30-9-22

Leibnitz's theorem And its applications

Successive Differentiation

$y = f(x)$

$\frac{dy}{dx} = ?$.

$y = \sin(\sin x)$ Prove that

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$$

$y = \sin(\sin x)$.

$\frac{dy}{dx} = -\cos(\sin x) \cdot -\cos x = \cos(\sin x) \cos x$.

$\frac{d^2y}{dx^2} = \cos(\sin x) \cdot -\sin x + \cos x \cdot -\sin(\sin x) \cdot \cos x$

$\cos(\sin x) - \sin x - \cos^2 x \, \underline{\sin(\sin x)}$.

$\cancel{\cos(\sin x) \sin x}$

$\frac{d^2y}{dx^2} = -\sin x \cos(\sin x) - y \cos^2 x$

$\frac{d^2y}{dx^2} + y \cdot \cos^2 x = -\sin x \cos(\sin x)$

$= -\frac{\sin x}{\cos x} \cdot \cos x \cos(\sin x)$

$\frac{d^2y}{dx^2} + y \cos^2 x = -\tan x \frac{dy}{dx}$.

$\frac{d^2y}{dx^2} + y \cos^2 x + \tan x \frac{dy}{dx} = 0$.

Proved

n^{th} Derivatives

$y = x^m$.

$y_1 = \frac{dy}{dx} = y' = m x^{m-1}$

$y_2 = m(m-1) x^{m-2}$

$y_3 = m(m-1)(m-2) x^{m-3}$

$y_n = m(m-1)(m-2)(m-3) \ldots [m-(n-1)] x^{m-n}$

$y_n = m(m-1)(m-2) \ldots [m-n+1) x^{m-n}$

1) If $n = m$

$y_n = m(m-1)(m-2) \ldots [m-m+1] x^{m-m}$

$y_n = m(m-1)(m-2) \ldots x^0$

$\boxed{y_n = m!}$

Ex $y_{30} = y = x^{30}$

$= 30!$

2) $y = e^{ax}$

$y_1 = a e^{ax}$

$y_2 = a^2 e^{ax}$

$\boxed{y_n = a^n e^{ax}}$

(3) $y = a^{mx}$

$y_n = a^{x} (\log_e a)^n$

$$\boxed{\frac{d}{dx}(a^x) = a^x \log_e a}$$

(4) $y = (ax+b)^{-1}$ or $\frac{1}{ax+b}$

$y_n = (-1)^n (ax+b)^{-n-1} a^n . n!$

(5) $y = (ax+b)^{-m}$

$$\boxed{y_n = \frac{(-1)^n (m+n-1)!\, a^n}{(m-1)!\, (ax+b)^{m+n}}}$$

(6) $y = \log(ax+b)$

$$\boxed{y_n = \frac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$$

Q. Find the n^{th} derivative of $\log(ax+x^2)$

$y = \log(ax+x^2)$

$y = \log x(a+x) \sim \log(ax+b)$

$a = 1$

$b = x$

$\cancel{y_n}$ $y = \log x + \log(a+x)$

$$y_n = \frac{(-1)^{n-1} (n-1)!\, 1^n}{x^n} + \frac{(-1)^{n-1} (n-1)!\, 1^n}{(a+x)^n}$$

Q

$y = \sin(ax+b)$

$y_1 = a\cos(ax+b)$

$= a\left[\sin(ax+b) + \frac{\pi}{2}\right]$

$y_2 = \left[a^2\cos(ax+b) + \frac{\pi}{2}\right]$

$y = a^2\sin\left[ax+b+\frac{2\pi}{2}\right]$

$y_n = a^n\sin\left[ax+b+\frac{n\pi}{2}\right]$

or

If $y = \cos(ax+b)$

$y_n = a^n\cos\left[ax+b+\frac{n\pi}{2}\right]$

$y = a^{mx}$

$y_n = m^n a^{mx}(\log a)^n$

If $m = 1$,

$y = a^x$

$y_n = a^x(\log a)^n$

$y = \sin 2x \sin 3x$ find $y_n = ?$

Form

$y = \sin(ax+b)$

$y_n = \left(a^n \sin\left(ax+b+\frac{n\pi}{2}\right)\right)$

Use

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \cos(A+B)$

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$$y = \frac{1}{2}\left[\cos x - \cos 5x\right]$$

$$\cancel{y = \cos(ax+b)}$$

$$\cancel{y_n = a^n \cos\left[ax+b+\frac{n\pi}{2}\right]}$$

$\left.\begin{array}{l} a=1 \\ b=0 \end{array}\right]$ $\cos x = \cos(ax+b)$

$\left.\begin{array}{l} a=5 \\ b=0 \end{array}\right\}$ $\cos 5x = \cos(ax+b)$

$$y_n = \frac{1}{2}\left[\left\{1^n \cos\left(x+\frac{n\pi}{2}\right)\right\} - 5^n \cos\left\{5x+\frac{n\pi}{2}\right\}\right]$$

$$y = \cos(ax+b)$$

$$y_n = a^n \cos\left\{ax+b+\frac{n\pi}{2}\right\}$$

Q $y = \sin^3 x$

$$\cancel{y} = 3\sin x - 4\sin^3 x = \sin 3x$$

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$$

$$y_n = \frac{1}{4}\left[3 \cdot 1^n \sin\left(x+\frac{n\pi}{2}\right) - 3^n \sin\left(3x+\frac{n\pi}{2}\right)\right]$$

Use of Partial Fraction

Working rule :

① $$\frac{Px}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

(2) $$\frac{px}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

(3) $$\frac{px}{(x-a)^3(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{x-b}$$

(4) $$\frac{px}{(x-a)(x-b)(px^2+qx+r)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx+d}{px^2+qx+r}$$

Q. $$y = \frac{1}{6x^2-5x+1}$$

Step 1 — factorise

$6x^2-3x-2x+1$

$3x(2x-1)-1(2x-1)=0$

$(2x-1)(3x-1)$

$$= \frac{1}{(2x-1)(3x-1)}$$

$$= \frac{A}{2x-1} + \frac{B}{3x-1}$$

$$\frac{1}{(2x-1)(3x-1)} = \frac{A}{(2x-1)} + \frac{B}{3x-1}$$

div $$\frac{(2x-1)(3x-1)}{(2x-1)(3x-1)} = \frac{A\,\cancel{(2x-1)}(3x-1)}{\cancel{(2x-1)}} + \frac{B(2x-1)\cancel{(3x-1)}}{\cancel{3x-1)}}$$

$$1 = A(3x-1) + B(2x-1)$$

if $x = \frac{1}{3}$

$$1 = A(0) + B\left(\frac{2}{3}-1\right)$$

$$1 = B\left(\frac{2-3}{3}\right)$$

$$1 = B\left(\frac{-1}{3}\right)$$

$$\boxed{B = -3}$$

~~A~~

If $x = \frac{1}{2}$

$$1 = A\left(3 \times \frac{1}{2} - 1\right) + 0$$

$$1 = A\left(\frac{3-2}{2}\right)$$

$$1 = A\left(\frac{1}{2}\right)$$

$$\boxed{A = 2}$$

$$\therefore \frac{2}{2x-1} + \frac{-3}{3x-1}$$

$$\frac{1}{6x^2-5x+1} = \frac{2}{2x-1} + \frac{-3}{3x-1}.$$

$$2(2x-1)^{-1} - 3(3x-1)^{-1}$$

~~using formula~~

$$= 2\cdot(-1)^n[2x-1]^{-n-1}\cdot 2^n\cdot n! - 3\times(-1)^n(3x-1)^{-n-1}3^n\cdot n!$$

$$= 2\cdot(-1)^n[2x-1]^{-n-1}\cdot 2^n\cdot n! - 3^{n+1}(-1)^n(3x-1)^{-n-1}\cdot n!$$

$$= (-1)^n 2^{n+1}(2x-1)^{-n-1} n! - 3^{n+1}(-1)^n(3x-1)^{-n-1}\cdot n!$$

$(ax+b)^{-1}$

$$y_n = (-1)^n(ax+b)^{-n-1}\cdot a^n\cdot n!$$

n^{th} derivative of $e^{ax}\sin(bx+c)$

$$y = e^{ax}\sin(bx+c)$$

$$y_n = e^{ax} r^n \sin(bx+c+n\alpha)$$

Where, $r^2 = a^2+b^2$ & $\tan\alpha = \frac{b}{a}$

Similarly,

$$y = e^{ax}\cos(bx+c)$$

$$y_n = e^{ax} r^n \cos(bx+c+n\alpha)$$

Where, $r^2 = a^2+b^2$ & $\tan\alpha = \frac{b}{a}$

If $y = e^x \sin^3 x$.

find y_n..

$= \sin 3x = 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin 3x$

$\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$

$y = e^x \frac{1}{4}[3\sin x - \sin 3x]$

$= \frac{3}{4}e^x \sin x - \frac{1}{4}e^x \sin 3x$

$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}}\sin(bx+c+n\alpha)$

$= e^x(1+1)^{\frac{n}{2}}\sin(x+c+n\alpha)$

$= e^x 2^{\frac{n}{2}}\sin(x+c+n\frac{\pi}{4})$

$r^2 = a^2+b^2$

$r = (1+9)^{\frac{1}{n}}$

$\sqrt{10}$

$\alpha = \tan^{-1}\frac{3}{1}$

$a=1$, $b=1$, $c=0$ | $a=1$, $b=3$, $c=0$

$r^2 = a^2+b^2$

$r = (a^2+b^2)^{\frac{1}{2}}$

$r^n = (a^2+b^2)^{\frac{n}{2}}$

$\alpha = \tan^{-1}\frac{b}{a}$

$\alpha = \tan^{-1}\frac{1}{1}$

$\alpha = \frac{\pi}{4}$

$$y_n = \left[\left(\frac{3}{4} \cdot e^x 2^{\frac{n}{2}} \sin\left(x + c + \frac{n\pi}{4}\right) - \frac{1}{4} e^x 10^{\frac{n}{2}} \sin(3x + c + n\tan^{-1}3)\right)\right]$$

$$y_n = \frac{3}{4} e^x 2^{\frac{n}{2}} \sin\left(x + \frac{n\pi}{4}\right) - \frac{1}{4} e^x 10^{\frac{n}{2}} \sin(3x + n\tan^{-1}3)$$

Q. Find the n^{th} derivative of

$$\frac{x^2}{(x+2)(2x+3)}$$

$(x+2)(2x+3)$

$2x^2 + 4x + 3x + 6$

$2x^2 + 7x + 6$

* If the degree of the numerator is equal or greater than the degree of denominator then we have to divide the num. by the den.

$$\frac{x^2}{2x^2+7x+6}$$

$$\begin{array}{r|l} & \frac{1}{2} \\ \hline 2x^2+7x+6 & x^2 \\ & x^2 + \frac{7}{2}x + \cancel{\frac{6}{2}}3 \\ & - \quad - \\ \hline & -\frac{7}{2}x - 3 \end{array}$$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} \frac{-\left[\frac{7}{2}x + 3\right]}{2x^2+7x+6}$$

$$= \frac{1}{2} - \boxed{\frac{\frac{7}{2}x - 3}{(x+2)(2x+3)}}$$ Do it Partial fraction.

$$\frac{-\frac{7}{2}x - 3}{(x+2)(2x+3)} = \frac{A}{x+2} + \frac{B}{2x+3}$$

$$-\frac{7}{2}x - 3 = A(2x+3) + B(x+2)$$

~~take B = 0~~

$-\frac{7}{2}x - 3 = A(2x+3)$ $\quad \frac{1}{2}$

$A = \cancel{\dfrac{\frac{-7x}{2} - 3}{2x+3}}$

$\cancel{A = \dfrac{-7x-6}{2(2x+3)} = \dfrac{-7x-6}{4x+3}}$ $\quad \cancel{x = -2}$

$A = \dfrac{-7x-2-6}{4x-2+3} = \dfrac{14-6}{-8+3} = \dfrac{8}{-5}$

If $x = -2$

$B = 0.$

$-\frac{7}{2}(-2) - 3 = A(2\times(-2)+3) + B(-2+2)$

$\Rightarrow 7 - 3 = A(-4+3) + 0$

$4 = A(-1)$

$A = -4$

If $x = -\frac{3}{2}$

$A = 0.$

$-\frac{7}{2}\times\left(-\frac{3}{2}\right) - 3 = A(0) + B\left(-\frac{3}{2}+2\right)$

$\frac{21}{4} - 3 = 0 + B\left(\frac{-3+4}{2}\right)$

$\frac{21-12}{\cancel{4}_{2}} = B\left(\frac{1}{\cancel{2}}\right)$

$\frac{9}{2} = B.$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2} \cdot \frac{1}{2x+3}$$

$(x+2)^{-1}$

y_n

$(ax+b)^{-1}$

$$\frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2} \times \frac{1}{2x+3}$$

$$= \frac{1}{2} - 4(x+2)^{-1} + \frac{9}{2}(2x+3)^{-1}$$

$$y_n = \frac{1}{2} (-4)(-1)^n (x+2)^{-n-1} (1)^n \cdot n! + \frac{9}{2} (-1)^n (2x+3)^{-n-1} (2)^n n!$$

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Leibnitz's theorem.

If $u + v$ are functions of x, then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_n u v_n$$

$$= u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2} u_{n-2} v_2 + \ldots + u.v_n$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$${}^nC_n = 1$$

$${}^nC_0 = 1$$

$${}^nC_1 = n$$

$${}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

Differentiate n times

Q. $x^2 y_2 + x y_1 + y = 0$.

$D^n(x^2 y_2) + D^n(x y_1) + D^n(y) = 0$

$D^n(x^2 y_2)$

$U = y_2$, $V = x^2$. $U_n \cdot V$ $\quad U_{n-1} V_1$ $\quad U_{n-2} V_2$

$U_{n-2} = y_n$. $\quad V = x^2$

$U_{n-1} = y_{n+1}$ $\quad V_1 = 2x$

$U_n = y_{n+2}$ $\quad V_2 = 2$.

$D^n(x^2 y_2) = y_{n+2} x^2 + n y_{n+1} 2x + \cancel{y_{n-2}} \frac{n(n-1)}{\cancel{2}} y_n \cancel{2}$

$D^n(x y_1) \Rightarrow$ $U = y_1$ $\quad V = x$.

$V_1 = 1$

$V_2 = 0$

$U_{n-1} = y_n$

$U_n = y_{n+1}$.

$D^n(x y_1) = y_{n+1} x + n y_n$.

$D^n(y) =$

$U = y$ $\quad V = 1$.

$V_1 = 0$

$\cancel{U_{n-1}}$

$U_n = y_n$.

$D^n(y) = y_n +$

$$x^2 y_{n+2} + 2xn y_{n+1} + n(n-1) y_n + y_{n+1} x + n y_n + y_n = 0$$

$$x^2 y_{n+2} + x y_{n+1}(2n+1) + [n^2 - n + n + 1] y_n = 0.$$

$$\boxed{x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0}$$ — n th Derivative.

Q. If $y = a\cos(\log x) + b\sin(\log x)$

Show that

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$$

$$y = a\cos(\log x) + b\sin(\log x)$$

$$y_1 = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}.$$

$$y_1 x = -a\sin(\log x) + b\cos(\log x)$$

$$y_1 + x y_2 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x) \times \frac{1}{x}.$$

$$x y_1 + x^2 y_2 = -[a\cos(\log x) + b\sin(\log x)]$$

$$= -y$$

$$x^2 y_2 + x y_1 + y = 0$$

[Check 1st Question Same thing. Solve as you Solved 1st Question]

Q. Differentiate n times

~~Q.~~ If $y = x^n \log x$

Prove that

$$y_{n+1} = \frac{n!}{x}$$

$\Rightarrow$ $y = x^n \log x$

$y_1 = x^n \frac{1}{x} + \log x \, n x^{n-1}$

$xy_1 = x^n + x \log x \, n x^{n-1}$

$xy_1 = x^n + \log \, n x^n$

$= x^n + n \boxed{x^n \log x}$

$y_1 x = x^n + ny$

Apply liebnitz's theorem

$y_{n+1} x + \cancel{n y_n} = n! + \cancel{n y_n}$

$y_{n+1} x = n!$

$$\boxed{y_{n+1} = \frac{n!}{x}}$$ Proved

$u = y_1$ | $v = x$, $v_1 = 1$

$u_{n-1} = y_n$

$u_n = y_{n+1}$

$x^n = n!$ | $x^n = n!$

Q. If $y = \sin(m \sin^{-1} x)$ Prove that

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$.

Partial Differentiation :

$y = f(x)$.

$z = f(x, y) \rightarrow \{x \text{ or } y \text{ is independent Variable}\}$

$\downarrow$

Dependent Variable.

$\frac{\partial z}{\partial y} \rightarrow$ Differentiate with respect to (y) & Keeping (x) as Constant.

$\frac{\partial z}{\partial x} \rightarrow$ Differentiate with respect to (x) & Keeping (y) as Constant.

Ex.1 $z = x^2 + y^2$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $z = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = 2x + 0 = 2x.$$

$$\frac{\partial z}{\partial y} = 0 + 2y = 2y.$$

Ex.2 $z = 3x^2 y$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $z = 3x^2 y$

$$\frac{\partial z}{\partial x} = 3y \cdot 2x = 6xy.$$

$$\frac{\partial z}{\partial y} = 3x^2 \cdot (1) = 3x^2.$$

Ex.3 If $z = x^2 + y^2 - 3xy$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$;

Solution $z = x^2 + y^2 - 3xy$.

$$\frac{\partial z}{\partial x} = 2x + 0 - 3y = 2x - 3y$$

$$\frac{\partial z}{\partial y} = 0 + 2y - 3x. = 2y - 3x.$$

Ex. 4. If $z = \log(x^2+y^2)$ find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

Solution $\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \times 2y = \frac{2y}{x^2+y^2}$$

Ex. 5 $F = \sin(5x^2y + 7xy^2)$ find $\frac{\partial F}{\partial x}$ or $\frac{\partial F}{\partial y}$?

Solution $F = \sin(5x^2y + 7xy^2)$

$$\frac{\partial F}{\partial x} = \cos(5x^2y + 7xy^2).$$

$$= \cos\{5y \cdot 2x + 7y^2 \cdot (1)\}$$

$$\frac{\partial F}{\partial x} = \cos[5x^2y + 7xy^2](10xy + 7y^2)$$

$$\frac{\partial F}{\partial y} = \cos(5x^2y + 7xy^2)$$

$$= \cos(5x^2 + 2y \cdot 7x)$$

$$\frac{\partial f}{\partial y} = \cos[5x^2y + 7xy^2](5x^2 + 14xy)$$

Ex. 6 If $f = \underset{u}{x^4} \, \underset{v}{\sin(xy^3)}$ find $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$?

Solution $f = x^4 \sin(xy^3)$

$$\frac{\partial f}{\partial x} = x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3.$$

$$\frac{\partial f}{\partial y} = x^4 \cos(xy^3) \cdot 3y^2 \cdot x \cdot \cancel{+ \sin(xy^3) \cdot x^4 \cdot 0}$$

$$\frac{\partial f}{\partial y} = 3x^5y^2 \cos(xy^3)$$

$$\left\{\because d(uv) = u \cdot d(v) + v \cdot d(u)\right\}$$

Q.1. If $U = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$.

find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$?.

Solution $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times y \times \frac{-1}{x^2}$.

Differentiation.

$\tan^{-1} x$.

$\frac{\partial}{\partial x}(\tan^{-1}x) = \frac{1}{1+x^2}$

Differentiation.

$\sin^{-1} x$.

$\frac{\partial}{\partial x}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{\frac{x^2+y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}.$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{--- eq}^n \ 1$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\, x \times \frac{-1}{y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}.$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times x \times \frac{-1}{y^2} + \frac{x^2}{x^2+y^2} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{1}{\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{--- eq}^n \ 2$$

∴ Now adding eq^n (1) & eq^n (2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= 0.$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$$

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Homogeneous function:

a function $f(x,y)$ is said to be Homogeneous function in which the power of each term is same.

a function $f(x)$ is a homogeneous function of order "n" if degree of each of its term in x & y is equal to n.

$$a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + \dots a_{n-1}x^1y^{n-1} + a_ny^n.$$

is homogeneous function of order n.

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots\dots + a_n\left(\frac{y}{x}\right)^n\right] = x^n\phi\left(\frac{y}{x}\right)$$

$x^{n-(n+1)}$ $n-n-1$

for ex. $$\frac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \frac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= \frac{x^{\frac{1}{2}}\left[1+\sqrt{\frac{y}{x}}\right]}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}$$

$$= x^{-\frac{3}{2}}\left[\frac{1+\sqrt{\frac{y}{x}}}{1+\left(\frac{y}{x}\right)^2}\right]$$

$$\text{Order} = n = -\frac{3}{2}$$

***

Euler's theorem on homogeneous function:

If z is homogeneous function is x, y order n.

then $$\frac{x\partial z}{\partial x} + \frac{y\partial z}{\partial y} = nz.$$

Verify Euler's theorem for the function.

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$$

Solution

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times -\frac{1}{x^2}\cdot y.$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{\frac{x^2+y^2}{x^2}} \times -\frac{1}{x^2}\cdot y$$

$$= \frac{1}{\sqrt{y^2-x^2}} + \frac{1\times -y}{x^2+y^2}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}.$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{—— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \times x\left(\frac{-1}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x}$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \times -\frac{x}{y^2} + \frac{1}{\frac{x^2+y^2}{x^2}} \times \frac{1}{x}.$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$x\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{—— (ii)}$$

Add (i) & (ii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} + \left(\cancel{\frac{-x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}\right)$$

$$= 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$$

$$= x^0 \sin^{-1}\frac{x}{y} + x^0 \tan^{-1}\frac{y}{x}$$

$n = 0$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Deduction from Euler's theorem ***** VVI

If z is a homogeneous function of x, y of n degree and $z = f(u)$; then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ Prove that

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1}$$

$$\left\{ \because \begin{array}{l} u = \log x, \\ x = e^u \end{array} \right\}$$

<u>Solution</u> $u = \log\left(\frac{x^2+y^2}{x+y}\right)$

$$= \frac{x^2+y^2}{x+y} = e^u.$$

$$z = e^u = \frac{x^2+y^2}{x+y}.$$

$$= \frac{x^2\left[1+\left(\frac{y^2}{x^2}\right)\right]}{x\left[1+\frac{y}{x}\right]}$$

$$= x\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}}\right]$$

→ Yes it is Homogeneous

$n = 1$.

By Euler's deduction formula,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{fu}{f'(u)} = \cancel{x}\cdot\frac{e^{u}}{e^{u}} = 1$$

2nd deduction formula.

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

Where,

$$g(u) = \frac{nf(u)}{f'(u)}$$

Ist deduction formula:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nfu}{f'(u)}$$

If $u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$ find the value of.

1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

2. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Q $u = \tan^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$

$$\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}} = \tan u$$

$$\frac{x^3\left(1+\frac{y^3}{x^3}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right.} = \tan u$$

$$x^{3-\frac{1}{2}}\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right] \qquad n=\frac{5}{2}.$$

$$x^{\frac{5}{2}}\left[\frac{1+\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right]$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\frac{\tan u}{\sec^2 u}$$

$$=\frac{5}{2}\sin u\cos u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\times\frac{1}{2}\times 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{4}\sin 2u.$$

Now $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$$

$$=\frac{5}{4}\sin 2u\left[\frac{5}{2}\cos 2u\times 2-1\right]$$

$$=\frac{25}{8}\sin 2u\cos u-\frac{5}{4}\sin 2u$$

Ans

Total derivative

(I) Differentiation $\qquad y=f(x).$

(II) Partial differentiation $\qquad \frac{dy}{dx} \cdot \frac{dz}{dx} \cdot \frac{dz}{dy}.$

(III) total differentiation.

If $z = f(x, y)$.

$x = \phi(t)$

$y = \phi(t)$.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}}$$

$v = x^2$

$\frac{dv}{dx} = 2x$

$\cancel{\frac{\partial v}{\partial y} = 2x}$.

Q If $z = x^2 y$ Where $x = t^2$, $y = t^3$

find $\frac{dz}{dt}$.

Solution $z = x^2 y$.

$\frac{\partial z}{\partial x} = 2xy$; $\frac{\partial z}{\partial y} = x^2$

then, $x = t^2$; $y = t^3$

$\frac{dx}{dt} = 2t$; $\frac{dy}{dt} = 3t^2$.

Now $\frac{dz}{dt} = \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt}$

$= 2xy \cdot 2t + x^2 \cdot 3t^2$

$= 4xyt + 3x^2 t^2$

Now put the value of x & y.

$\frac{dz}{dt} = 4(t^2)(t^3)t + 3(t^2)^2 t^2$

$4t^6 + 3t^6 = 7t^6$. Ans

Q If $U = x^2 + y^2$.

Where, $x = at^2$ $y = 2at$. find $\frac{du}{dt}$.

$U = x^2 + y^2$. ; $\frac{\partial u}{\partial y} = 0 + 2y$.

$\frac{\partial u}{\partial x} = 2x + 0$

then, $x = at^2$; $y = 2at$

$\frac{dx}{dt} = a \cdot 2t = 2at$ $\frac{dy}{dt} = 2a$.

Now, $\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$

$\frac{du}{dt} = 2x \times 2at + 2y \times 2a$

$= 4atx + 4ay$.

Put the values of x & y

$\frac{du}{dt} = 4at(at^2) + 4a(2at)$

$\boxed{\frac{du}{dt} = 4a^2t^3 + 8a^2t}$. Ans.

Q. If $U = x^2 + y^2 + z^2$. Where, $x = e^t$

$y = e^t \sin t$

$z = e^t \cos t$

Prove that.

$\frac{du}{dt} = 4e^{2t}$.

Formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt}.$$

Solution → $U = x^2 + y^2 + z^2$.

$\frac{\partial u}{\partial x} = 2x$ | $\frac{\partial u}{\partial y} = 0 + 2y$ | $\frac{\partial u}{\partial z} = 2z$

$x = e^t$ $\quad y = e^t \cdot \sin t$ $\quad z = e^t \cos t$

$\frac{dx}{dt} = e^t$

$\frac{dy}{dt} = \cancel{e^t \sin t \cdot \cos t}$

$e^t \cdot \cos t + \sin t \, e^t$

$= e^t \cos t + e^t \sin t$

$\frac{dz}{dt} = e^t \cdot (-\sin t) + \cos t \, e^t$

$= e^t \sin t + e^t \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt}$$

$$= 2x \, e^t + 2y \left(e^t \cos t + e^t \sin t \right) + 2z \left(-e^t \sin t + e^t \cos t \right)$$

$$= 2xe^t + 2ye^t \left[\cos t + \sin t \right] - 2ze^t \left[\sin t - \cos t \right]$$

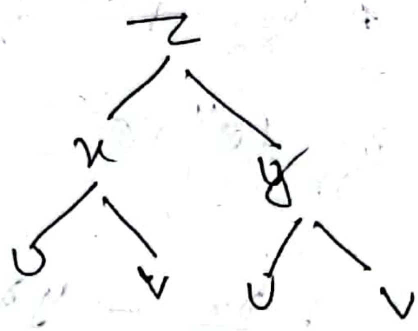
20-10-22

Change in the independent Variables $x + y$ by other two Variables $u + v$.

$$z = f(x,y).$$

Where. $x = \phi(u,v)$ & $y = \psi(u,v)$

$\cancel{\frac{\partial z}{\partial v}}$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}.$$

If $z = f(x,y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

Show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

= $x = e^u + e^{-v}$

$\frac{\partial x}{\partial u} = e^u$ | $\frac{\partial x}{\partial v} = -e^{-v}$

$y = e^{-u} - e^v$

$\frac{\partial y}{\partial u} = -e^{-u}$ | $\frac{\partial y}{\partial v} = -e^v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times (e^u) + \frac{\partial z}{\partial y} \times (-e^{-u})$$

$$= e^u \frac{\partial z}{\partial x} - e^{-u} \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{v}\right)$$

$$= -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} - \left[-e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}\right]$$

$$= e^{u}\frac{\partial z}{\partial x} + e^{-v}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{v}\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial x}\underbrace{\left(e^{u}+e^{-v}\right)}_{x} - \frac{\partial z}{\partial y}\underbrace{\left(e^{-u}-e^{+v}\right)}_{y}$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$ Proved.

If $u = u(y-z,\ z-x,\ x-y)$

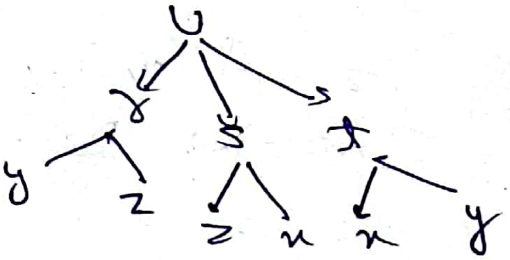
Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let

$r = y - z$

$s = z - x$

$t = x - y$



$u = u(r, s, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial t}\times\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial z}$$

$r = y - z$

$\frac{\partial r}{\partial y} = 1$

$\frac{\partial r}{\partial z} = -1$

$s = z - x$

$\frac{\partial s}{\partial z} = 1$

$\frac{\partial s}{\partial x} = -1$

$t = x - y$

$\frac{\partial t}{\partial x} = 1$

$\frac{\partial t}{\partial y} = -1$

$$\frac{\partial u}{\partial x} = \cancel{\frac{\partial u}{\partial s}(-1)} + \cancel{\frac{\partial u}{\partial t}(1)} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial u}{\partial r}(1)} + \cancel{\frac{\partial u}{\partial t}(-1)} \quad \text{— (ii)}$$

$$\frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial r}(-1)} + \cancel{\frac{\partial u}{\partial s}(1)} \quad \text{— (iii)}$$

Add (i) + (ii) + (iii).

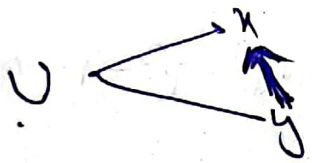
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \quad \text{Proved}$$

Implicit function

We cannot subtract $x-y$ because $x^3 = 1 - y^3 - 3xy$

Q. If $u = x \log(xy)$, $x^3 + y^3 + 3xy = 1$.

find $\frac{du}{dx}$.



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

for implicit function.

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{xy} y + \log(xy) \times 1$$

$$= 1 + \log xy$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x = \frac{x^2}{xy}$$

$$x^3 + y^3 + 3xy = 1.$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0.$$

$$\frac{dy}{dx} = \frac{-x^2 + y}{x + y^2}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 1 + \log xy + \left(\frac{x^2}{xy}\right)\left(\frac{-x^2+y}{x+y^2}\right)$$

Let $z = f(x, y)$

$$\frac{dy}{dx} = \frac{-\partial f}{\partial x} \bigg/ \frac{\partial f}{\partial y}.$$

$\frac{\partial f}{\partial x} = p$ $\quad \frac{\partial f}{\partial y} = q$ $\quad \frac{\partial^2 f}{\partial x^2} = r$ $\quad \frac{\partial^2 f}{\partial x \partial y} = s$.

$\frac{\partial^2 f}{\partial y^2} = t$.

$$\boxed{\frac{d^2y}{dx^2} = \frac{-q^2 r - 2pqs + p^2 t}{q^3}.}$$

Q. If $x^3 + 3x^2y + 6xy^2 + y^3 - 1 = 0$, find $\frac{dy}{dx}$.

$f = x^3 + 3x^2y + 6xy^2 + y^3 - 1$.

$\frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$

$\frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \bigg/ \frac{\partial f}{\partial y}$$

$$= -\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}$$

$$= -\left(\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}\right)$$

Some Typical Cases.

Q.1 If $x = u^2 - v^2$ and $y = uv$
find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

$x = u^2 - v^2$ — ①

diff w.r.t x

$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$.

$y = uv$ — ②

$\cancel{\frac{\partial y}{\partial x}} = 0 = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$.

from eq-n ①

$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}$ — ③. $\times u$.

$0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$ — ④ $\times 2v$.

$u = 2u^2\frac{\partial u}{\partial x} - \cancel{2uv\frac{\partial v}{\partial x}}$

$+ \quad 0 = 2v^2\frac{\partial u}{\partial x} + \cancel{2uv\frac{\partial v}{\partial x}}$.

$u + 0 = 2u^2\frac{\partial u}{\partial x} + 2v^2\frac{\partial u}{\partial x}$

$u = (2u^2 + 2v^2)\frac{\partial u}{\partial x}$.

$$\boxed{\frac{\partial u}{\partial x} = \frac{u}{2u^2 + 2v^2}}$$

Now

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \quad — (3) \quad \times v$$

$$0 = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} \quad — (4) \quad \times 2u.$$

$$v = \cancel{2uv\frac{\partial u}{\partial x}} - 2v^2\frac{\partial v}{\partial x}$$

$$— \quad 0 = \cancel{2uv\frac{\partial u}{\partial x}} + 2u^2\frac{\partial v}{\partial x}$$

$$v - 0 = -2v^2\frac{\partial v}{\partial x} - 2u^2\frac{\partial v}{\partial x}.$$

$$v = \cancel{-2v^2\partial x}\left(-2v^2 - 2u^2\right)\frac{\partial v}{\partial x}.$$

$$\boxed{\frac{\partial v}{\partial x} = \frac{v}{-2v^2 - 2u^2}}$$

$$\frac{\partial v}{\partial x} = -\frac{v}{2v^2 + 2u^2}$$

Again

$$x = u^2 - v^2$$

diff with respect to y

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y}. \quad — (5)$$

$$y = u.v.$$

diff w.r.t to y

$$1 = u.\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}. \quad — (6).$$

$$\cancel{0 = 2u\partial} \quad 1 = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} \quad — (6). \quad \cancel{\times 2v} \quad \times 2v$$

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} \quad — (7) \quad \cancel{\times v} \quad \times u$$

$$\cancel{0 = 2uv\frac{\partial u}{\partial y} - 2v^2\frac{\partial v}{\partial y}}$$

$$\cancel{2v = 2uv\frac{\partial v}{\partial y} + 2v^2\frac{\partial u}{\partial y}}$$

$$2v = \cancel{2uv\frac{\partial v}{\partial y}} + 2v^2\frac{\partial u}{\partial y}$$

$$0 = \cancel{-2uv\frac{\partial v}{\partial y}} + 2u^2\frac{\partial u}{\partial y}.$$

$$2v = (2v^2+2u^2)\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{2v}{2v^2+2u^2} = \frac{v}{u^2+v^2}.$$

Again from equation 5 & 6.

$$0 = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} \quad \times v$$

$$1 = \cancel{\frac{u\,\partial v}{\partial y}}\; v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}. \quad \times 2u.$$

$$0 = 2uv\frac{\partial u}{\partial y} - 2v^2\frac{\partial v}{\partial y}.$$

$$2u = 2uv\frac{\partial u}{\partial y} + 2u^2\frac{\partial v}{\partial y}.$$

$$-2u = -2v^2\frac{\partial v}{\partial y} - 2u^2\frac{\partial v}{\partial y}$$

$$2u = 2v^2\frac{\partial v}{\partial y} + 2u^2\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{2u}{2v^2+2u^2} = \frac{u}{u^2+v^2}$$

CamScanner

Equation to tangent Plane.

$$(x-x_1)\frac{\partial F}{\partial x}+(y-y_1)\frac{\partial F}{\partial y}+(z-z_1)\frac{\partial F}{\partial z}=0.$$

Equation of normal to the plane.

$$\frac{x-x_1}{\frac{\partial F}{\partial x}}=\frac{y-y_1}{\frac{\partial F}{\partial y}}=\frac{z-z_1}{\frac{\partial F}{\partial z}}$$

Find the equation of the tangent plane and normal line to the Surface.

$$x^2+2y^2+3z^2=12 \quad \text{at } (1,2,-1)$$

$$F = x^2+2y^2+3z^2-12=0.$$

$\frac{\partial F}{\partial x}=2x$ at $(1,2,-1)$ $\quad \frac{\partial F}{\partial z}=6z$

$\frac{\partial F}{\partial y}=4y$

$(x_1,y_1,z_1)=(1,2,-1)$

$\frac{\partial F}{\partial x}$ at $(1,2,-1)$ $=2$

$\frac{\partial F}{\partial y}$ at $(1,2,-1)$ $=8$

$\frac{\partial F}{\partial z}$ at $(1,2,-1)$ $=-6$

equation to tangent :

$$(x-1)\times 2+(y-2)8+(z+1)\times -6=0$$

$$2x-2+8y-16-6z-6=0$$

$$2x+8y-6z-24=0$$

$$x+4y-3z=12$$

Equation to normal :

$$\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6}$$

$$\frac{x-1}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$$

Taylor's Series

A function $f(x)$ Can be expanded as a Series of the point $x = x_0$. by

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \dots\dots$$

Maclaurin's Series

$$x_0 = 0.$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Q. Find the maclaurin's Series of $f(x) = \frac{1}{1-x}.$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1} \qquad f(0) = \frac{1}{1-0} = 1$$

$$f'(x) = (-1)(1-x)^{-2} \times -1 = (1-x)^{-2} \qquad f'(0) = 1$$

$$f''(x) = (-2)(1-x)^{-3} \times -1 = \cancel{(1-x)^{-3}} \qquad f''(0) = 2.$$

$$= \frac{2}{(1-x)^3}$$

$$f'''(0) = 3$$

$$f'''(x) = 2 \cdot (-3)(1-x)^{-4} \times (-1)$$

$$\frac{+6}{(1-x)^4}$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\frac{1}{1-x} = 1 + x \times 1 + \frac{x^2}{2} \times 2 + \frac{x^3}{3!} \times 6 + \ldots$$

$- \sin 0°$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Q. $f(x) = \sin x$.

~~$f(0) = \sin 0° = 0$~~

$f'(x) = \cos x$ ~~$f'(0) = 0$~~

$f''(x) = -\sin x$. $f''(0) = 0$.

$f'''(x) = -\cos x$ $f'''(0) = 0$

$\sin x = 1$:

Sol^n

	$x=0$
$f(x) = \sin x$	
$f'(x) = \cos x$	$\cos(0) = 1$.
$f''(x) = -\sin x$	$-\sin(0) = 0$.
$f'''(x) = -\cos x$	$-\cos(0) = -1$
$f^{iv}(x) = \sin x$	$\sin(0) = 0$
$f^{(v)} x = \cos x$	$\cos(0) = 1$

$480 / 360$

$\frac{3 \times 180}{4}$ 45, 135

$\sin(135°)$

$\sin(90+45)$

$\cos 45°$

$\frac{5 \times 180}{4}$ 45, 225

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

Q. Expand $e^x \sin x$. by macullians Series.

$f(x) = e^x \sin x$ ~~$f(x) = e^x \sin x$~~ $x=0$

$f'(x) = e^x . \cos x + \sin x \, e^x$ ~~$f'(x) = e^x . \cos x + \sin x$~~

$= e^x \cos x + e^x \sin x$

$f'(0) = e^x \cos(0) + e^x \sin(0)$

$= e^x + 0 = e^x$.

Other method

$$y = e^{ax} \sin(bx+c)$$

$$y_n = e^{ax} \cdot r^n \sin(bx+c+n\alpha)$$

$$r^2 = a^2+b^2, \quad \tan\alpha = \frac{b}{a}$$

$r = \sqrt{a^2+b^2}$

$= \sqrt{1+1} = \sqrt{2}$

$\tan\alpha = \frac{b}{a}$

$\alpha = \tan^{-1}\frac{b}{a}$

$\alpha = \tan^{-1} 1$

$\alpha = \frac{\pi}{4}$

$y = e^x \sin x$

$y_n = e^x (\sqrt{2})^n \sin(x + n\frac{\pi}{4})$

$\cancel{y_1 = e^x \sqrt{2} \sin(x+\frac{\pi}{4})}$

$\cancel{= e^x \sqrt{2} \cos\frac{\pi}{4}}$

$\cancel{= e^x \sqrt{2} \times \frac{1}{\sqrt{2}} = e^x}$

$f^n(x)$.

$f(x) = e^x \sin x$

$f'(x) = e^x(\sqrt{2}) \sin(x+\frac{\pi}{4})$

$f''(x) = e^x(\sqrt{2})^2 \sin(x+2\frac{\pi}{4})$

$f'''(x) = e^x(\sqrt{2})^3 \sin(x+\frac{3\pi}{4})$

$f^{iv}(x) = e^x(\sqrt{2})^4 \sin(x+\frac{4\pi}{4})$

$f^{(v)}(x) = e^x(\sqrt{2})^5 \sin(x+\frac{5\pi}{4})$

$f(0) = 0$.

$f'(0) = e^0 \sqrt{2} \sin\frac{\pi}{4} = \sqrt{2} \times \frac{1}{\sqrt{2}}$

$= 1$

$f''(0) = e^0 (\sqrt{2})^2 \sin(2\frac{\pi}{4})$

$= e^0 \times 2 \times 1 = 2$

$f'''(0) = e^0 (2\sqrt{2}) \sin(\frac{3\pi}{4})$

$= e^0 (2\sqrt{2}) \times \frac{1}{\sqrt{2}} = 2$

$f^{iv}(0) = e^0 \cdot (2\times 2) \cdot \sin(\pi)$

$= 0$

$f^{(v)}(0) = e^0 \, 4\sqrt{2} \sin(\frac{5\pi}{4})$

$= e^0 \times 4\sqrt{2} \times -\frac{1}{\sqrt{2}}$

$= -4$

$$e^x \sin x = \cancel{4+x} \cdot x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots$$

Taylor's Series

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \dots\dots$$

Find the taylor Series for $f(x) = \cos x$ about $x = \frac{\pi}{2}$

Solution

	$x_0 = \frac{\pi}{2}$
$f(x) = \cos x$	$\cos(\frac{\pi}{2}) = 0$
$f'(x) = -\sin x$	$-\sin(\frac{\pi}{2}) = -1$
$f''(x) = -\cos x$	$-\cos(\frac{\pi}{2}) = 0$
$f'''(x) = \sin x$	$\sin(\frac{\pi}{2}) = 1$
$f^{iv}(x) = \cos x$	$\cos(\frac{\pi}{2}) = 0$
$f^{v}(x) = -\sin x$	$-\sin(\frac{\pi}{2}) = -1$

$$\cos x = 0 + (x - \tfrac{\pi}{2}) \times (-1) + \frac{(x-\frac{\pi}{2})^2}{2!} \times 0 + \frac{(x-\frac{\pi}{2})^3}{3!} \times 1$$

$$+ \frac{(x-\frac{\pi}{2})^4}{4!} \times 0 + \frac{(x-\frac{\pi}{2})^5}{5!} \times (-1)$$

$$= -(x - \tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \dots\dots$$

Q. Expand $\log x$ in powers of $(x-1)$ by Taylors Series.

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!}$$

$$f'''(x_0) + \ldots\ldots\ldots\ldots$$

Solution.

$(x-1)$ $\qquad x-1=0$, $x=1$.

$x_0 = 1$

$f(x) = \log x.$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f'''(x) = \cancel{-(x^2)^{-1}} \quad \frac{2}{x^3}$

$= \cancel{(x^2)}$

$f^{iv}(x) = \frac{-6}{x^4}$

$x_0 = 1$, $\log(1) = 0$

$f'(1) = 1$

$f''(1) = \frac{-1}{1} = -1$

$f'''(1) = 2$

$f^{iv}(1) = -6$

$$\log(x) = 0 + \frac{(x-1)}{1!}(1) + \frac{(x-1)^2}{2!} \times (-1) + \frac{(x-1)^3}{3!} \times 2$$

$$+ \frac{(x-1)^4}{4!} \times -6$$

$$\log(x) = x-1 + \frac{(x-1)^2}{2} + \frac{(x-3)^3}{3} - \frac{(x-1)^4}{4} + \ldots\ldots\ldots$$

Taylor's Series for two Variables :

~~f(a+h)~~

$$f(a+h,\ b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots$$

$$= f(a,b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \frac{1}{3!}\left[h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right] + \ldots$$

$$= f(a,b) + hf_x + kf_y + \frac{1}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right] + \frac{1}{3!}\left[h^3 f_{xxx} + 3h^2k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right]$$

Put $a=0,\ b=0,\ h=x,\ k=y$

$$= f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f_{(a,b)} + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f_{(a,b)} + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(a,b) + \ldots$$

$$= f_{(0,0)} + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f_{(0,0)} + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f_{(0,0)} + \frac{1}{3!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^3 f_{(0,0)} + \ldots$$

Q: Expand $e^x \sin y$ in powers of x & y.

$f(x,y) = e^x \sin y$

$f_x = e^x \sin y$

$f_y = e^x \cos y$

$f_{xx} = e^x \sin y$

$f_{xy} = e^x \cos y$

$f_{yy} = -e^x \sin y$

$f_{xxx} = e^x \sin y$

$f_{xxy} = e^x \cos y$

$f_{xyy} = -e^x \sin y$

$f_{yyy} = -e^x \cos y$

$x = 0 \quad y = 0$

$e^0 (\sin(0)) = 0$

$f_x(0,0) = 0$

$f_y(0,0) = e^0 \cos(0) = 1$

$f_{xx}(0,0) = e^0 \sin(0) = 0$

$f_{xy}(0,0) = e^0 \cos 0 = 1$

$f_{yy}(0,0) = -e^0 \sin 0 = 0$

$f_{xxx}(0,0) = e^0 \sin(0) = 0$

$f_{xxy}(0,0) = e^0 \cos 0 = 1$

$f_{xyy}(0,0) = -e^0 \sin 0 = 0$

$f_{yyy}(0,0) = -e^0 \cos 0 = -1$

~~$e^x \sin y = 0 +$~~

$$f(x,y) = f(0,0) + x f_x + y f_y + \frac{1}{2!}\left[x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right] + \frac{1}{3!}\left[x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right] + \dots$$

$$e^x \sin y = 0 + y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \dots$$

Q. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ Using taylors theorem.

Solution

$$f(a+h, b+k) = f(a,b) + hf_x(a,b) + kf_y(a,b) + \frac{1}{2!}\left[h^2 f_{xx}(a,b)\right.$$

$$\left. + 2hkf_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b)\right.$$

$$\left. + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

		$(a,b) = (1,-2)$
$f(x,y)$	$x^2y + 3y - 2$	-10
f_x	$2xy$	-4
f_y	$x^2 + 3$	4
f_{xx}	$2y$	-4
f_{xy}	$2x$	2
f_{yy}	0	0
f_{xxy}	2	2
f_{xyy}	0	0
f_{yyy}	0	0
f_{xxx}	0	0

$\therefore f(x,y) = x^2y + 3y - 2$

Substitute

$a + h = x$, $\quad h = x - a = x - 1$

$b + k = y$ $\quad k = y - b = y + 2$

$$f(x,y) = x^2y + 3y - 2 = -10 + (x-1)\times(-4) + (y+2)\times 4 + \frac{1}{2!}(x-1)^2\times(-4) + 2(x-1)(y+2)\times 2 + (y+2)^2 \times 0 + \frac{1}{3!}(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3\times 0 + \cdots$$

$$f(x,y) = x^2y + 3y - 2 = -10 + (x-1)(-4) + (y+2)(4) + \frac{1}{2!}\left[(x-1)^2(-4) + 4(x-1)(y+2)\right] + \frac{1}{6}\left[6(x-1)^2(y+2)\right]$$

$$\cancel{= -10 - 4x + 4 + 4y + 8 + \frac{1}{2}\left[x^2 + 1 - 2x\,(-4)\right] +}$$

Jacobian

If u, v are function of x & y

$$\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial y}.$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right)$$

(jacobian of u, v w.r.t to x & y)

for u, v, w are function x, y, z

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

CamScanner

Q. $u = x^2 - y^2$
$v = 2xy$

find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y$ $\qquad$ $\frac{\partial v}{\partial y} = 2x$

Now

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = (2x)(2x) - [2y(-2y)]$$
$$4x^2 + 4y^2 = 4\cdot[x^2+y^2]$$

Q. If $x = r\cos\theta$
$y = r\sin\theta$

Evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$

Solution $\frac{\partial x}{\partial r} = \cos\theta$ $\qquad$ $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$ $\qquad$ $\frac{\partial y}{\partial \theta} = r(\cos\theta)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r[\cos^2\theta + \sin^2\theta] = r. \quad \text{Ans}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}. \quad \text{Ans.}$$

2 If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$.

Show that $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = 4$.

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$\cancel{\partial y_1} \quad y_1 = \frac{x_2x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_1} = x_2x_3\left(-\frac{1}{x_1^2}\right) = -\frac{x_2x_3}{x_1^2}$$

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1} \qquad y_2 = \frac{x_3x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = x_3x_1\left(-\frac{1}{x_2^2}\right) = -\frac{x_3x_1}{x_2^2}$$

$y_3 = \frac{x_1 x_2}{x_3}$

$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$ | $\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$ | $\frac{\partial y_3}{\partial x_3} = \frac{-x_1 x_2}{x_3^2}$

Now

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \left(\frac{1}{x_1^2 x_2^2 x_3^2}\right) \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_2 x_1 \\ x_3 x_2 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$(x_3 x_1) \frac{(x_2 x_3)(x_2 x_1)}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$(1)\ -1[1-1] - 1[-1-1] + 1[1+1]$

$(1) \times 0 - 1(-2) + 2$

$0\ 2+2$

$= 4$

Properties of Jacobian

① If u, v are the functions of x & y.

then, $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Eg if $x = uv$.

$y = \frac{u+v}{u-v}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

$x = uv$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{+2v}{(u-v)^2} \end{vmatrix}$$

Rough

$\partial\left(\frac{u}{v}\right) = \frac{v\,\partial u - u\partial v}{v^2}$

$y = \frac{u+v}{u-v}$.

$$\frac{\partial y}{\partial u} = \frac{(u-v)\frac{\partial}{\partial u}(u+v) - (u+v)\frac{\partial}{\partial u}(u-v)}{(u-v)^2}$$

$$= \frac{(u-v) - (u+v)}{(u-v)^2} = \frac{\cancel{u}-v-\cancel{u}-v}{(u-v)^2} = \frac{-2v}{(u-v)^2}$$

$$\frac{\partial y}{\partial v} = (u-v)\frac{\partial}{\partial v}(u+v) - (u+v)\frac{\partial}{\partial v}(u-v) \Big/ (u-v)^2$$

$$= (u-v) - (u+v) = [u-v-(u-v)\times(-1)] = \frac{+2u}{(u-v)^2}$$

$(u-\cancel{v}+u+\cancel{v}) =$

$$= VU \begin{vmatrix} 1 & 1 \\ \frac{-2}{(U-V)^2} & \frac{2}{(U-V)^2} \end{vmatrix}$$

$$\frac{UV}{(U-V)^2} \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix}$$

$$\frac{UV}{(U-V)^2} \left| 2 + 2 \right|$$

$$\frac{4UV}{(U-V)^2} \text{ Ans} \leftarrow$$

$$\therefore \frac{\partial(U,V)}{\partial(x,y)} = \frac{(U-V)^2}{4UV}. \text{ Ans} \leftarrow$$

(2) $\frac{\partial(U,V)}{\partial(x,y)} = \frac{\partial(U,V)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

Find the value of $\frac{\partial(U,V)}{\partial(r,\theta)}$ where, $\left.\begin{array}{l} U = x^2 - y^2 \\ V = 2xy. \end{array}\right] \frac{\partial(U,V)}{\partial(x,y)}$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right] \frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4x^2 + 4y^2 = 4(x^2+y^2).$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r(-\sin\theta) \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(1) = r$ Ans.

$x = r\cos\theta$ } so
$y = r\sin\theta$

$x^2 = r^2\cos^2\theta$
$y^2 = r^2\sin^2\theta$ $\Rightarrow$ $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$
$x^2+y^2 = r^2.$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$= 4r^2 \cdot r$

$= \underline{\underline{4r^3}}$ Ans

14/11/22

Functional dependence and Independence.

Q. If u, v are functions of two Variable x & y this for functional dependence $J\left(\frac{u,v}{x,y}\right) = 0$.

or $\frac{\partial(u,v)}{\partial(x,y)} = 0$

If for 3 Variables,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

If $u = xy + yz + zx$,

$v = x^2+y^2+z^2$

$w = x+y+z$, determine whether ther is functional relationship between u, v, w & If so find it.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \Rightarrow$$ it is fonctional dependent
There is relation betw them

Solution

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = y+z \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = x+z \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = y+x \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & x+z & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$R_1 \to R_1 + R_2$$

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$