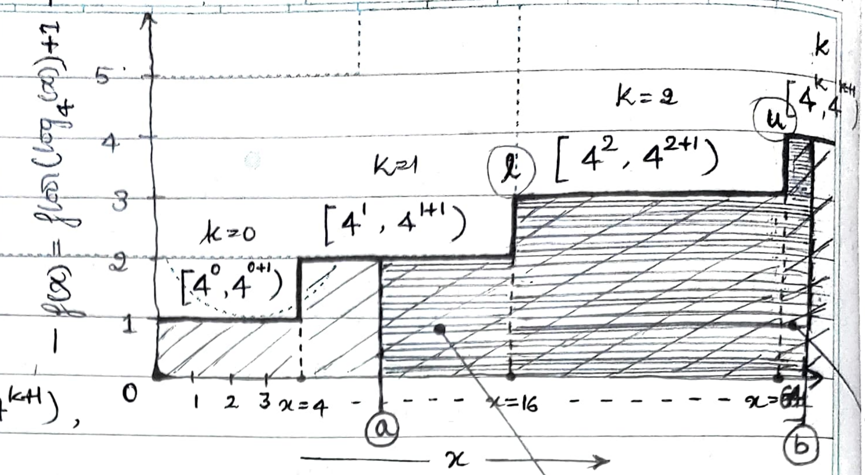
$f(x) = \text{floor}(\log_4(x)) + 1$, We need $\int_a^b f(x) \cdot dx$.

* Based on plot,
  - It's a type of step function
  - Steps up by 1, for every $4^{k+1} - 4^k$ numbers, at every power of 4.



* At each step, i.e in $[4^k, 4^{k+1})$,
  - $\log_4(x) = k$
  - $\text{floor}(\log_4(x)) = k$ (Since, floor function gives nearest integer less than 'x' as output)
  - Hence, $f(x) = \text{floor}(\log_4(x)) + 1$

$$\boxed{f(x) = k+1 \text{ , where } x \in [4^k, 4^{k+1}) \text{ and } k \in \mathbb{Z}^+} \quad \text{— (1)}$$

* So, Area under each slab of $[4^k, 4^{k+1})$ (or step, whatever), is given by,
  - $A = \int_{4^k}^{4^{k+1}} f(x) \cdot dx$

~~Since we know by (1), change of variable from 'k' to 'x', we get,~~ ~~(expressing limits in terms of 'x')~~

- $\cancel{A = \int_{\log_4 x}^{\log_4(x+1)-1} (k+1) \cdot dx}$
- $\cancel{A = (k+1) \cdot x \Big|_{\log_4 x}^{\log_4(x+1)-1}}$
- $\cancel{A = (k+1)(\log_4(x+1) - 1) - (k+1)(\log_4 x)}$
- $\cancel{A = (k+1)[\log_4(x+1) - 1 - \log_4 x]}$

~~(Lower limit)~~
- $\cancel{x = 4^k}$
- $\cancel{\log_4 x = k}$

~~(Higher limit)~~
- $\cancel{4^{k+1} - 1 = x}$
- $\cancel{4^{k+1} = x+1}$
- $\cancel{k+1 = \log_4(x+1)}$
- $\cancel{k = \log_4(x+1) - 1}$

* $A = \int_{4^k}^{4^{k+1}} (k+1) \cdot dx$ from eqn —(1)

Then,

$$\boxed{A_k = (k+1) \cdot x \Big|_{4^k}^{4^{k+1}} \quad \text{for all } k \in \mathbb{Z}^+}$$

(2) → Gives area under any step 'k' in graph shown above

$$\boxed{\text{Which is, } A_k = (k+1)[4^{k+1} - 4^k]} \quad \forall k \in \mathbb{Z}^+ \text{ — (3)}$$

- What we need is summation of area of this $f(x)$ from any point 'a' till any point 'b', that is area of $f(x)$ under curve in interval $[a,b]$.
- Through use of Integral Tools, we try to find a formula with least computations, which can be computed faster than traditional iterative methods that are simpler but do lot more computations.
- Upon observing graph, we see that, area under $f(x)$ curve in interval $[a,b]$ can be expressed as or equivalent to area under multiple 'k' steps,
- Then Sum of area under multiple 'k' steps that contribute to area under of $f(x)$ curve in $[a,b]$ gives total area in $[a,b]$.
- That is, first to find partial areas under a step,
- We know from ②, $A = (k+1)\cdot x \Big|_{4^k \to \text{lower}}^{4^{k+1} \to \text{upper}}$
- If its partial, either upper or lower limit changes,
- For lower limit 'a', (partial area)
  - $A_L = (k+1)\cdot x \Big|_{a \to \text{partial lower limit}}^{4^{k+1}}$
  - $\boxed{A_L = (k+1)[4^{k+1} - a]}$ — ④
- For upper limit 'b', (partial area)
  - $A_U = (k+1)\cdot x \Big|_{4^k}^{b \to \text{partial upper limit}}$
  - $\boxed{A_U = (k+1)[b - 4^k]}$ — ⑤
- We know full step Area is given by eqn ②
- Adding All these partial and full areas under $f(x)$ in $[a,b]$ gives $\int_a^b f(x)\,dx$, that is $A_L + A_K + A_U$. There will be $\log_4(b-a)+1$ segments to add at most for any interval $[a,b]$, since each segment is $4^{k+1} - 4^k$ wide, depending on 'k'.
- Time complexity of each $[a,b]$ interval sum of $f(x)$ function would be $O(\log_4(b-a)+1)$.
- $\boxed{A_{custom} = (k+1)[b-a]}$