Vector Calculus

$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\vec{a} = a_1 \hat{i} + a_3 \hat{j} + a_4 \hat{k}$$

$$\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$a \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

$$a \times b =$$

(Position Vector)

$$P(x, y, z) \longrightarrow \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(Speed Vector)

$$\frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

(6pg) Q At any point on the curve

$x = 3\cos t \quad y = 3\sin t \quad z = 4t$

Find i) Tangent Vector ii) Unit Tangent Vector

Soln $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$

i) Tangent $\dfrac{dr}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$

ii) Magnitude of Tangent Vector $\Rightarrow$

$\sqrt{9\sin^2 t + 9\cos^2 t + 16}$

$= \sqrt{25}$

$= 5$

Unit Tangent Vector $= \dfrac{-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}}{5}$

Scaler & Vector Point Function :-

Point Function → A variable quantity whose value at any point in a space depends on the position of the point is called the point Function.

There are two types →

- Scaler Point ~~vector~~ Fnⁿ (only magnitude)

  eg Temperature

- Vector Point Function (magnitude + directⁿ)

  eg Velocity

Vector Differential Operator :-

denoted by del — ∇

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

$\phi(x, y, z)$ is a scaler point ~~vector~~ functⁿ

$$\boxed{\text{grad } \phi = \nabla \cdot \phi}$$

Grad ϕ is vector normal to the surface ϕ

Q If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point $(1,-2,-1)$

Soln grad $\phi = \nabla \phi$

$$= \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right) \cdot \left(3x^2y - y^3z^2\right)$$

$$= \hat{i}\frac{d}{dx}\left(3x^2y - y^3z^2\right) + \hat{j}\frac{d}{dy}\left(3x^2y - y^3z^2\right)$$

$$+ \hat{k}\frac{d}{dz}\left(3x^2y - y^3z^2\right)$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Directional Derivative :-

↳ The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\boxed{\nabla\phi \cdot \vec{d}}$

It is called the directional derivative of ϕ in the directⁿ of $\vec{d}$

Q Find the directional derivative of $x^2y^2z^2$ at the point $(1, 1, -1)$ in the directⁿ of tangent to the curve:-

$$x = e^t \;;\; y = \sin 2t + 1 \;;\; z = 1 - \cos t$$

at $t = 0$

Soln

$$\phi = x^2 y^2 z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\left(x^2y^2z^2\right)$$

$$\nabla\phi = 2xy^2z^2\,\hat{i} + 2x^2yz^2\,\hat{j} + 2x^2y^2z\,\hat{k}$$

$$\nabla\phi(1,1,-1) = 2\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = e^t\hat{i} + (\sin 2t + 1)\hat{j} + (1-\cos t)\hat{k}$$

Tangent $\frac{dr}{dt} = e^t\,\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

$$\left(\frac{dr}{dt}\right)_{t=0} = \hat{i} + 2\hat{j}$$

(8) $\vec{d} = \hat{i} + 2\hat{j}$

direct^n derivative $= \nabla\phi \cdot \vec{d} = 2(\hat{i}+\hat{j}-\hat{k})(\hat{i}+2\hat{j})$

$= 2(1+2)$

Ans $= 6$

Divergence ~~Theorem~~ of a Vector:—

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \left(\frac{\partial F_1}{dx} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

→ It is a Scalar Function

- If div $\vec{F} = 0$; it is called Solenoid Funct^n Vector

Curl:–

$$\text{curl } F = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- If curl $\vec{F} = 0$; The field $\vec{F}$ is irrotational

Q Find divergence & curl of funct^n.

$$\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$$

at $(2, -1, 1)$

Sol: $\text{div } \vec{V} = \nabla \cdot \vec{V}$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= \left(yz + 3x^2 + (2xz - y^2)\right)_{at\ (2,-1,1)}$$

$$= -1 + 12 + 4 - 1$$

$$\text{div } \vec{V} = 14$$

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$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xyz) & (3x^2y) & (xz^2-y^2z) \end{vmatrix}$$

$$= \left[\hat{i}(-2yz - 0) - \hat{j}(z^2 - xy) + \hat{k}(6xy - xz)\right]_{\text{at } (2,-1,\dots)}$$

$$= 2\hat{i} - 3\hat{j} + \hat{k}(-12 - 2)$$

$$= (2\hat{i} - 3\hat{j} - 14\hat{k})$$

Vector Integration :-

line integral $\int_C \vec{F} \cdot d\vec{r}$

Work Done $= \int_A^B \vec{F} \cdot d\vec{r}$

Circulation : If $\vec{V}$ represents the velocity of a liquid. Then $\oint \vec{V} \cdot d\vec{r}$ is called the circulation of $\vec{V}$ round the closed curve C. If circular is zero; then $\vec{V}$ is said to be irrotational

Q If a Force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ (9)

denotes a particle in xy plane from (0,0) to (1, 4) along a curve $y = 4x^2$. Find the work done

Soln

$$\text{Work done} = \int_A^B \vec{F}\cdot d\vec{x}$$

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$WD = \int_A^B (2x^2y\,\hat{i} + 3xy\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_A^B 2x^2y\,dx + 3xy\,dy$$

Now, $y = 4x^2$

$dy = 8x\cdot dx$

$$\frac{128}{5} \Uparrow \quad = \int_0^1 8x^4\,dx + 12x^3\cdot(8x\cdot dx)$$

$$\frac{256}{10} \Uparrow \quad = \left[8\frac{x^5}{5} + \frac{\cancel{96}^{24}\,x^4}{\cancel{4}}\right]_0^1 = \cancel{\frac{48 \ldots}{20}}$$

$$\frac{512}{20} \Leftarrow \frac{32+480}{20} \Leftarrow \cancel{\frac{136}{5}} = \frac{136}{5} = \cancel{\frac{544}{20}}\ 136$$

(9)

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$$\text{Work done} = \int_A^B \vec{F}\cdot d\vec{r}$$

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(10) Q Suppose $\vec{F}(x,y,z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field. Find the work done by $\vec{F}$ along the line from (1,2,3) to (3,5,7)

Soln Work Done $= \int \vec{F} \cdot \vec{dr}$.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$W.D = \int_{(1,2,3)}^{(3,5,7)} (x^3\hat{i} + y\hat{j} + z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_{(1,2,3)}^{(3,5,7)} x^3\,dx + y\,dy + z\,dz$$

$$\int_1^3 x^3\,dx + \int_2^5 y\,dy + \int_3^7 z\,dz$$

$$\left[\frac{x^4}{4}\right]_1^3 + \left[\frac{y^2}{2}\right]_2^5 + \left[\frac{z^2}{2}\right]_3^7$$

$$\Rightarrow \frac{80}{4} + \frac{21}{2} + \frac{40}{2}$$

$$\Rightarrow \frac{101}{2}$$

Q If $A = (3x^2+6y)\hat{i}\ (-14yz)\hat{j} + (20xz^2)\hat{k}$

Evaluate the line integral $\oint \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve

$$x = t\ ;\ y = t^2\ ;\ z = t^3$$

Soln $\oint \vec{A}\cdot d\vec{r} = \int (3x^2+6y)\hat{i} - (14yz)\hat{j} + (20xz^2)\hat{k}$

$$\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_{(0,0,0)}^{(1,1,1)} dx(3x^2+6y) - (14yz)\,dy + (20xz^2)\,dz$$

when, $x=0\ ;\ t=0$
$x=1\ ;\ t=1$ - - - - - - - $(\because x=t)$

$x = t$
$dx = dt$
$y = t^2$
$dy = 2t\,dt$
$z = t^3$
$dz = 3t^2\,dt$

$$\int_0^1 (3x^2+6y)dt - (14yz)(2t)dt + (20xz^2)(3t^2\,dt)$$

$$\Rightarrow \int_0^1 (3t^2+6t^2)dt - (28t^6)dt + (60t^9)dt$$

$$\Rightarrow \left[3t^3\right]_0^1 - \left[4t^7\right]_0^1 + \left[6t^{10}\right]_0^1$$

$$\Rightarrow$$

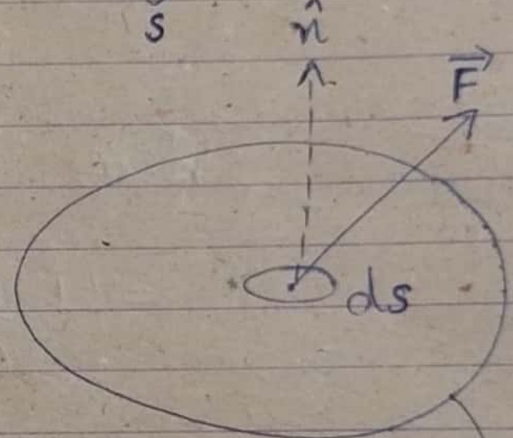
$$3 - 4 + 6 \Rightarrow 5 \text{ Ans}$$

$=5$

Surface Integral :—

Any integral which is to be evaluated over a surface is called Surface Integral.

Surface integral of F over $S = \iint_S (\vec{F} \cdot \hat{n})\, ds$



$$\text{Flux} = \iint_S (\vec{F} \cdot \hat{n})\, dS,$$

where $\vec{F}$ represents the velocity of a liquid

If $\iint_S (F \cdot \hat{n})\, dS = 0$, then $\vec{F}$ is said to be Solenoidal Vector.

$$\text{Surface Integral} = \iint_S (F \cdot \hat{n})\, dS$$

- $\hat{n} = \dfrac{\text{grad } F}{|\text{grad } F|}$
- $dS = \dfrac{dx\,dy}{\hat{n} \cdot \hat{k}}$ → because its the (X-Y) Plane

for Y-Z plane

$$dS = \frac{dy \cdot dz}{\hat{n} \cdot \hat{i}}$$

for X-Z plane

$$dS = \frac{dx \cdot dz}{\hat{n} \cdot \hat{j}}$$

Q Evaluate $\iint (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot dS$

where S is the surface of the sphere

$x^2 + y^2 + z^2 = a^2$ in the first octant.

Soln

$$\vec{n} = \nabla\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\hat{n} = \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

Now,

$$\vec{F}\cdot\hat{n} = \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)(yz\hat{i} + zx\hat{j} + xy\hat{k})$$

$$= \frac{3xyz}{a}$$

(12) $\hat{n}\cdot\hat{k} = \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right)\cdot\hat{k}$

$\hat{n}\cdot\hat{k} = \frac{z}{a}$

Now, $\iint \frac{3xyz}{a} \cdot \frac{dx\,dy}{(\hat{n}\cdot\hat{k})}$

$\iint \frac{3xy\cancel{z}}{\cancel{a}} \frac{dx\,dy}{\cancel{z}} \cancel{a}$

$\iint 3xy \, dx\,dy$ $\Rightarrow \frac{3a^4}{8}$

$\int_0^a \int_0^{\sqrt{a^2-x^2}} 3xy \, dx\,dy$

$\left[\frac{3}{2}xy^2\right]_0^{\sqrt{a^2-x^2}}$

$\int_0^a \frac{3x}{2}(a^2-x^2)$

$\frac{3}{2}\int_0^a x(a^2-x^2)$

$\Rightarrow \frac{3}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\cdot\frac{a^4}{4}$

$= 3a^4/8$

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Q Evaluate $\iint \vec{A} \cdot \hat{n} \cdot dS$ where

$\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$

S is the ~~~~ surface of the cylinder $x^2+y^2 = 16$ included in the Ist octant between $z=0$ and $z=5$

Soln dS can be in →

$\text{Ans} = 90$

x-y plane ×

y-z plane ✓

z-x plane ✓

$$\int_{y=0}^{y=4}\int_{z=0}^{z=5} (z+y)\,dy\,dz$$

$\vec{n} = \nabla\phi$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-16)$

$\vec{n} = 2x\hat{i} + 2y\hat{j}$

$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2x\hat{i}+2y\hat{j}}{2\sqrt{x^2+y^2}}$

$$\hat{n} = \frac{x\hat{i} + y\hat{j}}{4}$$

$$\vec{F} \cdot \hat{n} = (z\hat{i} + x\hat{j} - 3y^2 z\hat{k})\left(\frac{x\hat{i} + y\hat{j}}{4}\right)$$

$$= \frac{xz + xy}{4}$$

$$= \frac{x}{4}(y+z)$$

z

16)

$z\hat{k})$

Volume Integral

Let $\vec{F}$ be vector point function and Volume V enclosed by a closed surface.

The volume Integral $= \iiint_V \vec{F} \cdot dV$

Q If $\vec{F} = (2z\hat{i} - x\hat{j} + y\hat{k})$

Evaluate $\iiint \vec{F} \cdot dV$ where V is the region bounded by the surface

$x=0 \quad y=0 \quad x=2 \quad y=4 \quad z=x^2 \quad z=2$

Ans $\boxed{\frac{32}{15}(3\hat{i} + 5\hat{k})}$

Soln

$$\iiint_V \vec{F} \cdot dV$$

$4\hat{i} - 2x\hat{j} + 2y\hat{k}$

$$\int_0^2 \int_0^4 \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$$

$x^4\hat{i} - x^3\hat{j} + x^2y\hat{k}$

$$\left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2$$

$\boxed{(4-x^4)\hat{i} + (x^3-2x)\hat{j} + (2y-x^2y)\hat{k}}$

$$\int_0^2 \int_0^4 (z^2\hat{i} - x^3\hat{j} + x^2y\hat{k})\, dx\, dy$$

$$\int_0^2 (4z^2\hat{i} - 4x^3\hat{j} + 2x^2\hat{k})\, dx$$