

$f(x) = \text{floor}(\log_4(x)) + 1$, We need $\int_a^b f(x) \cdot dx$.

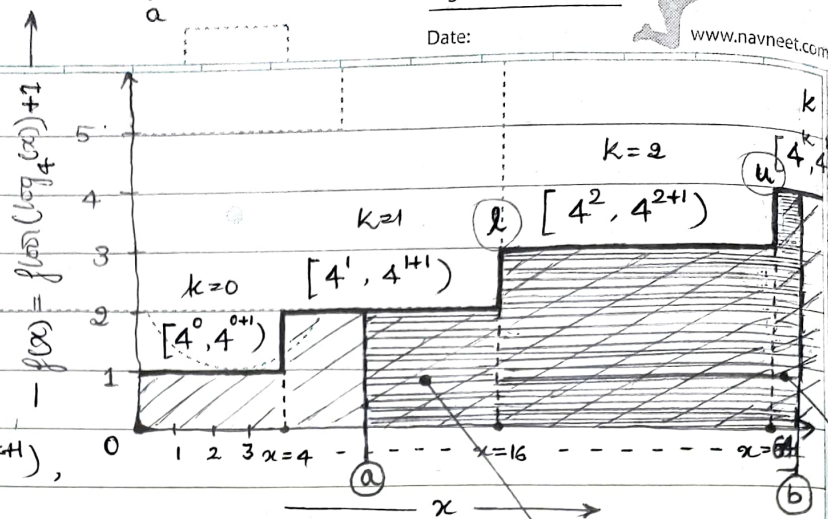
Page No.:

Date:



* Based on plot,

- It's a type of step function
- Steps up by 1, for every $4^{k+1} - 4^k$ numbers, at every power of 4.



* At each step, i.e in $[4^k, 4^{k+1})$,

- $\log_4(x) = k$
- $\text{floor}(\log_4(x)) = k$ (Since, floor function gives nearest integer less than 'x' as output)
- Hence, $f(x) = \text{floor}(\log_4(x)) + 1$

$$f(x) = k+1, \text{ where } x \in [4^k, 4^{k+1}) \text{ and } k \in \mathbb{Z}^+ \quad \text{--- (1)}$$

* So, Area under each slab of $[4^k, 4^{k+1})$ (or step, whatever), is given by,

$$A = \int_{4^k}^{4^{k+1}-1} f(x) \cdot dx$$

(expressing limits in terms of 'x')
Since we know by (1), change of variable from 'k' to 'x', we get,

$$A = \int_{\log_4 x}^{\log_4(x+1)-1} (k+1) \cdot dx$$

$$A = (k+1) \cdot x \Big|_{\log_4 x}^{\log_4(x+1)-1}$$

$$A = (k+1) (\log_4(x+1) - 1) - (k+1) (\log_4 x)$$

$$A = (k+1) [\log_4(x+1) - 1 - \log_4 x]$$

(Lower Limit)

$$x = 4^k$$

$$\log_4 x = k$$

(Higher Limit)

$$4^{k+1}-1 = x$$

$$4^{k+1} = x+1$$

$$k+1 = \log_4(x+1)$$

$$k = \log_4(x+1) - 1$$

$$A = \int_{4^k}^{4^{k+1}-1} (k+1) \cdot dx \text{ from eqn (1)}$$

Then,

$$A_k = (k+1) \cdot x \Big|_{4^k}^{4^{k+1}-1} \text{ for all } k \in \mathbb{Z}^+$$

Given area under
→ any step 'k' in
graph shown above

$$\text{Which is, } A_k = (k+1) [4^{k+1} - 1 - 4^k] \quad \forall k \in \mathbb{Z}^+ \quad \text{--- (3)}$$

- What we need is summation of area of this $f(x)$ from any point 'a' till any point 'b', that is

Page No.:

Date:



www.navneet.com

area of $f(x)$ under curve in interval $[a, b]$.

- Through Use of Integral Tools, we try to find a formula with least computations, which can be computed faster than traditional iterative methods that are simpler but do lot more computations.
- Upon observing graph, we see that, area under $f(x)$ curve in interval $[a, b]$ can be expressed as or equivalent to area under multiple 'k' steps,
- Then Sum of area under multiple 'k' steps that contribute to area under of $f(x)$ curve in $[a, b]$ gives total area in $[a, b]$.

- That is, first to find partial areas under a step,

We know from (2), $A = (k+1) \cdot x$ | $4^{k+1} \rightarrow \text{upper}$

$4^k \rightarrow \text{lower}$

- If its partial, either upper or lower limit changes,

- For lower limit 'a', (partial area)

- For upper limit 'b', (partial area)

$$A_L = (k+1) \cdot x \Big|_a^{4^{k+1}-1}$$

$a \rightarrow \text{partial lower limit}$

$$A_U = (k+1) \cdot x \Big|_{4^k}^b$$

$b \rightarrow \text{partial upper limit}$

$$A_L = (k+1) [4^{k+1} - 1 - a] \quad - (4)$$

$$A_U = (k+1) [b - 4^k] \quad - (5)$$

- We know full step Area is given by eqn (2)

- Adding All these partial and full areas under $f(x)$ in $[a, b]$ gives

$\int_a^b f(x) \cdot dx$, that is $A_L + A_k + A_U$. There will be $\log_4(b-a) + 1$ segments to add at most for any interval $[a, b]$, since each segment is $4^{k+1} - 4^k$ wide, depending on 'k'.

- Time Complexity of each $[a, b]$ interval sum of $f(x)$ function would be $O(\log_4(b-a) + 1)$.

$$A_{\text{Custom}} = (k+1) [b-a]$$