1. A parabola is defined as the set of points in the plane that are equidistant from a fixed point F (called the focus) and a fixed line l (called the directrix). The vertex V of the parabola lies halfway between the focus and the directrix. The axis of symmetry is a line perpendicular to the directrix that passes through the focus. The general form of the parabola can be written in one of the following ways depending on the nature of the parabola:

   (a) $(y-k)^2 = 4a(x-h)$ with vertex (h,k), focus $(h+a,k)$ and directrix $x = h-a$.

   (b) $(y-k)^2 = -4a(x-h)$ with vertex (h,k), focus $(h-a,k)$ and directrix $x = h+a$.

   (c) $(x-h)^2 = 4a(y-k)$ with vertex (h,k), focus $(h,k+a)$ and directrix $y = k-a$.

   (d) $(x-h)^2 = -4a(y-k)$ with vertex (h,k), focus $(h,k-a)$ and directrix $y = k+a$.

   We know that the quadratic function is a general example of a parabola. You are going to take the following quadratic equations and show the vertex, focus and directrix. Also verify that the points on the parabola have the same distance between the focus and the directrix.

   (a) $y = x^2 - 5x + 6$

   (b) $y = -x^2 - x + 6$

   (c) $x = y^2 - 5y + 6$

   (d) $x = -y^2 - y + 6$

2. The n^{th} root of unity, n a positive integer is a number z satisfying the equation $z^n = 1$. The n^{th}roots of unity are

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1.$$

   These points lie on a circle of unit radius. They can be used as the vertices of the regular polygon and other interesting geometries.

   (a) Construct some regular polygons using the n^{th} roots of unity with $k = 0, 1, 2, \ldots n$.

   (b) Also vary the radius (lengths) of the n^{th}roots to create some interesting geometric figures.

   (c) Show that as $n \to \infty$, the geometric figure tends to a circle and the perimeter approaches $2\pi r$.

3. Given the circle $x^2 + y^2 = r^2$.

   (a) Given a point $(x_0, 0)$ not on the circle, there are two tangents from this point to the circle. These tangents touch the circle at the point $P(a, b)$ with

$$a = \frac{r^2}{x_0}, \quad b = \pm\sqrt{r^2 - a^2}.$$

   (b) Can you find an equivalent expression for a point $(0, y_0)$ not on a circle.

Verify these expressions using python. Also show that there are always two point on the circle that have the same tangent slope.

4. The Taylor expansion of a function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + f'(x)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \frac{f^{(iv)}(x)}{4!}(x-a)^4 + \ldots$$

Using the function $f(x) = e^{-\sin(x)}$ with $x = 0$, show how the Taylor polynomial approximates the function using approximations of order $1, 2, 3, 4$. Use the approximations to find the derivative of $f(x)$ at $x = 0$ and $x = 1$. How accurate are the approximations.

5. The Taylor expansion of a function $f(x)$ about a point $x = a$ is given by

$$f(x) = f(a) + f'(x)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \frac{f'''(x)}{3!}(x-a)^3 + \frac{f^{(iv)}(x)}{4!}(x-a)^4 + \ldots$$

Using the function $f(x) = \sin(2x)$ with $x = 0$, show how the Taylor polynomial approximates the function using approximations of order $1, 2, 3, 4$. Use the approximations to find the integral of $f(x)$ from $x = -1$ to $x = 1$. How accurate are the approximations as we use higher order Taylor polynomials.

6. The natural logarithm is defined as

$$\ln x = \int_1^x \frac{1}{t} dt.$$

By the laws of integrals, we see that $\ln 1 = 0$. We can approximate the value of $\ln x$ by approximating the derivative. We can do that by dividing the interval $[1, x]$ into n partitions with $1 = x_0 < x_1 < x_2 < \ldots < x_n = x$. Then the integral

$$\int_1^x \frac{1}{t} dt \approx h \left(\frac{\frac{1}{x^n} + 1}{2} + \sum_{k=1}^{n-1} \frac{1}{x^i} \right)$$

where $h = \dfrac{x-1}{n}$ is the length $|x_i - x_{i-1}|$.

   (a) Approximate the value of $\ln x$ for different values of x. You can use $x = 1, 5, 10, 200$. Show what happens to your approximation as n gets larger.

7. You want to explore transformations of the function $f(x)$. Using the function $f(x) = x^2 - x - 6$, perform the transformations $f(x \pm h)$, $f(hx)$, $f\left(\frac{x}{h}\right)$, $f(x) \pm h$. Disscuss that happens to the stationary point and roots of $f(x)$.

8. Given the line $ax + by + c = 0$, the distance from the point $P(x_0 y_0)$ to the line is given by

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

This measures the distance from the point to the foot of the perpendicular from the point to the line. Given the line $3x + 4y + 1 = 0$ and the point $P(1, 5)$,

(a) find the foot of the perpendicular from the point to the line.

(b) Measure the distance from the point to the foot.

(c) Show that as we move away from the foot of the perpendicular, the distance between points on the line and P gets larger.

9. Polar Coordinates: A lot of our work in studying mathematics is performed in the rectangular coordinate system. One other coordinate system that sometimes eases our work in calculus and differential equations is the polar coordinate system where coordinates are represented as the ordered pair (r, θ) where r is the magnitude of the vector from the origin to the point and θ is the angle the point makes with the positive $x-$axis.

   (a) The equation of a circle in polar coordinates is given by

   $$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

   Plot the equation of the circle with $\theta \in [0, 2\pi]$ and $r = [1, 5]$. This allows us to think of polar coordinates as a giant radar screen with direction controlled by θ and how far towards the edge of the screen controlled by r.

   (b) It is possible to combine these conversions to create some very wonderful results. Plot the following functions.

      i. $r = 4\cos\theta + 2$

      ii. Cochleod: $r = a\sin(\theta)/\theta, \quad a = 3$

      iii. Conchoid of de Sluze: $r = b\cos(\theta)/a - a/\cos(\theta), \quad a = 1, \quad b = 2.$

      iv. Rhodonea Curves: $r = a\sin(b\theta), \quad a = 1, \quad b = \pi \quad \theta \in [0, 20\pi]$..

      v. Butterfly Curve: $r = e^{\cos(\theta)} - 2\cos(4\theta) + \sin^5((2\theta - \pi)/25)$

   (c) Plot the Chrysanthemum Curve

   $$r = 5(1 + \sin(11\theta/5)) - \left(4\sin^4(17\theta/3)\right)\left(\sin^8(2\cos(3\theta) - 28\theta)\right), \ \theta \in [0, 21\pi]$$

   Showing how the curve evolves with θ.

10. A periodic function is a function that repeats its values in regular intervals or periods. Trigonometric functions are an important example of periodic functions.

   (a) Two periodic functions will usually sum to give another periodic function.

      i. Plot the functions $\sin 2x$, $\sin 6x$ and plot their sum.

      ii. Plot the functions $\sum_{i=1}^{5} \sin ix$

   (b) Two periodic functions will however not be periodic if their periods do not have a common integer multiple.

i. Plot the function $\sin x + \sin \pi x$

ii. Plot the function $\sin x + \sin\left(\sqrt{2}x\right)$

(c) Give that the function $f(x) = x \ (-\pi < x < \pi)$ is periodic,

i. Plot the function from $-3\pi < x < 3\pi$.

ii. This function has the Fourier series representation $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$. Note that this function has discontinuities at the end points.

(d) The function $f(x) = x\,(\pi - x)\,,\ (-\pi < x < \pi)$ is also periodic.

i. Plot the function from $-3\pi < x < 3\pi$.

ii. The Fourier series representation of this function is $\frac{8}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\,x}{(2n-1)^3}$. Plot that function also.

11. Given the circle of radius r, we can approximate the circle as the union (partitioning) into equal triangles with a common vertex being the centre. In this case, we want to approximate the value of π so we are going to work in degrees. The math module has a function radians that converts a number from degrees to radians. We will be using that. We will be using the quarter circle in the first quadrant of the plane therefore the angle $\theta \in [0, 90]$. We Will divide this angle into $n + 1$ points. This essentially approximates the circle by a polygon. We know that the circumference of a circle which we will approximate by the perimeter of the polygon is given as $C = 2\pi r$. Thus for the quarter circle we have that $C = 1/2\pi r$. By dividing the interval $\theta \in [0, 90]$ into n intervals and $n + 1$ points, the corresponding point on the circle can be written as

$$x_i = r\cos\theta_i$$
$$y_i = r\sin\theta_i.$$

The length of a section of the polygon can be written as

$$l_i = \sqrt{(x_{i+1} - x_i)^2 - (y_{i+1} - y_i)^2}.$$

The sum of the lengths of the line segments is an approximation of C. Use that information to find an approximation for π. Find approximations for π for $N = 10, 100, 1000, 10000, 100000, 1000000$ triangles. Also explore what happens to the approximation are r increases.

12. (a) If a number is not prime, it is divisible by a combination of primes. Write a function that finds all prime factors of a number.

(b) Pythagoreans classified all numbers as deficient, abundant or perfect. Given a number, find all its proper factors, that is, all numbers that go into it except the number itself. If the sum of the proper factors of n is less than n, then it is said to be deficient. It the sum exceeds n, it is abundant. If the sum equals n, then n is said to be perfect.

i. For the integers between 1 and 10,000,000. Classify them into deficient, abundant and perfect.

ii. It is conjectured that the even perfect numbers are of the form $2^{p-1}(2^p - 1)$. Can you verify this with the perfect numbers you have found?

iii. What other perfect numbers can you find using the fornuma directly?

(c) For your numbers that are primes, can the combination $2^p - 1$ also be prime? These are called Mersene primes. How many Mersene primes did you get, and how does this relate to the number of perfect numbers you got?

(d) A number is called semi-perfect if it is the sum of some (but not all) of its proper divisors. Find the semi perfect numbers in your list.

(e) Plot the distribution of abundant and deficient numbers.

13. The general form of an autonomous linear system for x_n and y_n is

$$\begin{aligned} x_{n+1} &= ax_x + by_n + h \\ y_{n+1} &= cx_n + dy_n + k \end{aligned}$$

where the parameters a, b, c, d, h, k are any real constants. If $h = k = 0$, the system is homogeneous, otherwise it is non-homogeneous.

(a) Generate a table and time series graph for the first 20 iterates of the predator-prey model

$$\begin{aligned} P_{n+1} &= 1.2P_n - 0.1Q_n \\ Q_{n+1} &= 0.1P_n + Q_n \end{aligned}$$

beginning with $P_0 = 1200$ and $Q_0 = 1000$. Explain what is hapenning.

(b) Create a table and time series graph of the competition model

$$\begin{aligned} P_{n+1} &= 0.7P_n - 0.05Q_n \\ Q_{n+1} &= -0.05P_n + 0.8Q_n \end{aligned}$$

begining with $P_0 = 5000$ and $Q_0 = 2000$. Explain what is hapenning.

(c) A space solution graph or phase plane graph is a two dimensional plot of the set (x_i, y_i) for an iterated system. Explain the behaviour of the following systems using a space solution graph:

i. The predator-prey model with constant immigration

$$\begin{aligned} P_{n+1} &= 0.8P_n - 0.3Q_n + 8000 \\ Q_{n+1} &= 0.2P_n + 0.9Q_n + 2000 \end{aligned}$$

with $P_0 = 1000,\ Q_0 = 15,000$ and 50 iterations.

ii. The price-demand model

$$\begin{aligned} P_{n+1} &= P_n + 0.4D_n - 20 \\ D_{n+1} &= -0.3P_n + D_n + 5 \end{aligned}$$

with $P_0 = 15,\ D_0 = 50$ and 50 iterations.

iii. The model for two competing species

$$\begin{aligned} P_{n+1} &= P_n - 0.2Q_n + 1500 \\ Q_{n+1} &= -0.4P_n + Q_n + 2000 \end{aligned}$$

create solution space curves with different initial condition and 50 iterations. What is the story?

(d) Can you find the fixed points of the systems above? Classify them as sinks, sources and saddles.

14. Consider the series

$$y = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

This series represents the rational function $1/(1-x)$. We want to test the series for the values $x = 0.2, 0.5, 0.8, 0.9, 0.99, 1.01, 1.1, 1.5$. We also want to use partial sums $N = 100, 1000, 10000, 100000, 1000000$. For each of these partial sums, we test the last 10 elements and check the values $|y_{i+1} - y_i|$. You have to determine whether it is decreasing or increasing for each of them. Also test the ratio $|y_{i+1}/y_i|$. What can you say about these values in relation to convergence and divergence of the series.

15. Aliasing:

(a) Create a sine wave with frequency 60Hz as follows:

i. Set $f = 60$. Set $t_{\min} = -0.05, \quad t_{\max} = 0.05$. Create a linearly spaced interval $t = [t_{\min}, t_{\max}]$ with 400 node points.

ii. Create $x = \cos(2\pi f t)$

iii. Plot t against x.

(b) Subsample this function with sampling frequency 800 Hz as follows:

i. Set $T = 1/800,\ n_{\min} = \lceil t_{\min}/T \rceil,\ n_{\max} = \lfloor t_{\max}/T \rfloor$. Create a linearly spaced interval $n = [n_{\min}, n_{max}]$ with interval 1.

ii. Create $x_1 = \cos(2\pi f n T)$

iii. Plot t against x_1. This is an oversampled version of x.

(c) Create another oversampled version of x with sampling frequency 400 Hz.

(d) Create another critically sampled version of x with sampling frequency 120 Hz.

(e) Create another undersampled version of x with sampling frequency 70 Hz. How well do these functions represent the original function x.

16. Plasma Effect: Plasma effects are often used to create wobbly animations in demo products like screen savers. Using mathematical functions like sine, tan, etc, is one way of creating such effects.

    (a) Create a grid with $x, y \in [-\pi, \pi]$ (you want read about numpy.meshgrid for this). Plot the function $f(x, y) = \sin(5 * x)$. This an over simplified plasma.

    (b) Now Plot the function $f(x, y) = \left(\sin^2(3x) + \sin^2(3y)\right)/2$. You can try changing the powers of sine and the frequency to see how it evolves.

    (c) Plot $f(x, y) = \left|\cos\left(20\left(\cos^2(20) + \sin^2(20)\right)\right)\right|$

    (d) Now create a colour image with the following parameters

    $$\begin{aligned} r &= \cos(x) + \sin(y) \\ \text{red} &= \cos(yr) \\ \text{green} &= \cos(xyr) \\ \text{blue} &= \sin(xr). \end{aligned}$$

    Take the absolute value of the result to avoid working with negative numbers.

17. Exploring Pascal's triangle: This project seeks to understand the complexity of Pascal's triangle.

    (a) In a 500×500 array, generate the first 500 Pascal's triangle entries.

    (b) What do these numbers represent.

    (c) What is the relationship between the non-one numbers on a row and the first number on a row whose 1st element (0th element is 1) is prime.

    (d) Show all odd entries in Pascal's triangle.

    (e) What patterns do you get when you take Pascal's triangle mod:

        i. 3
        ii. 4
        iii. 5
        iv. 6
        v. 7
        vi. 8
        vii. 9
        viii. 10
        ix. 11

x. 12