Vector Calculus

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$P(x, y, z) =$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\frac{dr}{d\theta}$.

At any point of the curve

$x = 3\cos t$, $y = 3\sin t$, $z = 4t$

find the

(i) Tangent Vector

(ii) Unit tangent Vector

Solution

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$

$$\frac{d\vec{r}}{d\theta} = 3(-\sin\theta)\cdot\hat{i} + 3\cos\theta\,\hat{j} + 4\hat{k}$$

$$= -3\sin\theta\,\hat{i} + 3\cos\theta\,\hat{j} + 4\hat{k}.$$

for unit Vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Magnitude.

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{9\sin^2\theta + 9\cos^2\theta + 4^2}$$

$$= \sqrt{9(\sin^2\theta + \cos^2\theta) + 16}$$

$$= 5$$

Unit tangent Vector. $\therefore \hat{r} = \frac{\vec{r}}{|\vec{r}|}$

$$= \frac{-3\sin\theta\,\hat{i} + 3\cos\theta\,\hat{j} + 4\hat{k}}{5}$$

Scalar and Vector point function.

Point function: A Variable quantity whose Value depends on the position at any point in a Space of the point is Called point function.

(i) Scalar point function
e.g. → temperature

(ii) Vector point function.
e.g. → Velocity.

Vector differential operator denoted by ∇ (del)

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.}$$

If $\phi(x,y,z)$ is a Scalar point function then

$$\boxed{\text{Gradient of } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}}$$

$$\boxed{\text{Grad } \phi = \nabla \phi}$$

→ Grad ϕ is a Vector normal to the Surface ϕ

Q If $\phi = 3x^2y - y^3z^2$ find grad ϕ at the point $(1,-2,-1)$

→ grad $\phi = \nabla \phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) 3x^2y - y^3z^2$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\, 6xy + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(0 - y^3 \cdot 2z)$$

$$= \hat{i}\, 6xy + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$(\nabla\phi)_{(1,-2,-1)} = 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2zy^3(\hat{k})$$

$$= 6(1)(-2)\hat{i} + [3(1) - 3(4)(1)]\hat{j} - 2(-1)(-8)\hat{k}$$

$$= -12\hat{i} + [3 - 12]\hat{j} - 16\hat{k}$$

$$= -12\hat{i} - 8\hat{j} - 16\hat{k}$$

Directional Derivative

The Component of $\nabla\phi$ in the direction of a Vector $\vec{d}$ is equal to $\boxed{\nabla\phi \cdot \vec{d}}$. It is the Called the directional derivative of ϕ in the direction of $\vec{d}$

Find the directional derivative of $x^2 y^2 z^2$ at the point $(1,1,-1)$ in the direction of tangent to the Curve.

$x = e^t, \quad y = \sin 2t + 1, \quad z = 1 - \cos t$ at $t = 0$

$\phi = x^2 y^2 z^2$ (given)

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 y^2 z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y^2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2y^2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= \hat{i}(2xy^2z^2) + \hat{j}(x^2z^2 2y) + \hat{k}(x^2y^2 2z)$$

$$= 2xy^2z^2\,\hat{i} + 2y\,x^2z^2\,\hat{j} + 2zx^2y^2\,\hat{k}$$

$$\nabla\phi_{(1,1,-1)} = 2(1)(1)(1)\hat{i} + 2(1)(1)(1)\hat{j} + 2(1)(1)(-1)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= e^t\hat{i} + (\sin 2t + 1)\hat{j} + (1 - \cos t)\hat{k}$$

$$\text{tangent} = \frac{d\vec{r}}{dt}$$

$$= e^t\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$$

at $t=0$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=0} = e^0\hat{i} + \cancel{\hat{j} + 2\cos 0}\,\hat{j} + 0$$

$$= \hat{i} + 2\hat{j}$$

Directional derivative $= \cancel{(2\hat{i}+2\hat{j}-2\hat{k})\cdot(\hat{i}+2\hat{j})}$

$$= \cancel{2+4-2}$$

$$=$$

$$= \Delta\phi \cdot \hat{a}$$

$$\hat{a} = \frac{\hat{i}+2\hat{j}}{\sqrt{1^2+2^2}} = \frac{\hat{i}+2\hat{j}}{\sqrt{5}}$$

$$\therefore \Delta\phi\cdot\hat{a} = (2\hat{i}+2\hat{j}-2\hat{k})\left(\frac{\hat{i}+2\hat{j}}{\sqrt{5}}\right)$$

$$= \frac{2+4}{\sqrt{5}} = \frac{6}{\sqrt{5}} \quad \text{Ans}$$

Divergence of a Vector function.

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + k\frac{d}{dz}\right)\cdot(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$ → It is a Scalar function

If $\text{div}\,\vec{F} = 0$, it is called Solenoidal Vector function.

Curl of a Vector function.

$$\text{Curl } \vec{F} = \Delta \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If Curl $\vec{F} = 0$ the field F is irrotational.

Q. Find the divergence and Curl of

$\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at

$(2,-1,1)$

Solution $\text{div } \vec{V} = \Delta \cdot \vec{V}$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + (2zx - y^2)$$

at $(2,-1,1)$ $= (-1)(1) + 3(4) + [2(1)(2) - 1]$

$$= -1 + 12 + [4-1]$$

$$= -1 + 12 + 3$$

$$= 15 - 1 = 14$$

$$\text{Curl } \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial r} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\hat{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2z)\right] - \hat{j}\left[\frac{\partial}{\partial x}\right]$$

$$= \hat{i}\begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & xz^2-y^2z \end{bmatrix} - \hat{j}\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xyz & xz^2-y^2z \end{bmatrix}$$

$$+ \hat{k}\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xyz & 3x^2y \end{bmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xz^2-y^2z) - \frac{\partial}{\partial x}3x^2y\right] + \hat{j}\left[\frac{\partial}{\partial x}(xz^2-y^2z) - \frac{\partial}{\partial x}(xyz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}3x^2y - \frac{\partial}{\partial y}xyz\right]$$

$$= \hat{i}\left[0(-2yz - 6xy)\right] + \hat{j}\left[(2z - \cancel{0}) - yz)\right]$$

$$+ \hat{k}\left[6xy - xz\right]$$

$= \cancel{\hat{i}(-2yz+2xy)}$

$$= (-2yz - 6xy)\hat{i} - (2z - yz)\hat{j} + (6xy - xz)\hat{k}$$

$-12 \quad 2$

at $(2,-1,1)$

$$\hat{i}(-2(-1)(1) - [2(1) - (-1)(1)]\hat{j} + [6(2)(-1) - (2)(1)]\hat{k}$$

$$\boxed{2\hat{i} + 3\hat{j} - 14\hat{k}} \quad \text{A} \leftarrow$$

Vector Integration.

Line integral $\int_C \vec{F} \cdot d\vec{r}$

Work done $= \int_A^B \vec{F} \cdot d\vec{r}$

Circulation If $\vec{V}$ represents the velocity of a liquid then $\oint \vec{V} \cdot d\vec{r}$ is called the Circulation of v round the Close Curve C. If Circulation is Zero, then v is said to be irrotational.

Q. If a force $\vec{F} = 2x^2 y \hat{i} + 3xy \hat{j}$ displaces a particle in xy plane from (0,0) to (1,4) along a Curve $y = 4x^2$. find the work done.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

$$= \text{workdone} = \int_A^B \vec{F} \cdot d\vec{r}$$

$$y = 4x^2$$

$$\frac{dy}{dx} = 8x$$

$$dy = 8x\,dx$$

$$= \int_A^B (2x^2 y)\hat{i} + (3xy)\hat{j} \cdot \hat{i}\,dx + \hat{j}\,dy$$

$$\int_A^B 2x^2 \cdot y \; dx + (3xy)dy$$

$$\int_A^B 2 \cdot x^2 \cdot 4x^2 \, dx + 3 \cdot x \cdot 4x^2 \cdot 8x\,dx$$

$$\int_0^1 2\cdot x^2 \cdot 4x^2\, dx + 3\cdot x\cdot 4x^2\; 8x\, dx$$

$$\int_0^1 8x^4\, dx + 96x^4\, dx$$

$$8\cdot\left[\frac{x^5}{5}\right]_0^1 + 96\left[\frac{x^5}{5}\right]_0^1$$

$$8\left[\frac{1}{5}\right] + 96\left[\frac{1}{5}\right]$$

$$\frac{1}{5}[8+96] = \frac{1}{5}\times 104 = \frac{104}{5} \quad \text{Ans}$$

Suppose $F(x,y,z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field.
Find the work done by $\vec{F}$ along the line from $(1,2,3)$ to $(3,5,7)$

$=$ Workdone $= \vec{F}\cdot d\vec{r}$

$$= (x^3\hat{i} + y\hat{j} + z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= x^3\cdot dx + y\,dy + z\,dz$$

Now

$$\int_{(1,2,3)}^{(3,5,7)} (x^3\, dx + y\,dy + z\,dz)$$

$$\int x^3\, dx + \int y\,dy + \int_{(1,2,3)}^{(3,5,7)} z\,dz$$

$\cancel{3,5,7}$

$\boxed{\cancel{\frac{x^4}{4}}}$

$\cancel{(1,2,3)}$

$$\int_1^3 x^3\,dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

$$\left[\frac{x^4}{4}\right]_1^3 + \left[\frac{y^2}{2}\right]_2^5 + \left[\frac{z^2}{2}\right]_3^7$$

$$\left(\frac{81}{4}-\frac{1}{4}\right)+\left(\frac{25}{2}-\frac{4}{2}\right)+\left(\frac{49}{2}-\frac{9}{2}\right)$$

160

$\cancel{202}$

$$\frac{80}{4} + \frac{21}{2} + \frac{40}{2}$$

$\cancel{101}$ $$\frac{80+42+80}{4} = \frac{\cancel{202}^{101}}{\cancel{4}_2} = \frac{101}{2} \text{ Ans}$$

Q If $A = (3x^2+6y)\hat{i} - 14yz\,\hat{j} + 20x^2z^2\hat{k}$

Evaluate the line integral $\oint \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x = t$, $y = t^2$, $z = t^3$.

line integral $= \oint \vec{A}\cdot d\vec{r}$

$$= \int \left[(3x^2+6y)\hat{i} - 14yz\,\hat{j} + 20x^2z^2\hat{k}\right]\cdot\left[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right]$$

$$= \int (3x^2+6y)\,dx - 14yz\,dy + 20x^2z^2\,dz$$

$$= \int (3t^2+6t^2)\,dt - 14t^2t^3\,2t\,dt + 20t^2t^6\,3t^2\,dt$$

$$\int_0^1 (3t^2+6t^2)\,dt - 14(t^2)(t^3).2t\,dt$$
$$+ 20t(t^6)\,3t^2\,dt$$

$$\int_0^1 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$9\left[\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$$

$$\frac{9}{3} - \frac{28}{7} + \frac{60}{10}$$

$$\frac{\cancel{630 - 84 + 72}}{210}$$

Rough

$x = t$

$\frac{dx}{dt} = 1$

$\boxed{dx = dt}$

$y = t^2$

$\frac{dy}{dt} = 2t$

$dy = 2t\,dt$

$z = t^3$

$\frac{dz}{dt} = 3t^2$

$dz = 3t^2\,dt$

$\frac{7\,3}{630}$

$\frac{7\,3\,2}{84}$

$\frac{21\;60}{1260}$

$\frac{630 -}{210}$

Surface Integral.

Any integral which is to be evaluated over a Surface is called Surface integral.

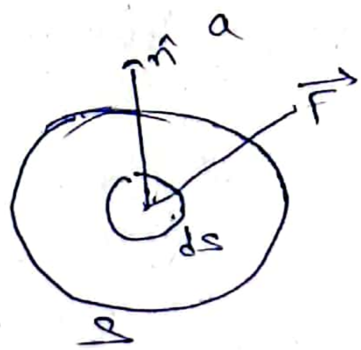
Surface integral of F over S = $\iint_S (\vec{F} \cdot \hat{n})\, ds$

Flux = $\iint_S (\vec{F} \cdot \hat{n})\, ds$ where $\vec{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F} \cdot \hat{n})\, ds = 0$, then $\vec{F}$ is said to be Solenoidal Vector.

Surface integral = $\iint (\vec{F} \cdot \hat{n})\, ds$

$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}$ $\quad ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$



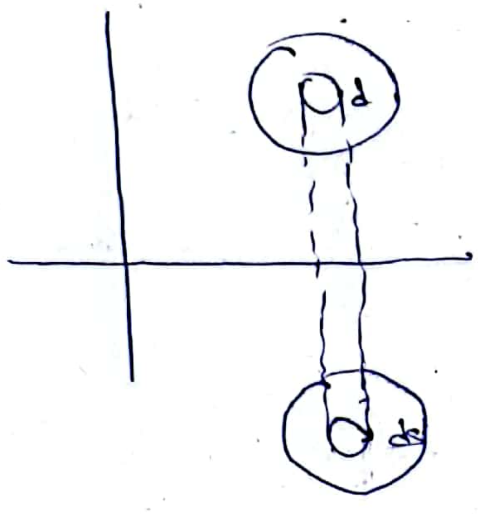
Surface integral.

Surface integral = $\iint_S \vec{F} \cdot \hat{n}\, ds$

$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

xy plane
yz plane
xz plane

$yz = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$ $\quad xz = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$



Evaluate $\iint yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k} \cdot ds$

Where, S is the Surface of the Sphere $x^2+y^2+z^2 = a^2$ in the ~~first~~ octant.

∴ 3 dimensional figure - octant
2 dimension figure - quadrant.

Solution

$\phi = x^2 + y^2 + z^2 - a^2$

$\vec{n} = \nabla\phi$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2)$

$= \frac{\partial}{\partial x}x^2\,\hat{i} + \frac{\partial}{\partial y}y^2\,\hat{j} + \frac{\partial}{\partial z}z^2\,\hat{k} \cancel{-a^2}$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 0$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\cdot\sqrt{x^2+y^2+z^2}}$$

$\hat{i}\cdot\hat{i} = 1$

$\cos 0^\circ = 1$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$\vec{F}\cdot\hat{n} = (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)$$

$$= \frac{xyz + xyz + xyz}{a}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \frac{3xyz}{a}$$

Now

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint \frac{3xyz}{a}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n}\cdot\hat{k} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\cdot\hat{k}$$

$$= \frac{z}{a}$$

for limit of y

$x^2+y^2+z^2 = a^2$

let $z = 0$

$x^2+y^2 = a^2$

$y^2 = a^2 - x^2$

$y = \sqrt{a^2-x^2}$

for limit of x

Put $y^2 = 0$, $x^2 \cdot 0$

$x^2 = a^2$

$x = a$

Now

$$\iint \frac{3xy\cancel{z}}{\cancel{a}}\,\frac{dx\,dy}{\frac{\cancel{z}}{\cancel{a}}}$$

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} 3xy\cdot dx\,dy$$

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} 3xy\,dy\,dx$$

$$3\int_0^a \cdot x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx \Rightarrow 3\int_0^a x\left[\frac{a^2-x^2}{2}\right] dx$$

$$\frac{3}{2}\int_0^a a^2x - x^3 \; dx$$

$$\frac{3}{2}\; a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a$$

$$\frac{3}{2}\left[a^2\left[\frac{a^2}{2}\right] - \frac{a^4}{4}\right]$$

$$\frac{2-1}{2}$$

$$\frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$\frac{3}{2}a^4\left[\frac{1}{1} - \frac{1}{2}\right] \Rightarrow \frac{3}{2}a^4\left[\frac{1}{4}\right]$$

$$= \frac{3}{8}a^4 \quad \text{Ans}$$

Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ Where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$

S is the Surface of the Cylinder $x^2+y^2=16$ included in the Ist octant between $z=0$ & $z=5$.

Hint: ds. ~~xy~~ ×
yz =
xz

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$\phi = x^2+y^2$$

$$\hat{n} = \nabla\cdot\phi =$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x^2+y^2)$$

$$= \frac{\partial}{\partial x}x^2\,\hat{i} + \frac{\partial}{\partial y}y^2\,\hat{j}$$

$$= 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i} + y\hat{j}}{4}$$

<u>Now</u>

$$\iint \vec{A} \cdot \vec{n}\, ds$$

$$= \iint_S \left(z\hat{i} + x\hat{j} - 3y^2z\hat{k}\right) \cdot \frac{x\hat{i} + y\hat{j}}{4}\, ds$$

$$\cancel{= \iint} \quad ds = \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|} = \frac{dy\,dz}{\frac{x}{4}}$$

$$\hat{n} \cdot \hat{i} = \frac{x\hat{i} + y\hat{j}}{4} \cdot \hat{i}$$

$$= \frac{x}{4}$$

$$\cancel{\text{Now} \iint \left(z\hat{i} + x\hat{j} - 3y^2z\hat{k}\right) \cdot \left(\frac{x\hat{i} + y\hat{j}}{4}\right) \cdot \frac{dy\,dz}{\frac{x}{4}}}$$

$$\frac{xz + xy}{4}$$

$x^2 + y^2 = 16$

let $x = 0$

$y^2 = 16$

$y = 4$

$$\iint \frac{\cancel{x}z + \cancel{x}y}{\cancel{4}} \cdot \frac{dy\,dz}{\frac{\cancel{x}}{\cancel{4}}}$$

$$\int_0^4 \int_0^5 (z+y)\, dy\,dz$$

$$\int_0^4 \left[\frac{z^2}{2}\right]_0^5 dy \Rightarrow \int_0^4 \frac{25}{2}\, dy$$

$\Rightarrow$

$$\int_0^4\int_0^5 (z+y)\,dy\,dz$$

$x^2+y^2 = 16$

put $x=0$

$y^2 = 16$

$y = \pm 4$

$$\cancel{\int_0^4\int_0^5 (z+y)\,dz\,dy}$$

$$\int_0^4\int_0^5 z\cdot dy\,dz + y\,dy\,dz$$

$$\cancel{\int_0^4 \left[\frac{z^2}{2}\right]_0^5 + \left[\frac{y^2}{2}\right]_0^4}$$

$$\cancel{\frac{25}{2} + \frac{16}{2}}$$

$$\cancel{\int_0^4 y\cdot\left[\frac{z^2}{2}\right]_0^5 dy}$$

$$\cancel{\int_0^4 y\cdot\left[\frac{25}{2}\right]dy}$$

$$\cancel{\frac{25}{2}\int_0^4\left[\frac{y^2}{2}\right]_0^4} \Rightarrow \cancel{\frac{25}{2}\times\frac{16}{2}} = \cancel{\frac{41}{2}}$$

$$\int_0^4\left[\int_0^5 (z+y)\,dz\right]dy$$

$$\int_0^4\left[\frac{z^2}{2}+yz\right]_0^5 dy$$

$$\int_0^4\left[\frac{25}{2}+5y\right]dy$$

$$\int_0^4 \frac{25+10y}{2}\,]dy \Rightarrow \frac{5}{2}\int_0^4 (5+2y)\,dy$$

$$\frac{5}{2}\cdot 5[y]_0^4 + 2\left[\frac{y^2}{2}\right]_0^4$$

$$\frac{5}{2}\times \cancel{36}\ 18 \quad 90 \quad \frac{5}{2}\times\left[20+\cancel{2\cdot\frac{16}{2}}\right]$$

Volume Integrated.

Let $\vec{F}$ be Vector point function and Volume V enclosed by a closed Surface.

The volume integral $= \iiint_V F\,dv$

If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$,

Evaluate $\iiint_V \vec{F}\,dv$ where V is the region bounded by the Surface

$x=0$, $y=0$, $x=2$, $y=4$, $z=x^2$, $z=2$.

Solution $dv = dx\,dy\,dz$

$$\iiint_V \vec{F}\cdot dv = \int_0^2\int_0^4\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$\int_0^2\int_0^4 2\cdot\left[\frac{z^2}{2}\right]_0^2\hat{i} - \hat{j}x[z]_0^2 + \hat{k}\,y[z]_0^2\,dy\,dz$$

$$\int_0^2\int_0^4 2\cdot 2\hat{i} - 2x\hat{j} + 2y\hat{k}\,dy\,dz$$

$$\int_0^2 4[y]_0^4\hat{i} - 2x[y]_0^4\hat{j} + 2\hat{k}\left[\frac{y^2}{2}\right]_0^4\,dz$$

$$\int_0^2 16\hat{i} - 8x\hat{j} + 16\hat{k}\,dz$$

$$\cancel{16[x]_0^2\,\hat{i} - 8\cdot\left[\frac{x^2}{2}\right]_0^2\hat{j} + 16[z]_0^2\,\hat{k}}$$

$$\cancel{32\hat{i} - 16\hat{j} + 32\hat{k}}$$

$$\int_0^2\int_0^4\int_{x^2}^2 \left(2z\hat{i} - x\hat{j} + y\hat{k}\right) dx\,dy\,dz$$

$$\int_0^2\int_0^4 2\cdot\left[\frac{z^2}{2}\right]_{x^2}^2 \hat{i} - x[z]_{x^2}^2\hat{j} + y[z]_{x^2}^2\hat{k}\ dx\,dy$$

$$\int_0^2\int_0^4 2\left[2 - \frac{x^4}{2}\right]\hat{i} - x\left[2 - x^2\right]\hat{j} + y\left[2 - x^2\right]\hat{k}\ dx\,dy$$

$$\int_0^2\int_0^4 (4 - x^4)\hat{i} - (2x + x^3)\hat{j} + (2y - x^2y)\hat{k}\ dx\,dy$$

$$\int_0^2 \left[4[y]_0^4 - x^4\cdot[y]_0^4\right]\hat{i} - \left[2x\cdot[y]_0^4 + x^3[y]_0^4\right]\hat{j} +$$

$$\left[2\left[\frac{y^2}{2}\right]_0^4 - x^2\left[\frac{y^2}{2}\right]_0^4\hat{k}\right] dx$$

$$\int_0^2 (16 - 4x^4)\hat{i} - 8x\hat{j} + 4x^3\hat{j} + (16 - 8x^2)\hat{k}\ dx$$

$$\left[16\cdot[x]_0^2 - 4\left[\frac{x^5}{5}\right]_0^2\right]\hat{i} - \left[8\cdot\left[\frac{x^2}{2}\right]_0^2 + 4\left[\frac{x^4}{4}\right]_0^2\right]\hat{j} +$$

$$\left[16\cdot[x]_0^2 - 8\cdot\left[\frac{x^3}{3}\right]_0^2\right]\hat{k}$$

$$\left(32 - 4\cdot\frac{32}{5}\right)\hat{i} - (16 \cancel{+} 16)\hat{j} + \left(32 - \frac{64}{3}\right)\hat{k}$$

$$\left(32 - 4\cdot\frac{32}{5}\right)\hat{i} + \left(32 - \frac{64}{3}\right)\hat{k}$$

$$\left(32 - \frac{128}{5}\right)\hat{i} + \left(\frac{96 - 64}{3}\right)\hat{k}$$

$$\left(\frac{160 - 128}{5}\right)\hat{i} + \left(\frac{96-64}{3}\right)\hat{k}$$

$$\frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}$$

$$\cancel{\frac{96\hat{i} + 32\hat{k}}{15}} \qquad \frac{96\hat{i} + 160\hat{k}}{15}$$

$$\frac{32}{15}\left[3\hat{i} + 5\hat{k}\right]$$ Ans

16-1-23

Maths

Green's theorem. [For a plane]

Statement : If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$ & $\frac{\partial \psi}{\partial x}$ be Continuous function over a region R bounded by Simple Closed Curve C in xy - Plane then

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy .$$

A Vector field $\vec{F}$ is given by $\vec{F} = \sin y\, \hat{i} + x(1+\cos y)\hat{j}$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where, C is the Circular path given by $x^2 + y^2 = a^2$

Solution

$$\int \vec{F} \cdot d\vec{r} = \int_C \sin y\, \hat{i} + x(1+\cos y)\,\hat{j} \cdot (\hat{i}\, dx + \hat{j}\, dy)$$

$$= \int_C \sin y\, dx + x(1+\cos y)\, dy$$

$\phi = \sin y$, $\psi = x(1+\cos y)$

$\frac{\partial \phi}{\partial y} = \cos y$ $\qquad$ $\frac{\partial \psi}{\partial x} = (1+\cos y)$

$$\iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \iint_R (1 + \cos y - \cos y)\, dx\, dy$$

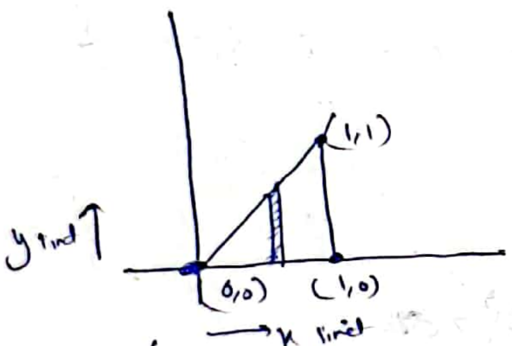
$\iint_R dx\,dy$ ∵ area का formula $\iint \partial x\,\partial y$.

$\iint dx\,dy = \pi a^2$

Using Green's theorem Evaluage

$\int_C (x^2 y\,dx + x^2\,dy)$ Where, C is the boundary described Counter ~~to~~ Clock wise of a triangle with Vertices (0,0), (1,0), (1,1)

⇒



$\phi = x^2 y$ $\quad \dfrac{\partial \psi}{\partial x} = 2x$

$\psi = x^2$ $\quad \dfrac{\partial \phi}{\partial y} = x^2$

$$\oint (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^x (2x - x^2)\,dx\,dy$$

x's limit $\quad$ y's limit

$$= \int_0^1 \cancel{2\left[\frac{x^2}{2}\right]_0^x}\; 2x.[y]_0^x - x^2[y]_0^x$$

$$= \int_0^1 2x^2 - x^3$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]$$

$$\frac{2}{3} - \frac{1}{4} = \frac{6-1}{12} = \frac{5}{12} \text{ Ans}$$

Stoke's theorem (Relation between line integral and Surface integral)

Statement: Surface integral of the Component of Curl $\vec{F}$ along the normal is the Surface S, taken over the Surface S bounded by Curve C is equal to the line integral of the Vector point function $\vec{F}$ taken along the Closed Curve C.

Mathematically

$$\boxed{\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \hat{n}\, ds}$$

Where $\hat{n}$ is a unit external is any Surface ds.

Q. Using Stoke's theorem

Evaluate $\int (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$

Where C is the Circle $x^2+y^2=1$ Corresponding to the Surface of Sphere of unit radius.

$$\Rightarrow \int \left((2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$\text{Curl}\,\vec{F} = \nabla\cdot F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$\hat{i}\begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz^2 & -y^2z \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2x-y & -y^2z \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x-y & -yz^2 \end{vmatrix}$$

$$\hat{i}\left[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right]$$

$$\Rightarrow \hat{i}\left[-\cancel{2yz} + \cancel{2zy}\right] - \hat{j}\left[0 - 0\right] + \hat{k}\left[0+1\right]$$

$$= \hat{k} \quad \text{Ans} \leftarrow$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$\Rightarrow \iint \hat{k}\cdot\hat{n}\,ds$$

$$\Rightarrow \iint \hat{k}\cdot\cancel{\hat{n}}\cdot\frac{dx\,dy}{\cancel{\hat{n}\cdot\hat{k}}}$$

$$\Rightarrow \iint dx\,dy \rightarrow \text{area of Circle}$$

radius = 1

$$\text{Area} = \pi r^2 = \pi\cdot 1\cdot 1 = \pi. \quad \text{Ans} \leftarrow$$

Q Evaluate $\oint \vec{F}\cdot d\vec{r}$ where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at $(0,0,0)$, $(1,0,0)$, & $(1,1,0)$.

$$\Rightarrow \text{Curl } \vec{F} = \Delta\cdot F$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$\Rightarrow \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -(x+z) \end{vmatrix} - \hat{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y^2 & -(x+z) \end{vmatrix} + \hat{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 & x^2 \end{vmatrix}$$

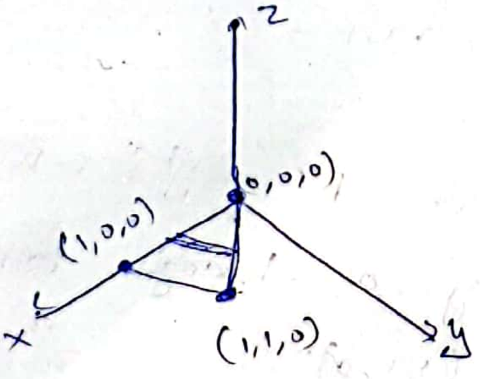
$$\Rightarrow \hat{i} \left[\frac{\partial}{\partial y} \{-(x+z)\} - \frac{\partial}{\partial z}(x^2) \right] - \hat{j} \left[\frac{\partial}{\partial x} \{-(x+z)\} - \frac{\partial}{\partial z} y^2 \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2 \right]$$

$$\Rightarrow \hat{i} [0 + 0] - \hat{j} [-1 + 0 - 0] + \hat{k} [2x - 2y]$$

$$\Rightarrow \hat{j} + 2(x-y)\hat{k} \quad \text{Ans}$$

Now.



Triangle is in XY plane because $z = 0$

$$\frac{ds}{\hat{n} \cdot \hat{k}}$$

$$= \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$= dx\,dy.$$

$$\hat{n} = \hat{k}$$

$$\iint (\hat{j} + 2(x-y)\hat{k}) \cdot \hat{k} \; dx\,dy$$

$$= \int_0^1 \int_0^x 2(x-y)\, dx\, dy$$

$$\int_0^1 2\left[x[y]_0^x - \left[\frac{y^2}{2}\right]_0^x\right] dx$$

$$\int_0^1 2\left[x^2 - \frac{x^2}{2}\right] dx$$

$$\int_0^1 2\left[\frac{2x^2 - x^2}{\cancel{2}}\right] dx$$

$$\int_0^1 x^2 .$$

Gauss divergence theorem.

(Relation between Surface integral & Volume integral).

Statement:

The Surface integral of the normal Component of a ~~function~~ Vector function F taken around a Closed Surface S is equal to the integral of the divergence of F taken over Volume V enclosed by the Surface

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div} \vec{F}\, dv$$

Q Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$

Where, $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ and S is the Surface of the Cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

$$= \iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv = \iiint_V (\nabla\cdot\vec{F})\, dv$$

$$= \int_0^1\int_0^1\int_0^1 \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}\right]\cdot dv$$

$dv = dx\,dy\,dz$

$$\cancel{= \int_0^1\int_0^1\int_0^1 \left[\hat{i}\frac{\partial}{\partial x}(4xz\hat{i} - y^2\right.}$$

$$= \int_0^1\int_0^1\int_0^1 \left[\frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}yz\right].$$

$$\Rightarrow \int_0^1\int_0^1\int_0^1 (yz\hat{i} - 2y\hat{j} + y\hat{k}) \cdot dx\,dy\,dz$$

$$\Rightarrow \int_0^1\int_0^1 y\left[\frac{z^2}{2}\right]_0^1 - 2y[z]_0^1 + y[z]_0^1$$

$$\int_0^1\int_0^1 \left(\frac{y}{2} - 2y + y\right) dx\,dy$$

$$\int_0^1 2[y]_0^1 - 2\left[\frac{y^2}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 \quad dx$$

$$\int_0^1 \left(2 - \left(2\times\frac{1}{2}\right) + \frac{1}{2}\right) dx$$

$$\int_0^1 \left(2 - 1 + \frac{1}{2}\right) dx \Rightarrow \int_0^1 \left(1 + \frac{1}{2}\right) dx$$

$$\int_0^1 \frac{3}{2}\,dx \Rightarrow \frac{3}{2}[x]_0^1 = \frac{3}{2} \quad A \leftarrow$$

Q Evaluate $\iint \vec{F} \cdot \hat{n}\, ds$

Where, $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$.

Where r is the surface of sphere having centre $(3, -1, 2)$ & radius 3.

$$= \iint \vec{F} \cdot \hat{n}\, ds = \iiint \operatorname{div} \vec{F} \cdot dv = \iiint (\nabla \cdot \vec{F})\, dv.$$

$$= \iiint \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right] \cdot dv$$

$$\iiint \frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}(y^2+2z)$$

$= \iiint 2-1+2 \, dv$

$\iiint 3 \, dv$

Volume of sphere $= \frac{4}{3}\pi r^3$

$$\boxed{\begin{array}{c} dx\,dy\,dz = dv \\ \text{Volume of Sphere} \end{array}}$$

~~V = dx~~

$3V$

$3 \times \frac{4}{3} \pi r^3$

put $r = 3$

$3 \times \frac{4}{\cancel{3}} \times \pi \times \cancel{27}^{9}$

108π.