$\leftarrow$

	-1.9	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-1.9		$\leftarrow$

POLICY EVALUATION

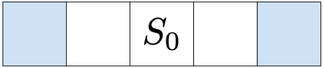
$$\arg\max_a q(s,a)$$

K=1		$\leftarrow\rightarrow$	$\leftarrow\rightarrow$	$\leftarrow\rightarrow$	
K=2		$\leftarrow$	$\leftarrow\rightarrow$	$\rightarrow$	

Question: Have we converged?

Example

Simple MDP



$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

VALUE ITERATION

$$q(s,a) \leftarrow r(s,a) + \gamma \max_{a'} q(s',a')$$

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$q(s, a)$

VALUE ITERATION

$$q(s,a) \leftarrow r(s,a) + \gamma \max_{a'} q(s', a')$$

K=1

	-1.0	-1.0	-1.0		$\rightarrow$
	-1.0	-1.0	-1.0		$\leftarrow$

K=2

	-1.9	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-1.9		$\leftarrow$

Question: Have we converged?

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$q(s, a)$

VALUE ITERATION

$\arg\max_a q(s, a)$

	-1.0	-1.0	-1.0		$\rightarrow$
	-1.0	-1.0	-1.0		$\leftarrow$

K=1

	$\leftarrow\rightarrow$	$\leftarrow\rightarrow$	$\leftarrow\rightarrow$	

	-1.9	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-1.9		$\leftarrow$

K=2

	$\leftarrow$	$\leftarrow\rightarrow$	$\rightarrow$	

Example (Other value functions?)

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$q(s, a)$

	-1.3	-1.3	-1.0		$\rightarrow$
	-1.0	-1.3	-1.3		$\leftarrow$

	-1.8	-1.9	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

Example (Other value functions?)

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$
$$R_t = -1$$

$q(s, a)$

	-1.3	-1.3	-1.0		$\rightarrow$
	-1.0	-1.3	-1.3		$\leftarrow$

	-1.8	-1.7	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-1.9	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

Performance of a Greedy Policy

Theorem (Value of greedy policy)

Consider a MDP. Let $q : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ be an arbitrary function and let π be the greedy policy associated with q, then:

$$\|q^* - q^\pi\|_\infty \leq \frac{2\gamma}{1-\gamma}\|q^* - q\|_\infty$$

where q^* is the optimal value function associated with this MDP.

Performance of a Greedy Policy (Proof)

Statement: $\|q^* - q^\pi\|_\infty \leq \frac{2\gamma}{1-\gamma}\|q^* - q\|_\infty$

Proof.

$$
\begin{aligned}
\|q^* - q^\pi\|_\infty &= \|q^* - T^\pi q + T^\pi q - q^\pi\|_\infty && (15)\\
&\leq \|q^* - T^\pi q\|_\infty + \|T^\pi q - q^\pi\|_\infty && (16)\\
&= \|T^* q^* - T^* q\|_\infty + \|T^\pi q - T^\pi q^\pi\|_\infty && (17)\\
&\leq \gamma\|q^* - q\|_\infty + \gamma \underbrace{\|q - q^\pi\|_\infty}_{\leq \|q - q^*\|_\infty + \|q^* - q^\pi\|_\infty} && (18)\\
&\leq 2\gamma\|q^* - q\|_\infty + \gamma\|q^* - q^\pi\|_\infty && (19)
\end{aligned}
$$

Re-arranging: $(1-\gamma)\|q^* - q^\pi\|_\infty \leq 2\gamma\|q^* - q\|_\infty$. □

Example

Simple MDP

		S_0		

$$\mathcal{A} = \{\leftarrow, \rightarrow\}$$

$$R_t = -1$$

$q^*(s, a)$

	-2.71	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-2.71		$\leftarrow$

$q(s, a)$

	-1.8	-1.9	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

Example

Simple MDP

		S_0		

$q^*(s, a)$

	-2.71	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-2.71		$\leftarrow$

$q(s, a)$

	-1.8	-1.9	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

$$\|q^* - q\|_\infty$$

$$\|q^* - q^\pi\|_\infty$$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

Example

Simple MDP

		S_0		

$q^*(s,a)$

	-2.71	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-2.71		$\leftarrow$

$q(s,a)$

	-1.8	-1.9	-1.1		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

$\|q^* - q\|_\infty$	$\|q^* - q^\pi\|_\infty$
small	0
HIGH	0

Example

Simple MDP

		S_0		

$q^*(s,a)$

	-2.71	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-2.71		$\leftarrow$

$q(s,a)$

	+100	-1.9	-100		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

$$\|q^* - q\|_\infty$$

$$\|q^* - q^\pi\|_\infty$$

Example

Simple MDP

		S_0		

$q^*(s,a)$

	-2.71	-1.9	-1.0		$\rightarrow$
	-1.0	-1.9	-2.71		$\leftarrow$

$q(s,a)$

	+100	-1.9	-100		$\rightarrow$
	-1.0	-1.9	-1.7		$\leftarrow$

	-1.8	-100	-1.1		$\rightarrow$
	-1.0	+100	-1.7		$\leftarrow$

$\lVert q^* - q \rVert_\infty$	$\lVert q^* - q^\pi \rVert_\infty$
HIGH	HIGH
$\wr\wr$	
HIGH	0

(Reminder) Policy Iteration

Policy Iteration

- Start with π_0.
- Iterate:
  - Policy Evaluation: $q_i = q_{\pi_i}$
  - Greedy Improvement: $\pi_{i+1} = \arg\max_a q_{\pi_i}(s, a)$

As $i \to \infty$, $q_i \to_{\|.\|_\infty} q^*$. Thus $\pi_i \to \pi^*$.

Approximate Policy Iteration

Approximate Policy Iteration

- Start with π_0.
- Iterate:
  - Policy Evaluation: $q_i = \mathcal{A} q_{\pi_i}$ $\quad (q_i \approx q_{\pi_i})$
  - Greedy Improvement: $\pi_{i+1} = \arg\max_a q_i(s, a)$

Question 1: As $i \to \infty$, does $q_i \to_{\|.\|_\infty} q^*$? **X**
Answer: In general, no.

Question 2: Or does π_i converge to the optimal policy? **X**
Answer: In general, no.

Hopeless? In some cases, no, depending on the nature of $\mathcal{A}$. (More: Next lecture)

(Summary) Approximate Dynamic Programming

Approximate Value Iteration

- Start with v_0.
- Update values: $v_{k+1} = \mathcal{A} T^* v_k$. $\quad (v_{k+1} \approx T^* v_k)$

Approximate Policy Iteration

- Start with π_0.
- Iterate:
  - Policy Evaluation: $q_i = \mathcal{A} q_{\pi_i}$ $\quad (q_i \approx q_{\pi_i})$
  - Greedy Improvement: $\pi_{i+1} = \arg\max_a q_i(s, a)$

Questions?

The only stupid question is the one you were afraid to ask but never did.
-Rich Sutton

For questions that arise outside of class, please use Moodle!