MATHS

ASSIGNMENT

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Assignment - 1

Matrix

1. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -4 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -4 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 4R_1$

$R_3 \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 12 & 7 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1$

$C_3 \rightarrow C_3 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 7 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 \rightarrow \frac{C_2}{12}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & -1/6 & 1 & -8 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{6} R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 13/6 & -8 \end{bmatrix}$$

$C_3 \to C_3 - 7C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 13/6 & -8 \end{bmatrix}$$

$C_4 \leftrightarrow C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 13/6 \end{bmatrix}$$

$C_3 \to \frac{-C_3}{8}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 13/6 \end{bmatrix}$$

$C_4 \to C_4 - \frac{13}{6} C_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$A = [I_3 \ 0]$

Rank of matrix $A = 3$

2. Explain the following System of Vectors for linear dependent, if dependent find the relation.

$x_1 = (1, 1, 1, 3)$

$x_2 = (1, 2, 3, 4)$

$x_3 = (2, 3, 4, 7)$

Solution: For dependend Vectors,

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$

$\lambda_1 (1,1,1,3) + \lambda_2 (1,2,3,4) + \lambda_3 (2,3,4,7) = 0$

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$

$\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$

$\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0$

$3\lambda_1 + 4\lambda_2 + 7\lambda_3 = 0$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$R_4 \rightarrow R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$R_4 \rightarrow R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$
$$0\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$0\lambda_1 + 0\lambda_2 + 0\lambda_3 = 0$$
$$0\lambda_1 + 0\lambda_2 + 0\lambda_3 = 0$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$
$$\lambda_2 + \lambda_3 = 0$$

Let $\lambda_2 = t$

$$\lambda_2 + \lambda_3 = 0$$
$$t + \lambda_3 = 0 \Rightarrow \lambda_3 = -t$$
$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$
$$\lambda_1 + t - 2t = 0$$
$$\lambda_1 = t$$

~~Now, Put the value of λ_1, λ_2 & λ_3~~

$$\cancel{\lambda_1 + \lambda_2 + 2\lambda_3 = 0}$$
$$\cancel{t + t - 2t = 0}$$
$$\cancel{2t - 2t = 0}$$

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0$$
$$t x_1 + t x_2 - t x_3 = 0$$

$t = 1$

$$\boxed{x_1 + x_2 - x_3 = 0}$$

3. Using matrix method Show that test the Consistent Solution for x, y & z

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Solution :

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$$AX = B$$
$$C = AB$$

$$\left[\begin{array}{ccc:c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$$R_2 \rightarrow \frac{R_2}{-3}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$$R_3 \rightarrow R_3 - 10R_2$$
$$R_4 \rightarrow R_4 + 7R_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{array}\right]$$

$$R_3 \rightarrow \frac{R_3}{29/3}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -17/3 & 68/3 \end{array}\right]$$

$R_4 \rightarrow R_4 + \frac{17}{3} R_4$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2y = 4$

$y - \frac{2z}{3} = \frac{11}{3}$

$\boxed{z = -4}$

$y - \frac{2(-4)}{3} = \frac{11}{3}$

$y + \frac{8}{3} = \frac{11}{3}$

$y = \frac{3}{3} = 1$

$\boxed{y = 1}$

$x + 2y = 4$

$x + 2 = 4$

$\boxed{x = 2}$

$\therefore x = 2, \; y = 1, \; z = -4.$

4. Determine the Value of a & b for which the System :

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$$

(i) Has a unique Solution
(ii) Has no Solution.
(iii) Has an infinite Solution.

Solution :
$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$$

$$AX = B$$

$$C = AB$$

$$\left[\begin{array}{ccc:c} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right]$$

$R_1 \rightarrow R_1 - R_3$

$$\left[\begin{array}{ccc:c} 1 & -3 & (1-a) & (b+1) \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & -1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 5R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & -3 & (1-a) & (b+1) \\ 0 & 7 & (4+5a) & (-2-5b) \\ 0 & 7 & (-6-2a) & (3b-1) \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & -3 & (1-a) & (b+1) \\ 0 & 7 & (4+5a) & (-2-5b) \\ 0 & 0 & (-6-2a) & (3b-1) \end{array}\right]$$

$\cancel{-6-2a = 3b-1}$

$\cancel{-6-2a-3b+1=0}$

Rank (A) = 3 , Rank (AB) = 3

(i) for a unique Solution

$r[A] = r[AB] = n = 3$

$(-6-2a) \neq 0$

$a \neq -3$, b Can take any Value

(ii) for no Solution

$r[A] \neq r[AB]$

$(-6-2a) = 0$

$a = -3$,

$b \neq \frac{1}{3}$

(iii) for infinite Solution

$r[A] = r[AB] < n = 3$

$a = -3$, $b = \frac{1}{3}$

5. Find the Characteristic equation of Matrix A.

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$ Hence Find A^{-1}

Solution: $$(A - \lambda I) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$4-\lambda \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} - 3 \begin{vmatrix} 2 & -2 \\ 1 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & 1-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$(4-\lambda)\left[(1-\lambda)^2 + 4\right] - 3\left[2 - 2\lambda + 2\right] + 1\left[4 - 1 + \lambda\right] = 0$$

$$(4-\lambda)\left[1 + \lambda^2 - 2\lambda + 4\right] - 3\left[4 - 2\lambda\right] + 1\left[3 + \lambda\right] = 0$$

$$(4-\lambda)\left[\lambda^2 - 2\lambda + 5\right] - 12 + 6\lambda + 3 + \lambda = 0$$

$$4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda - 12 + 6\lambda + 3 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$$

$$\lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0 \quad \text{—} \quad (1)$$

Put the value of $\lambda = A$ in equation (i)

$$A^3 - 6A^2 + 6A - 11 = 0$$

Premultiplies of A^{-1}

by using Cayley hamilton theorem.

$$A^{-1}A^3 - 6A^{-1}A^2 + 6A^{-1}A - 11A^{-1} = 0$$

$$A^2 - 6A + 6I = 11A^{-1} \quad \text{—— (ii)}$$

Now,

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 16+6+1 & 12+3+2 & 4-6+1 \\ 8+2-2 & 6+1-4 & 2-2-2 \\ 4+4+1 & 3+2+2 & 1-4+1 \end{bmatrix}$$

$$\begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$6A = 6 \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix}$$

$$6I = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 6 \end{bmatrix}$$

Putting all the values of A^2, $6A$ & $6I$ in equation (ii)

$$A^2 - 6A + 6I = 11A^{-1}.$$

$$11A^{-1} = A^2 - 6A + 6I$$

$$11A^{-1} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 23-24+6 & 17-18+0 & -1-6+0 \\ 8-12+0 & 3-6+6 & -2+12+0 \\ 9-6+0 & 7-12+0 & -2-6+6 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

6. Find the similarity transformation that diagonalization matrix A.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution: Step 1 : Find the eigen Values
Step 2 : Find the eigen Vectors
Step 3 : Form the Modal Matrix
$P = [x_1 \ x_2 \ x_3]$
Step 4 : Find the inverse of P
$P^{-1} = \frac{adj\, P}{|P|}$
Step 5 : Find $D = P^{-1}AP$

Step 1

$$\lambda^3 - p\lambda^2 + q\lambda - |A| = 0. \quad \text{— (1)}$$

p = Sum of diagonal elements
$= -2 + 1 + 0 = -1$

q = Sum of Minors diagonal element

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$q = (0-12) + (0-6) + (-4+1)$
$-12 - 6 - 3$
$q = -21$

$$|A| = -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$= -2[0-12] - 2[0-6] - 3[-4+1]$
$= -2[-12] - 2[-6] - 3[-3]$
$|A| = 24 + 12 + 9 = 45$

Now, Put the Value of p & q.

$\lambda^3 - p\lambda^2 + q\lambda - |A| = 0$

$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$(\lambda - 5)$ is the factor of given equation.

$$
\begin{array}{l}
\lambda - 5 \,\big)\, \cancel{\lambda^3} + \lambda^2 - 21\lambda - 45 \,\big(\, \lambda^2 + 6\lambda + 9 \\
\quad\quad \underline{-\lambda^3 \mp 5\lambda^2} \\
\quad\quad\quad 6\cancel{\lambda^2} - 21\lambda - 45 \\
\quad\quad\quad \underline{\cancel{6\lambda^2} \mp 30\lambda} \\
\quad\quad\quad\quad 9\lambda - \cancel{45} \\
\quad\quad\quad\quad \underline{9\cancel{\lambda} - 45} \\
\quad\quad\quad\quad\quad 0
\end{array}
$$

$(\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0$

$\lambda = 5$

$\lambda^2 + 6\lambda + 9 = 0$

$\lambda^2 + 3\lambda + 3\lambda + 9 = 0$

$\lambda(\lambda + 3) + 3(\lambda + 3) = 0$

$\lambda = -3, \quad \lambda = -3$

$\therefore [\lambda = 5], [\lambda = -3], [\lambda = -3]$

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

Step 2

Case 1 $\lambda = 5$ & $(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$
$$-x_1 - 2x_2 - 5x_3 = 0$$

Take any two equation

$$-7x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 - 4x_2 - 6x_3 = 0$$

$$x_1 = x_2 = x_3$$

$$\begin{bmatrix} 2 & -3 \\ -4 & -6 \end{bmatrix} \quad \begin{bmatrix} -3 & -7 \\ -6 & 2 \end{bmatrix} \quad \begin{bmatrix} -7 & 2 \\ 2 & -4 \end{bmatrix}$$

$$(-12-12) \quad (-6-42) \quad (28-4)$$

$$-24 \qquad -48 \qquad 24$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -24 \\ -48 \\ 24 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Case 2 $\lambda = -3$

$(A - \lambda I)\, x = 0$

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & +3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

$$x_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Case 3 $\lambda = -3$, Just Case no 2 = $\lambda = -3$

So,

$$x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Step 3

$$P = [x_1 \; x_2 \; x_3]$$

$$P = \begin{bmatrix} -1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4

$$P^{-1} = \frac{\text{adj } P}{|P|}$$

$$|P| = -1 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} -2 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= -1[-1] - 2[-2-0] + 3[+1]$$

$$= 1 + 4 + 3 = 8$$

$$\text{adj } P = \begin{array}{ccccc} -1 & 2 & 3 & -1 & 2 \\ -2 & -1 & 0 & -2 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ -1 & 2 & 3 & -1 & 2 \\ 2 & -1 & 0 & -2 & -1 \\ \hline -1-0 & & 0+2 & & 0+3 \\ 0+2 & & -1-3 & & -6 \\ 0+1 & & 2-0 & & 1+4 \end{array}$$

$$\cancel{\text{adj } P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & -4 & -6 \\ 1 & 2 & +4 \end{bmatrix}}$$

$$\text{adj } P = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P^{-1} = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & 5 \end{bmatrix}$$

<u>Step 5</u>

$$D = P^{-1} A P.$$

$$D = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ 2 & -4 & -6 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \frac{1}{8} \begin{bmatrix} -5 & -10 & 15 \\ -6 & -12 & 18 \\ 3 & -6 & -15 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Ans

7. Write a note on Complex Matrix.

Complex Matrix.

A Matrix whose elements may Contain Complex numbers is Called Complex matrix.

The Matrix product of two 2x2 Complex Matrices is given by

$$\begin{bmatrix} x_{11}+y_{11}i & x_{12}+y_{12}i \\ x_{21}+y_{21}i & x_{22}+y_{22}i \end{bmatrix} \begin{bmatrix} u_{11}+v_{11}i & u_{12}+v_{12}i \\ u_{21}+v_{21}i & u_{22}+v_{22}i \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} + i \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$$

Where,

$R_{11} = u_{11}x_{11} + u_{21}x_{22} - v_{11}y_{11} - v_{21}y_{12}$

$R_{12} = u_{12}x_{11} + u_{22}x_{12} - v_{12}y_{11} - v_{22}y_{12}$

$R_{21} = u_{11}x_{21} + u_{21}x_{22} - v_{11}y_{21} - v_{21}y_{22}$

$R_{22} = u_{12}x_{21} + u_{22}x_{22} - v_{12}y_{21} - v_{22}y_{22}$

$I_{11} = v_{11}x_{11} + v_{21}x_{12} + u_{11}y_{11} + u_{21}y_{12}$

$I_{12} = v_{12}x_{11} + v_{22}x_{12} + u_{12}y_{11} + u_{22}y_{12}$

$I_{21} = v_{11}x_{21} + v_{21}x_{22} + u_{11}y_{21} + u_{21}y_{22}$

$I_{22} = v_{12}x_{21} + v_{22}x_{22} + u_{12}y_{21} + u_{22}y_{22}$.

for ex $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2-3i & 7i \\ 5 & 1+i \end{bmatrix}$

7. (a) Conjugate of Complex

A Conjugate Matrix is a matrix of a Complex matrix, which is formed by Changing the Sign between two terms in a Complex number.

for ex → $A = \begin{bmatrix} 4-7i & 8 \\ 9 & 6+3i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 4+7i & 8 \\ 9 & 6-3i \end{bmatrix}$

(b) Transpose of Conjugate of Matrix.

The Complex Conjugate transpose of a matrix, interchanges the row & Column index for each element, reflecting the elements across the main diagonal. The operation also negates the imaginary part of any Complex numbers.

For ex : if $B = A'$ & $A(1,2)$ is $(1+1i)$ then, the element $B(2,1)$ is $(1-1i)$.

8. Write note on Hermitian matrix & Skew Hermitian matrix.

Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called Hermitian if

$$\bar{A}^T = A, \text{ that is } \bar{a}_{ji} = a_{ij}$$

for ex $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$$

Skew Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called Skew-Hermitian if

$$\bar{A}^T = -A, \text{ that is } \bar{a}_{ji} = -a_{ij}$$

for ex

$$A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & +i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix} = -\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = -A.$$

9.(i) Periodic Matrix:

A square matrix A such that the matrix power $A^{K+1} = A$ for K, a positive integer is called a periodic matrix. if K is the least such integer, then the matrix is said to have period K.

if $K = 1$

$$A^{K+1} = A^2 = A.$$

for example

$$A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 16-12 & -4+3 \\ 48-36 & -12+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A.$$

(ii) Idempotent Matrix:

an idempotent matrix is a matrix which, when multiplied by itself, yields itself. i.e the matrix A is idempotent if and only if $\boxed{A^2 = A}$ for this product A^2 to be defined. A must be square matrix.

for example

$$A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6 & -18+12 \\ 3-2 & -6+4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

$$\boxed{A^2 = A}$$

$\therefore$ Idempotent matrix is an identity matrix.

Ex

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) Unitary Matrix

A square matrix $A = [a_{ij}]$ is called Unitary if

$$\bar{A}^T = A^{-1}$$

for ex

$$A = \begin{bmatrix} \frac{1}{2} i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} i \end{bmatrix}$$

10. Algebraic Multiplicity :

Let A be a $k \times k$ Matrix. Denoted by $\lambda_1 \ldots \lambda_k$ the k Possibly repeated eigenvalues of A, Which Solve the Characteristic equation.

$$\det(\lambda I - A) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_k) = 0$$

for ex

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$$

$$f(\lambda) = \left(\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \right)$$

$$= \left(\det \begin{bmatrix} \lambda - 4 & -2 \\ -1 & \lambda - 2 \end{bmatrix} \right)$$

$$= (\lambda - 4)(\lambda - 2) - (-2) \cdot (-1)$$

$$= \lambda^2 - 2\lambda - 4\lambda + 8 - 2$$

$$= \lambda^2 - 6\lambda + 6$$

The roots of the polynomial, i.e Solution of $f(\lambda) = 0$ are,

$$\lambda_1 = 3 + \sqrt{3}$$

$$\lambda_2 = 3 - \sqrt{3}$$

Thus, A has two distinct eigen Values. their algebraic multiplicities are

$$\mu(\lambda_1) = 1$$

$$\mu(\lambda_2) = 1.$$

Geometric Multiplicity

Let A be $k \times k$ Matrix, let λ_k be one of the eigenvalues of A & denote its associated eigenspace by E_k.

for ex $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$$f(\lambda) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \lambda-2 & 0 \\ -1 & \lambda-1 \end{bmatrix}\right)$$

$$= (\lambda-2)(\lambda-1) - 0\cdot(-1)$$

$$= (\lambda-2)(\lambda-1)$$

$\lambda_1 = 2$

$\lambda_2 = 1$

$$x_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1-2 & 0 \\ -1 & \lambda_1-1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_{11} = x_{21}$

$$x_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

Assignment - 2.

1. Find the nth derivative of.

(a) $\frac{x+1}{x^2-4}$

$= \frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)}$

$= \frac{x+1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$

$= \frac{x+1}{(x+2)\cancel{(x-2)}} = \frac{A(x-2)+B(x+2)}{\cancel{(x+2)}(x-2)}$

$x+1 = A(x-2) + B(x+2)$

If $x = -2$

$x+1 = A(-2-2) + \cancel{B}(-2+2)$

$x+1 = A(-4) + B(0)$

$x+1 = -4A$

$-2+1 = -4A$

$-1 = -4A$

$A = \frac{1}{4}$

If $x = 2$

$x+1 = A(x-2) + B(x+2)$

$2+1 = A(2-2) + B(4)$

$B = \frac{3}{4}$

$A = \frac{1}{4}, \quad B = \frac{3}{4}$

$$\frac{x+1}{(x+2)(x-2)} = \frac{\frac{1}{4}}{x+2} + \frac{\frac{3}{4}}{x-2}$$

$$= \frac{1}{4}(x+2)^{-1} + \frac{3}{4}(x-2)^{-1}$$

$$y_n = \frac{1}{4}\left[(-1)^n (x+2)^{-n-1} (1)^n n!\right] + \frac{3}{4}\left[(-1)^n (x-2)^{-n-1} (1)^n n!\right]$$

$$\therefore y_n = \frac{1}{4}\left[(-1)^n (x+2)^{-n-1} (1)^n n!\right] + \frac{3}{4}\left[(-1)^n (x-2)^{-n-1} (1)^n n!\right]$$

(b) $\cos^2 x$

$= \cos 2x = 2\cos^2 x - 1$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \frac{1}{2}\left[1 + \cos 2x\right]$

$$y_n = \frac{1}{2}\left[0 + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$

$\because \cos(ax+b) =$

$y_n = a^n \cos\left\{ax + b + \frac{n\pi}{2}\right\}$

(c) $e^x \cos x$

$y = e^x \cos x$

as we know

$e^{ax} \cos(bx+c)$

$$y_n = e^{ax} \cdot r^n \cos(bx + c + n\alpha)$$

Where,

$$r = (a^2 + b^2)^{\frac{1}{2}}$$

&

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right).$$

$a = 1$, $b = 1$, $r = (1^2 + 1^2)^{\frac{1}{2}} = (2)^{\frac{1}{2}}$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1} 1$

$\alpha = \frac{\pi}{4}$.

Now,

$y = e^x \cdot \cos x$

$y_n = e^x \cdot (2)^{\frac{n}{2}} \cos\left(x + \frac{n\pi}{4}\right)$

2. If $y = \sin(m \sin^{-1} x)$ so prove that

$[(1-x^2) y_{n+2} - (2x+1) x y_{n+1} + (m^2 - n^2) y_n = 0$

$= y = \sin(m \sin^{-1} x)$ — ①

$y_1 = \cos(m \sin^{-1} x)\left[m \cdot \frac{1}{\sqrt{1-x^2}}\right]$

multiply both sides by $\sqrt{1-x^2}$

$\sqrt{1-x^2}\, y_1 = m \cos(m \sin^{-1} x)$.

Squaring both sides

$(1-x^2) y_1^2 = m^2 \cos^2(m \sin^{-1} x)$

$(1-x^2) y_1^2 = m^2 [1 - \sin^2(m \sin^{-1} x)]$

$$(1-x^2)\,y_1^2 = m^2[1-y^2] \quad \text{— (II)}$$

differentiation with respect to x

$$(1-x^2)\cdot 2y_1\cdot y_2 + (-2x)\,y_1^2 = m^2(-2yy_1)$$

$$(1-x^2)\,y_2 - xy_1 + m^2y = 0 \quad \text{— (III)}$$

differentiate n^{th} time with respect to x

$$\therefore [uv]_n = uv_n + nu_1v_{n-1} + \frac{n(n-1)}{2!}u_2v_{n-2} + \cdots$$

$$[\underset{u}{(1-x^2)}\,\underset{v}{y_2}]_n = (1-x^2)(y_2)_n + n(-2x)(y_2)_{n-1} + \frac{n(n-1)}{2!}(-2)\left(\tfrac{1}{2}\right)_{n-2} + 0$$

$$\therefore = (1-x^2)\,y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$$

$$[xy_1]_n = xy_{n+1} + n(1)y_n + 0$$

$$[m^2y]_n = m^2y_n$$

from equation (III).

$$\therefore (1-x^2)y_2 - xy_1 + m^2y = 0$$

diff w.r.to x, n^{th} times

$$[(1-x^2)y_2]_n - [xy_1]_n + m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$(1-x^2)y_{n+2} + xy_{n+1}(-2n-1) + y_n(-n(n-1) - n + m^2) = 0$$

$$(1-x^2)y_{n+2} - (2n-1)x\,y_{n+1} + (m^2-n^2)y_n = 0.$$

Proved

3. If $y = \log(x+\sqrt{1+x^2})$ Prove that
$[(1+x^2)y_{n+2}] + (2n+1)x\,y_{n+1} + n^2 y(n) = 0.$

Solution $y = \log(x+\sqrt{1+x^2})$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}} \times \left(1 + \frac{1}{2\sqrt{1+x^2}} \times 2x\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$y_1 = \frac{1}{\sqrt{1+x^2}}$$

$$\sqrt{1+x^2}\,y_1 = 1.$$

Squaring both Sides

$$(1+x^2)\,y_1^2 = 1$$

n^{th} differentiating

$$(1+x^2).\,2y_1y_2 + (2x)\,y_1^2 = 0$$

$$2y_1[(1+x^2)y_2 + xy_1] = 0$$

$$(1+x^2)y_2 + xy_1 = 0 \quad \text{—— (i)}$$

$$[(1+x^2)y_2]_n = (1+x^2)(y_2)_n + n(2x)(y_2)_{n-1} + \frac{n(n-1)}{2!}(2)(y_2)_{n-2} + 0$$

$$= (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)(y)_n$$

$$[\underset{u}{x}\,\underset{v}{y_1}]_n = x y_{n+1} + n(1) y_n = 0$$

$$= x y_{n+1} + n y_n$$

from equation (1), n^{th} derivative

$$[(1+x^2) y_2]_n + [x y_1]_n = 0$$

$$(1+x^2) y_{n+2} + n(2x) y_{n+1} + n(n-1) y_n + y_{n+1} x + n y_n = 0$$

$$y_{n+2}(1+x^2) + y_{n+1}[2n+1] + y_n[n^2 - \cancel{n} + \cancel{n}] = 0$$

$$y_{n+2}(1+x^2) + (1+2n) x y_{n+1} + n^2 y_n = 0$$

Proved

4. If $x = r\cos\theta$, $y = r\sin\theta$ find

(i) $\left(\frac{dy}{d\theta}\right)_r$

Solution: $y = r\sin\theta$

$$\left(\frac{dy}{d\theta}\right)_r = r \cdot \cos\theta = x$$

$\because \left(\frac{dy}{d\theta}\right)_r$ means partial derivative of y with respect to θ & take r as Constant.

(ii) $\left(\frac{dr}{dx}\right)_y$

Given: $x = r\cos\theta$ — (i)

$y = r\sin\theta$ — (ii)

add eq (I) & eq (II)

$(I)^2 + (II)^2$

$x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta$

$x^2 + y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$\left(\frac{dr}{dx}\right)_y$ means partial derivative of r with respect to x by taking y as Constant.

$$\left(\frac{dr}{dx}\right)_y = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

5. If $u = \log(\tan x + \tan y)$ prove that,

$$\sin \cdot \frac{\partial u}{\partial x} + \sin 2y \frac{\partial x}{\partial y} = 2$$

Solution: $u = \log(\tan x + \tan y)$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y} \times \sec^2 x + 0$$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y} \quad \text{— (I)}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y} \quad \text{— (II)}$$

6. Solution

L.H.S

Now, $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y}$

$$\sin 2x \left(\frac{\sec^2 x}{\tan x + \tan y} \right) + \sin 2y \left(\frac{\sec^2 y}{\tan x + \tan y} \right)$$

$$\frac{2 \cdot \sin x \cancel{\cos x} \times \frac{1}{\cos^{\cancel{2}} x}}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}} + \frac{2 \cdot \sin y \cancel{\cos y} \times \frac{1}{\cos^{\cancel{2}} y}}{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}$$

$$\left[\frac{2 \frac{\sin x}{\cancel{\cos x}}}{\frac{\sin x \cos y + \sin y \cos x}{\cancel{\cos x} \cos y}} \right] + \left[\frac{2 \frac{\sin y}{\cancel{\cos y}}}{\frac{\sin x \cos y + \sin y \cos x}{\cos x \cancel{\cos y}}} \right]$$

$$\frac{2 \sin x \cos y}{\sin x \cos y + \sin y \cos x} + \frac{2 \sin y \cos x}{\sin x \cos y + \sin y \cos x}$$

$$\frac{2 \sin x \cos y + 2 \sin y \cos x}{\sin x \cos y + \sin y \cos x}$$

$$2 \frac{\cancel{(\sin x \cos y + \sin y \cos x)}}{\cancel{\sin x \cos y + \sin y \cos x}}$$

$$= 2 = \text{R.H.S}$$

Proved.

6. If $V = \frac{x^3y^3}{x^3+y^3}$, Show that $x.\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3V$

Solution: $V = \frac{x^3y^3}{x^3+y^3}$

$$\frac{\partial v}{\partial x} = \frac{(x^3+y^3).\ 3x^2y^3 - x^3y^3\,(3x^2+0)}{(x^3+y^3)^2}$$

$$= \frac{(x^3+y^3)\ 3x^2y^3 - 3x^5y^3}{(x^3+y^3)^2}$$

$$\cancel{x.\frac{\partial v}{\partial x}} = \frac{3x^5y^3 + 3x^2y^6 - 3x^5y^3}{(x^3+y^3)^2}$$

$$x.\frac{\partial v}{\partial x} = \frac{3x^6y^3 + 3x^3y^6 - 3x^6y^3}{(x^3+y^3)^2} \quad \text{— (i)}$$

$$\frac{\partial v}{\partial y} = \frac{(x^3+y^3)\ x^3.3y^2 - x^3y^3\,(0+3y^2)}{(x^3+y^3)^2}$$

$$= \frac{3y^2x^6 + 3y^5x^3 - 3y^5x^3}{(x^3+y^3)^2}$$

$$y.\frac{\partial v}{\partial y} = \frac{3y^3x^6 + 3y^6x^3 - 3y^6x^3}{(x^3+y^3)^2} \quad \text{— (ii)}$$

add equation (i) & (ii)

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{3x^6y^3 + 3x^3y^6 - 3x^6y^3}{(x^3+y^3)^2}$$

$$+ \frac{3y^3x^6 + 3y^6x^3 - 3x^3y^6}{(x^3+y^3)^2}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{3x^3y^3\left[x^3 + \cancel{y^3} - \cancel{x^3} + \cancel{x^3} + y^3 - \cancel{y^3}\right]}{(x^3+y^3)^2}$$

$$= \frac{3x^3y^3\,\cancel{(x^3+y^3)}}{(x^3+y^3)\cancel{(x^3+y^3)}}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{3x^3y^3}{x^3+y^3} = 3v$$

Proved

7. If $u = \log\frac{x^3+y^3}{x^2+y^2}$, Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$$

Solution: $u = \log\frac{x^3+y^3}{x^2+y^2}$

$$\therefore \frac{x^3+y^3}{x^2+y^2} = e^u.$$

$$\cancel{u} = e^u = \frac{x^3+y^3}{x^2+y^2}$$

$$= \frac{x^3\left(1 + \frac{y^3}{x^3}\right)}{\cancel{x^2}\left(1 + \frac{y^2}{x^2}\right)}$$

$$= x\left[\frac{1 + \left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)^2}\right]$$

$n = 1$

So, by Euler's deduction formula,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$= \frac{e^u}{e^u} = 1.$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1}$$

8. If $u = \tan^{-1}\dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}$, find the Value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

Solution: $u = \tan^{-1}\dfrac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}.$

$$\tan u = \frac{\sqrt{x^3+y^3}}{\sqrt{x}+\sqrt{y}}.$$

$$= \frac{\sqrt{x^{\cancel{3}2}}\left(1+\sqrt{\frac{y^3}{x^3}}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$= x\left[\frac{1+\left(\frac{y^3}{x^3}\right)^{\frac{1}{3}}}{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}}\right]$$

$$= x\left[\frac{1+\sqrt{\left(\frac{y}{x}\right)^3}}{1+\sqrt{\frac{y}{x}}}\right]$$

$n = 1$.

by euler's deduction formula

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\frac{f(v)}{f'(v)}$$

$$= 1 \cdot \frac{\tan v}{\sec^2 v}$$

$$= 1 \cdot \frac{\frac{\sin v}{\cos v}}{\frac{1}{\cos^2 v}}$$

$$= 1 \cdot \frac{\sin v}{\cancel{\cos v}} \times \cos^{\cancel{2}} v$$

$$= 1 \cdot \sin v \cos v$$

$$= \frac{1}{2} \cdot 2 \sin v \cos v$$

$$= \frac{\sin 2v}{2}$$

2nd deduction formula

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = g(v)\left[g'(v) - 1\right]$$

Where, $g(v) = \frac{n f(v)}{f'(v)}$

$$= 1 \cdot \frac{\sin 2v}{2}$$

$$\therefore x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{\sin 2v}{2}\left[\frac{1}{\cancel{2}}\cos 2v \times \cancel{2} - 1\right]$$

$$= \frac{1}{2}\sin 2v \cos 2v - \frac{\sin 2v}{2}$$

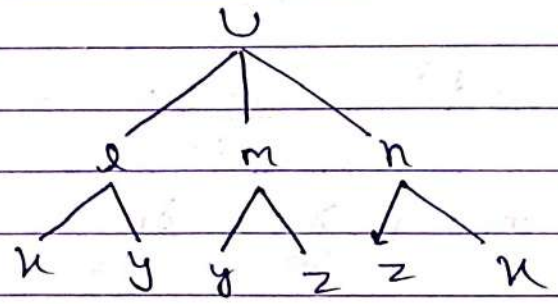
$$= \frac{1}{4}\sin 4v - \frac{1}{2}\sin 2v$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\sin 4u - \frac{1}{2}\sin 2u.$$

9. If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$, then prove that, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution:

- Let $l = e^{x-y}$
  $m = e^{y-z}$
  $n = e^{z-x}$



- $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial x} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial x}$$

$l = e^{x-y}$

$\frac{\partial l}{\partial x} = e^{x-y} \cdot 1 = e^{x-y}$

$n = e^{z-x}$

$\frac{\partial n}{\partial x} = e^{z-x} \cdot (-1) = -e^{z-x}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} e^{x-y} + \frac{\partial u}{\partial n} \cdot (-e^{z-x})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} e^{x-y} - \frac{\partial u}{\partial n} e^{z-x} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \times \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial y}$$

$$l = e^{x-y}$$

$$\frac{\partial l}{\partial y} = e^{x-y} \cdot (-1) = -e^{x-y}$$

$$m = e^{y-z}$$

$$\frac{\partial m}{\partial y} = e^{y-z} \cdot 1 = e^{y-z}$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial l} e^{x-y} + \frac{\partial u}{\partial m} \cdot e^{y-z}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial m} e^{y-z} - \frac{\partial u}{\partial l} e^{x-y} \quad \text{— (ii)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial m} \times \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \times \frac{\partial n}{\partial z}$$

$$m = e^{y-z}$$

$$\frac{\partial m}{\partial z} = e^{y-z} \cdot (-1) = -e^{y-z}$$

$$n = e^{z-x}$$

$$\frac{\partial n}{\partial z} = e^{z-x} \cdot 1 = e^{z-x}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial n} e^{z-x} - \frac{\partial u}{\partial m} e^{y-z} \quad \text{— (iii)}$$

Add eq (i), (ii) & (iii)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial l} e^{x-y}} - \cancel{\frac{\partial u}{\partial n} e^{z-x}} +$$

$$\cancel{\frac{\partial u}{\partial m} e^{y-z}} - \cancel{\frac{\partial u}{\partial l} e^{x-y}} + \cancel{\frac{\partial u}{\partial n} e^{z-x}} - \cancel{\frac{\partial u}{\partial m} e^{y-z}}$$

$$= 0$$

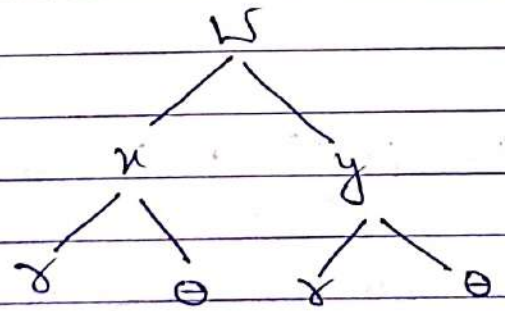
$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Proved

10. If $w = f(x, y)$ $x = r\cos\theta$, $y = r\sin\theta$.
Show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

Solution:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r}$$

$$= \frac{\partial w}{\partial x}(\cos\theta) + \frac{\partial w}{\partial y}(\sin\theta)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$\text{L.H.S} = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

$$= \left[\cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial w}{\partial y}\right]^2 + \frac{1}{r^2}\left[-r\sin\theta \frac{\partial w}{\partial x} + r\cos\theta \frac{\partial w}{\partial y}\right]^2$$

$$= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + 2\sin\theta\cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$+ \frac{1}{r^2}\cdot\left[r^2\cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + r^2\sin^2\theta\left(\frac{\partial w}{\partial x}\right)^2 - 2\sin\theta \cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

$$= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + 2\sin\theta\cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$+ \frac{1}{\cancel{r^2}}\cdot\cancel{r^2}\left[\cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial x}\right)^2 - 2\sin\theta \cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]$$

$$= \cos^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial w}{\partial y}\right)^2 +$$

$$\cos^2\theta\left(\frac{\partial w}{\partial y}\right)^2 + \cancel{2\sin\theta\cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}} - \cancel{2\sin\theta\cos\theta \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}$$

$$= \left(\frac{\partial w}{\partial x}\right)^2\left[\cos^2\theta + \sin^2\theta\right] + \left(\frac{\partial w}{\partial y}\right)^2\left[\sin^2\theta + \cos^2\theta\right]$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \text{R.H.S}$$

$$\therefore \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

Proved

Assignment -3

1. Expand the following in power of x by Maclauren's theorem as far as x^5

(i) $e^x \sin x$

Solution: $y = e^x \sin x$

$\because y = e^{ax} \sin(bx+c)$

$y_n = e^{ax} \cdot (\gamma)^n \sin(bx+c+n\alpha)$

where,

$\gamma^2 = a^2+b^2$, $\tan\alpha = \frac{b}{a}$

So,

$a = 1$, $b = 1$, $c = 0$

$\tan\alpha = \frac{b}{a} = \frac{1}{1}$

$\alpha = \tan^{-1} 1$

$\alpha = \frac{\pi}{4}$, $\gamma^2 = a^2+b^2$

$\gamma = \sqrt{1+1} = \sqrt{2}$

$y_n = e^{ax} \cdot (\sqrt{2})^n \sin\left(x + n\frac{\pi}{4}\right) = f(x)$

$y_1 = e^{ax} \sqrt{2} \sin\left(x + \frac{\pi}{4}\right) = f'(x)$

$y_2 = e^{ax} (\sqrt{2})^2 \sin\left(x + 2\frac{\pi}{4}\right) = f''(x)$

$= e^{ax}\, 2 \sin\left(x + 2\frac{\pi}{4}\right)$

$y_3 = e^{ax}\, 2\sqrt{2} \sin\left(x + 3\frac{\pi}{4}\right) = f'''(x)$

$y_4 = e^{ax} \cdot 4 \sin\left(x + \frac{4\pi}{4}\right) = f^{iv}(x)$

$y_5 = e^{ax} \cdot 4\sqrt{2} \sin\left(x + \frac{5\pi}{4}\right) = f^{(v)}(x)$

for $x_0 = 0$

$f(0) = e^0 \sin 0 = 0$

$f'(0) = e^0 (\sqrt{2})' \sin\left(0 + \frac{\pi}{4}\right)$

$= 1 \cdot \cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} = 1$

$f''(0) = e^0 \cdot 2 \sin\left(\frac{2\pi}{4}\right)$

$= 1 \cdot 2 \cdot 1 = 2$

$f'''(0) = e^0 \cdot 2\sqrt{2} \sin\left(\frac{3\pi}{4}\right)$

$= 1 \cdot 2\cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} = 2$

$f^{(iv)}(0) = e^0 \, 2 \times 2 \sin\left(\frac{\cancel{4}\pi}{\cancel{4}}\right)$

$= 1 \cdot 4 \cdot 0 = 0$

$f^{v}(0) = e^0 \, 4\sqrt{2} \cdot \sin\left(\frac{5\pi}{4}\right)$

$= e^0 \cdot 4\cancel{\sqrt{2}} \cdot \frac{-1}{\cancel{\sqrt{2}}}$

$= 1 \cdot 4 \cdot (-1) = -4$

Now,

$\cancel{e^x \sin x = x +}$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \dots$$

$$e^x \sin x = 0 + x(1) + \frac{x^2}{\cancel{2}} \times \cancel{2} + \frac{x^3}{\cancel{6}_3} \times \cancel{2} + \frac{x^4}{24} \times 0 + \frac{x^5}{\cancel{120}_{30}} \times \cancel{-4}$$

$$\therefore e^x \sin x = x + x^2 + \frac{x^3}{3} + 0 - \frac{x^5}{30}$$

(ii) $e^{\sin x}$

Solution: $\cancel{y = e^{f_i}}$

$$f(x) = e^{\sin x}$$
$$f'(x) = e^{\sin x} \cdot \cos x$$
$$f'(x) = f(x) \cdot \cos x$$
$$f''(x) = f(x) \cdot (-\sin x) + \cos x\, f'(x)$$
$$= -f(x) \sin x + f'(x) \cos x$$
$$f'''(x) = -[f(x) \cos x + \sin x\, f'(x)] + f'(x)(-\sin x) + \cos x\, f''(x)$$
$$= -f(x) \cos x - f'(x) \sin x - f'(x) \sin x + f''(x) \cos x$$
$$= -f(x) \cos x - 2f'(x) \sin x + f''(x) \cos x$$
$$f^{iv}(x) = -[f(x)(-\sin x) + \cos x\, f'(x)] - 2[f'(x) \cdot \cos x + \sin x\, f''(x)] + f''(x)(-\sin x) + \cos x\, f'''(x)$$
$$= f(x) \sin x - f'(x) \cos x - 2f'(x) \cos x - 2f''(x) \sin x - f''(x) \sin x + f'''(x) \cos x$$

$$f^{iv}(x) = f(x)\sin x - 3f'(x)\cos x - 3f''(x)\sin x + f'''(x)\cos x$$

$$f^{v}(x) = f(x)\cdot\cos x + \sin x f'(x) - 3\left[f'(x)\cdot(-\sin x) + \cos x f''(x)\right] - 3\left[f''(x)\cdot\cos x + \sin x\cdot f'''(x)\right] + f'''(x)\cdot(-\sin x) + \cos x f^{iv}(x)$$

$$= f(x)\cos x + f'(x)\sin x + 3f'(x)\sin x - 3f''(x)\cos x - 3f''(x)\cos x - 3f'''(x)\sin x - f'''(x)\sin x + f^{iv}(x)\cos x$$

$$= f(x)\cos x + 4f'(x)\sin x - 6f''(x)\cos x - 4f'''(x)\sin x + f^{iv}(x)\cos x$$

for $x = 0$

$$f(x) = e^{\sin x}$$

$$f(0) = e^{\sin 0} = e^{0} = 1$$

$$f'(x) = f(x)\cos x$$

$$f'(0) = f(0).\cos 0 = 1.1 = 1$$

$$f''(x) = -f(x)\sin x + f'(x)\cos x$$

$$f''(0) = -f(0)\sin 0 + f'(0)\cos 0$$

$$= -1.0 + 1.1 = 1$$

$$f'''(x) = -f(x)\cos x - 2f'(x)\sin x + f''(x)\cos x$$

$$f'''(0) = -f(0)\cos 0 - 2f'(0)\sin 0 + f''(0)\cos 0$$

$$= -1.1 - 2(1)(0) + 1.1$$

$$= -1 + 1 = 0$$

$$f^{iv}(x) = f(x)\sin x - 3f'(x)\cos x - 3f''(x)\sin x + f'''(x)\cos x$$

$$f^{iv}(0) = f(0)\sin 0 - 3f'(0)\cos 0 - 3f''(0)\sin 0 + f'''(0)\cos 0$$

$= 1.0 - 3.1.1 - 3.1.0 + 0.0$

$= -3$

$f^{v}(x) = f(x)\cos x + 4f'(x)\sin x - 6f''(x)\cos x - 4f'''(x)\sin x + f^{iv}(x)\cos x$

$f^{v}(0) = f(0)\cos 0 + 4f'(0)\sin 0 - 6f''(0)\cos 0 - 4f'''(0)\sin 0 + f^{iv}(0)\cos 0$

$= 1.1 + 4.1.0 - 6.1.1 - 4.0.0 + (-3).1$

$= 1 - 6 - 3 = 1 - 9 = -8$

Now, According to ~~Taylor's~~ MacLaurin's Series

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \dots$$

$$= 1 + x + \frac{x^2}{2}(1) + \frac{x^3}{6}.0 + \frac{x^4}{\cancel{24}_{8}}\cancel{(-3)} + \frac{x^5}{\cancel{120}_{15}} \times \cancel{-8}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \times 0 - \frac{x^4}{8} - \frac{x^5}{15}$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} \quad \text{Ans} \leftarrow$$

(iii) $\log(1 + \sin x)$

Solution

$f(x) = y = \log(1 + \sin x)$

$$f'(x) = \frac{1}{1+\sin x}(0 + \cos x) = \frac{\cos x}{1 + \sin x}$$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(0+\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-\cancel{[1+\sin x]}}{(1+\sin x)^{\cancel{2}}}$$

$$= \frac{-1}{1+\sin x}$$

$$f''(x) = (-1)(1+\sin x)^{-1}$$

$$f'''(x) = -1\cdot(-1)(1+\sin x)^{-2}\cdot(0+\cos x)$$

$$= \frac{\cos x}{(1+\sin x)^2}$$

$$f^{iv}(x) = \frac{(1+\sin x)^2\cdot(-\sin x) - [\cos x\cdot 2\cdot(1+\sin x)\cdot(0+\cos x)]}{(1+\sin x)^4}$$

$$= \frac{\cancel{(1+\sin x)}\,[(1+\sin x)(-\sin x) - 2\cos^2 x]}{(1+\sin x)^{\cancel{4}\,3}}$$

$$= \frac{-\sin x - \sin^2 x - 2\cos^2 x}{(1+\sin x)^3}$$

$$f(x) = \log(1+\sin x)$$
$$f(0) = \log(1+\sin 0) = \log(1+0) = 0$$

$$f'(x) = \frac{\cos x}{1+\sin x}$$

$$f'(0) = \frac{\cos 0}{1+\sin 0} = \frac{1}{1} = 1$$

$$f''(x) = \frac{-1}{1+\sin x}$$

$$f''(0) = \frac{-1}{1+\sin 0} = -1$$

$$f'''(x) = \frac{\cos x}{(1+\sin x)^2}$$

$$f'''(0) = \frac{\cos 0}{(1+\sin 0)^2} = 1$$

$$f^{iv}(x) = \frac{-\sin x - \sin^2 x - 2\cos^2 x}{(1+\sin x)^3}$$

$$f^{iv}(0) = \frac{-\sin 0 - \sin^2 0 - 2\cos^2 0}{(1+\sin 0)^3}$$
$$= -2$$

Now,

According to Maclaurin's Series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)$$

$$+ \frac{x^4}{4!} f^{iv}(0) \cdots$$

$$= 0 + x + \frac{x^2}{2} \times (-1) + \frac{x^3}{6} \times 1 + \frac{x^4}{\cancel{24}\,12} \times \cancel{-2}$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

2. Expand by Taylor's theorem.

(1) $\sin x$ in powers of $(x - \frac{\pi}{2})$

$$x - \frac{\pi}{2} = 0$$

$$x = \frac{\pi}{2}$$

$$f(x) = \sin x$$

at $x_0 = \frac{\pi}{2}$ $\quad f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$f'(x) = \cos x$$

$$f'(\tfrac{\pi}{2}) = \cos\frac{\pi}{2} = 0$$

$$f''(x) = -\sin x$$

$$f''(\tfrac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$$

$f'''(x) = -\cos x$

$f'''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f^{iv}(x) = \sin x$

$f^{iv}(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$f^{vi}(x) = -\sin x$.

$f^{vi}(\frac{\pi}{2}) = -\sin\frac{\pi}{2}$

$= -1$

$f^{v}(x) = \cos x$

$f^{v}(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

Now, According to taylor's theorem.

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0)$$

$$+ \frac{(x-x_0)^3}{3!} f'''(x_0) + \frac{(x-x_0)^4}{4!} f^{iv}(x_0) + \frac{(x-x_0)^5}{5!} f^{v}(x_0)$$

$$+ \frac{(x-x_0)^6}{6!} f^{vi}(x_0)$$

$x_0 = \frac{\pi}{2}$.

$$\sin x = 1 + \left(x-\frac{\pi}{2}\right)\cdot 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2} \times -1 + \frac{\left(x-\frac{\pi}{2}\right)^3}{6} \times 0$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^4}{24} \times 1 + \frac{\left(x-\frac{\pi}{2}\right)^5}{120} \times 0 + \frac{\left(x-\frac{\pi}{2}\right)^6}{720} \times -1$$

$$\sin x = 1 - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x-\frac{\pi}{2}\right)^6$$

(ii) x^3 in power of $x-1$

$x-1=0 \Rightarrow x=1$

<u>Solution</u> $f(x) = x^3$

at $x_0 = 1$

$f(1) = 1^3 = 1$

$f'(x) = 3x^2$

$f'(1) = 3(1) = 3$

$f''(x) = 6x$

$f''(1) = 6(1) = 6$

$f'''(x) = 6$

$f'''(1) = 6$

$f^{iv}(x) = 0$

$f^{iv}(1) = 0$

Here, $x_0 = 1$

According to taylor's theorem

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \frac{(x-x_0)^4}{4!} f^{iv}(x_0) + \cdots$$

$$= 1 + (x-1)3 + \frac{(x-1)^2}{\cancel{2}} \times \overset{3}{\cancel{6}} + \frac{(x-1)^3}{\cancel{6}} \times \cancel{6} + 0 + \cdots$$

$$= 1 + 3x - 3 + (x-2)^2 \times 3 + (x-1)^3 + \cdots$$

$$= 1 + 3(x-1) + 3(x-2)^2 + (x-1)^3 + \cdots$$

(iii) $\log \cos x$ about the power $\frac{\pi}{3}$.

<u>Solution</u>

at $x_0 = \frac{\pi}{3}$

$f(x) = \log \cos x$

$f\left(\frac{\pi}{3}\right) = \log \cos \frac{\pi}{3}$

$= \log \cancel{\cos} \frac{1}{2}$

$= -\log 2$

$f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

$f'\left(\frac{\pi}{3}\right) = -\tan \frac{\pi}{3}$

$= -\sqrt{3}$

$f''(x) = -\sec^2 x$

$f''\left(\frac{\pi}{3}\right) = -\sec^2\left(\frac{\pi}{3}\right) = -4$

$f'''(x) = -2 \sec x \cdot \sec x \cdot \tan x$

$= -2 \sec^2 x \cdot \tan x$

$f'''\left(\frac{\pi}{3}\right) = -2 \sec^2\left(\frac{\pi}{3}\right) \cdot \tan \frac{\pi}{3}$

$= -2 \cdot (+4) \cdot \sqrt{3}$

$= -8\sqrt{3}$

$$f^{iv}(x) = -2\left[(\sec^2 x \cdot \sec^2 x + \tan x) \cdot (2\sec x \cdot \sec x \cdot \tan x)\right]$$

$$= -2\left[\sec^4 x + \tan x \cdot 2\sec^2 x \cdot \tan x\right]$$

$$= -2\tan x\left[\sec^4 x + 2\sec^2 x\right]$$

According to taylor's theorem

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0)$$

$$+ \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots$$

Here $x_0 = \frac{\pi}{3}$

$$\log \cos x = \log \frac{1}{2} + \left(x - \frac{\pi}{3}\right)(-\sqrt{3}) + \frac{\left(x - \frac{\pi}{3}\right)^2}{\cancel{2}}(-4)$$

$$+ \frac{\left(x - \frac{\pi}{3}\right)^3}{\cancel{6}\,3} \times -\cancel{8}^{4}\sqrt{3}$$

$$\log \cos x = -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2$$

$$- \frac{4}{3}\left(x - \frac{\pi}{3}\right)^3$$

Ans

3. Show that [use taylor's theorem]

(i) $e^y \log(1+x) = x + xy - \frac{x^2}{2}$

Solution

$f(a,b) = e^y \log(1+x)$

$f(0,0) = e^0 \log(1+0) = 1.0 = 0$

$f_x(a,b) = e^y \times \frac{1}{1+x} = \frac{e^y}{1+x} = e^y (1+x)^{-1}$

$f_x(0,0) = \frac{e^0}{1+0} = 1$

$f_y(a,b) = e^y \log(1+x)$

$f_y(0,0) = e^0 \log(1+0) = 0$

$f_{xx}(a,b) = \frac{-e^y}{(1+x)^2} = -e^y (1+x)^{-2}$

$f_{xx}(0,0) = \frac{-e^0}{(1+0)^2} = -1$

$f_{yy}(a,b) = e^y \log(1+x)$

$f_{yy}(0,0) = 0$

$f_{xy}(a,b) = \frac{e^y}{1+x}$

$f_{xy}(0,0) = \frac{e^0}{1} = 1$

$$f_{xxx}(a,b) = -e^y(-2)(1+x)^{-3}$$
$$= \frac{2e^y}{(1+x)^3}$$

$$f_{xxx}(0,0) = \frac{2e^0}{(1+0)^3} = 2$$

$$f_{yyy}(a,b) = e^y \log(1+x)$$
$$f_{yyy}(0,0) = e^0 \log(1+0) = 0$$

$$f_{xxy}(a,b) = \frac{-e^y}{(1+x)^2}$$

$$f_{xxy}(0,0) = -1$$

$$f_{xyy}(a,b) = \frac{e^y}{1+x}$$

$$f_{xyy}(0,0) = \frac{e^0}{1+0} = 1$$

According to taylor's Series for two Variables

$$\Rightarrow f(a,b) + [hf_x(a,b) + kf_y(a,b)] + \frac{1}{2!}$$
$$[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b) + 2hk f_{xy}(a,b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + k^3 f_{yyy}(a,b) + 3h^2 k f_{xxy}(a,b)$$
$$+ 3hk^2 f_{xyy}(a,b)]$$

Put $a=0$, $b=0$, $h=x$, $k=y$

$$f(x,y) = f(0,0) + [xf_x + yf_y] + \frac{1}{2!}[x^2 f_{xx} + y^2 f_{yy} + 2xy f_{xy}] + \frac{1}{3!}[x^3 f_{xxx} + y^3 f_{yyy} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy}]$$

$$= [0 + (x \times 1) + (y \times 0)] + \frac{1}{2!}[-x^2 + 0 + 2xy] + \frac{1}{3!}[2x^3 + 0 + 3x^2 y(-1) + 3xy^2(1)]$$

$$\therefore f(x,y) = x - \frac{x^2}{2} + xy + \frac{x^3}{3} - \frac{x^2 y}{2} + \frac{xy^2}{2} \ldots\ldots$$

(ii) $\sin(x+y) = x + y - \frac{(x+y)^3}{3!} + \ldots\ldots$

<u>Solution</u>

$f(a,b) = \sin(x+y)$

$f(0,0) = \sin(0+0) = 0$

$f_x(a,b) = \cos(x+y)$

$f_x(0,0) = \cos(0+0) = 1$

$f_y(a,b) = \cos(x+y)$

$f_y(0,0) = \cos(0+0) = 1$

$f_{xx}(a,b) = -\sin(x+y)$

$f_{xx}(0,0) = -\sin 0 = 0$

$f_{yy}(a,b) = -\sin(x+y)$
$f_{yy}(0,0) = 0$

$f_{xy}(a,b) = -\sin(x+y)$
$f_{xy}(0,0) = 0$

$f_{xxx}(a,b) = -\cos(x+y)$
$f_{xxx}(0,0) = -\cos 0 = -1$

$f_{yyy}(a,b) = -\cos(x+y)$
$f_{yyy}(0,0) = -1$

$f_{xxy}(a,b) = -\cos(x+y)$
$f_{xxy}(0,0) = -1$

$f_{xyy}(a,b) = -\cos(x+y)$
$f_{xyy}(0,0) = -1$

Now, According to taylor's Series for two Variables.

$$f(x,y) = f(0,0) + [x f_x + y f_y + 2xy f_{xy}] + \frac{1}{2!}[x^2 f_{xx} + y^2 f_{yy} + 2xy f_{xy}] + \frac{1}{3!}[x^3 f_{xxx} + y^3 f_{yyy} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy}]$$

$$\sin(x+y) = 0 + x + y + 0 + 0 + 0 + \frac{1}{6}\left[-x^3 - y^3 - 3x^2y - 3xy^2\right]$$

$$= x + y - \frac{(x+y)^3}{3!} + \cdots\cdots$$

4. Divide 120 into three part so that the sum of their products taken two at a time shall be maximum.

Solution.

Let, The three part are x, y & z

$$120 = x + y + z$$

$$z = 120 - x - y$$

Sum of three product $= xy + yz + zx$

$$S = xy + yz + zx \quad \text{— (i)}$$

Put the value of z in equation (i)

$$S = xy + y(120 - x - y) + (120 - x - y)x$$

$$S = xy + 120y - xy - y^2 + 120x - x^2 - xy$$

$$S = 120x + 120y - xy - x^2 - y^2$$

$$\frac{\partial S}{\partial x} = 120 - y - 2x$$

$$\frac{\partial S}{\partial x} = 0$$

$$120 - y - 2x = 0 \quad \text{— (ii)}$$

$$\frac{\partial s}{\partial y} = 120 - x - 2y$$

$$\frac{\partial s}{\partial y} = 0$$

$$120 - x - 2y = 0 \quad \text{— (iii)}$$

from eq^n (ii) & (iii)

$$\cancel{120} - y - 2x = \cancel{120} - x - 2y$$

$$-y - 2x + x + 2y = 0$$

$$-x + y = 0$$

$$y = x$$

Now, from equation

$$S = 120x + 120y - xy - x^2 - y^2$$

Put the value of y

$$S = 120x + 120x - x^2 - x^2 - x^2$$

$$S = 240x - 3x^2$$

$$\frac{\partial s}{\partial x} = 240 - 6x$$

$$\frac{\partial s}{\partial x} = 0$$

$$240 - 6x = 0$$

$$240 = 6x$$

$$x = 40$$

$$\frac{\partial^2 s}{\partial x^2} = -6 \quad \text{(-ve hence Maximum Value)}$$

$$x = 40$$

$$y = 40 \quad (\because x = y)$$

$$Z = 120 - x - y = 120 - 80 = 40$$

$\therefore$ 40, 40, 40 are three parts of 120 so that the sum of three product taken two at a time is maximum.

5. A rectangular box, which is open at the top, has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box. use Lagarange's method of multiplier's to obtain the solution.

Solution :

Let $x \to$ length

$y \to$ breadth

$z \to$ height

$$\text{Volume} = xyz$$

$$256 = xyz$$

$$xyz - 256 = 0 \quad -(i)$$

$$\phi(x,y,z) = xyz - 256$$

When box is open

$$S = xy + 2[yz + zx]$$

$$\frac{\partial s}{\partial x} = y + 2z$$

$$\frac{\partial s}{\partial y} = x + 2z$$

$$\frac{\partial s}{\partial z} = 2y + 2x$$

$\phi(x,y,z) = xyz - 256 \Rightarrow xyz = 256$

$\frac{\partial \phi}{\partial x} = yz$

$\frac{\partial \phi}{\partial y} = xz$

$\frac{\partial \phi}{\partial z} = xy$

By lagaranges method of multiplier, we have

$\frac{\partial s}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$y + 2z + \lambda(yz) = 0$

$y + 2z + \lambda yz = 0$ — (ii)

$\frac{\partial s}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

$x + 2z + \lambda xz = 0$ — (iii)

$\frac{\partial s}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

$2y + 2x + \lambda xy = 0$ — (iv)

Multiply x in equation (ii)

$xy + 2xz + \lambda xyz = 0$

$xy + 2xz + 256\lambda = 0$ — (v)

Multiply y in eq^{-n} (iii)

$$xy + 2yz + \lambda xyz = 0$$

$$xy + 2yz + 256\lambda = 0 \quad \text{--- (vi)}$$

Multiply z in eq^{-n} (iv)

$$2yz + 2xz + \lambda xyz = 0$$

$$2yz + 2xz + 256\lambda = 0 \quad \text{--- (vii)}$$

From eq^{-n} (v) & (vi)

$$\cancel{xy} + 2xz + \cancel{256\lambda} = \cancel{xy} + 2yz + \cancel{256\lambda}$$

$$x = y$$

From eq^{-n} (vi) & (vii)

$$xy + \cancel{2yz} + \cancel{256\lambda} = \cancel{2yz} + 2xz + \cancel{256\lambda}$$

$$y = 2z$$

$$z = \frac{y}{2}$$

From eq^{-n} (i)

$$xyz = 256$$

$$y \times y \times \frac{y}{2} = 256$$

$$y^3 = 512$$

$$y^3 = 8^3$$

$$y = 8$$

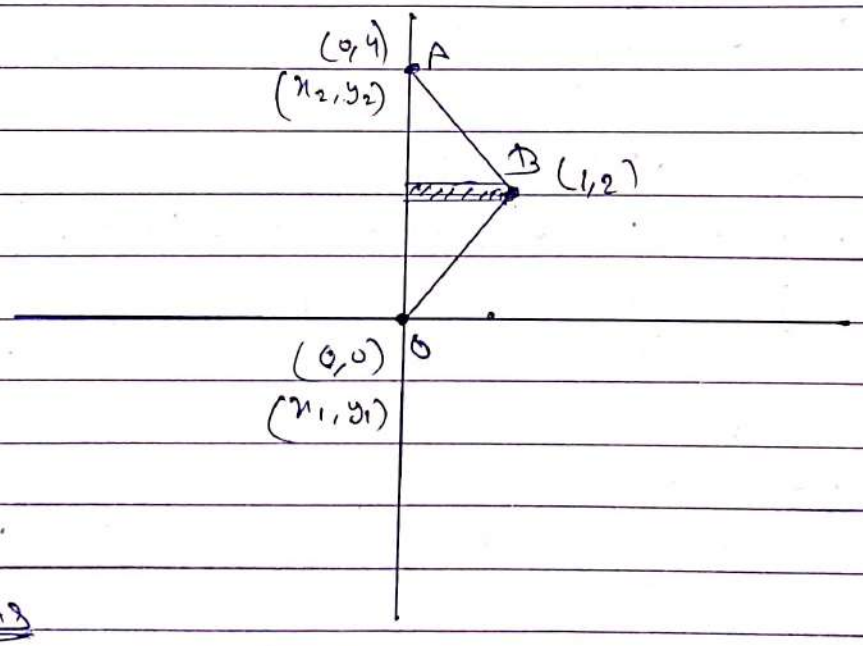
$x = 8$, $y = 8$, $z = 4$

Assignment - 4

1. Evaluate $\iint_R x(x-y)\,dx\,dy$ where R is the triangle with vertices $(0,0)$, $(1,2)$, $(0,4)$.

Sol^n: Given :

$$\iint_R (x^2 - xy)\,dx\,dy$$



<u>limits</u>

$y = 0$ to $y = 4$

Equation of OA $= \dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{y - 0}{x - 0} = \dfrac{4 - 0}{0}$

$= \dfrac{y}{x} = 4$

$x = \dfrac{y}{4}$

$\therefore\ x = 1$ to $x = \frac{y}{4}$

Now $\displaystyle\int_0^4 \int_1^{\frac{y}{4}} (x^2 - xy)\, dx\, dy$

$$\int_0^4 \left[\frac{x^3}{3}\right]_1^{\frac{y}{4}} - y\left[\frac{x^2}{2}\right]_1^{\frac{y}{4}} dy$$

$$\int_0^4 \left(\frac{y^3}{64\times 3} - \frac{1}{3}\right) - y\left(\frac{y^2}{16\times 2} - \frac{1}{2}\right) dy$$

$$\int_0^4 \left(\frac{y^3 - 64}{192}\right) - y\left(\frac{y^2 - 16}{32}\right) dy$$

$$\int_0^4 \left(\frac{y^3 - 64}{192}\right) - \left(\frac{y^3 - 16y}{32}\right) dy$$

$$\int_0^4 \left(\frac{y^3 - 64 - 6y^3 + 96y}{192}\right) dy$$

$$\frac{1}{192}\int_0^4 (y^3 - 64 - 6y^3 + 96y)\, dy$$

$$\frac{1}{192}\left[\left[\frac{y^4}{4}\right]_0^4 - 64\left[y\right]_0^4 - 6\left[\frac{y^4}{1}\right]_0^4 + 96\left[\frac{y^2}{2}\right]_0^4\right]$$

$$\frac{1}{192}\left[64 - (64\times4) - (6\times64) + 96\times8\right]$$

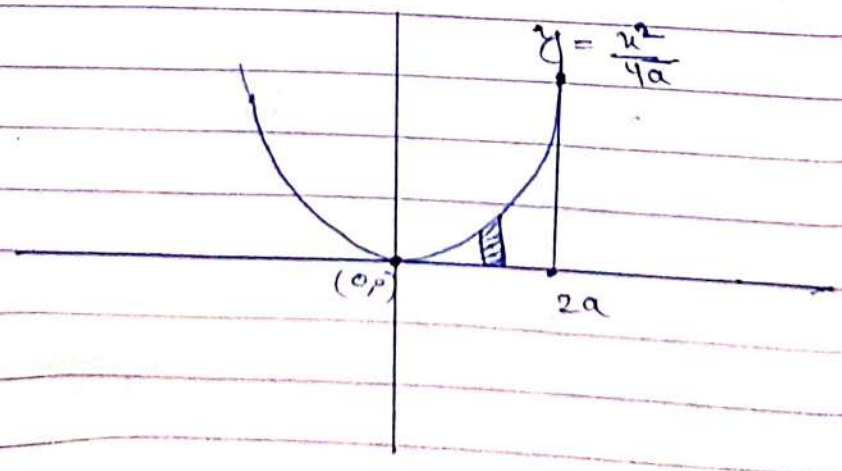
$$\frac{64}{192}\left[1-4-6+12\right]$$

$$\frac{64}{192}\left[-3+6\right]$$

$$\frac{\cancel{64}}{\cancel{192}_{3}} \times 3 = 1$$

2. Evaluate $\iint_A xy\,dy\,dx$ where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Solution : $\iint_A xy\,dx\,dy$



Limits

$y = 0$ to $y = \frac{x^2}{4a}$

$x = 0$ to $x = 2a$

Now

$$= \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy \, dx\, dy$$

$$= \int_0^{2a} x \, dx \int_0^{\frac{x^2}{4a}} y \, dy$$

$$= \int_0^{2a} x \cdot dx \left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}}$$

$$= \int_0^{2a} x \cdot \frac{\left(\frac{x^2}{4a}\right)^2}{2} \, dx$$

$$= \int_0^{2a} x \cdot \frac{x^4}{32a^2} \, dx = \frac{1}{32a^2}\int_0^{2a} x^5 \, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2} \times \frac{(2a)^6}{6}$$

$$= \frac{1}{\cancel{32a^2}} \times \frac{\cancel{64}^{2}\, \cancel{a^6}^{a^4}}{\cancel{6}_{3}} = \frac{a^4}{3} \text{ Ans} \leftarrow$$

3. Change the order of integration and Evaluate,

$$\int_0^a \int_0^y \frac{x\,dx\,dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

Solution

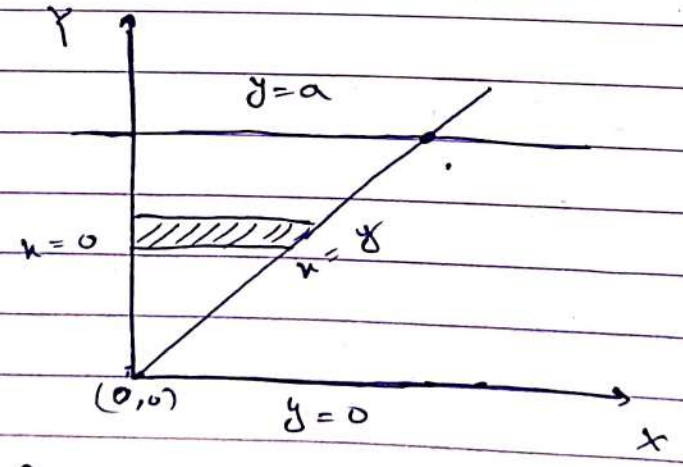
limits of x

$x = 0$ to $x = y$

limits of y

$y = 0$ to $y = a$

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$



$$\int_0^a \int_x^a \frac{x\,dx}{\sqrt{a^2-x^2}} \cdot \frac{dy}{\sqrt{(a-y)(y-x)}}$$

Let $y - x = t^2$

$$\frac{dy}{dt} =$$

$$\therefore \; dy = 2t\,dt$$

If $y = x$, $t = 0$;

$y = a$, $t = \sqrt{a-x}$

$$= \int_0^a \int_0^{\sqrt{a-x}} \frac{x\,dx}{\sqrt{a^2-x^2}} \cdot \frac{2t\,dt}{\sqrt{[(a-x)-t^2]\,t^2}}$$

$$= 2\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{t}{\sqrt{(a-x)}}\right)\right]_0^{\sqrt{a-x}}$$

$$= 2\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{\sqrt{a-x}}{\sqrt{a-x}} - \sin^{-1} 0\right]$$

$$= 2\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \left(\frac{\pi}{2}\right)$$

$$a^2 - x^2 = v$$

$$\therefore \; -2x\,dx = dv$$

$$x\,dx = -\frac{1}{2}\,dv$$

$$2 \times \frac{\pi}{2} \int_0^a \frac{1}{\sqrt{v}} \times -\frac{1}{2}\,dv$$

$$-\frac{\pi}{2} \int_0^a \frac{1}{\sqrt{v}}\,dv$$

$$-\frac{\pi}{2}\left[\frac{(v)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^a$$

$$-\pi\left[\sqrt{a^2-x^2}\right]_0^a$$

$$-\pi\left[\sqrt{a^2-a^2}-\sqrt{a^2-0}\right]$$

$$-\pi\left[-\sqrt{a^2}\right]$$

$$a\pi$$

4. Evaluate the area enclosed between the parabola $y = x^2$ and the straight line $y = x$.

$\Rightarrow$ Solution $\Rightarrow$ $y = x^2$ — (i)

$y = x$ — (ii)

limits of x from eq^n (i) & (ii)

$x = 0$, $y = 0$

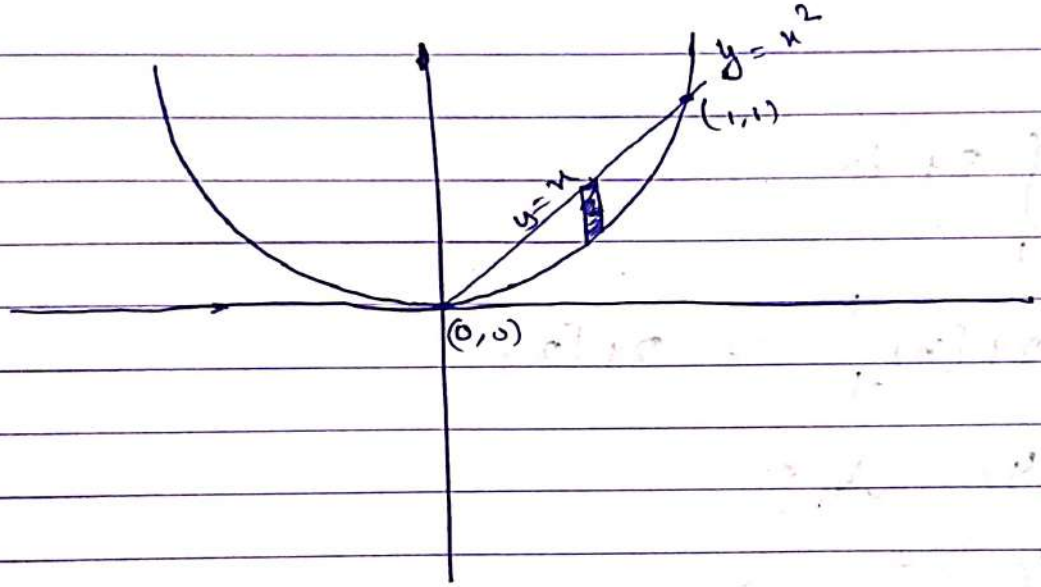
$x = 1$, $y = 1$

$$\iint \partial x\,\partial y = \int_0^1 \partial x \int_{x^2}^{x} \partial y$$

$$= \int_0^1 \partial x\,[y]_{x^2}^{x}$$

$$= \int_0^1 [x - x^2]\,\partial x$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6} \text{ Ans}$$



5. $$\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} \partial x\, \partial y\, \partial z$$

$$\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} \partial z\, \partial y\, \partial x$$

$$\int_{-1}^{1}\int_{-2}^{2} [z]_{-3}^{3}\, \partial y\, \partial x \Rightarrow \int_{-1}^{1}\int_{-2}^{2} 6\, \partial y\, \partial x$$

$$\int_{-1}^{1}\int_{-2}^{2} 6\, \partial y\, \partial x$$

$$\int_{-1}^{1} 6[y]_{-2}^{2} \, dx$$

$$\int_{-1}^{1} 6\,[2+2] \, dx$$

$$\int_{-1}^{1} 24 \, dx$$

$$24\,[x]_{-1}^{1} \Rightarrow 24\,[2]$$

$$\Rightarrow 48$$

6. $$\int_{0}^{4}\int_{0}^{x}\int_{0}^{x+y} z \, dz\, dy\, dx$$

$$\Rightarrow \int_{0}^{4}\int_{0}^{x} \left[\frac{z^2}{2}\right]_{0}^{x+y}$$

$$\int_{0}^{4}\int_{0}^{x} \frac{x^2+y^2+2xy}{2} \, dy\, dx$$

$$\int_{0}^{4} \frac{1}{2}\left[x^2\,[y]_{0}^{x} + \left[\frac{y^3}{3}\right]_{0}^{x} + 2x\left[\frac{y^2}{2}\right]_{0}^{x}\right] dx$$

$$\int_{0}^{4} \frac{1}{2}\left[x^3 + \frac{x^3}{3} + \frac{2x^3}{2}\right] dx$$

$$\int_0^4 \frac{1}{2\times 3}\left[3x^3 + x^3 + 3x^3\right] \partial x$$

$$\frac{1}{6}\int_0^4 7x^3 \, \partial x$$

$$\frac{1}{6} \times 7 \left[\frac{x^4}{4}\right]_0^4$$

$$\frac{1}{6} \times 7 \left[\frac{(4)^4}{4} - 0\right]$$

$$\frac{1}{6} \times 7 \times \frac{\cancel{256}\ 64}{\cancel{4}}$$

$$\frac{64 \times 7}{6} = 74.66$$

7. Find the unit normal to the Surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

$$\vec{n} = \nabla \cdot \phi$$

$$\vec{n} = (xy^3z^2 - 4)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$\hat{i}(xy^3z^2 - 4)\frac{\partial}{\partial x} + \hat{j}(xy^3z^2 - 4)\frac{\partial}{\partial y} + \hat{k}(xy^3z^2 - 4)\frac{\partial}{\partial z}$$

$$\hat{i}(y^3z^2) + \hat{j}(3y^2xz^2) + \hat{k}(2zxy^3)$$

$$\vec{n} \text{ at } (-1,-1,2) = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

$$\vec{n} = \hat{n} \cdot |\vec{n}|$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}$$

$\therefore$ $\hat{n}$ = normal vector

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}}$$

$$= \frac{-\cancel{4}(\hat{i} + 3\hat{j} - \hat{k})}{\cancel{4}\sqrt{11}}$$

$$= \frac{-\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{11}}$$ Ans

8. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).

Directional derivative $= \nabla \cdot \phi \cdot \hat{d}$

$$\Delta \cdot \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2 - y^2 + 2z^2)$$

$$= \frac{\partial}{\partial x}(x^2 - y^2 + 2z^2)\hat{i} + \frac{\partial}{\partial y}(x^2 - y^2 + 2z^2)\hat{j}$$

$$+ \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\therefore$ ~~Direction derivative~~

$\Delta \cdot \phi$ at $(1,2,3)$ $= 2(1)\hat{i} - 2(2)\hat{j} + 4(3)\hat{k}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k}$

$P \text{ (1,2,3)} \longrightarrow Q \text{ (5,0,4)}$

$\vec{PQ} = \vec{Q} - \vec{P}$

$= (5,0,4) - (1,2,3)$

$= (4,-2,1)$

$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$.

$\hat{PQ} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}}$

$= \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}} = \hat{d}$

$\therefore$ Directional derivative $= \Delta \cdot \phi \cdot \hat{d}$.

$\cancel{(2x\hat{i} - 2y\hat{j} + 4z\hat{k}) \cdot \left(\dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}\right)}$

$\cancel{8 +}$.

$\Rightarrow \Delta \cdot \phi \cdot \hat{d}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k} \cdot \left(\frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}} \right)$

$= \frac{8 + 8 + 12}{\sqrt{21}}$

$= \frac{28}{\sqrt{21}}$ Ans $\leftarrow$

9. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

Let, $u = (y^2 - z^2 + 3yz - 2x)\hat{i}$

$v = (3xz + 2xy)\hat{j}$

$w = (3xy - 2xz + 2z)\hat{k}$.

$\frac{\partial u}{\partial x} = -2$

$\frac{\partial v}{\partial y} = 2x$

$\frac{\partial w}{\partial z} = -2x + 2$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

for solenoidal.

Now, Put the Value of $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ & $\frac{\partial w}{\partial z}$

$\Rightarrow -2+2x-2x+2$

$\Rightarrow 0$

It is Solenoidal Vector

For irrotational, Curl $\vec{F} = 0$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-z^2+3yz-2x & 3xz+2xy & 3xy-2xz+2z \end{vmatrix}$$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(3xy-2xz+2z) - \frac{\partial}{\partial z}(3xz+2xy)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3xy-2xz+2z) - \frac{\partial}{\partial z}(y^2-z^2+3yz-2x)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xz+2xy) - \frac{\partial}{\partial y}(y^2-z^2+3yz-2x)\right]$$

$$\Rightarrow \hat{i}(3x-3x) - \hat{j}(3y-2z+2z-3y) + \hat{k}(3z+2y-2y-3z)$$

$$0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$0$$

$\therefore$ It is irrotational.

10. Evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds$, where

$\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and
S is the surface of the plane
$2x + y + 2z = 6$ in the first octant.

Solution ⇒ $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$.

$\phi = 2x + y + 2z - 6$

$$\nabla \cdot \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2x+y+2z-6)$$

$$\nabla \cdot \phi = \frac{\partial}{\partial x}(2x+y+2z-6)\hat{i} + \frac{\partial}{\partial y}(2x+y+2z-6)\hat{j}$$

$$+ \frac{\partial}{\partial z}(2x+y+2z-6)\hat{k}.$$

$$\Delta \cdot \phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{\Delta \cdot \phi}{|\Delta \cdot \phi|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}}$$

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Now, $\vec{A} \cdot \hat{n}$

$$\vec{A} \cdot \hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\,\hat{k}\right] \cdot \left[\frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\right]$$

$$= \frac{1}{3}\left[2(x+y^2) - 2x + 4yz\right]$$

$$= \frac{1}{3}\left[\cancel{2x} + 2y^2 - \cancel{2x} + 4yz\right]$$

$$= \frac{1}{3}\left[2y^2 + 4yz\right]$$

$$\therefore \iint_S \vec{A} \cdot \hat{n} \cdot ds = \iint_S \frac{1}{3}\left[2y^2 + 4yz\right] \cdot ds \quad \text{—(i)}$$

$$ds = \frac{dy\,dz}{|\hat{n} \cdot \hat{i}|}$$

$$\hat{n} \cdot \hat{i} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} \cdot \hat{i}$$

$$= \frac{2}{3}$$

$$ds = \frac{dy\,dz}{\frac{2}{3}}$$

Now, from eq^n (i)

$$= \iint_S \frac{1}{3}\left[2y^2 + 4yz\right] \frac{\partial y\, \partial z}{\frac{2}{3}}$$

$$= \iint \frac{\cancel{2}}{3}\left[y^2 + 2yz\right] \frac{\partial y\, \partial z}{\frac{\cancel{2}}{3}}$$

$$= \iint (y^2 + 2yz)\, \partial y\, \partial z$$

$2x + y + 2z = 6$

Let $x = 0$

$y + 2z = 6$

$2z = 6 - y$

$z = \frac{6-y}{2}$

$z = 0$ to $z = \frac{6-y}{2}$

Let $x = 0$, $z = 0$

$y = 6$

$y = 0$ to $y = 6$

$$\int_0^6 \int_0^{\frac{6-y}{2}} (y^2 + 2yz)\, \partial y\, \partial z$$

$$\int_0^6 \int_0^{\frac{6-y}{2}} (y^2 + 2yz)\, \partial z\, \partial y$$

$$\int_0^6 y^2 [z]_0^{\frac{6-y}{2}} + 2y\left[\frac{z^2}{2}\right]_0^{\frac{6-y}{2}} \, dy.$$

$$\int_0^6 y^2 \frac{(6-y)}{2} + \frac{2y}{2}\left(\left(\frac{6-y}{2}\right)^2 - 0\right) dy$$

$$\int_0^6 \frac{6y^2 - y^3}{2} + y\frac{[36+y^2-12y]}{4} \, dy$$

$$\int_0^6 \frac{12y^2 - 2y^3 + 36y + y^3 - 12y^2}{4} \, dy$$

$$\frac{1}{4}\int_0^6 \cancel{12y^2} - y^3 + 36y \cancel{-12y^2} \, dy$$

$$\frac{1}{4}\int_0^6 -y^3 + 36y$$

$$\frac{1}{4}\left\{\left[-\frac{y^4}{4}\right]_0^6 + 36\left[\frac{y^2}{2}\right]_0^6\right\}$$

$$\frac{1}{4}\left[-\frac{1296}{4} + \frac{1296}{2}\right]$$

$$\frac{1}{4}(1296)\left[\frac{-1}{4} + \frac{1}{2}\right]$$

$$324\left[\frac{-1+2}{4}\right] \Rightarrow \overset{81}{\cancel{324}} \times \frac{1}{\cancel{4}}$$

$$= 81 \quad \text{Ans} \leftarrow$$

THANK

YOU