Ayush Gaggar MAE 107 Assignment 7 12/11/20

1. ① $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \frac{f'''(x_i)h^3}{3!} + O(h^4)$

② $f(x_{i+2}) = f(x_i) + f'(x_i)(2h) + \frac{4f''(x_i)h^2}{2!} + \frac{9f'''(x_i)h^3}{3!} + O(h^4)$

③ $f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2!} - \frac{f'''(x_i)h^3}{3!} + O(h^4)$

④ $f(x_{i-2}) = f(x_i) - f'(x_i)(2h) + \frac{4f''(x_i)h^2}{2!} - \frac{9f'''(x_i)h^3}{3!} + O(h^4)$

② $-2$① : $f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)h^2 + \frac{7}{6}f'''(x_i)h^3 - O(h^4)$

forward $\rightarrow$ $f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} - \frac{7}{6}f'''(x_i)h + O(h^2)$

④ $-2$③ : $f(x_{i-2}) - 2f(x_{i-1}) = -f(x_i) + f''(x_i)h^2 - \frac{7}{6}f'''(x_i)h^3 - O(h^4)$

backward $\rightarrow$ $f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} + \frac{7}{6}f'''(x_i)h + O(h^2)$

$$f'''(x_i) = \frac{f''(x_i)_{\text{forward}} - f''(x_i)_{\text{backward}}}{2h}$$

$$= \frac{f(x_{i+2}) - 2f(x_{i+1}) - f(x_{i-2}) + 2f(x_{i-1})}{2h^3} - \frac{7}{3}f'''(x_i)h + O(h^2)$$

Since $f'''(x_i)$ is cont., this term is a constant and is smaller than $O(h^2)$

$$\boxed{f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3} + O(h^2)}$$

2. Three-point Gauss Quadrature

$I_g(f) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$

$I_g(f) = I(f)$ for $f(x) = 1, x, x^2, x^3, x^4$, and x^5

1) $I_g(f(x)=1) = \int_{-1}^{1} (1)\,dx = 2 \longrightarrow 2 = c_0(1) + c_1(1) + c_2(1)$

$2 = c_0 + c_1 + c_2$ ①

2) $I_g(x) = \int_{-1}^{1} x\,dx = 0 \longrightarrow 0 = c_0 x_0 + c_1 x_1 + c_2 x_2$ ②

3) $I_g(x^2) = \int_{-1}^{1} x^2 dx = 2/3 \rightarrow 2/3 = c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2$ ③

4) $I_g(x^3) = \int_{-1}^{1} x^3 dx = 0 \rightarrow 0 = c_0 x_0^3 + c_1 x_1^3 + c_2 x_2^3$ ④

5) $I_g(x^4) = \int_{-1}^{1} x^4 dx = 2/5 \rightarrow 2/5 = c_0 x_0^4 + c_1 x_1^4 + c_2 x_2^4$ ⑤

6) $I_g(x^5) = \int_{-1}^{1} x^5 dx = 0 \longrightarrow 0 = c_0 x_0^5 + c_1 x_1^5 + c_2 x_2^5$ ⑥

$x_0 = 0.5556 = 5/9$ $\qquad c_0 = 0.7746 = \sqrt{3/5}$

$x_1 = 0.8889 = 8/9$ $\qquad c_1 = 0$

$x_2 = 0.5556 = 5/9$ $\qquad c_2 = -0.7746 = -\sqrt{3/5}$

4. $\frac{dy}{dx} = (1+2x)\sqrt{y}$, $y(0) = 1$, $h = \Delta x = 0.25$, $x \in [0,1]$

a) $\frac{dy}{\sqrt{y}} = (1+2x)\,dx$

$\int_{y(0)}^{y(1)} y^{-1/2}\,dy = \int_0^1 (2x+1)\,dx$

$2y^{1/2}\Big|_{y(0)}^{y(1)} = x^2 + x\Big|_0^1$

$2\sqrt{y(1)} - 2\sqrt{1} = 1 + 1 - 0$

$2\sqrt{y(1)} = 4 \Rightarrow y(1) = 2$

General Solution

$2\sqrt{y}\Big|_1^y = x^2 + x\Big|_0^x$

$\sqrt{y} - 1 = \frac{1}{2}(x^2+x)$

$y = \left(\frac{1}{2}(x^2+x)+1\right)^2$