$= r\cos^2\theta + r\sin^2\theta$

$= r(1) = r$ Ans.

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ so

$x^2 = r^2\cos^2\theta$
$y^2 = r^2\sin^2\theta$ $\Rightarrow$ $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$
$x^2+y^2 = r^2.$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= 4r^2 \cdot r$$

$$= \underline{\underline{4r^3}} \text{ Ans}$$

14/11/22

Functional dependence and Independence.

Q. If u, v are functions of two Variable x & y this for functional dependence $J\left(\frac{u,v}{x,y}\right) = 0$.

or $\frac{\partial(u,v)}{\partial(x,y)} = 0$

If for 3 Variables,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

If $u = xy + yz + zx$,

$v = x^2+y^2+z^2$

$w = x+y+z$, determine whether ther is functional relationship between u, v, w & if so find it.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \Rightarrow$$ it is functional dependent
There is relation b/w them

Solution

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = y+z \qquad \frac{\partial v}{\partial x} = 2x \qquad \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = x+z \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial w}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = y+x \qquad \frac{\partial v}{\partial z} = 2z \qquad \frac{\partial w}{\partial z} = 1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & x+z & y+x \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \rightarrow R_1 + R_2$

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad w = x+yz$$

$$w^2 = x^2+y^2+z^2+2xy+2yz+2zx.$$

$$\boxed{w^2 = v+2u.}$$

Q Show that the function $u = \frac{x+y}{1-xy}$.

$v = \tan^{-1}x + \tan^{-1}y$ are not independent

and find the relation between them

$$= \frac{\partial(u,v)}{\partial(x,y)} = 0 \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\cancel{u = \frac{x+y}{1-xy}}$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{1}{-y} = \frac{-1}{y}}$$

$$\cancel{\frac{\partial u}{\partial y} = \frac{-1}{x}}$$

$$\cancel{v = \tan^{-1}x + \tan^{-1}y}$$

$$\cancel{\frac{\partial v}{\partial x} = \frac{1}{1+x^2}}$$

$$u = \frac{x+y}{1-xy}.$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)\frac{\partial(x+y)}{\partial x} - (x+y)\frac{\partial(1-xy)}{\partial x}}{(1-xy)^2}$$

$$= \frac{(1-xy) - (x+y)(-1)y}{(1-xy)^2} = \frac{1-xy-[(x+y)(-y)]}{(1-xy)^2}$$

$$\frac{1-xy-[-yx-y^2]}{(1-xy)^2}$$

$$\frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)\frac{\partial(x+y)}{\partial y} - (x+y)\frac{\partial(1-xy)}{\partial y}}{(1-xy)^2}$$

$$= \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2}$$

$$= \frac{1-xy - [-x^2 - yx]}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy} + x^2 + \cancel{yx}}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$V = \tan^{-1}x + \tan^{-1}y$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

$$(1-xy)^2 \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

$$(1-xy)^2 \left[\cancel{1+y^2} \times \frac{1}{\cancel{1+y^2}} - \frac{1}{\cancel{1+x^2}} \times \cancel{1+x^2} \right] = 0$$

$$(1-xy)^2 [1-1] = 0$$

$V = \tan^{-1} x + \tan^{-1} y$:

$V = \tan^{-1}\left[\frac{x+y}{1-xy}\right] =$

$V = \tan^{-1} U$

$$\boxed{U = \tan V}$$

Jacobian of implicit functions

The Variables x, y, u, v are Connected by implicit functions $f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$ where u, v are implicit function of x & y.

$$\left\{ \begin{aligned} \frac{\partial(u,v)}{\partial(x,y)} &= \frac{(-1)^2 \partial(f_1,f_2) / \partial(x,y)}{\partial(f_1,f_2) / \partial(u,v)} \\ \frac{\partial(u,v,w)}{\partial(x,y,z)} &= \frac{(-1)^3 \partial(f_1,f_2,f_3) / \partial(x,y,z)}{\partial(f_1,f_2,f_3) / \partial(u,v,w)} \end{aligned} \right\}$$

Q If $x^2 + y^2 + u^2 - v^2 = 0$
$uv + xy = 0$ Prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2 - y^2}{u^2 + v^2}$.

$f_1 = x^2 + y^2 + u^2 - v^2$

$f_2 = uv + xy$.

$\frac{\partial f_1}{\partial x} = 2x$.

$\frac{\partial f_1}{\partial y} = 2y$

$\frac{\partial f_2}{\partial x} = y$

$\frac{\partial f_2}{\partial y} = x$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2.$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}.$$

$$= 2u^2 + 2v^2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\cancel{2}(x^2-y^2)}{\cancel{2}(u^2+v^2)}$$

$$= \frac{x^2-y^2}{u^2+v^2}$$

Q If $x+y+z=u$

$y+z=uv$.

$z=uvw$.

Show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

$f_1 = x+y+z-C_1 = 0$

$f_2 = y+z-uv$

$f_3 = z-uvw$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\cancel{\partial(x,y,}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1,f_2,f_3) / \partial(\cancel{x},\cancel{y},\cancel{z})}{\partial(f_1,f_2,f_3) / \partial(\cancel{u},\cancel{v},\cancel{w})} \cdot \frac{(u,v,w)}{(x,y,z)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{f x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{z} \end{vmatrix}$$

$=$

Extrema of functions of Several Variables.

Maximum ~~Variables~~ Value :

A function $f(x,y)$ said is have a maximum Value at $x=a,\ y=b$ if there exist a small neighbourhood of (a,b) such that.

$$\boxed{f(a,b) > f(a+h,\ b+k)}$$

Minimum Value.

if $\boxed{f(a,b) < f(a+h,\ b+k)}$

Working rule to find extremum Value.

① $f(x,y)$ be the function

$\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y^2}$

② Put $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ & Solve for (x,y)

Let the Value be (a,b)

③ Evaluate $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$ at the point (a,b)

④ if $rt - s^2 > 0$ and if $r < 0$ then $f(x,y)$ has maximum Value.

If $r > 0$ then $f(x,y)$ has minimum Value.

⑤ If $rt - s^2 < 0$ then $f(x,y)$ has not extremum values at (a,b).

(6) If $rt - s^2 = 0$ then the Case is doubtful.

Q. Discuss the maximum and Minimum Value of $x^2 + y^2 + 6x + 12$:

$f(x,y) = x^2 + y^2 + 6x + 12$

$\frac{\partial f}{\partial x} = 2x + 6$

$\frac{\partial f}{\partial y} = 2y$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0 \Rightarrow 2x = -6 \Rightarrow x = -3$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$

Point is $(-3, 0)$

$r = \frac{\partial^2 f}{\partial x^2} = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

$\Rightarrow rt - s^2$

$\Rightarrow 2(2) - 0$

$= 4 > 0$

$r > 0$

function has minimum value at $(-3, 0)$

Minimum value

$f(x,y) = x^2 + y^2 + 6x + 12$

$= (-3)^2 + 0 + 6(-3) + 12$

$= 9 - 18 + 12$

$= 21 - 18$

$= 3$

$\therefore$ The minimum value of the given function is 3.

Q. $f(x,y) = y^2 + 4xy + 3x^2 + x^3$.

$$\frac{\partial f}{\partial x} = 4y + 6x + 3x^2$$

$$\frac{\partial f}{\partial y} = 2y + 4x$$

$\frac{\partial f}{\partial x} = 0$

$4y + 6x + 3x^2 = 0$.

Put value of y

$4(-2x) + 6x + 3x^2 = 0$

$-8x + 6x + 3x^2 = 0$

$3x^2 - 2x = 0$

$\cancel{3x^2 = 2x}$

$\cancel{3x^2 = 2x}$

$\cancel{3x = 2}$

$\cancel{x = \frac{2}{3}}$

$x(3x-2)=0$

$x = 0$

$3x - 2 = 0$.

$x = \frac{2}{3}$

$\frac{\partial f}{\partial y} = 0$

$2y + 4x = 0$

$2y = -4x$.

$y = -2x$

Put value of x

at $x = \frac{2}{3}$

$y = -2\left(\frac{2}{3}\right)$

$= \frac{-4}{3}$

at $x = 0$.

$y = -2(0)$

$= 0$

$x = 0, \quad y = 0$

$x = \frac{2}{3}, \quad y = \frac{-4}{3}$ $\Rightarrow$ Points are $(0,0)$, $\left(\frac{2}{3}, \frac{-4}{3}\right)$

$$\frac{\partial f}{\partial x} = 4y + 6x + 3x^2$$

$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x$.

$\cancel{\frac{\partial f}{\partial y} = 2y + 4x}$

$s = \frac{\partial^2 f}{\partial x \partial y} = 4$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

$r = 6 + 6x$

$s = 4$

$t = 2$

∂f

	$(0,0)$	$\left(\frac{2}{3}, -\frac{4}{3}\right)$
$r = 6x + 6$	6	$6 \times \frac{2}{3} + 6 = 10$
$s = 4$	4	4
$t = 2$	2	2
$rt - s^2$	$\cancel{(6x+6)(2) - (4)^2}$ $\cancel{(6x+6)2 - 16}$ $\cancel{12x + 12 - 16}$ $\cancel{12x - 4}$ -4	$+4$

$$rt - s^2 > 0 \quad \text{at} \quad \frac{2}{3}, \frac{-4}{3}$$

$$r > 0.$$

function has minimum value at $\left(\frac{2}{3}, -\frac{4}{3}\right)$

Minimum Value.

$$f(x,y) = y^2 + 4xy + 3x^2 + x^3$$

$$= \left(\frac{-4}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(-\frac{4}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$$

$$= \frac{16}{9} - \frac{32}{9} + \frac{12}{9} + \frac{8}{27}$$

$$= \frac{48-}{27}$$

Show that the minimum value of

$f(x,y) = xy + a^3\left(\frac{1}{x}+\frac{1}{y}\right)$ is $3a^2$

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$

$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$

$\frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$

$\frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$

$\frac{\partial f}{\partial x} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0$

$\frac{\partial f}{\partial y} = 0$, $x - \frac{a^3}{y^3} = 0.$

$y - \frac{a^3}{x^2} = 0 \quad -①$

$x - \frac{a^3}{y^2} = 0. \quad -②$

$x^2 y - a^3 = 0$

$xy^2 - a^3 = 0$

$x = \frac{a^3}{y^2}$

Put $x = \frac{a^3}{y^2}$

$\left(\frac{a^3}{y^2}\right)^2 y - a^3 = 0$

$\cancel{\frac{a^6 y}{y^4} - a^3 = 0}$

$\cancel{a^6 y - a^3 y^4 = 0}$

$\cancel{a^3 y (a^3 - y^3) = 0}$

$\frac{a^6}{y^3} - a^3 = 0$

$\frac{a^6}{y^3} = a^3$

$y^3 = a^3$

$y = \pm a$

$\cancel{x - \frac{a^3}{y^2} = 0}$

$x = \frac{a^3}{y^2}$

$x = \frac{a^3}{a^2} = a$

	(a,a)	$(-a,a)$
$r = \frac{2a^3}{x^3}$	2	-2
$s = 1$	1	1
$t = \frac{2a^3}{y^3}$	2	2
$rt - s^2$	3	-5

$rt - s^2 > 0$ at (a,a)

$r > 0$.

function has minimum Value at (a,a)

Minimum Value

$$f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$= (a \times a) + a^3\left(\frac{1}{a} + \frac{1}{a}\right)$$

$$= \cancel{a^2 + a^3\left(\frac{2}{a^2}\right)} \quad a^2 + a^3\left(\frac{2}{a}\right)$$

$$a^2 + 2a^2 = 3a^2 \text{ Ans}$$

$$= \cancel{a^2}$$

Lagrange's Method of Multipliers

It is mainly used to find maxima & Minima Value of function of several (2 or More) Variabl in which all Variables are not Independent but Connected by a relation.

Working rule:

Let $f(x,y)$ be an objective function of 3 Variables x, y, z and Variables be Connected by the Connected by the Constraint.

Step I : Identify the objective function.

Step II : Identify the Constraint function.

Step III : Find the lagranges function

$$F(x,y) = f(x,y,z) + \lambda \phi(x,y,z) \quad — (1)$$

$\lambda \rightarrow$ a non zero lagrange multiplies

Step IV : find $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$.

Step V : Solve it find the Stationary function

1. Divide a number 'b' into 3 parts such that the product will be maximum

Let the parts be x, y, z.

$x + y + z = b \rightarrow$ Constraint $\phi(x,y,z) = 0$

Product $= xyz \rightarrow$ objective function $= f(x,y,z)$

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$= xyz + \lambda(x+y+z-b)$$

$$=$$

$$\frac{\partial F}{\partial x} = yz + \lambda = 0 \quad — (i)$$

$$\frac{\partial F}{\partial y} = xz + \lambda = 0 \quad — (ii)$$

$$\frac{\partial F}{\partial z} = xy + \lambda = 0 \quad — (iii)$$

from (i) & (ii)	from (ii) & (iii)
$\lambda = -yz$	$\lambda = -xz$
$\lambda = -xz$	$\lambda = -xy$
$-y\cancel{z} = -x\cancel{z}$	$-\cancel{x}z = -\cancel{x}y$
$y = x$	$z = y$

~~from (1)~~

$$\boxed{y = x = z}$$

$x+y+z=b$

$x+x+x=b$

$3x=b$

$x=\frac{b}{3}$

Similarly, $y=\frac{b}{3}$ & $z=\frac{b}{3}$

18-11-22

Find the dimension of rectangular box of Maximum Capacity whose surface area is given. When;

(a) box is open at the top.

(b) box is closed.

Solution Let $x \to$ length

$y \to$ breadth

$z \to$ heigh.

Volume $= xyz$

Total $S = 2[xy+yz+zx]$

$S = xy + 2[yz + zx]$

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

When box is open.

$S = xy + 2[yz + zx]$

$F(x,y,z) = xyz + \lambda[xy + 2(yz+zx)]$

$\frac{\partial F}{\partial x} = yz + \lambda[y + 2z] = 0 \quad \times x$

$\frac{\partial F}{\partial y} = xz + \lambda[x + 2z] = 0 \quad \times y$

$\frac{\partial F}{\partial z} = xy + \lambda[2y + 2x] = 0 \quad \times z$

$$xyz + \lambda[yx + 2zx] = 0 \quad \text{— (iv)}$$

$$xyz + \lambda[yx + 2zy] = 0 \quad \text{— (v)}$$

$$xyz = \lambda[2yz + 2xz] = 0 \quad \text{— (vi)}$$

Subtract (iv) & (v)

$$\cancel{xyz} + \lambda[\cancel{yx} + 2zx] = 0$$

$$\cancel{xyz} + \lambda[\cancel{yx} + 2yz] = 0$$

$$\therefore 2zx\lambda - 2zy\lambda = 0$$

$$2z\lambda[x-y] = 0 \qquad \because \phi[x-y] = 0, \quad \phi \neq 0$$

$$x - y = 0$$

$$\boxed{x = y}$$

From eq (v) & (vi)

$$\cancel{xyz} + \lambda[\cancel{xy} + 2zy] = 0$$

$$\cancel{xyz} + \lambda[2yz + 2xz] = 0$$

$$\cancel{-\lambda yz + 2\lambda zy - 2\lambda xz = 0}$$

$$\lambda[xy - 2xz] = 0$$

$$\lambda x[y - 2z] = 0$$

$$y - 2z = 0$$

$$y = 2z$$

$$\boxed{x = y = 2z}$$

When the box is closed

$S = 2[xy + yz + zx]$

$F(x,y,z) = xyz + \lambda[2(xy + yz + zx)]$

$\frac{\partial F}{\partial x} = yz + \lambda[2(y + z)] = 0 \quad \times x \quad$ — (i)

$\frac{\partial F}{\partial y} = xz + \lambda[2(x + z + 0)] = 0 \quad \times y \quad$ — (ii)

$\frac{\partial F}{\partial z} = xy + \lambda[2(y + x)] = 0 \quad \times z \quad$ — (iii)

$yz + \lambda[2(y + yz + z)] = 0 \quad \times x$

$xz + \lambda[2(x + z)] = 0 \quad \times y$

$xy + \lambda[2(y + x)] = 0 \quad \times z$

$\cancel{xyz = x\lambda[2y + 2yz + 2z] = 0}$

$\cancel{xyz} = xyz + \lambda[2(\cancel{xy} + xyz + \cancel{xz})] = 0$

$xyz + \lambda[2(\cancel{yx} + \cancel{zy})] = 0$

$\cancel{2\lambda xyz} - 2\lambda$

$2\lambda xyz + xz - 2\lambda zy = 0$

$\cancel{2\lambda zy[x - 1]} = -xz$

$\cancel{2\lambda yz[}$

~~from~~ (i) & (ii)

$xyz + 2\lambda[xy + xz]$

$xyz + 2\lambda[xy + yz]$

$2\lambda[xz - yz]$

$2\lambda z[x - y] = 0$

$x = y$

From equation (ii) & (iii)

$$\cancel{xyz} + 2\lambda\,[xy + yz] = 0$$

$$\underline{-\,\cancel{xyz} + 2\lambda\,[yz + xz] = 0}$$

$$2\lambda x\,[y - z] = 0$$

$$y = z$$

Finally $\boxed{x = y = z}$

Prove that the Volume of greatest rectangular parallelopiped that can be inscribed in ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{is} \quad \frac{8abc}{3\sqrt{3}}$$

Solution:

Let $l = 2x$
$b = 2y$
$h = 2z$

$$V = lbh$$
$$= 8xyz$$

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$= 8xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right] = 0$$

$$\frac{\partial F}{\partial x} = 8yz + \lambda\left[\frac{2x}{a^2}\right] = 0 \quad \times x \qquad -(i) \qquad -(i)$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda\left[\frac{2y}{b^2}\right] = 0 \quad \times y \qquad -(ii) \qquad -(ii)$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda\left[\frac{2z}{c^2}\right] = 0 \quad \times z \qquad -(iii) \qquad -(iii)$$

from (i) & (ii)

$$8xyz + \lambda\left[\frac{2x^2}{a^2}\right] = 0$$

$$8xyz = \lambda\left[\frac{2\cdot y^2}{b^2}\right] = 0$$

$$\lambda\left[\frac{2x^2}{a^2} - \frac{2y^2}{b^2}\right] = 0$$

$$2\lambda\left[\frac{x^2}{a^2} - \frac{y^2}{b^2}\right] = 0$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

(i) + (ii) + (iii)

$$24xyz + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

$$24xyz + 2\lambda = 0$$

$$\lambda = -12xyz$$

Put λ in eq (i).

$$8xyz + \lambda\left[\frac{2x^2}{a^2}\right] = 0.$$

$$8xyz + (-12xyz)\left[\frac{2x^2}{a^2}\right] = 0$$

$$8yz - \frac{24x^2yz}{a^2} = 0$$

$$8yz\left[1 - \frac{3x^2}{a^2}\right] = 0$$

$$1 - \frac{3x^2}{a^2} = 0.$$

$$\frac{3x^2}{a^2} = 1 \Rightarrow 3x^2 = a^2$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Put λ in eq (ii)

$$8xz + \lambda\left[\frac{2y}{b^2}\right] = 0$$

$$8xz - 12xyz\left[\frac{2y}{b^2}\right] = 0$$

$$8xz - \frac{24xy^2z}{b^2} = 0$$

$$8xz\left[1 - \frac{3y^2}{b^2}\right] = 0$$

$$1 = \frac{3y^2}{b^2}$$

$$3y^2 = b^2$$

$$y^2 = \frac{b^2}{3}$$

$$y = \frac{b}{\sqrt{3}}$$

Similarly
Put y in eq (iii)

$$8xy + \lambda\left[\frac{2z}{b^2}\right] = 0$$

$$8xy - 12xyz\left[\frac{2z}{b^2}\right] = 0$$

$$8xy - \frac{24xyz^2}{b^2} = 0$$

$$8xy\left[1 - \frac{3z^2}{b^2}\right] = 0$$

$$3z^2 = b^2$$

$$z = \frac{b}{\sqrt{3}}$$

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}} \times \frac{b}{\sqrt{3}} \times \frac{c}{\sqrt{2}}\right)$$

$$= \frac{8\,abc}{3\sqrt{3}}$$

Ans

Maths

21-11-22.

Double Integration with Constant limit.

Q.

$$\int_0^2 \int_0^1 (x+y)\,dx\,dy$$

$$\int_0^2 \left[\frac{x^2}{2} + yx\right]_0^1 dy$$

$$\int_0^2 \left(\frac{1}{2} + y) - (0)\right] dy$$

$$\int_0^2 \left(\frac{1}{2}+y\right) dy$$

$$\left[\frac{1}{2}y + \frac{y^2}{2}\right]_0^2$$

$$(1 + 2) - (0)$$

$$\underline{3}\ \text{As} \leftarrow$$

Q.

$$\int_0^a \left[\int_0^b (xy + y)\,dy\right] dx.$$

$$\int_0^a \left[x\frac{y^2}{2} + \frac{y^2}{2}\right]_0^b dx$$

$$\int_0^a \left(x\frac{b^2}{2} + \frac{b^2}{2}\right) - (0)\Big]\,dx$$

$$\int_0^a \left(\frac{xb^2}{2} + \frac{b^2}{2}\right) dx$$

$$\left[\frac{x^2}{2}\times\frac{b^2}{2} + \frac{b^2}{2}\cdot x\right]_0^a$$

$$\left[\frac{a^2b^2}{4} + \frac{ab^2}{2}\right].$$

$$\frac{b^2}{2}\left[\frac{a^2}{2} + a\right]$$

Q/

$$\int_1^a \left[\int_1^b \frac{dx}{xy}\right] dy$$

$$= \int_1^a \int_1^b \frac{1}{xy}\, dx\, dy$$

$$= \int_1^a \int_1^b \frac{1}{x} \cdot \frac{1}{y}\, dx\, dy$$

$$= \int_1^a \left[\frac{\log x}{y}\right]_1^b dy$$

$$\int_1^a \left(\frac{\log b}{y}\right) - \left(\frac{\log 1}{y}\right) dy$$

$$\int_1^a \left(\frac{\log b}{y}\right) dy$$

$$\log b \int_1^a \frac{1}{y}\, dy$$

$$\log b \left[\log y\right]_1^a$$

$$\log b \cdot \left[\log a - \log 1\right]$$

$$\log b \cdot \log a \quad \text{A} \leftarrow$$

$$\cancel{\log(a+b)}$$

Q $\int_0^{\log 3}\int_0^{\log 2} e^{x+y}\,dx\,dy$.

$= \int_0^{\log 3}\left[\int_0^{\log 2} e^{x}.e^{y}\,dx\right]dy$

$= \int_0^{\log 3}\left[e^{y}.e^{x}\right]_0^{\log 2}dy$

$= \int_0^{\log 3}\left[(e^{y}.e^{\log 2}) - (e^{y}.e^{0})\right]dy$

$\int_0^{\log 3}\left[e^{y}.e^{\log 2} - e^{y}\right]dy$

$\int_0^{\log 3} e^{y}\left[e^{\log 2} - 1\right]dy$

$(e^{\log 2} - 1)\int_0^{\log 3} e^{y}\,dy$

$\left[e^{y}\right]_0^{\log 3}$

$(e^{\log 3} - e^{0})$

$3 - 1 = 2$ A

$$\int_0^{\log 3}\int_0^{\log 2} e^{2x+3y}\,dx\,dy$$

$$\int_0^{\log 3}\int_0^{\log 2} e^{2x}\cdot e^{3y}\,dx\,dy$$

$$\int_0^{\log 3}\left[\frac{e^{2x}}{2}e^{3y}\right]_0^{\log 2} dy$$

$$\int_0^{\log 3}\left[\left(\frac{e^{2(\log 2)}}{2}e^{3y}\right)-\left(\frac{e^0}{2}\cdot e^{3y}\right)\right]dy$$

$\log 2^2$

$$\int_0^{\log 3}\left(\frac{e^{(\log 2)^2}}{2}-e^{3y}\right)-\left(\frac{1}{2}\cdot e^{3y}\right)\Big]dy$$

$$\cancel{\int_0^{\log 3}\left[-\frac{e^{3y}}{2}\left(e^{(\log 2)^2}+1\right)\right]dy}$$

$$\cancel{\frac{1}{2}\left(e^{(\log 2)^2}+1\right)\int_0^{\log 3} -\frac{e^{3y}}{3}\,dy}$$

$$\cancel{-\frac{1}{2}\left(e^{(\log 2)^2}+1\right)\left[\frac{e^{3y}}{3}\right]_0^{\log 3}}$$

$$\cancel{-\frac{1}{2}\left(e^{(\log 2)^2}+1\right)\left[\frac{e^{3(\log 3)}}{3}-\frac{e^0}{3}\right]}$$

$$\int_0^{\log 3}\left(2-e^{3y}\right)-\left(\frac{1}{2}\cdot e^{3y}\right)dy$$

$$\int_0^{\log 3} -e^{3y}\left(2+\frac{1}{2}\right)$$

$$\frac{5}{2}\int_0^{\log 3} -e^{3y}$$

$$-\frac{5}{2}\left[\frac{e^{3\log 3}}{3}-\frac{e^0}{3}\right]$$

$$-\frac{5}{2}\left(3-\frac{1}{3}\right)$$

$$-\frac{5}{2}\left(\frac{9-1}{3}\right)$$

$$-\frac{5}{2}\times\frac{8}{3}$$

$\frac{-40}{6}$

$$\int_0^{\log 3} e^{3y}\left[\frac{e^{2x}}{2}\right]_0^{\log 2} dy$$

$$\int_0^{\log 3} e^{3y}\left[\frac{e^{2\log 2}}{2} - \frac{e^0}{2}\right] dy.$$

$$\int_0^{\log 3} e^{3y}\left[2 - \frac{1}{2}\right] dy.$$

$$\frac{3}{2}\int_0^{\log 3} e^{3y}\, dy$$

$$\frac{3}{2}\left[\frac{e^{3\log 3}}{3} - \frac{e^0}{3}\right]$$

$$\cancel{\frac{3}{2}\left[27 - \frac{1}{3}\right]} \Rightarrow \cancel{\frac{3}{2} \times \frac{81-1}{3}}$$

$$\cancel{\frac{3}{2} \times \frac{80}{3}}$$

$$\cancel{\frac{3}{2} \times \frac{8}{3}}$$

$$\frac{1}{2}\left[27 - 1\right]$$

$$\frac{1}{2} \times 26 = 13 \quad \text{Ans}$$

Q/ $$\int_0^2 \int_0^1 y \sin x \, dy \, dx$$

$$\int_0^2 \left[\frac{y^2}{2} \sin x\right]_0^1 dx$$

$$\int_0^2 \sin x \left[\frac{y^2}{2}\right]_0^1 dx \Rightarrow \int_0^2 \sin x \left(\frac{1}{2}\right) - 0 \Big] dx$$

$$\frac{1}{2}\int_0^2 \sin x \, dx \qquad (-\cos\theta) \quad \cos$$

$$\frac{1}{2}\left[\cos x\right]_0^2$$

$$\frac{1}{2}\left[\cos 2 - (\cos 0)\right]$$

$$\frac{1}{2}\left[\cos 2 - \cos 0\right]$$

$$\frac{1}{2}\left[\cos 2 - 1\right]$$

$$\frac{1}{2}(\cos 2 - 1)$$

$$\frac{1}{2}\int_0^2 -\cos x \, dx$$

$$\frac{1}{2}\left[-\cos x\right]_0^2$$

$$\frac{1}{2}\left[(-\cos 2) - (-\cos 0)\right]$$

$$\frac{1}{2}\left[-\cos 2 + \cos 0\right]$$

$$\frac{1}{2}\left[-\cos 2 + 1\right]$$

$$\frac{1}{2}\left[1 - \cos 2\right]$$

Double Integrated with Variable limit.

Note If the inner integral limit is in terms of x first, find the integration w.r.t y.

If the inner integral limit is in terms of y first find the integration w.r.t to x

Evaluate $\displaystyle\int_0^1 \int_x^{x^2} y^2 x \, dx \, dy$

$$= \int_0^1 \left[\int_x^{x^2} y^2 x \, dy\right] dx$$

$$= \int_0^1 x \left[\frac{y^3}{3}\right]_x^{x^2} dx$$

$$\int_0^1 x \cdot \left[\frac{(x^2)^3}{3} - \frac{x^3}{3}\right] dx$$

$$\int_0^1 \left[x\left(\frac{x^6}{3} - \frac{x^3}{3}\right)\right] dx = \frac{1}{3}\int_0^1 (x^7 - x^4)\, dx$$

$$\cancel{\int_0^1 x \cdot \frac{x^3}{3} \cdot x^7}$$

$$= \frac{1}{3}\left[\frac{x^8}{8} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left[\left(\frac{1}{8} - \frac{1}{5}\right) - (0 - 0)\right]$$

$$= \frac{1}{3}\left[\frac{1}{8} - \frac{1}{5}\right] = \frac{1}{3}\left[\frac{5-8}{40}\right] = \frac{-1}{40} \text{ Ans}$$

Q. $$= \int_0^1 \int_0^y (x+y)\, dx\, dy.$$

$$= \int_0^1 \int_0^y (x+y)\, dy\, dx = \cancel{\int_0^1 \left[\frac{y^2}{2}\right]_0^y dx}$$

$$= \cancel{\int_0^1 \left[\frac{y^2}{2}\right] - (0)\, dx} \Rightarrow \frac{1}{2}\int_0^1 y^2\, dx =$$

$$= \cancel{\int_0^1 x +}$$

$$= \frac{1}{2}$$

$$= \int_0^1 \int_0^y (x+y)\, \underline{dx}\, dy = \int_0^1 \left[\frac{x^2}{2} + yx\right]_0^y$$

$$= \int^1 \left[\frac{y^2}{2} + y^2\right] - (0)$$

$$\int_0^1 \left[\frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1}{6} + \frac{1}{3}\right] = \frac{3}{6} = \frac{1}{2} \text{ Ans}$$

$$\therefore \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int_1^2 \int_0^x \frac{dy\,dx}{x^2+y^2}$$

$$= \int_1^2 \left[\int_0^x \frac{1}{x^2+y^2}\,dy\right] dx$$

$$= \int_1^2 \left(\int_0^x \frac{dy}{y^2+x^2}\right) dx = \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_1^2 \left(\frac{1}{x}\tan^{-1}\frac{x}{x}\right) - \left(\frac{1}{x}\tan^{-1}\frac{0}{x}\right) dx$$

$$= \int_1^2 \left(\frac{1}{x}\tan^{-1} 1\right) dx = \cancel{\int_1^2 \frac{1}{2}\left(\frac{\pi}{4}\right) dx}$$

$$= \cancel{\frac{1}{4}\int_1^2 \frac{\pi}{4} dx} = \int_1^2 \frac{1}{x}\frac{\pi}{4}\,dx$$

$$\frac{\pi}{4}\int_1^2 \left[\log x\right]_1^2 = \frac{\pi}{4}\log 2 - \log 1$$

$$\frac{\pi}{4}\log 2 \quad \text{Ans}$$

$\underline{\underline{G}}$

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{\underset{\text{constant}}{1+x^2}+y^2}\, dy\right] dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1} \cancel{\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}}\right) - \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{0}{\sqrt{1+x^2}}\right) dx$$

$$\because \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1} \frac{x}{a}$$

$$= \int_0^1 \left[\left(\frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4}\right)\right)\right] dx$$

$$= \frac{\pi}{4} \int_0^1 \left(\frac{1}{\sqrt{1+x^2}}\right) dx$$

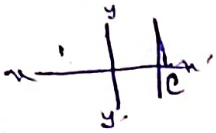
$$\Rightarrow \frac{\pi}{4} \left[\sinh^{-1} \frac{x}{1}\right]_0^1$$

$$\Rightarrow \frac{\pi}{4} \left[\sinh^{-1} 1\right]$$

Calculate the integral over a given region

1. $y = 0$ x-axis 

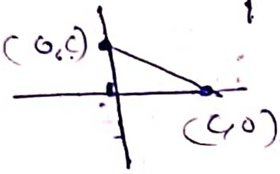
2. $x = 0$ y-axis 

3. $x = c$ 

x intercepts

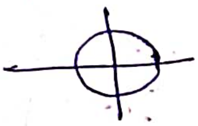
   $y = c$ 

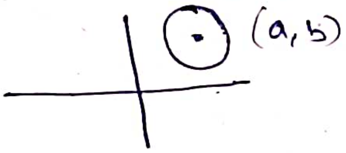
y intercepts

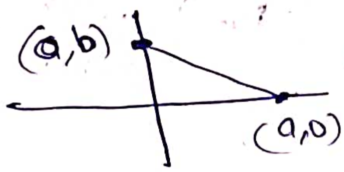
4. $x + y = c$ 

5. $x = y$ 

   $x = -y$

6. $x^2 + y^2 = r^2$ 

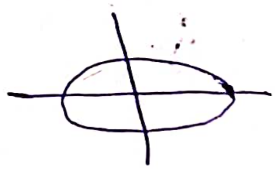
7. $(x-a)^2 + (y-b)^2 = r^2$ 

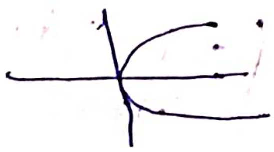
8. $\frac{x}{a} + \frac{y}{b} = 1$ 

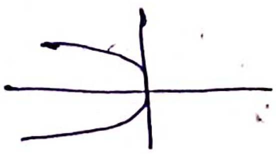
$x + y = c$

$\frac{x}{c} + \frac{y}{c} = 1$

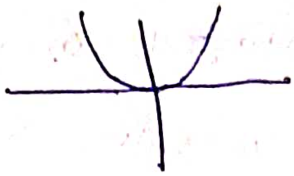
$(a,0)$. $(0,b)$

9. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 

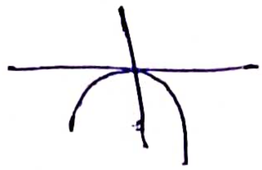
10. $y^2 = 4ax$ 

11. $y^2 = -4ax$ 

(12) $x^2 = 4ay$



(13) $x^2 = -4ay$

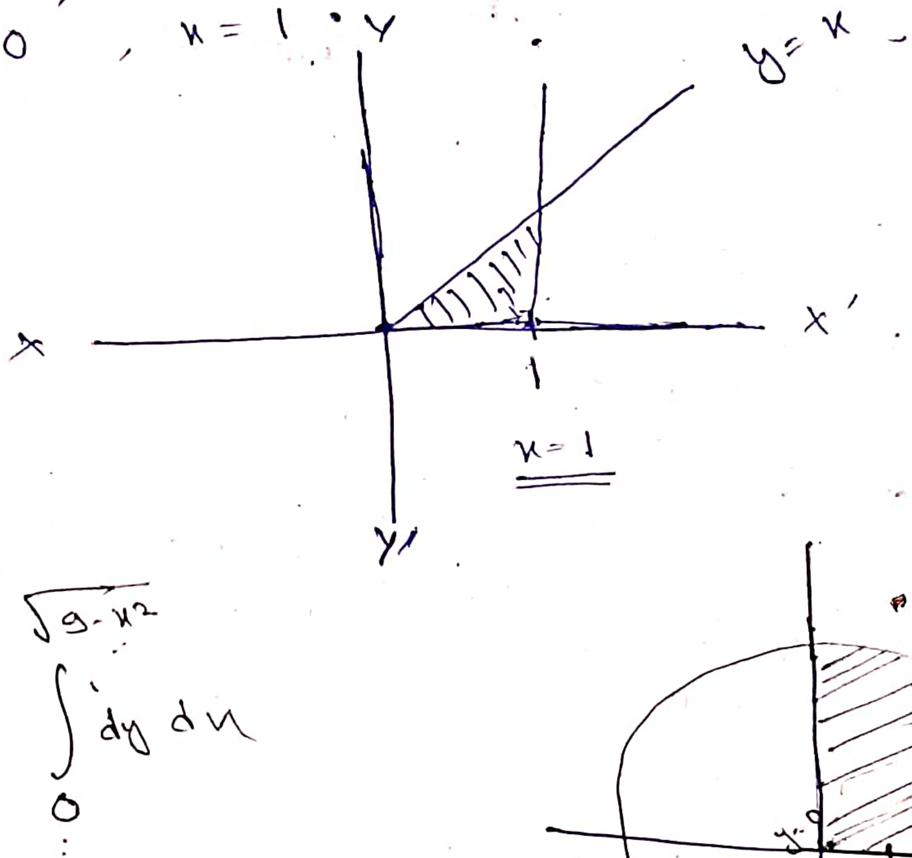


Q. Draw the region of integrales

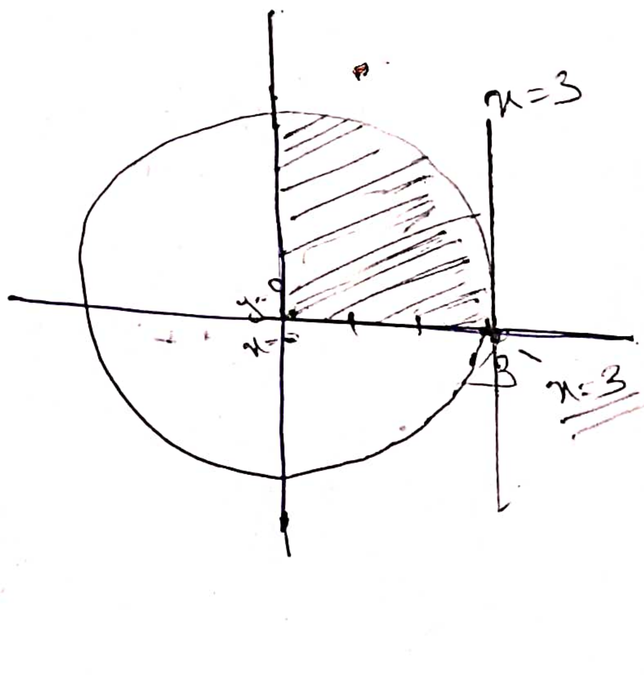
$$\int_0^1 \int_0^x \frac{x}{x+y}\, dy\, dx$$

$y = 0, \quad y = x$

$x = 0, \quad x = 1$



Q. $$\int_0^3 \int_0^{\sqrt{9-x^2}} dy\, dx$$



$y = 0$

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$y^2 + x^2 = 9$

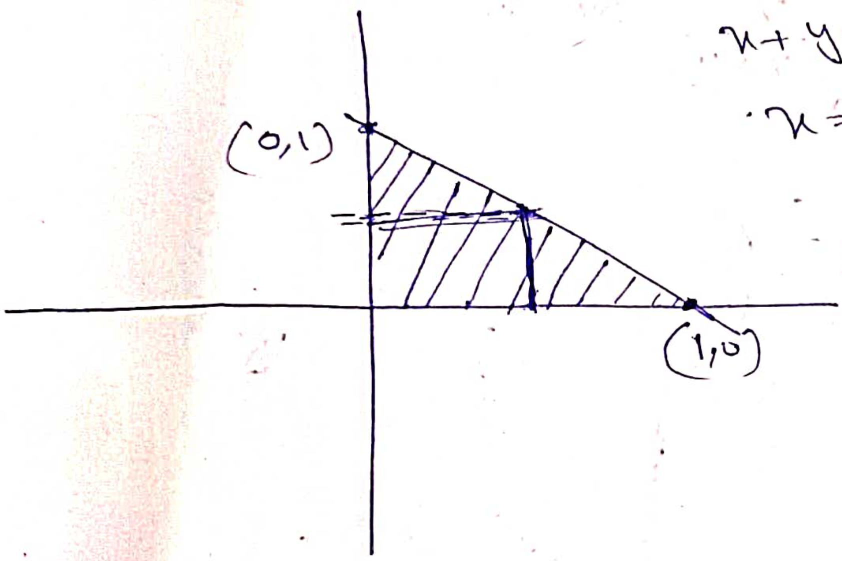
$x^2 + y^2 = 3^2$

$x = 0, \quad x = 3$

To Calculate the integral over a given region.

Evaluate $\iint_R x\,dx\,dy$ over the region in the Ist quadrant with $x+y \le 1$.

<u>Solution</u> $x+y=1$.



$x+y=1$

$x=1-y$.

$$\int_0^1 \int_0^{1-y} x\,dx\,dy.$$

$$\int_0^1 \left[\frac{x^2}{2}\right]_0^{1-y} dy = \int_0^1 \left[\frac{(1-y)^2}{2} - 0\right] dy$$

$$\int_0^1 \left[\frac{1+y^2-2y}{2}\right] dy$$

$$\frac{1}{2}\int_0^1 1\,dy + y^2\,dy - 2y\,dy$$

$$= \frac{1}{2}\left[[y]_0^1 + \left[\frac{y^3}{3}\right]_0^1 - \cancel{2}\left[\frac{y^2}{\cancel{2}}\right]_0^1\right]$$

$$\cancel{\frac{1}{6}\left[1+1-8\right]} = \frac{1}{6} \text{ Ans}$$

$$\cancel{\frac{1}{2}\left[1-0\right]+\left[\frac{1}{3}\right]-2\left[1-0\right]}$$

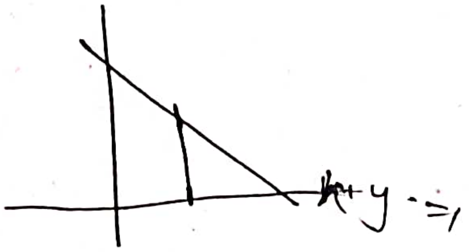
$$\cancel{\frac{1}{2}+\frac{1}{3}-2 = \frac{3+2-12}{6}}$$

$$\frac{1}{2}\left[1\right] \cancel{*} \frac{1}{3} \cancel{-1} = \frac{1}{6} \text{ Ans}$$

<u>Another</u>

$x+y=1$

$y=1-x$



$$\Rightarrow \int_0^1 \int_0^{1-x} x\, dx\, dy = \int_0^1 \int_0^{1-x} x\, dy\, dx$$

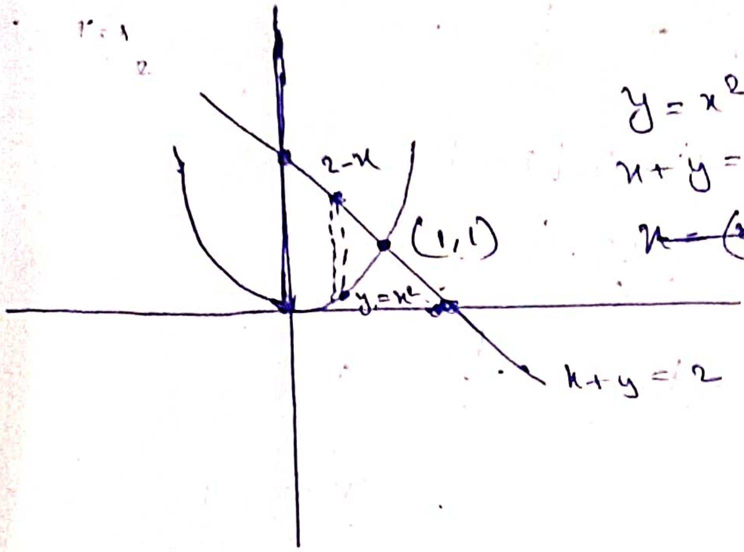
$$\cancel{\Rightarrow \int_0^1 \left[\frac{x^2}{2}\right]} \quad \int_0^1 x.\left[y\right]_0^{1-x} dx = \int_0^1 x.\left[1-x\right] dx$$

$$\int_0^1 x - x^2\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3}$$

$$\frac{3-2}{6} = \frac{1}{6} \text{ Ans}$$

Q/ $\iint y\, dx\, dy$ over the area bounded by $x=0$, $y=x^2$, $x+y=2$ in the I^{st} Quadrant.

$\frac{1}{2} = \frac{1}{2}$ $r = \frac{1}{2}$



$y = x^2$

$x + y = 2$

$\cancel{x = (2-y)}$

$x + x^2 = 2$

$x + x^2 - 2 = 0$

$x^2 + x - 2 = 0$

$\cancel{y = (2-y)^2}$

$\cancel{y = 4 + y^2 - 4y}$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x+2)(x-1) = 0$

$\underline{\underline{x = 1}}, \; x = -2 \; \times$

$$\int_0^1 \int_{x^2}^{2-x} y \, dx \, dy$$

$$\int_0^1 \int_{x^2}^{2-x} y \, dy \, dx \Rightarrow \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$\Rightarrow \int_0^1 \left[\frac{(2-x)^2}{2}\right] - \left[\frac{(x^2)^2}{2}\right] dx$$

$$= \int_0^1 \left(\frac{4 + x^2 - 2x}{2}\right) - \left(\frac{x^4}{2}\right) dx$$

$$= \frac{1}{2} \int_0^1 (x^2 - 2x + 4) - (x^4) \, dx$$

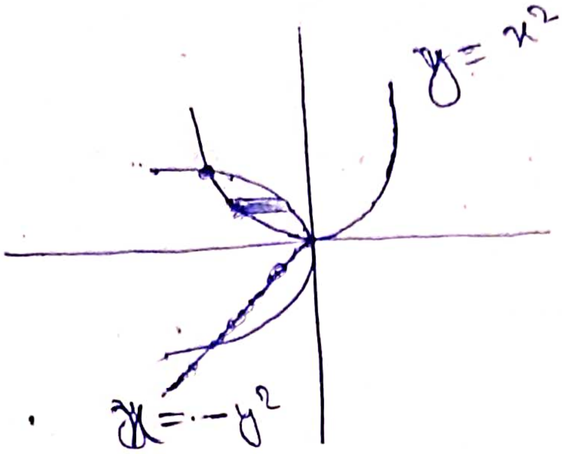
$$= \frac{1}{2}\left[\frac{x^3}{3} - \cancel{2} \cdot \frac{x^2}{\cancel{2}} + 4x - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - 1 + 4 - \frac{1}{5}\right] \Rightarrow \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5} + 3\right]$$

$$= \cancel{\frac{1}{2}\left[\frac{5 - 3 + 45}{15}\right]}$$

$$\cancel{\frac{1}{2}\left[\frac{47}{15}\right]} \quad \frac{1}{2}\left[\right.$$

Evaluate $\iint xy\,dx\,dy$ over the area bounded by the parabolas $y = x^2$ and $x = -y^2$



$y = x^2$
$x = -y^2$

$y = (-y^2)^2$
$y = y^4$
~~$y^4 - y + 1 = 0$~~
~~$y^4 - y = 0$~~
~~$y(y^3 - 1) = 0$~~
~~$y = 0$~~
$y^3 - 1 = 0$

$y^3 = 1$
$y = 1$
$x = -y^2$
$x = -(1)^2$
$x = -1$

lower limit

$y = x^2$

$x = \pm\sqrt{y}$

$x^2 = y$

$x = \pm\sqrt{y}$

$x = -\sqrt{y}$

upper limit

$x = -y^2$

$x = -y^2$

If the strip is ∥ to x axis the limit will be in terms of y otherwise from the eqn you have to find out x value.

$$\int_0^1 \int_{-\sqrt{y}}^{-y^2} xy\,dx\,dy$$

$$\Rightarrow \int_0^1 \left[\frac{y x^2}{2}\right]_{-\sqrt{y}}^{-y^2} dy = \int_0^1 y \cdot \left[\frac{(-y^2)^2}{2} - \frac{(-\sqrt{y})^2}{2}\right] dy$$

$$\frac{1}{2}\int_0^1 \left[y^5 - y^2\right] dy = \frac{1}{2}\int_0^1 (y^5 - y^2)\,dy = \frac{1}{2}\left[\frac{y^6}{6} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{2} \times \frac{1}{3}\left[\frac{y^6}{2} - y^3\right]_0^1$$

$$= \frac{1}{6}\left[\frac{1}{2} - 1\right] - (0)$$

$$= \frac{1}{6} \times \left(\frac{1-2}{2}\right) = \frac{-1}{12} \quad A \leftarrow$$

Q. Evaluate $\iint xy\, dx\, dy$ over the region $x^2+y^2-2x=0$, $y^2 = 2x$, $y = x$.

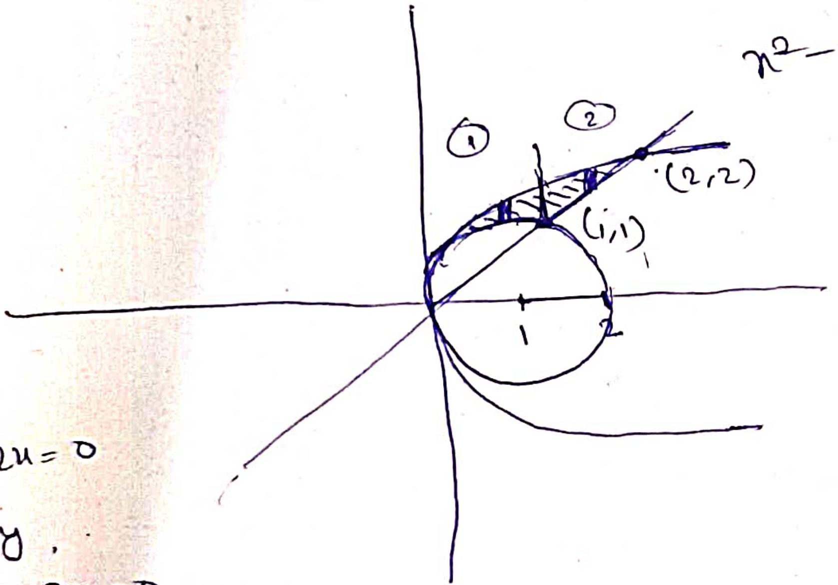
$x^2+y^2-2x=0$

$x^2+y^2 = r^2$

$x^2-2x+1-1+y^2=0$

$(x-1)^2+y^2=1$

$(1,0)$ $r=1$



$x^2+y^2-2x=0$

$x=y$

$y^2+y^2-2y=0$

$2y^2-2y=0$

$2y(y-1)=0$

$y=1 = (1,1)$

$x=y=1$

$y^2=2x$

$y=x$

$y^2 = 2y$

$y=2$

$x=2$

y limit

lower limit

$x^2-2y+y^2=0$

$y^2 = 2x-x^2$

$y = \sqrt{2x-x^2}$

Upper limit

$y^2=2x$

$y=\sqrt{2x}$

x limit

1

0

$$\Rightarrow \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy \, dx \, dy + \int_1^2 \int_x^{\sqrt{2x}} xy \, dx \, dy$$

$$\Rightarrow \int_0^1 x \left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left[\frac{y^2}{2}\right]_x^{\sqrt{2x}} dx$$

$$= \int_0^1 x \cdot \left(\frac{2x}{2} - \frac{2x-x^2}{2}\right) dx + \int_1^2 x \cdot \left(\frac{2x}{2} - \frac{x^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - 2x^2 + x^3) \, dx + \frac{1}{2}\int_1^2 (2x^2 - x^3) \, dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - 2 \cdot \frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[2 \cdot \frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{2}{3} + \frac{1}{4}\right] + \frac{1}{2}\left[\left(2 \cdot \frac{8}{3} - \frac{16}{4}\right.\right.$$

02-12-22

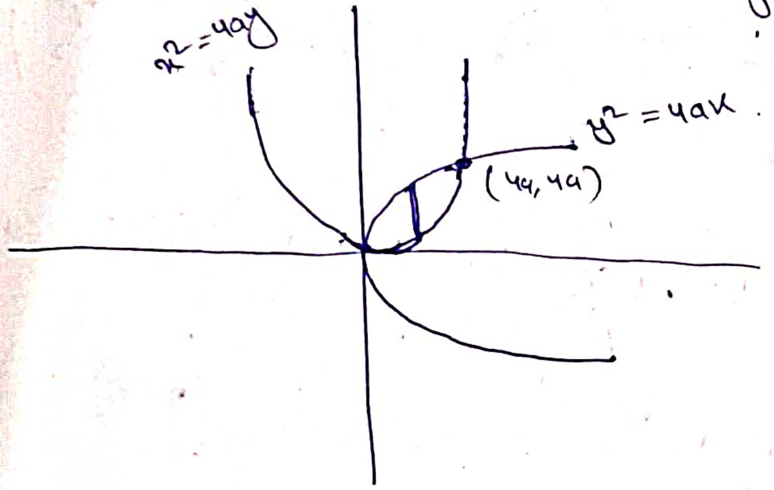
Area of double intigration:

$A = \iint dx\,dy$.

$\iint dA$.

Find the area bounded by the region $y^2 = 4ax$.
& $x^2 = 4ay \rightarrow x = 2\sqrt{ay}$

$y^2 = 4ax$
$y^2 = 4a \cdot 2\sqrt{ay}$
$y^2 = 8a\sqrt{ay}$
$y^4 = 64a^2 y$.
$y^3 = 64a^2$



$y^2 = 4ax \longrightarrow x = \frac{y^2}{4a}$

$x^2 = 4ay$

$\left(\frac{y^2}{4a}\right)^2 = 4ay$.

$\frac{y^4}{16a^2} = 4a\cancel{y}$.

$\frac{y^3}{16a^3} = 4 \rightarrow y^3 = 64a^3$

$y = 4a$

$\cancel{y^3 = 4 \times 16a^3}$

$\cancel{y = 4 \times (4a)^3}$

$x = \frac{y^2}{4a} = \frac{(4a)^2}{4a}$

$= \frac{16a^2}{4a}$

$x = 4a$

Lower limit | Upper limit

$x^2 = 4ay$ | $y^2 = 4ax$

$y = \frac{x^2}{4a}$ | $y = 2\sqrt{ax}$

$\sqrt{x}$ $\quad x^{\frac{1}{2}}$

$\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx\,dy \Rightarrow \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$$

$$\Rightarrow \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}}\,dx \Rightarrow \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$

$$\Rightarrow \cancel{2\left[\sqrt{ax}\right]_0^{4a}} \quad \int_0^{4a}\left(2\cdot(ax)^{\frac{1}{2}} - \frac{x^2}{4a}\right)dx$$

$$\left[\frac{2\sqrt{a}\,x^{\frac{1}{2}+1}}{\frac{3}{2}} - \frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a}$$

$$= \cancel{\frac{2\sqrt{a}\cdot(4a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\cdot\frac{(4a)^3}{3}}$$

$$= \cancel{\frac{4}{3}\sqrt{a}\cdot 4a\sqrt{4a} - \frac{1}{12}}$$

$$\left[\frac{4}{3}\sqrt{a}\cdot x\sqrt{x} - \frac{1}{12a}x^3\right]_0^{4a}$$

$$\frac{4}{3}\sqrt{a}\cdot 4a\sqrt{4a} - \frac{1}{12a}(4a)^3$$

$$\frac{4}{3}a\cdot 4a\cdot 2 - \frac{1}{12a}\times 64a^{3}$$

$$\frac{32}{3}a^2 - \frac{64}{12}a^2 \rightarrow \frac{128a^2 - 64a^2}{12} = \frac{64}{12}$$

$$= 16a^2 \text{ Ans}$$

Volume double integration

$A = \iint z \, dx\, dy$

Use double integral is find the Volume under this Surface

$z = x^2 + 3x^2 y$ over the rectangle

$-1 \leq x \leq 3 \qquad 0 \leq y \leq 2$

$$\int_{-1}^{3} \int_{0}^{2} x^2 + 3x^2 y \, dy\, dx$$

$$\cancel{\int^{3} \left[\frac{x^3}{3} + 3 \cdot y \cdot \frac{x^3}{3} \right]_0^2 dy}$$

$$\int_{-1}^{3} \left[x^2 \cdot y + 3 \cdot x^2 \cdot \frac{y^2}{2} \right]_0^2 dx$$

$$\int_{-1}^{3} x^2 \cdot [2] + 3x^2 \left[\frac{4}{2} \right] dx$$

$$\int_{-1}^{3} 2x^2 + 6x^2 \, dx$$

$$\left[2 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^3}{3} \right]_{-1}^{3}$$

$-1 \quad -3$

$$\frac{2}{3} \left[x^3 + 3x^3 \right]_{-1}^{3}$$

$$\frac{2}{3} \left[(27 + 3 \cdot 27) - (-1 + 3(-1)) \right]$$

$$\frac{2}{3} \left[(27 + 81) - (-4) \right]$$

$$\frac{2}{3} \left[27 + 81 \right.$$

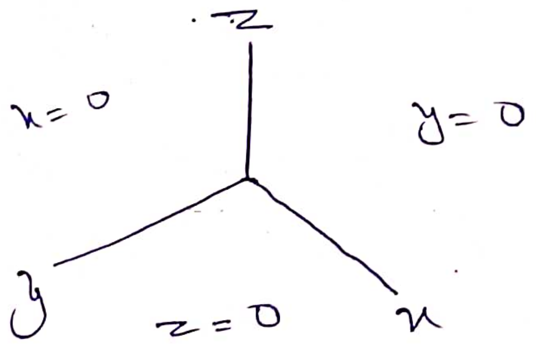
$\frac{2}{3}\cdot[108+4]$

$\frac{2}{3}\times 112 \rightarrow$

$= \frac{224}{3}$ A ←

Q Find the Volume of tetrahedron bounded by Coordinate planes & $z = 4-4x-2y$.

$\Rightarrow \quad V = \iint z \, dx\, dy$.



$z = 4-4x-2y$

Put $z=0$

$0 = 4-4x-2y$

$2y = 4-4x$

$y = \frac{4-4x}{2}$

$y = 2(1-x)$

$z=0,\ y=0$

$2-2x=0$

$2 = 2x \Rightarrow \boxed{x=1}$

$$\int_0^1\int_0^{(2-2x)} (4-4x-2y)\, dx\, dy.$$

$$\int_0^1\int_0^{2-2x} (4-4x-2y)\, dy\, dx \Rightarrow$$

$$\int_0^1 \left[\cancel{4y}^{\,0} - x\cdot 4y - 2\cdot\frac{y^2}{2}\right]_0^{2-2x} dx$$

$$\int_0^1 \left[4\cdot\cancel{(2-2x)}^{\,0} - 4x(2-2x) - \cancel{2}\cdot\frac{(2-2x)^2}{\cancel{2}} \right] dx$$

$$\int_0^1 \left(\cancel{8-8x}^{\,0} - 8x + 8x^2 - (4+4x^2-8x)\right) dx$$

$$\int_0^1 \cancel{8}\left[\overset{0\,-\,\cancel{8x}}{(8-16x)} + 8x^2 - 4 - 4x^2 + \cancel{8x}\right] dx$$

$$\times\left\{ \int_0^1 4 - 8x + \cancel{4}x^2 \Rightarrow 4 - 8 + 4 \quad 8 - 8 = 0 \right\}$$

2) $\int_0^1 (4x^2 - 4)\, dx$

Evaluate of double Integral in polar Co-ordinate

Evaluate $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr\, d\theta$.

Solution

$$\int_0^{\pi} \sin\theta \, d\theta \int_0^{a(1-\cos\theta)} r^2 \, dr$$

$$\int_0^{\pi} \sin\theta \, d\theta \left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)}$$

$$\int_0^{\pi} \sin\theta \, d\theta \; \frac{1}{3}\left[a^3 (1-\cos\theta)^3\right]$$

$$\frac{a^3}{3}\int_0^{\pi} (1-\cos\theta)^3 \sin\theta \, d\theta$$

Let $1-\cos\theta = t$

$\sin\theta \, d\theta = dt$

$\theta = 0 \;,\; t = 0$

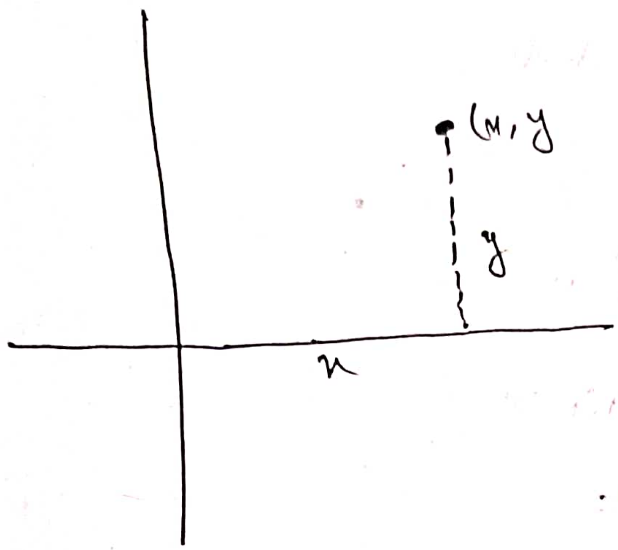
$\theta = \pi \;,\; t = 2$

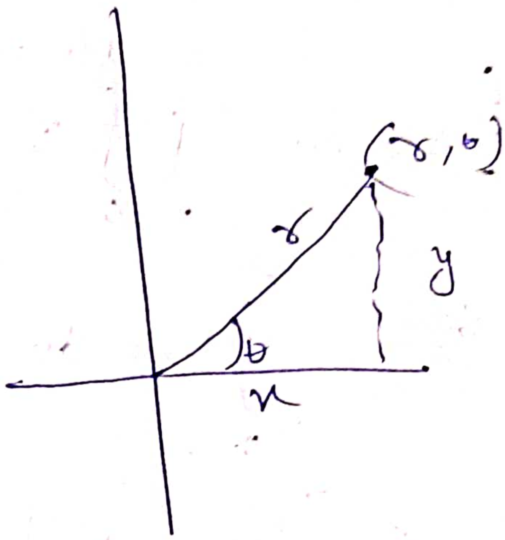
$$\frac{a^3}{3}\int_0^{2} t^3 \, dt \Rightarrow \frac{a^3}{3}\left[\frac{t^4}{4}\right]_0^2$$

$$\frac{a^3}{3}\left[\frac{2^4}{4}\right] \Rightarrow \cancel{\frac{a^3}{3}\times\frac{16}{16}} \cdot \frac{a^3 \times \cancel{16}^{\,4}}{\cancel{12}_{\,3}}$$

$$\frac{4a^3}{3}$$ Ans

Integrating by Changing into its polar Co-ordinate





$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$ $\qquad$ $y = r\sin\theta$

$x = r\cos\theta$ $\qquad$ $dy\,dx = r\cdot dr\,d\theta$

Q $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

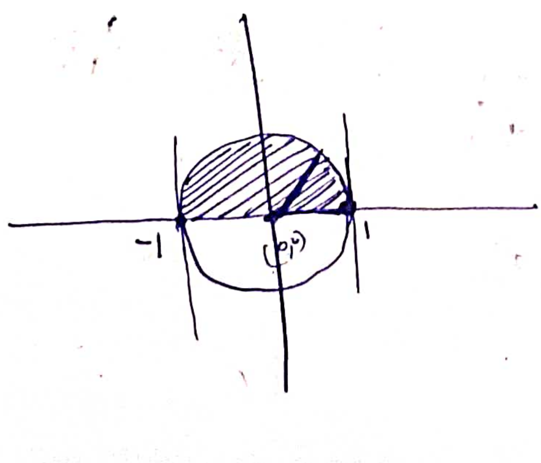
Limit $\quad y = 0$

y varies from $y = 0$ to $y = \sqrt{1-x^2}$.

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

$x = -1$



$$\int_0^{\pi}\int_0^{1} r^3 . r . dr\, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$(x^2+y^2)^{3/2}$

$(r^2\cos^2\theta + r^2\sin^2\theta)^{3/2}$

$(r^2)^{3/2} = r^3$

$$\int_0^{\pi}\left[\frac{r^5}{5}\right]_0^1 d\theta$$

$$\int_0^{\pi}\frac{1}{5}\, d\theta$$

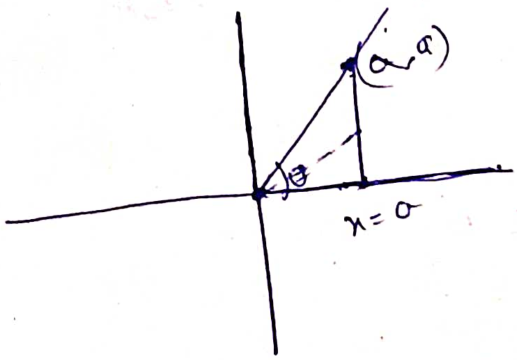
$$\frac{1}{5}\int_0^{\pi} d\theta = \frac{1}{5}[\theta]_0^{\pi} = \frac{1}{5}[\pi - 0] = \frac{1}{5}\pi \quad \text{Ans}$$

Q. $$\int_0^{a}\int_0^{x}\frac{x}{x^2+y^2}\, dy\, dx$$

$y = 0$, $y = x$

y varies from 0 to x.

$x = 0$ to a, $x = a$



$$\frac{x}{x^2+y^2} = \cancel{\frac{r\cos\theta}{r^2}} = \cancel{\frac{r}{r^2}} = \cancel{\frac{1}{r}}$$

$x = r\cos\theta$

$a = r\cos\theta$

$r = \frac{a}{\cos\theta}$

$$\cancel{\int_0^{\pi/4}\int_0^{a/\cos\theta}\frac{1}{r}. r\, dr\, d\theta} \quad \cancel{\int_0^{\pi/4}\int_0^{a/\cos\theta} dr\, d\theta} = \cancel{\int_0^{\pi/4}\left[r\right]_0^{a/\cos\theta} d\theta}$$

$$\cancel{\int_0^{\pi/4}\left[\theta\right]_0^{a/\cos\theta} d\theta} \Rightarrow$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \left(\frac{a}{\cos\theta} - 0\right) d\theta$$

$$\int_0^{\frac{\pi}{4}} \frac{a}{\cos\theta} \, d\theta$$

$$= a \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \, d\theta$$

$$\Rightarrow a.$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{\cos\theta}} \frac{r\cos\theta}{r^2} \cdot r \, dr \cdot d\theta$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{\cos\theta}} \cos\theta \, dr \, d\theta$$

$$\int_0^{\frac{\pi}{4}} \cos\theta \left[r\right]_0^{\frac{a}{\cos\theta}} \Rightarrow \int_0^{\frac{\pi}{4}} \cancel{\cos\theta} \times \frac{a}{\cancel{\cos\theta}} \, d\theta$$

$$a \left[\theta\right]_0^{\frac{\pi}{4}} \qquad a\frac{\pi}{4} \qquad A \leftarrow$$