Ayush Gaggar — Assignment 3 — 10/22/20

1. $f(x) = 27x^3 - 4x^2 + 6x - 12$

$f'(x) = 81x^2 - 8x + 6 \rightarrow f'(3) = 81(9) - 8(3) + 6 = 711$

FDA: $$f'(x) = \frac{27(x+h)^3 - 4(x+h)^2 + 6(x+h) - 12 - 27x^3 + 4x^2 - 6x + 12}{(x+h) - x} + O(h)$$

$$= \frac{1}{h}\left(27(x+h)^3 - 4(x+h)^2 + 6(x+h) - 27x^3 + 4x^2 - 6x\right) + O(h)$$

$$f'(3) = \frac{1}{0.1}\left(27(3.1)^3 - 4(3.1)^2 + 6(3.1) - 27(3)^3 + 4(9) - 18\right)$$

$$= 735.17$$

$$\varepsilon_T = \left|\frac{(711 - 735.17)}{711}\right| \times 100\% = 3.40\%$$

The Taylor Series approximation for Forward Difference Approx. will have a remainder term of magnitude $O(h)$. Given that the function is INCREASING at x, it makes sense that the FDA approximation is an overestimation.

BDA: $$f'(x) = \frac{27x^3 - 4x^2 + 6x - 12 - 27(x-h)^3 + 4(x-h)^2 - 6(x-h) + 12}{x - (x-h)} + O(h)$$

$$= \frac{1}{h}\left(27x^3 - 4x^2 + 6x - 27(x-h)^3 + 4(x-h)^2 - 6(x-h)\right) + O(h)$$

$$f'(3) = \frac{1}{0.1}\left(27(3)^3 - 4(9) + 18 - 27(2.9)^3 + 4(2.9)^2 - 6(2.9)\right)$$

$$= 687.37$$

$$\varepsilon_T = |(711 - 687.37)|/711 \times 100\% = 3.32\%$$

The Taylor Series approximation for Backward Difference Approx. will have a remainder term of magnitude $O(h)$. Given that the function is INCREASING at x, it makes sense that the approximation is an underestimation.

CDA: $f'(x) = \dfrac{27(x+h)^3 - 4(x+h)^2 + 6(x+h) - 27(x-h)^3 + 4(x-h)^2 - 6(x-h)}{2h} - O(h^2)$

$$f'(3) = \frac{27(3.1)^3 - 4(3.1)^2 + 6(3.1) - 27(2.9)^3 + 4(2.9)^2 - 6(2.9)}{0.2}$$

$$= 711.27$$

$$\varepsilon_T = \frac{|711.27 - 711|}{711} \times 100\% = 0.038\%$$

The Taylor Series approximation for Backward Difference Approx. will have a remainder term of magnitude $O(h^2)$, rather than $O(h)$ like the FDA & BDA. Since the error magnitude is $O(h^2)$, the error will be significantly smaller than when $O(h)$. As we can see, the CDA was by far the best approx. of the three.

2. $f'' = \dfrac{f'_{FDA} - f'_{BDA}}{h} = \dfrac{1}{h}\left[\dfrac{1}{h}\left(27(x+h) - 4(x+h)^2 + 6(x+h) - 27x^3 + 4x^2 - 6x\right) - \dfrac{1}{h}\left(27x^3 - x^2 + 6x - 27(x-h)^3 + 4(x-h)^2 - 6(x-h)\right)\right]$

$$f''(3) \approx \left[\frac{1}{0.1}(735.17 - 687.37)\right] = 478$$

$$f''(3)_{actual} = 162(3) - 8 = 478$$

$$\varepsilon_T = \frac{|478 - 478|}{478} \times 100\% = 0\% \text{ Error}$$

3. $CN = \frac{1}{f(\tilde{x})} (f'(\tilde{x})\ \tilde{x})$; CN is used to show how well you can approximate f at $\tilde{x}$.

a) $f(x) = 5$, $\tilde{x} = 2$

$f'(x) = 0$

$CN = \frac{1}{f(2)} (f'(2)(2)) = \frac{1}{5}(2 \cdot 0) = 0 \Rightarrow CN = 0$

Any condition number > 1 means that the function's error will amplify. A condition number of 0 tells us that the output value will <u>not</u> change for a small change in the input

b) $f(x) = e^{-x} + 10$, $\tilde{x} = 5$

$f'(x) = -e^{-x} \Rightarrow f'(5) = -e^{-5}$

$CN = \frac{1}{e^{-5}+10} (-5e^{-5}) = -0.003367$

A small change in the input would not change the output by much. In other words, the function is not very sensitive.

c) $f(x) = \tan(x)$, $\tilde{x} = 0.9999 \frac{\pi}{2}$

$f'(x) = \sec^2 x$; $f'(\tilde{x})$ approaches infinity

$CN = \frac{1}{\tan(\tilde{x})} (\tilde{x}\ f'(\tilde{x}))$ approaches infinity

With such a large CN, a small change in input would result in a massively different output value. At this x, the function is very sensitive; if an approximation were used, the error would continue to amplify.