PH108 : Electricity & Magnetism : Problem Set 2

Only * problems are to be solved in the tut session

Topics Covered : Stokes and Divergence Theorems; Curvilinear Vector Calculus

1. Consider a planar surface S parallel to $x-y$ plane, bounded by a closed curve C. Consider a vector field

$$\vec{F} = 0.5(-y\hat{i} + x\hat{j})$$

Prove that the value of the integral of this vector field along the curve C is exactly equal to the area of the planar surface S. Using this find the area inside the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

2. A ship sails from the southernmost point of India $(6.75^o N, 93.84^o E)$ to the southernmost point of Africa $(34.5^o S, 20^o E)$ following the shortest possible path.
(a) Given that the radius of the earth is 6400Km, what is the distance it has covered?
(b) If instead of sailing one had travelled in an aeroplane, by what percentage would the shortest possible distance change?

3.* Compute the divergence of the vector field given by:

$$\vec{v} = r\cos\theta\,\hat{r} + r\sin\theta\,\hat{\theta} + r\sin\theta\cos\phi\,\hat{\phi}$$

Check the divergence theorem for this using the volume of an inverted hemisphere of radius R, resting on the xy plane and centered at the origin.

4. Compute the line integral of $\vec{F} = (2xz + y, 2yz + 3x, x^2 + y^2 + 5)$. Use Stokes theorem to compute

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$$

C is the positively oriented curve on the surface of the circular cylinder of radius 1 with it's axis along the +ve z axis and with one face as the xy plane with O as centre.

5.* If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors,

$$\phi(r) = (\mathbf{a} \times \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{r})$$

is the potential over all space, then find the electric field and charge density in spherical co-ordinates. [Hint: $\vec{E} = -\nabla\phi$, $\rho = \epsilon_0 \nabla \cdot \vec{E}$]

6.* A vector field is given by

$$\vec{v} = ay\hat{i} + bx\hat{j}$$

where a, b are constants.

(a) Find the line integral of this field over a circular path of radius R, lying in the xy plane and centered at the origin using (i) plane polar and then (ii) Cartesian system.

(b) Imagine a right circular cylinder of length L with its axis parallel to the z axis standing on the circle. Use cylindrical co-ordinate system to show that the theorem is valid over its surface.

7.* Although the gradient, divergence and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_V (\nabla T)\, dV = \oint_S T\, d\vec{S}$ [Hint: Let $\vec{v} = \vec{c}T$ where $\vec{c}$ is a constant, in the divergence theorem]

(b) $\int_V (\nabla \times \vec{v})\, dV = -\oint_S \vec{v} \times d\vec{S}$ [Hint: Replace $\vec{v}$ by $\vec{v} \times \vec{c}$, where $\vec{c}$ is a constant in the divergence theorem]