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SURDS

EXPLAINED WITH WORKED EXAMPLES

By

Shefiu S. Zakariyah, *PhD*

PREFACE

In your hands is another booklet for potential mathematicians, scientists and engineers. This current work – *Surds Explained with Worked Examples* – offers 130+ worked examples accompanied with some background notes on the topic. The questions used in this work are of similar standard to those in mathematics and engineering textbooks designed for A-level, college and university students. Advanced learners, particularly those returning to study after a break from the academic environment, will also find this helpful. Additionally, it could be used as a reference guide by teaching staff/ tutors/teachers during classes and for assessment (home works and examinations).

Finally, many thanks to my colleagues who have offered suggestions and comments, especially Khadijah Olaniyan (Loughborough University, UK) and Teslim Abdul-Kareem (formerly with University of Dundee, UK).

Pertinent suggestions, feedbacks and queries are highly welcome and can be directed to the author at the address below.

Coming soon in this series are:

- Worked Examples on Mechanics
- Worked Examples on Circuit Theorems
- Worked Examples on Digital Electronics

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Email: shefiuz@theiet.org

Alternative Emails: S.Zakariyah@derby.ac.uk | S.S.Zakariyah@um.edu.my | Shefiu.Zakariyah@kaplan.com.

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Disclaimer

The author has exerted all effort to ensure an accurate presentation of questions and their associated solutions in this booklet. The author does not assume and hereby disclaims any liability to any party for any loss, damage, or disruption caused by errors or omissions, either accidently or otherwise in the course of preparing this booklet.

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CONTENTS

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FUNDAMENTALS OF SURDS

1. Introduction

Presenting numbers in surd form is quite common in science and engineering especially where a calculator is either not allowed or unavailable, and the calculations to be undertaken involve irrational values. Common applications of surds include solving a quadratic equation by formula and obtaining the values of trigonometric angles.

We will be looking at this form of representation of numbers and how to carry out their calculations. One thing I should add at this point, and which you will soon come to know better, is that surds share many things in common with complex numbers. Consequently, understanding one of the two will facilitate learning the other.

For this and other factors, attention will therefore be focused on the worked examples with minimum notes in-between calculations. If you are already familiar with surds, do proceed to the worked examples part of this booklet. Look out for footnotes strategically placed to provide further information.

2. What is Surd?

A number of the form $\pm\sqrt[n]{x}$, where $n \in \mathbb{Z}^+ - [0, 1]$ and $x \in \mathbb{Q}^+$, which cannot be expressed **exactly as a fraction of two integers** is called a **surd**. In other words, a surd is an irrational number of the form $\pm\sqrt[n]{x}$ such that $\pm\sqrt[n]{x}$ cannot be written as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. Therefore, $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$ and $\sqrt[5]{7}$ are surds because they are irrational numbers and cannot be written as exact ratio of two integers. Of course their approximation can still be obtained using a calculator or other suitable means. On the contrary, $\sqrt{36}$, $\sqrt{\frac{1}{4}}$, $\sqrt[3]{27}$ and $\sqrt[5]{32}$ are not surds. Why? Because they are rational numbers and can be expressed as ratios of two integers, i.e.

$$\sqrt{36} = 6 = \frac{6}{1}$$

$$\sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\sqrt[3]{27} = 3 = \frac{3}{1}$$

and

$$\sqrt[5]{32} = 2 = \frac{2}{1}$$

In surd operations, it is often common to see only cases where $n = 2$ in $\sqrt[n]{x}$, that is to say $\sqrt[2]{x}$ (or $\sqrt{x}$ as it is generally written). This, where $n = 2$, is an example of **quadratic surds** or **quadratic irrationals**. In similar terms, $\sqrt[3]{x}$ are called **cubic surds**.

2.1. Mixed or Compound Surds

You may recall that an improper fraction such as $\frac{7}{3}$ can be written as $2\frac{1}{3}$, this is referred to as a mixed fraction. Similarly when a number is made up of rational part and irrational part (surd), it is called a **mixed surd**. The general expression for a mixed surd is

$$a \pm b\sqrt{c}$$

where a, b and c are rational numbers but $\sqrt{c}$ is an irrational number, i.e. a surd. The above is an example of a **quadratic mixed surd**. In general, a mixed surd can be written as

$$a \pm b\sqrt[n]{c}$$

Again, n is any positive integer excluding 1 and 0.

On the other hand, a **compound surd** is made up of two irrational parts and takes the general form of

$$\sqrt[n]{a} \pm \sqrt[n]{b}$$

Analogically, a **quadratic compound surd** can be expressed as

$$\sqrt{a} \pm \sqrt{b}$$

However, sometime 'mixed surd' and 'compound surd' are used interchangeably and therefore refer to the same surd.

2.2. Similar Surds

Two quadratic surds $\pm\sqrt{a}$ and $\pm\sqrt{b}$ are said to be similar only if their radicands are the same, i.e. $a = b$. Consequently, two similar surds can be combined algebraically. For example, $\sqrt{7}$ and $-\sqrt{7}$ are similar surds. Similarly, $-3\sqrt{6}$ and $8\sqrt{6}$ are similar and can be combined to give a single surd as

$$-3\sqrt{6}+8\sqrt{6}=(-3+8)\sqrt{6}=(5)\sqrt{6}=5\sqrt{6}$$

What about $-3\sqrt[3]{6}$ and $\sqrt[3]{6}$? Yes they are similar surds because their radicands are the same and both are of cubic roots. This can therefore be combined, as in the previous example, to obtain a single surd $5\sqrt[3]{6}$. On the other hand, although $5\sqrt{6}$ and $5\sqrt[3]{6}$ have the same radicand and their coefficients are also the same, however they are not similar surds. This is because one is a square root while the other is a cubic root.

In general, two surds (quadratic or otherwise) $\pm\sqrt[n]{a}$ and $\pm\sqrt[m]{b}$ are said to be similar only if:

(i) their radicands are equal, i.e. $a=b$, and
(ii) $m=n$.

3. Rules of Surds

Essentially, there are two fundamental rules in surds

$$\sqrt{a}\times\sqrt{b}=\sqrt{a\times b} \quad \text{but} \quad \sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$$

and

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}} \quad \text{but} \quad \sqrt{a}-\sqrt{b}\neq\sqrt{a-b}$$

4. Conjugate Surds

Two surds are said to be conjugate of each other if their product gives rise to a non-surd. In other words, their product is a rational number. For example, given a mixed surd of the form

$$a+b\sqrt{c}$$

the conjugate of this is another surd that is the same as the above surd with a difference in their signs. In other words, the conjugate of $\boxed{a+b\sqrt{c}}$ is $\boxed{a-b\sqrt{c}}$. In the same vein, the conjugate of $\boxed{a-b\sqrt{c}}$ is $\boxed{a+b\sqrt{c}}$.

To sum it up, the conjugate of a quadratic compound surd having the general expression of $\sqrt{a}+\sqrt{b}$ and $a\sqrt{b}+c\sqrt{d}$ are of $\sqrt{a}-\sqrt{b}$ and $\sqrt{b}-c\sqrt{d}$ respectively.

Now it is your turn, what is the conjugate of $\boxed{-a-b\sqrt{c}}$? What is your answer? Its conjugate is $\boxed{-a+b\sqrt{c}}$. I hope you did not consider either $\boxed{a+b\sqrt{c}}$ or $\boxed{a-b\sqrt{c}}$ as both are incorrect. Why? This is simply because their products will not produce a rational number. In addition, remember that we only need to change the sign between the two terms of the surd to obtain its conjugate and nothing more.

I have got another question for you. Ready? What is the conjugate of $\boxed{\pm\sqrt{\boldsymbol{a}}}$? Puzzled! Well, we do not typically consider this as a problem. But going by our definition of conjugates, the conjugate of $\pm\sqrt{\boldsymbol{a}}$ is simply itself as this will produce a rational number. However, there can be other possibilities in this case. For example, the conjugate of $\sqrt{\mathbf{2}}$ is $\sqrt{\mathbf{2}}$ because $\boxed{\sqrt{\mathbf{2}} \times \sqrt{\mathbf{2}} = (\sqrt{\mathbf{2}})^{\mathbf{2}} = \mathbf{2}}$ but $\sqrt{\mathbf{8}}$ is a possibility since $\boxed{\sqrt{\mathbf{2}} \times \sqrt{\mathbf{8}} = \sqrt{\mathbf{2} \times \mathbf{8}} = \sqrt{\mathbf{16}} = \mathbf{4}}$. What do you think about $\sqrt{\mathbf{32}}$? Yes, it is also a conjugate of $\sqrt{\mathbf{2}}$ because $\boxed{\sqrt{\mathbf{2}} \times \sqrt{\mathbf{32}} = \sqrt{\mathbf{2} \times \mathbf{32}} = \sqrt{\mathbf{64}} = \mathbf{8}}$. In general, we only consider the surd itself in this case.

5. Operation of Surds

We can apply the four main operations on surds as will be discussed shortly. In this case, we need to distinguish between when the operations are to be applied to singular surds or mixed surds.

5.1. Addition and Subtraction

Addition / subtraction of two or more singular surds can only be carried out if they are similar as previously stated. No simplification is therefore possible if the case is otherwise. It should further be mentioned that

$$\sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}} \neq \sqrt{\boldsymbol{a} + \boldsymbol{b}}$$

and

$$\sqrt{\boldsymbol{a}} - \sqrt{\boldsymbol{b}} \neq \sqrt{\boldsymbol{a} - \boldsymbol{b}}$$

On the other hand, in a mixed surd, the rational parts are added together and the same applies to the irrational parts. For example,

$$\begin{aligned}(13 + 15\sqrt{3}) + (7 - 6\sqrt{3}) &= (13 + 7) + (15\sqrt{3} - 6\sqrt{3}) \\ &= (20) + (9\sqrt{3}) \\ &= \mathbf{20 + 9\sqrt{3}}\end{aligned}$$

5.2. Multiplication

Like with addition and subtraction, singular surds can only be multiplied if they are similar. The product of two singular surds can be obtained as

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

Similarly,

$$\sqrt[3]{a} \times \sqrt[3]{b} = \sqrt[3]{a \times b}$$

In general,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \times b}$$

The above rule applies when two or more surds are multiplied. For example,

$$\begin{aligned} \sqrt{6} \times \sqrt{3} \times \sqrt{2} &= \sqrt{6 \times 3 \times 2} \\ &= \sqrt{36} \\ &= \mathbf{6} \end{aligned}$$

However, given that $\boldsymbol{m} \neq \boldsymbol{n}$ then

$$\begin{aligned} \sqrt[m]{a} \times \sqrt[n]{b} &\neq \sqrt[m]{a \times b} \\ &\neq \sqrt[n]{a \times b} \end{aligned}$$

When the multiplication involves a singular surd and mixed or compound surds, we simplify open the brackets. This is the same when the product of two mixed or compound surds are to be computed. For example,

$$\begin{aligned} (2 + \sqrt{3})(1 + \sqrt{3}) &= 2(1 + \sqrt{3}) + \sqrt{3}(1 + \sqrt{3}) \\ &= (2 \times 1 + 2 \times \sqrt{3}) + (1 \times \sqrt{3} + \sqrt{3} \times \sqrt{3}) \\ &= (2 + 2\sqrt{3}) + (\sqrt{3} + 3) \\ &= (2 + 3) + (2\sqrt{3} + \sqrt{3}) \\ &= \mathbf{5 + 3\sqrt{3}} \end{aligned}$$

5.3. Division

Division is the most complicated one. Do not worry, we can 'chew and digest' it. The simplest case is when a singular surd is divided by another singular surd, which can be simplified as

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$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$$

Similarly,

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}}=\sqrt[3]{\frac{a}{b}}$$

In general,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

Again, given that $\boldsymbol{m} \neq \boldsymbol{n}$ then

$$\frac{\sqrt[m]{a}}{\sqrt[n]{b}} \neq \sqrt[m]{\frac{a}{b}} \neq \sqrt[n]{\frac{a}{b}}$$

Let us take an example,

$$\frac{\sqrt{18}}{\sqrt{2}}=\sqrt{\frac{18}{2}}$$

$$=\sqrt{9}$$

$$=\mathbf{3}$$

The above means of simplification is only suitable if the ratio of the two radicands will give an answer which is an integer, $\frac{\boldsymbol{a}}{\boldsymbol{b}}=\mathbb{Z}^{+}$, otherwise we need another means of simplification. For instance,

$$\frac{\sqrt{20}}{\sqrt{6}}=\sqrt{\frac{20}{6}}$$

$$=\sqrt{\frac{\mathbf{10}}{\mathbf{3}}}$$

The above is not a simplified form. Therefore in this and similar cases, we need a process called **rationalisation**.

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6. Rationalisation

Rationalisation is a method of simplifying a faction having a surd either as its denominator or as both the denominator and numerator such that it can be re-written without a surd in its denominator. The surd in the denominator can either be a singular, mixed or compound surd.

As shown above, a surd can be turn into a rational number by multiplying it with its conjugate. For this reason, this process is often referred to as '**rationalising the denominator**'. What we mean is that to simplify a fractional surd having a surd as its denominator, we multiply its numerator and denominator by the conjugate of the denominator.

Rationalising the denominator often takes the two following forms

$$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$$

$$= \frac{a\sqrt{b}}{b}$$

and

$$\frac{1}{a+\sqrt{b}} = \frac{1}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}}$$

$$= \frac{a-\sqrt{b}}{a^2 - \left(\sqrt{b}\right)^2}$$

$$= \frac{a-\sqrt{b}}{a^2 - b}$$

END OF FUNDAMENTALS OF SURDS

AND

BEGINNING OF WORKED EXAMPLES

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WORKED EXAMPLES

Section 1: Simplification (I)

1) Write each of the following surds as complete square roots, i.e. $\boxed{\sqrt{a}}$.

(a) $10\sqrt{2}$

Solution

$$\begin{aligned} 10\sqrt{2} &= \left(\sqrt{10}\right)^2 \times \sqrt{2} \\ &= \sqrt{100} \times \sqrt{2} \\ &= \sqrt{100 \times 2} \\ &= \boldsymbol{\sqrt{200}} \end{aligned}$$

(b) $7\sqrt{3}$

Solution

$$\begin{aligned} 7\sqrt{3} &= \left(\sqrt{7}\right)^2 \times \sqrt{3} \\ &= \sqrt{49} \times \sqrt{3} \\ &= \sqrt{49 \times 3} \\ &= \boldsymbol{\sqrt{147}} \end{aligned}$$

(c) $5\sqrt{6}$

Solution

$$\begin{aligned} 5\sqrt{6} &= \left(\sqrt{5}\right)^2 \times \sqrt{6} \\ &= \sqrt{25} \times \sqrt{6} \\ &= \sqrt{25 \times 6} \\ &= \boldsymbol{\sqrt{150}} \end{aligned}$$

(d) $3\sqrt{11}$

Solution

$$\begin{aligned} 3\sqrt{11} &= \left(\sqrt{3}\right)^2 \times \sqrt{11} \\ &= \sqrt{9} \times \sqrt{11} \\ &= \sqrt{9 \times 11} \\ &= \boldsymbol{\sqrt{99}} \end{aligned}$$

(e) $2\sqrt{12}$

Solution

$$\begin{aligned} 2\sqrt{12} &= \left(\sqrt{2}\right)^2 \times \sqrt{12} \\ &= \sqrt{4} \times \sqrt{12} \\ &= \sqrt{4 \times 12} \\ &= \boldsymbol{\sqrt{48}} \end{aligned}$$

2) Express the following in the simplest possible form, i.e. $\boxed{a\sqrt{b}}$.

(a) $\sqrt{27}$

Solution

$$\begin{aligned} \sqrt{27} &= \sqrt{9 \times 3} \\ &= \sqrt{9} \times \sqrt{3} \\ &= \sqrt{3^2} \times \sqrt{3} \\ &= 3 \times \sqrt{3} \\ &= \boldsymbol{3\sqrt{3}} \end{aligned}$$

(b) $\sqrt{96}$

Solution

$$\begin{aligned} \sqrt{96} &= \sqrt{16 \times 6} \\ &= \sqrt{16} \times \sqrt{6} \\ &= \sqrt{4^2} \times \sqrt{6} \\ &= 4 \times \sqrt{6} \\ &= \boldsymbol{4\sqrt{6}} \end{aligned}$$

NOTE

To simplify a surd, the radicand must be expressed in terms of a product of its largest perfect square, which is sometimes difficult to know. This can be overcome if the number is written as a product of its prime factors.

For example,

$$\mathbf{98 = 2 \times 2 \times 2 \times 2 \times 2 \times 3}$$

Since there are five twos, we can write

$$\mathbf{98} = (\mathbf{2 \times 2}) \times (\mathbf{2 \times 2}) \times (\mathbf{2 \times 3})$$
$$= (\mathbf{4}) \times (\mathbf{4}) \times (\mathbf{6})$$
$$= \mathbf{4^2 \times 6}$$
$$= \mathbf{16 \times 6}$$

(c) $\sqrt{44}$

Solution

$$\begin{aligned}\sqrt{44} &= \sqrt{4 \times 11}\\ &= \sqrt{4} \times \sqrt{11}\\ &= \sqrt{2^2} \times \sqrt{11}\\ &= 2 \times \sqrt{11}\\ &= \mathbf{2\sqrt{11}}\end{aligned}$$

(d) $\sqrt{250}$

Solution

$$\begin{aligned}\sqrt{250} &= \sqrt{25 \times 10}\\ &= \sqrt{25} \times \sqrt{10}\\ &= \sqrt{5^2} \times \sqrt{10}\\ &= 5 \times \sqrt{10}\\ &= \mathbf{5\sqrt{10}}\end{aligned}$$

(e) $\sqrt{52}$

Solution

$$\begin{aligned}\sqrt{52} &= \sqrt{4 \times 13}\\ &= \sqrt{4} \times \sqrt{13}\\ &= \sqrt{2^2} \times \sqrt{13}\\ &= 2 \times \sqrt{13}\\ &= \mathbf{2\sqrt{13}}\end{aligned}$$

(f) $\sqrt{175}$

Solution

$$\begin{aligned}\sqrt{175} &= \sqrt{25 \times 7}\\ &= \sqrt{25} \times \sqrt{7}\\ &= \sqrt{5^2} \times \sqrt{7}\\ &= 5 \times \sqrt{7}\\ &= \mathbf{5\sqrt{7}}\end{aligned}$$

(g) $\sqrt{54}$

Solution

$$\begin{aligned}\sqrt{54} &= \sqrt{9 \times 6}\\ &= \sqrt{9} \times \sqrt{6}\\ &= \sqrt{3^2} \times \sqrt{6}\\ &= 3 \times \sqrt{6}\\ &= \mathbf{3\sqrt{6}}\end{aligned}$$

(h) $\sqrt{72}$

Solution

$$\begin{aligned}\sqrt{72} &= \sqrt{36 \times 2}\\ &= \sqrt{36} \times \sqrt{2}\\ &= \sqrt{6^2} \times \sqrt{2}\\ &= 6 \times \sqrt{2}\\ &= \mathbf{6\sqrt{2}}\end{aligned}$$

(i) $\sqrt{80}$

Solution

$$\begin{aligned}\sqrt{80} &= \sqrt{16 \times 5}\\ &= \sqrt{16} \times \sqrt{5}\\ &= \sqrt{4^2} \times \sqrt{5}\\ &= 4 \times \sqrt{5}\\ &= \mathbf{4\sqrt{5}}\end{aligned}$$

(j) $\sqrt{243}$

Solution

$$\begin{aligned}\sqrt{243} &= \sqrt{81 \times 3}\\ &= \sqrt{81} \times \sqrt{3}\\ &= \sqrt{9^2} \times \sqrt{3}\end{aligned}$$

$$= 9 \times \sqrt{3}$$

$$= \mathbf{9\sqrt{3}}$$

Section 2: Addition and Subtraction of Surds

3) Simplify each of the following.

(a) $3\sqrt{5} + 5\sqrt{5}$

Solution

$$3\sqrt{5} + 5\sqrt{5} = \mathbf{8\sqrt{5}}$$

(b) $2\sqrt{6} - 4\sqrt{6}$

Solution

$$2\sqrt{6} - 4\sqrt{6} = \mathbf{-2\sqrt{6}}$$

(c) $3\sqrt{3} + 4\sqrt{3} + 5\sqrt{3}$

Solution

$$3\sqrt{3} + 4\sqrt{3} + 5\sqrt{3} = \mathbf{12\sqrt{3}}$$

(d) $25\sqrt{17} - 6\sqrt{17} - 12\sqrt{17}$

Solution

$$25\sqrt{17} - 6\sqrt{17} - 12\sqrt{17} = \mathbf{7\sqrt{17}}$$

(e) $3\sqrt{2} - 5\sqrt{3} - 2\sqrt{2} + 2\sqrt{3}$

Solution

$$3\sqrt{2} - 5\sqrt{3} - 2\sqrt{2} + 2\sqrt{3} = \mathbf{\sqrt{2} - 3\sqrt{3}}$$

NOTE

Remember that we can only add similar surds. $\mathbf{\sqrt{2}}$ is not similar to $\mathbf{\sqrt{3}}$, therefore no further simplification is possible in the question above.

4) Simplify each of the following.

(a) $\sqrt{5} + \sqrt{20}$

Solution

$$\begin{aligned}\sqrt{5} + \sqrt{20} &= \sqrt{5} + \sqrt{4 \times 5}\\ &= \sqrt{5} + \left(\sqrt{4} \times \sqrt{5}\right)\\ &= \sqrt{5} + \left(2 \times \sqrt{5}\right)\\ &= \sqrt{5} + 2\sqrt{5}\\ &= \mathbf{3\sqrt{5}}\end{aligned}$$

(b) $\sqrt{8} + \sqrt{8}$

Solution

$$\begin{aligned}\sqrt{8} + \sqrt{8} &= \sqrt{4 \times 2} + \sqrt{4 \times 2}\\ &= \left(\sqrt{4} \times \sqrt{2}\right) + \left(\sqrt{4} \times \sqrt{2}\right)\\ &= \left(2 \times \sqrt{2}\right) + \left(2 \times \sqrt{2}\right)\\ &= 2\sqrt{2} + 2\sqrt{2}\\ &= \mathbf{4\sqrt{2}}\end{aligned}$$

Alternatively,

$$\begin{aligned}\sqrt{8} + \sqrt{8} &= 2\sqrt{8}\\ &= 2\sqrt{4 \times 2}\\ &= 2 \times \sqrt{4} \times \sqrt{2}\\ &= 2 \times 2 \times \sqrt{2}\\ &= 4 \times \sqrt{2}\\ &= \mathbf{4\sqrt{2}}\end{aligned}$$

as before.

(c) $\sqrt{75} - \sqrt{27}$

Solution

$$\begin{aligned}\sqrt{75} - \sqrt{27} &= \sqrt{25 \times 3} - \sqrt{9 \times 3}\\ &= \left(\sqrt{25} \times \sqrt{3}\right) - \left(\sqrt{9} \times \sqrt{3}\right)\\ &= \left(5 \times \sqrt{3}\right) - \left(3 \times \sqrt{3}\right)\\ &= 5\sqrt{3} - 3\sqrt{3}\\ &= \mathbf{2\sqrt{3}}\end{aligned}$$

(d) $\sqrt{45} - \sqrt{125}$

Solution

$$\begin{aligned}\sqrt{45} - \sqrt{125} &= \sqrt{9 \times 5} - \sqrt{25 \times 5}\\ &= \left(\sqrt{9} \times \sqrt{5}\right) - \left(\sqrt{25} \times \sqrt{5}\right)\end{aligned}$$

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$$= (3 \times \sqrt{5}) - (5 \times \sqrt{5})$$
$$= 3\sqrt{5} - 5\sqrt{5}$$
$$\mathbf{= -2\sqrt{5}}$$

(e) $\sqrt{63} - \sqrt{112}$

Solution

$$\sqrt{63} - \sqrt{112} = \sqrt{9 \times 7} - \sqrt{16 \times 7}$$
$$= (\sqrt{9} \times \sqrt{7}) - (\sqrt{16} \times \sqrt{7})$$
$$= (3 \times \sqrt{7}) - (4 \times \sqrt{7})$$
$$= 3\sqrt{7} - 4\sqrt{7}$$
$$\mathbf{= -\sqrt{7}}$$

5) Evaluate each of the following.

(a) $\sqrt{48} + \sqrt{27} + \sqrt{12}$

Solution

$$\sqrt{48} + \sqrt{27} + \sqrt{12}$$
$$= \sqrt{16 \times 3} + \sqrt{9 \times 3} + \sqrt{4 \times 3}$$
$$= (\sqrt{16} \times \sqrt{3}) + (\sqrt{9} \times \sqrt{3}) + (\sqrt{4} \times \sqrt{3})$$
$$= (4 \times \sqrt{3}) + (3 \times \sqrt{3}) + (2 \times \sqrt{3})$$
$$= 4\sqrt{3} + 3\sqrt{3} + 2\sqrt{3}$$
$$\mathbf{= 9\sqrt{3}}$$

(b) $\sqrt{18} + \sqrt{32} + \sqrt{50}$

Solution

$$\sqrt{18} + \sqrt{32} + \sqrt{50}$$
$$= \sqrt{9 \times 2} + \sqrt{16 \times 2} + \sqrt{25 \times 2}$$
$$= (\sqrt{9} \times \sqrt{2}) + (\sqrt{16} \times \sqrt{2}) + (\sqrt{25} \times \sqrt{2})$$
$$= (3 \times \sqrt{2}) + (4 \times \sqrt{2}) + (5 \times \sqrt{2})$$
$$= 3\sqrt{2} + 4\sqrt{2} + 5\sqrt{2}$$
$$\mathbf{= 12\sqrt{2}}$$

(c) $\sqrt{12} + \sqrt{75} - \sqrt{192}$

Solution

$$\sqrt{12} + \sqrt{75} - \sqrt{192}$$
$$= \sqrt{4 \times 3} + \sqrt{25 \times 3} - \sqrt{64 \times 3}$$
$$= (\sqrt{4} \times \sqrt{3}) + (\sqrt{25} \times \sqrt{3}) - (\sqrt{64} \times \sqrt{3})$$
$$= (2 \times \sqrt{3}) + (5 \times \sqrt{3}) - (8 \times \sqrt{3})$$
$$= 2\sqrt{3} + 5\sqrt{3} - 8\sqrt{3}$$
$$\mathbf{= -\sqrt{3}}$$

(d) $\sqrt{1000} + \sqrt{1210} + \sqrt{1440}$

Solution

$$\sqrt{1000} + \sqrt{1210} + \sqrt{1440}$$
$$= \sqrt{100 \times 10} + \sqrt{121 \times 10} + \sqrt{144 \times 10}$$
$$= (\sqrt{100} \times \sqrt{10}) + (\sqrt{121} \times \sqrt{10}) + (\sqrt{144} \times \sqrt{10})$$
$$= (10 \times \sqrt{10}) + (11 \times \sqrt{10}) + (12 \times \sqrt{10})$$
$$= 10\sqrt{10} + 11\sqrt{10} + 12\sqrt{10}$$
$$\mathbf{= 33\sqrt{10}}$$

(e) $10\sqrt{20} - 4\sqrt{45} - \sqrt{80}$

Solution

$$10\sqrt{20} - 4\sqrt{45} - \sqrt{80}$$
$$= 10\sqrt{4 \times 5} - 4\sqrt{9 \times 5} - \sqrt{16 \times 5}$$
$$= (10 \times \sqrt{4} \times \sqrt{5}) - (4 \times \sqrt{9} \times \sqrt{5}) - (\sqrt{16} \times \sqrt{5})$$
$$= (10 \times 2\sqrt{5}) - (4 \times 3\sqrt{5}) - 4\sqrt{5}$$
$$= 20\sqrt{5} - 12\sqrt{5} - 4\sqrt{5}$$
$$\mathbf{= 4\sqrt{5}}$$

(f) $2\sqrt{18} - 4\sqrt{72} - \sqrt{50} + 3\sqrt{98}$

Solution

$$2\sqrt{18} - 4\sqrt{72} - \sqrt{50} + 3\sqrt{98}$$
$$= 2\sqrt{9 \times 2} - 4\sqrt{36 \times 2} - \sqrt{25 \times 2} + 3\sqrt{49 \times 2}$$
$$= (2 \times \sqrt{9} \times \sqrt{2}) - (4 \times \sqrt{36} \times \sqrt{2}) - (\sqrt{25} \times \sqrt{2}) + (3 \times \sqrt{49} \times \sqrt{2})$$
$$= (2 \times 3\sqrt{2}) - (4 \times 6\sqrt{2}) - 5\sqrt{2} + (3 \times 7\sqrt{2})$$

$$= 6\sqrt{2} - 24\sqrt{2} - 5\sqrt{2} + 21\sqrt{2}$$
$$\mathbf{= -2\sqrt{2}}$$

(g) $2\sqrt{48} + 5\sqrt{54} - \sqrt{75} - 2\sqrt{24}$

Solution

$$2\sqrt{48} + 5\sqrt{54} - \sqrt{75} - 2\sqrt{24}$$
$$= 2\sqrt{16 \times 3} + 5\sqrt{9 \times 6} - \sqrt{25 \times 3} - 2\sqrt{4 \times 6}$$
$$= (2 \times \sqrt{16} \times \sqrt{3}) + (5 \times \sqrt{9} \times \sqrt{6}) - (\sqrt{25} \times \sqrt{3}) - (2 \times \sqrt{4} \times \sqrt{6})$$
$$= (2 \times 4\sqrt{3}) + (5 \times 3\sqrt{6}) - (5\sqrt{3}) - (2 \times 2\sqrt{6})$$
$$= 8\sqrt{3} + 15\sqrt{6} - 5\sqrt{3} - 4\sqrt{6}$$
$$\mathbf{= 3\sqrt{3} + 11\sqrt{6}}$$

(h) $4\sqrt{8} - 2\sqrt{75} + \sqrt{200} - 3\sqrt{48} + 5\sqrt{45}$

Solution

$$4\sqrt{8} - 2\sqrt{75} + \sqrt{200} - 3\sqrt{48} + 5\sqrt{45}$$
$$= 4\sqrt{4 \times 2} - 2\sqrt{25 \times 3} + \sqrt{100 \times 2} - 3\sqrt{16 \times 3} + 5\sqrt{9 \times 5}$$
$$= (4 \times \sqrt{4} \times \sqrt{2}) - (2 \times \sqrt{25} \times \sqrt{3}) + (\sqrt{100} \times \sqrt{2}) - (3 \times \sqrt{16} \times \sqrt{3}) + (5 \times \sqrt{9} \times \sqrt{5})$$
$$= (4 \times 2\sqrt{2}) - (2 \times 5\sqrt{3}) + 10\sqrt{2} - (3 \times 4\sqrt{3}) + (5 \times 3\sqrt{5})$$
$$= 8\sqrt{2} - 10\sqrt{3} + 10\sqrt{2} - 12\sqrt{3} + 15\sqrt{5}$$
$$= 8\sqrt{2} + 10\sqrt{2} - 10\sqrt{3} - 12\sqrt{3} + 15\sqrt{5}$$
$$\mathbf{= 18\sqrt{2} - 22\sqrt{3} + 15\sqrt{5}}$$

Section 3: Multiplication of Surds

6) Evaluate each of the following.

Hint

In the questions to follow, we will be applying the fact that for $\boldsymbol{x \geq 0}$

$$\boxed{\sqrt{x} \times \sqrt{x} = x} \quad \text{and} \quad \boxed{\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x} = x}$$

In general,

$$\boxed{\begin{array}{c}\sqrt[n]{x_1} \times \sqrt[n]{x_2} \times \sqrt[n]{x_2} \times \dots \times \sqrt[n]{x_{n-1}} \times \sqrt[n]{x_n} = x \\ for\ x_1 = x_2 = x_3 \dots x_{n-1} = x_n\end{array}}$$

This is a special case of multiplication rule where the radicands are equal.

(a) $\sqrt{2} \times \sqrt{2}$

Solution

$$\sqrt{2} \times \sqrt{2} = (\sqrt{2})^2$$
$$\mathbf{= 2}$$

(b) $\sqrt{7} \times \sqrt{7}$

Solution

$$\sqrt{7} \times \sqrt{7} = (\sqrt{7})^2$$
$$\mathbf{= 7}$$

(c) $\sqrt{24} \times \sqrt{24}$

Solution

$$\sqrt{24} \times \sqrt{24} = (\sqrt{24})^2$$
$$\mathbf{= 24}$$

(d) $\sqrt{-1} \times \sqrt{-1}$

Solution

$$\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2$$
$$\mathbf{= -1}$$

NOTE

In complex numbers,

$$\boldsymbol{j = \sqrt{-1}}$$

and

$$\boldsymbol{j^2 = -1}$$

(e) $\sqrt[3]{4} \times \sqrt[3]{4} \times \sqrt[3]{4}$

Solution

$$\sqrt[3]{4} \times \sqrt[3]{4} \times \sqrt[3]{4} = \left(\sqrt[3]{4}\right)^3$$
$$= \mathbf{4}$$

(f) $\sqrt[4]{25} \times \sqrt[4]{25} \times \sqrt[4]{25} \times \sqrt[4]{25}$

Solution

$$\sqrt[4]{25} \times \sqrt[4]{25} \times \sqrt[4]{25} \times \sqrt[4]{25} = \left(\sqrt[4]{25}\right)^4$$
$$= \mathbf{25}$$

(g) $\sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16}$

Solution

$$\sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16} \times \sqrt[5]{16} = \left(\sqrt[5]{16}\right)^5$$
$$= \mathbf{16}$$

7) Simplify the following.

(a) $\sqrt{2} \times \sqrt{8}$

Solution

$$\sqrt{2} \times \sqrt{8} = \sqrt{2 \times 8}$$
$$= \sqrt{16}$$
$$= \sqrt{4^2}$$
$$= \mathbf{4}$$

(b) $\sqrt{2} \times \sqrt{18}$

Solution

$$\sqrt{2} \times \sqrt{18} = \sqrt{2 \times 18}$$
$$= \sqrt{36}$$
$$= \sqrt{6^2}$$
$$= \mathbf{6}$$

(c) $-\sqrt{3} \times \sqrt{12}$

Solution

$$-\sqrt{3} \times \sqrt{12} = -\sqrt{3 \times 12}$$
$$= -\sqrt{36}$$
$$= -\sqrt{6^2}$$
$$= \mathbf{-6}$$

(d) $\sqrt{2} \times \sqrt{32}$

Solution

$$\sqrt{2} \times \sqrt{32} = \sqrt{2 \times 32}$$
$$= \sqrt{64}$$
$$= \sqrt{8^2}$$
$$= \mathbf{8}$$

(e) $\sqrt{3} \times \sqrt{27}$

Solution

$$\sqrt{3} \times \sqrt{27} = \sqrt{3 \times 27}$$
$$= \sqrt{81}$$
$$= \sqrt{9^2}$$
$$= \mathbf{9}$$

8) Simplify each of the following.

(a) $3\sqrt{2} \times 5\sqrt{8}$

Solution

$$3\sqrt{2} \times 5\sqrt{8} = 3 \times 5 \times \sqrt{2} \times \sqrt{8}$$
$$= 15 \times \sqrt{2 \times 8}$$
$$= 15 \times \sqrt{16}$$
$$= 15 \times 4$$
$$= \mathbf{60}$$

(b) $10\sqrt{3} \times 6\sqrt{3}$

Solution

$$10\sqrt{3} \times 6\sqrt{3} = 10 \times 6 \times \sqrt{3} \times \sqrt{3}$$
$$= 60 \times \left(\sqrt{3}\right)^2$$
$$= 60 \times 3$$
$$= \mathbf{180}$$

(c) $2\sqrt{6} \times 5\sqrt{7}$

Solution

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$$2\sqrt{6} \times 5\sqrt{7} = 2 \times 5 \times \sqrt{6} \times \sqrt{7}$$
$$= 10 \times \sqrt{6 \times 7}$$
$$= 10 \times \sqrt{42}$$
$$\mathbf{= 10\sqrt{42}}$$

(d) $-2\sqrt{3} \times 4\sqrt{10}$

Solution

$$-2\sqrt{3} \times 4\sqrt{10} = -2 \times 4 \times \sqrt{3} \times \sqrt{10}$$
$$= -8 \times \sqrt{3 \times 10}$$
$$= -8 \times \sqrt{30}$$
$$\mathbf{= -8\sqrt{30}}$$

(e) $4\sqrt{10} \times 3\sqrt{8}$

Solution

$$4\sqrt{10} \times 3\sqrt{8} = 4 \times 3 \times \sqrt{10} \times \sqrt{8}$$
$$= 12 \times \sqrt{10 \times 8}$$
$$= 12\sqrt{80}$$
$$= 12 \times \sqrt{16 \times 5}$$
$$= 12 \times 4\sqrt{5}$$
$$\mathbf{= 48\sqrt{5}}$$

Section 4: Division of Surds (Basic)

9) Simplify each of the following leaving the answer in the simplified form, i.e. with a rational denominator.

(a) $\frac{\sqrt{8}}{\sqrt{2}}$

Solution

$$\frac{\sqrt{8}}{\sqrt{2}} = \sqrt{\frac{8}{2}}$$
$$= \sqrt{4}$$
$$\mathbf{= 2}$$

(b) $\frac{\sqrt{63}}{\sqrt{7}}$

Solution

$$\frac{\sqrt{63}}{\sqrt{7}} = \sqrt{\frac{63}{7}}$$
$$= \sqrt{9}$$
$$\mathbf{= 3}$$

(c) $\frac{\sqrt{125}}{\sqrt{5}}$

Solution

$$\frac{\sqrt{125}}{\sqrt{5}} = \sqrt{\frac{125}{5}}$$
$$= \sqrt{25}$$
$$\mathbf{= 5}$$

(d) $\frac{8\sqrt{12}}{2\sqrt{3}}$

Solution

$$\frac{8\sqrt{12}}{2\sqrt{2}} = \frac{8}{2} \times \sqrt{\frac{12}{3}}$$
$$= 4 \times \sqrt{4}$$
$$= 4 \times 2$$
$$\mathbf{= 8}$$

(e) $\frac{5\sqrt{35}}{15\sqrt{7}}$

Solution

$$\frac{5\sqrt{35}}{15\sqrt{7}} = \frac{5}{15} \times \sqrt{\frac{35}{7}}$$
$$= \frac{1}{3} \times \sqrt{5}$$
$$\mathbf{= \frac{1}{3}\sqrt{5}}$$

(f) $\sqrt{\frac{1}{4}}$

Solution

$$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}}$$
$$\mathbf{= \frac{1}{2}}$$

(g) $\sqrt{\frac{18}{50}}$

Solution

$$\sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}}$$
$$= \frac{\sqrt{9}}{\sqrt{25}}$$
$$= \mathbf{\frac{3}{5}}$$

10) Simplify each of the following.

Hint

In the questions to follow, we will need to rationalise the denominator.

(a) $\frac{1}{\sqrt{2}}$

Solution

$$\frac{1}{\sqrt{2}} = \left[\frac{1}{\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right]$$
$$= \frac{\sqrt{2}}{\left(\sqrt{2}\right)^2}$$
$$= \frac{\sqrt{2}}{2}$$
$$= \mathbf{\frac{1}{2}\sqrt{2}}$$

(b) $\frac{1}{\sqrt{13}}$

Solution

$$\frac{1}{\sqrt{13}} = \left[\frac{1}{\sqrt{13}}\right] \times \left[\frac{\sqrt{13}}{\sqrt{13}}\right]$$
$$= \frac{\sqrt{13}}{\left(\sqrt{13}\right)^2}$$
$$= \frac{\sqrt{13}}{13}$$
$$= \mathbf{\frac{1}{13}\sqrt{13}}$$

(c) $\frac{5}{6\sqrt{2}}$

Solution

$$\frac{5}{6\sqrt{2}} = \left[\frac{5}{6\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right]$$
$$= \frac{5\sqrt{2}}{6\left(\sqrt{2}\right)^2}$$
$$= \frac{5\sqrt{2}}{6 \times 2}$$
$$= \mathbf{\frac{5}{12}\sqrt{2}}$$

NOTE

In this and similar cases, do not use the whole of the denominator to rationalise. Instead, take only the surd part and leave out the coefficient while rationalising since the latter is already a rational number.

(d) $\frac{1}{5\sqrt{5}}$

Solution

$$\frac{1}{5\sqrt{5}} = \left[\frac{1}{5\sqrt{5}}\right] \times \left[\frac{\sqrt{5}}{\sqrt{5}}\right]$$
$$= \frac{\sqrt{5}}{5\left(\sqrt{5}\right)^2}$$
$$= \frac{\sqrt{5}}{5 \times 5}$$
$$= \mathbf{\frac{1}{25}\sqrt{5}}$$

(e) $\frac{10}{\sqrt{5}}$

Solution

$$\frac{10}{\sqrt{5}} = \left[\frac{10}{\sqrt{5}}\right] \times \left[\frac{\sqrt{5}}{\sqrt{5}}\right]$$
$$= \frac{10\sqrt{5}}{\left(\sqrt{5}\right)^2}$$

$$= \frac{10\sqrt{5}}{5}$$

$$= \mathbf{2\sqrt{5}}$$

(f) $\frac{21}{\sqrt{3}}$

Solution

$$\frac{21}{\sqrt{3}} = \left[\frac{21}{\sqrt{3}}\right] \times \left[\frac{\sqrt{3}}{\sqrt{3}}\right]$$

$$= \frac{21\sqrt{3}}{(\sqrt{3})^2}$$

$$= \frac{21\sqrt{3}}{3}$$

$$= \mathbf{7\sqrt{3}}$$

(g) $\frac{11}{\sqrt{11}}$

Solution

$$\frac{11}{\sqrt{11}} = \left[\frac{11}{\sqrt{11}}\right] \times \left[\frac{\sqrt{11}}{\sqrt{11}}\right]$$

$$= \frac{11\sqrt{11}}{(\sqrt{11})^2}$$

$$= \frac{11\sqrt{11}}{11}$$

$$= \mathbf{\sqrt{11}}$$

(h) $\frac{8\sqrt{3}}{\sqrt{2}}$

Solution

$$\frac{8\sqrt{3}}{\sqrt{2}} = \left[\frac{8\sqrt{3}}{\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right]$$

$$= \frac{8\sqrt{3 \times 2}}{(\sqrt{2})^2}$$

$$= \frac{8\sqrt{6}}{2}$$

$$= \mathbf{4\sqrt{6}}$$

(i) $\frac{4\sqrt{15}}{\sqrt{6}}$

Solution

$$\frac{4\sqrt{15}}{\sqrt{6}} = 4\sqrt{\frac{15}{6}} = 4\sqrt{\frac{5}{2}}$$

$$= \frac{4\sqrt{5}}{\sqrt{2}} = \left[\frac{4\sqrt{5}}{\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right]$$

$$= \frac{4\sqrt{5 \times 2}}{(\sqrt{2})^2}$$

$$= \frac{4\sqrt{10}}{2}$$

$$= \mathbf{2\sqrt{10}}$$

NOTE

In this type of questions, it is convenient to apply the rule shown below and then rationalise the denominator. This is because rationalising will lead to a big number as the radicand, which might then be cumbersome to simplify.

$$\frac{\sqrt{\boldsymbol{a}}}{\sqrt{\boldsymbol{b}}} = \sqrt{\frac{\boldsymbol{a}}{\boldsymbol{b}}}$$

(j) $\frac{\sqrt{15}}{\sqrt{6}}$

Solution

$$\frac{\sqrt{15}}{\sqrt{6}} = \left[\frac{\sqrt{15}}{\sqrt{6}}\right] \times \left[\frac{\sqrt{6}}{\sqrt{6}}\right]$$

$$= \frac{\sqrt{15 \times 6}}{(\sqrt{6})^2} = \frac{\sqrt{90}}{6}$$

$$= \frac{\sqrt{9 \times 10}}{6}$$

$$= \frac{\sqrt{9} \times \sqrt{10}}{6}$$

$$= \frac{3 \times \sqrt{10}}{6}$$

$$= \mathbf{\frac{1}{2}\sqrt{10}}$$

(k) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}}$

Solution

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}} = \left[\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}}\right] \times \left[\frac{\sqrt{3}}{\sqrt{3}}\right]$$

$$= \frac{\sqrt{3}(\sqrt{3}+\sqrt{2})}{(\sqrt{3})^2} = \frac{(\sqrt{3})^2 + (\sqrt{3}\times\sqrt{2})}{3}$$

$$= \frac{3+\sqrt{3\times 2}}{3}$$

$$= \mathbf{\frac{3+\sqrt{6}}{3}} \quad \text{or} \quad \mathbf{\frac{1}{3}(3+\sqrt{6})} \quad \text{or} \quad \mathbf{1+\frac{1}{3}\sqrt{6}}$$

(l) $\frac{2\sqrt{5}-15\sqrt{3}}{\sqrt{5}}$

Solution

$$\frac{2\sqrt{5}-15\sqrt{3}}{\sqrt{5}} = \left[\frac{2\sqrt{5}-15\sqrt{3}}{\sqrt{5}}\right] \times \left[\frac{\sqrt{5}}{\sqrt{5}}\right]$$

$$= \frac{\sqrt{5}(2\sqrt{5}-15\sqrt{3})}{(\sqrt{5})^2}$$

$$= \frac{2(\sqrt{5})^2 - 15(\sqrt{3}\times\sqrt{5})}{5}$$

$$= \frac{2(5) - 15\sqrt{3\times 5}}{5}$$

$$= \frac{10-15\sqrt{15}}{5}$$

$$= \frac{10}{5} - \frac{15\sqrt{15}}{5}$$

$$= \mathbf{2 - 3\sqrt{15}}$$

(m) $\sqrt{27} - \frac{3}{\sqrt{3}}$

Solution

$$\sqrt{27} - \frac{3}{\sqrt{3}} = \frac{\sqrt{27}\times\sqrt{3} - 3}{\sqrt{3}}$$

$$= \frac{\sqrt{27\times 3} - 3}{\sqrt{3}}$$

$$= \frac{\sqrt{81}-3}{\sqrt{3}} = \frac{9-3}{\sqrt{3}}$$

$$= \frac{6}{\sqrt{3}} = \left[\frac{6}{\sqrt{3}}\right] \times \left[\frac{\sqrt{3}}{\sqrt{3}}\right]$$

$$= \frac{6\times\sqrt{3}}{(\sqrt{3})^2} = \frac{6\sqrt{3}}{3}$$

$$= \mathbf{2\sqrt{3}}$$

NOTE

Alternatively,

$$\sqrt{27} - \frac{3}{\sqrt{3}} = \sqrt{9\times 3} - \left[\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right]$$

$$= 3\sqrt{3} - \left[\frac{3\sqrt{3}}{(\sqrt{3})^2}\right]$$

$$= 3\sqrt{3} - \frac{3\sqrt{3}}{3}$$

$$= 3\sqrt{3} - \sqrt{3}$$

$$= \mathbf{2\sqrt{3}}$$

as below.

(n) $\frac{\sqrt{7}-4}{\sqrt{7}}$

Solution

$$\frac{\sqrt{7}-4}{\sqrt{7}} = \left[\frac{\sqrt{7}-4}{\sqrt{7}}\right] \times \left[\frac{\sqrt{7}}{\sqrt{7}}\right]$$

$$= \frac{\sqrt{7}(\sqrt{7}-4)}{(\sqrt{7})^2}$$

$$= \frac{(\sqrt{7})^2 - 4\sqrt{7}}{7}$$

$$= \frac{7-4\sqrt{7}}{7}$$

$$= \frac{7}{7} - \frac{4\sqrt{7}}{7}$$

$$= \mathbf{1 - \frac{4}{7}\sqrt{7}}$$

Section 5: Powers of Surds

11) Find the square of each of the following surds.

(a) $2\sqrt{5}$

Solution

$$(2\sqrt{5})^2 = (2\times\sqrt{5})^2$$

$$= 2^2 \times (\sqrt{5})^2$$

$$= 4 \times 5$$
$$\boldsymbol{= 20}$$

(b) $7\sqrt{3}$

Solution

$$\left(7\sqrt{3}\right)^2 = \left(7 \times \sqrt{3}\right)^2$$
$$= 7^2 \times \left(\sqrt{3}\right)^2$$
$$= 49 \times 3$$
$$\boldsymbol{= 147}$$

(c) $10\sqrt{8}$

Solution

$$\left(10\sqrt{8}\right)^2 = \left(10 \times \sqrt{8}\right)^2$$
$$= 10^2 \times \left(\sqrt{8}\right)^2$$
$$= 100 \times 8$$
$$\boldsymbol{= 800}$$

(d) $9\sqrt{7}$

Solution

$$\left(9\sqrt{7}\right)^2 = \left(9 \times \sqrt{7}\right)^2$$
$$= 9^2 \times \left(\sqrt{7}\right)^2$$
$$= 81 \times 7$$
$$\boldsymbol{= 567}$$

(e) $6\sqrt{6}$

Solution

$$\left(6\sqrt{6}\right)^2 = \left(6 \times \sqrt{6}\right)^2$$
$$= 6^2 \times \left(\sqrt{6}\right)^2$$
$$= 36 \times 6$$
$$\boldsymbol{= 216}$$

(f) $\sqrt{7} \times 2\sqrt{11}$

Solution

$$\left(\sqrt{7} \times 2\sqrt{11}\right)^2 = \left(\sqrt{7} \times 2 \times \sqrt{11}\right)^2$$
$$= \left(\sqrt{7}\right)^2 \times 2^2 \times \left(\sqrt{11}\right)^2$$
$$= 7 \times 4 \times 11$$
$$\boldsymbol{= 308}$$

(g) $2\sqrt{5} \times 3\sqrt{2}$

Solution

$$\left(2\sqrt{5} \times 3\sqrt{2}\right)^2 = \left(2\sqrt{5}\right)^2 \times \left(3\sqrt{2}\right)^2$$
$$= 2^2 \times \left(\sqrt{5}\right)^2 \times 3^2 \times \left(\sqrt{2}\right)^2$$
$$= 4 \times 5 \times 9 \times 2$$
$$\boldsymbol{= 360}$$

12) Find the cube of each of the following singular surds.

(a) $2\sqrt[3]{3}$

Solution

$$\left(2\sqrt[3]{3}\right)^3 = \left(2 \times \sqrt[3]{3}\right)^3$$
$$= 2^3 \times \left(\sqrt[3]{3}\right)^3$$
$$= 8 \times 3$$
$$\boldsymbol{= 24}$$

(b) $5\sqrt[3]{4}$

Solution

$$\left(5\sqrt[3]{4}\right)^3 = \left(5 \times \sqrt[3]{4}\right)^3$$
$$= 5^3 \times \left(\sqrt[3]{4}\right)^3$$
$$= 125 \times 4$$
$$\boldsymbol{= 500}$$

(c) $3\sqrt[3]{13}$

Solution

$$\left(3\sqrt[3]{13}\right)^3 = \left(3 \times \sqrt[3]{13}\right)^3$$
$$= 3^3 \times \left(\sqrt[3]{13}\right)^3$$
$$= 27 \times 13$$
$$\boldsymbol{= 351}$$

(d) $10\sqrt[3]{5}$

Solution

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$$\left(10\sqrt[3]{5}\right)^3 = \left(10 \times \sqrt[3]{5}\right)^3$$
$$= 10^3 \times \left(\sqrt[3]{5}\right)^3$$
$$= 1000 \times 5$$
$$\mathbf{= 5\,000}$$

13) Simplify the following, giving the answer in the form of $a\sqrt[3]{b}$.

Hint

In general,

$$\sqrt[n]{a^n} = a$$

Also note that

$$\sqrt[n]{a^n} = \left(\sqrt[n]{a}\right)^n$$

(a) $\sqrt[3]{40}$

Solution

$$\sqrt[3]{40} = \sqrt[3]{8 \times 5}$$
$$= \sqrt[3]{8} \times \sqrt[3]{5}$$
$$= \sqrt[3]{2^3} \times \sqrt[3]{5}$$
$$= 2 \times \sqrt[3]{5}$$
$$\mathbf{= 2\sqrt[3]{5}}$$

(b) $\sqrt[3]{40} + \sqrt[3]{5}$

Solution

$$\sqrt[3]{40} + \sqrt[3]{5} = \sqrt[3]{8 \times 5} + \sqrt[3]{5}$$
$$= \left(\sqrt[3]{8} \times \sqrt[3]{5}\right) + \sqrt[3]{5}$$
$$= \left(\sqrt[3]{2^3} \times \sqrt[3]{5}\right) + \sqrt[3]{5}$$
$$= \left(2 \times \sqrt[3]{5}\right) + \sqrt[3]{5}$$
$$= 2\sqrt[3]{5} + \sqrt[3]{5}$$
$$\mathbf{= 3\sqrt[3]{5}}$$

(f) $\sqrt[3]{24} + \sqrt[3]{3}$

Solution

$$\sqrt[3]{24} + \sqrt[3]{3} = \sqrt[3]{8 \times 3} + \sqrt[3]{3}$$
$$= \left(\sqrt[3]{8} \times \sqrt[3]{3}\right) + \sqrt[3]{3}$$
$$= \left(2 \times \sqrt[3]{3}\right) + \sqrt[3]{3}$$
$$= 2\sqrt[3]{3} + \sqrt[3]{3}$$
$$\mathbf{= 3\sqrt[3]{3}}$$

(c) $\sqrt[4]{162}$

Solution

$$\sqrt[4]{162} = \sqrt[4]{81 \times 2}$$
$$= \sqrt[4]{81} \times \sqrt[4]{2}$$
$$= \sqrt[4]{3^4} \times \sqrt[4]{2}$$
$$= 3 \times \sqrt[4]{2}$$
$$\mathbf{= 3\sqrt[4]{2}}$$

(d) $\sqrt[4]{2} - \sqrt[4]{162}$

Solution

$$\sqrt[4]{2} - \sqrt[4]{162} = \sqrt[4]{2} - \left(\sqrt[4]{81 \times 2}\right)$$
$$= \sqrt[4]{2} - \left(\sqrt[4]{81} \times \sqrt[4]{2}\right)$$
$$= \sqrt[4]{2} - \left(\sqrt[4]{3^4} \times \sqrt[4]{2}\right)$$
$$= \sqrt[4]{2} - \left(3 \times \sqrt[4]{2}\right)$$
$$= \sqrt[4]{2} - 3\sqrt[4]{2}$$
$$\mathbf{= -2\sqrt[4]{2}}$$

(e) $\sqrt[5]{192}$

Solution

$$\sqrt[5]{192} = \sqrt[5]{32 \times 6}$$
$$= \sqrt[5]{32} \times \sqrt[5]{6}$$
$$= \sqrt[5]{2^5} \times \sqrt[5]{6}$$
$$= 2 \times \sqrt[5]{6}$$
$$\mathbf{= 2\sqrt[5]{6}}$$

(f) $\sqrt[5]{192} + \sqrt[5]{6}$

Solution

$$\sqrt[5]{192} = \sqrt[5]{32 \times 6} + \sqrt[5]{6}$$
$$= \left(\sqrt[5]{32} \times \sqrt[5]{6}\right) + \sqrt[5]{6}$$

$$= \left(\sqrt[5]{2^5} \times \sqrt[5]{6}\right) + \sqrt[5]{6}$$
$$= \left(2 \times \sqrt[5]{6}\right) + \sqrt[5]{6}$$
$$= 2\sqrt[5]{6} + \sqrt[5]{6}$$
$$= \mathbf{3\sqrt[5]{6}}$$

(g) $\sqrt[4]{64} + \sqrt[4]{4}$

Solution

$$\sqrt[4]{64} + \sqrt[4]{4} = \sqrt[4]{16 \times 4} + \sqrt[4]{4}$$
$$= \left(\sqrt[4]{16} \times \sqrt[4]{4}\right) + \sqrt[4]{4}$$
$$= \left(2 \times \sqrt[4]{4}\right) + \sqrt[4]{4}$$
$$= 2\sqrt[4]{4} + \sqrt[4]{4}$$
$$= \mathbf{3\sqrt[4]{4}}$$

Section 6: Simplification (II)

14) Simplify each of the following.

Hint

In the questions to follow, we can either simplify the fractions and then rationalise the denominator or vice versa. It appears that we may need to adopt the two approaches as deemed appropriate.

(a) $\frac{10}{\sqrt{2}} + \sqrt{8}$

Solution

$$\frac{10}{\sqrt{2}} + \sqrt{8} = \frac{10 + \sqrt{2} \times \sqrt{8}}{\sqrt{2}}$$
$$= \frac{10 + \sqrt{16}}{\sqrt{2}}$$
$$= \frac{10 + 4}{\sqrt{2}} = \frac{14}{\sqrt{2}}$$
$$= \left[\frac{14}{\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right] = \frac{14\sqrt{2}}{\left(\sqrt{2}\right)^2}$$
$$= \frac{14\sqrt{2}}{2}$$
$$= \mathbf{7\sqrt{2}}$$

NOTE

Alternatively,

$$\frac{10}{\sqrt{2}} + \sqrt{8} = \left[\frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right] + \sqrt{4 \times 2}$$
$$= \left[\frac{10\sqrt{2}}{\left(\sqrt{2}\right)^2}\right] + 2\sqrt{2}$$
$$= \frac{10\sqrt{2}}{2} + 2\sqrt{2}$$
$$= 5\sqrt{2} - \sqrt{2}$$
$$= \mathbf{3\sqrt{2}}$$

as below.

(b) $\frac{2}{\sqrt{8}} + \frac{1}{\sqrt{2}}$

Solution

$$\frac{2}{\sqrt{8}} + \frac{1}{\sqrt{2}} = \frac{2}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= \frac{2 + 2}{2\sqrt{2}} = \frac{4}{2\sqrt{2}}$$
$$= \left[\frac{4}{2\sqrt{2}}\right] \times \left[\frac{\sqrt{2}}{\sqrt{2}}\right]$$
$$= \frac{4\sqrt{2}}{2\left(\sqrt{2}\right)^2}$$
$$= \frac{4\sqrt{2}}{2 \times 2} = \frac{4\sqrt{2}}{4}$$
$$= \mathbf{\sqrt{2}}$$

(c) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$

Solution

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2 \times 3}}$$
$$= \frac{\sqrt{3} - \sqrt{2}}{\sqrt{6}}$$

$$= \left[\frac{(\sqrt{3}-\sqrt{2})}{\sqrt{6}}\right] \times \left[\frac{\sqrt{6}}{\sqrt{6}}\right]$$
$$= \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{6})^2}$$
$$= \frac{\sqrt{3\times 6}-\sqrt{2\times 6}}{6}$$
$$= \frac{\sqrt{18}-\sqrt{12}}{6}$$
$$= \frac{\sqrt{9\times 2}-\sqrt{4\times 3}}{6}$$
$$= \frac{(3\times\sqrt{2})-(2\times\sqrt{3})}{6}$$
$$\boldsymbol{= \frac{1}{6}(3\sqrt{2}-2\sqrt{3})}$$

(d) $\frac{\sqrt{5}}{2}+\frac{1}{\sqrt{3}}$

Solution

$$\frac{\sqrt{5}}{2}+\frac{1}{\sqrt{3}} = \frac{\sqrt{5}}{2}+\left[\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\right]$$
$$= \frac{\sqrt{5}}{2}+\frac{\sqrt{3}}{(\sqrt{3})^2}$$
$$= \frac{\sqrt{5}}{2}+\frac{\sqrt{3}}{3}$$
$$= \frac{(3\times\sqrt{5})+(2\times\sqrt{3})}{2\times 3}$$
$$\boldsymbol{= \frac{3\sqrt{5}+2\sqrt{3}}{6}}$$

(e) $\frac{5}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{6}}$

Solution

$$\frac{5}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{6}} = \left[\frac{5}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\right]-\left[\frac{\sqrt{3}}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}\right]$$
$$= \frac{5\sqrt{2}}{(\sqrt{2})^2}-\frac{\sqrt{3\times 6}}{(\sqrt{6})^2}$$
$$= \frac{5\sqrt{2}}{2}-\frac{\sqrt{18}}{6}$$
$$= \frac{5\sqrt{2}}{2}-\frac{\sqrt{9\times 2}}{6}$$
$$= \frac{5\sqrt{2}}{2}-\frac{3\sqrt{2}}{6}$$
$$= \frac{15\sqrt{2}-3\sqrt{2}}{6}$$
$$= \frac{12\sqrt{2}}{6}$$
$$\boldsymbol{= 2\sqrt{2}}$$

15) Given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$ and $\sqrt{7} = 2.646$, evaluate each of the following giving your answers correct to 4 significant figures.

(a) $\sqrt{20}$

Solution

$$\sqrt{20} = \sqrt{4\times 5} = 2\sqrt{5}$$
$$= 2(2.236)$$
$$\boldsymbol{= 4.472}$$

(b) $\sqrt{300}$

Solution

$$\sqrt{300} = \sqrt{100\times 3} = 10\sqrt{3}$$
$$= 10(1.732)$$
$$\boldsymbol{= 17.32}$$

(c) $\sqrt{112}$

Solution

$$\sqrt{112} = \sqrt{16\times 7} = 4\sqrt{7}$$
$$= 4(2.646)$$
$$\boldsymbol{= 10.58}$$

(d) $\sqrt{8}-3$

Solution

$$\sqrt{8}-3 = \sqrt{4\times 2}-3$$
$$= 2\sqrt{2}-3$$
$$= 2(1.414)-3$$
$$= 2.828-3 = -0.172$$
$$\boldsymbol{= -0.1720}$$

(e) $2\sqrt{27}+3\sqrt{80}$

Solution

$$
\begin{aligned}
2\sqrt{27}+3\sqrt{80} &= 2\sqrt{9\times 3}+3\sqrt{16\times 5}\\
&= \left(2\times 3\sqrt{3}\right)+\left(3\times 4\sqrt{5}\right)\\
&= 6\sqrt{3}+12\sqrt{5}\\
&= 6(1.732)+12(2.236)\\
&= 10.392+26.832=37.224\\
&= \mathbf{37.22}
\end{aligned}
$$

Section 7: Opening Brackets

16) Evaluate each of the following surds, leaving the answers in their forms without brackets.

(a) $\left(\sqrt{7}+2\right)^2$

Solution

$$
\begin{aligned}
\left(\sqrt{7}+2\right)^2 &= \left(\sqrt{7}+2\right)\left(\sqrt{7}+2\right)\\
&= \sqrt{7}\left(\sqrt{7}+2\right)+2\left(\sqrt{7}+2\right)\\
&= \sqrt{7}\left(\sqrt{7}\right)+2\left(\sqrt{7}\right)+2\left(\sqrt{7}\right)+2(2)\\
&= 7+4\left(\sqrt{7}\right)+4\\
&= \mathbf{11+4\sqrt{7}}
\end{aligned}
$$

(b) $\left(2\sqrt{2}+3\sqrt{3}\right)^2$

Solution

$$
\begin{aligned}
\left(2\sqrt{2}+3\sqrt{3}\right)^2 &= \left(2\sqrt{2}+3\sqrt{3}\right)\left(2\sqrt{2}+3\sqrt{3}\right)\\
&= \left[2\sqrt{2}\left(2\sqrt{2}+3\sqrt{3}\right)\right]\\
&\quad+\left[3\sqrt{3}\left(2\sqrt{2}+3\sqrt{3}\right)\right]\\
&= \left[2\sqrt{2}\left(2\sqrt{2}\right)\right]+\left[2\sqrt{2}\left(3\sqrt{3}\right)\right]\\
&\quad+\left[3\sqrt{3}\left(2\sqrt{2}\right)\right]+\left[3\sqrt{3}\left(3\sqrt{3}\right)\right]\\
&= \left[2^2\times\left(\sqrt{2}\right)^2\right]\\
&\quad+\left[(2\times 3)\left(\sqrt{2}\times\sqrt{3}\right)\right]\\
&\quad+\left[(2\times 3)\left(\sqrt{2}\times\sqrt{3}\right)\right]\\
&\quad+\left[3^2\times\left(\sqrt{3}\right)^2\right]\\
&= [4\times 2]+\left[6\left(\sqrt{2\times 3}\right)\right]\\
&\quad+\left[6\left(\sqrt{2\times 3}\right)\right]+[9\times 3]\\
&= 8+6\left(\sqrt{6}\right)+6\left(\sqrt{6}\right)+27\\
&= 8+27+12\sqrt{6}\\
&= \mathbf{35+12\sqrt{6}}
\end{aligned}
$$

(c) $\left(3\sqrt{6}-4\sqrt{5}\right)^2$

Solution

$$
\begin{aligned}
\left(3\sqrt{6}-4\sqrt{5}\right)^2 &= \left(3\sqrt{6}-4\sqrt{5}\right)\left(3\sqrt{6}-4\sqrt{5}\right)\\
&= \left[3\sqrt{6}\left(3\sqrt{6}-4\sqrt{5}\right)\right]-\left[4\sqrt{5}\left(3\sqrt{6}-4\sqrt{5}\right)\right]\\
&= \left[3\sqrt{6}\left(3\sqrt{6}\right)\right]+\left[3\sqrt{6}\left(-4\sqrt{5}\right)\right]-\left[4\sqrt{5}\left(3\sqrt{6}\right)\right]\\
&\quad-\left[4\sqrt{5}\left(-4\sqrt{5}\right)\right]\\
&= \left[3^2\times\left(\sqrt{6}\right)^2\right]-\left[(4\times 3)\left(\sqrt{6}\times\sqrt{5}\right)\right]\\
&\quad-\left[(4\times 3)\left(\sqrt{5}\times\sqrt{6}\right)\right]\\
&\quad+\left[4^2\times\left(\sqrt{5}\right)^2\right]\\
&= [9\times 6]-\left[12\left(\sqrt{5\times 6}\right)\right]-\left[12\left(\sqrt{5\times 6}\right)\right]\\
&\quad+[16\times 5]\\
&= 54-12\left(\sqrt{30}\right)-12\left(\sqrt{30}\right)+80\\
&= 54+80-24\sqrt{30}\\
&= \mathbf{134-24\sqrt{30}}
\end{aligned}
$$

17) By opening the brackets, simplify the following.

Hint

(a) $\sqrt{3}-\sqrt{2}\left(\sqrt{6}-\sqrt{24}\right)$

Solution

$$
\begin{aligned}
&\sqrt{3}-\sqrt{2}\left(\sqrt{6}-\sqrt{24}\right)\\
&= \left[\sqrt{3}\right]-\left[\sqrt{2}\left(\sqrt{6}\right)\right]-\left[\sqrt{2}\left(-\sqrt{24}\right)\right]\\
&= \sqrt{3}-\sqrt{2\times 6}+\sqrt{2\times 24}
\end{aligned}
$$

$$= \sqrt{3} - \sqrt{12} + \sqrt{48}$$
$$= \sqrt{3} - \sqrt{4 \times 3} + \sqrt{16 \times 3}$$
$$= \sqrt{3} - 2\sqrt{3} + 4\sqrt{3}$$
$$= \mathbf{3\sqrt{3}}$$

(b) $(5 - \sqrt{3})(\sqrt{3} + 6)$

Solution

$$(5 - \sqrt{3})(\sqrt{3} + 6)$$
$$= [5(\sqrt{3} + 6)] - [\sqrt{3}(\sqrt{3} + 6)]$$
$$= (5\sqrt{3}) + (5 \times 6) - (\sqrt{3} \times \sqrt{3}) - (6\sqrt{3})$$
$$= 5\sqrt{3} + 30 - 3 - 6\sqrt{3}$$
$$= \mathbf{27 - \sqrt{3}}$$

(c) $(\sqrt{15} - \sqrt{3})(\sqrt{6} + \sqrt{2})$

Solution

$$(\sqrt{15} - \sqrt{3})(\sqrt{6} + \sqrt{2})$$
$$= [\sqrt{15}(\sqrt{6} + \sqrt{2})] - [\sqrt{3}(\sqrt{6} + \sqrt{2})]$$
$$= [\sqrt{15}(\sqrt{6})] + [\sqrt{15}(\sqrt{2})] - [\sqrt{3}(\sqrt{6})] - [\sqrt{3}(\sqrt{2})]$$
$$= \sqrt{15 \times 6} + \sqrt{15 \times 2} - \sqrt{3 \times 6} - \sqrt{3 \times 2}$$
$$= \sqrt{90} + \sqrt{30} - \sqrt{18} - \sqrt{6}$$
$$= \sqrt{9 \times 10} + \sqrt{30} - \sqrt{9 \times 2} - \sqrt{6}$$
$$= 3\sqrt{10} + \sqrt{30} - 3\sqrt{9} - \sqrt{6}$$
$$= \mathbf{3\sqrt{10} + \sqrt{30} - 3\sqrt{9} - \sqrt{6}}$$

(d) $(\sqrt{5} + \sqrt{2})(\sqrt{2} - \sqrt{3})$

Solution

$$(\sqrt{5} + \sqrt{2})(\sqrt{2} - \sqrt{3})$$
$$= [\sqrt{5}(\sqrt{2})] - [\sqrt{5}(\sqrt{3})] + [\sqrt{2}(\sqrt{2})] - [\sqrt{2}(\sqrt{3})]$$
$$= \sqrt{5 \times 2} - \sqrt{5 \times 3} + \sqrt{2 \times 2} - \sqrt{3 \times 2}$$
$$= \sqrt{10} - \sqrt{15} + 2 - \sqrt{6}$$
$$= \mathbf{2 + \sqrt{10} - \sqrt{6} - \sqrt{15}}$$

(e) $(4\sqrt{3} - 2)(5\sqrt{10} + \sqrt{3})$

Solution

$$(4\sqrt{3} - 2)(5\sqrt{10} + \sqrt{3})$$
$$= [4\sqrt{3}(5\sqrt{10})] + [4\sqrt{3}(\sqrt{3})] - [2(5\sqrt{10})] - [2(\sqrt{3})]$$
$$= [(4 \times 5)\sqrt{3 \times 10}] + \left[4(\sqrt{3})^2\right] - [(2 \times 5)\sqrt{10}] - [2\sqrt{3}]$$
$$= (20\sqrt{30}) + (4 \times 3) - (10\sqrt{10}) - (2\sqrt{3})$$
$$= 20\sqrt{30} + 12 - 10\sqrt{10} - 2\sqrt{3}$$
$$= \mathbf{12 + 20\sqrt{30} - 10\sqrt{10} - 2\sqrt{3}}$$

Section 8: Conjugates & Rationalisation

18) Simplify the following.

Hint

In the questions to follow, we will be dealing with conjugates and therefore, it is to be remembered that

$$\boxed{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2}$$

or to be more general,

$$\boxed{(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d}) = (a\sqrt{b})^2 - (c\sqrt{d})^2}$$

This is an application of the 'difference of two squares'.

(a) $(1 - \sqrt{7})(1 + \sqrt{7})$

Solution

$$(1 - \sqrt{7})(1 + \sqrt{7}) = 1^2 - (\sqrt{7})^2$$
$$= 1 - 7$$
$$= \mathbf{-6}$$

(b) $(\sqrt{6} + 2)(\sqrt{6} - 2)$

Solution

$$(\sqrt{6}+2)(\sqrt{6}-2) = (\sqrt{6})^2 - 2^2$$
$$= 6 - 4$$
$$= \mathbf{2}$$

(c) $(3-\sqrt{17})(3+\sqrt{17})$

Solution

$$(3-\sqrt{17})(3+\sqrt{17}) = 3^2 - (\sqrt{17})^2$$
$$= 9 - 17$$
$$= \mathbf{-8}$$

(d) $(\sqrt{13}+\sqrt{5})(\sqrt{13}-\sqrt{5})$

Solution

$$(\sqrt{13}+\sqrt{5})(\sqrt{13}-\sqrt{5}) = (\sqrt{13})^2 - (\sqrt{5})^2$$
$$= 13 - 5$$
$$= \mathbf{8}$$

(e) $(4\sqrt{5}+\sqrt{11})(4\sqrt{5}-\sqrt{11})$

Solution

$$(4\sqrt{5}+\sqrt{11})(4\sqrt{5}-\sqrt{11}) = (4\sqrt{5})^2 - (\sqrt{11})^2$$
$$= 16(5) - 11$$
$$= 80 - 11$$
$$= \mathbf{69}$$

(f) $(8\sqrt{3}-2\sqrt{15})(8\sqrt{3}+2\sqrt{15})$

Solution

$$(8\sqrt{3}-2\sqrt{15})(8\sqrt{3}+2\sqrt{15}) = (8\sqrt{3})^2 - (2\sqrt{15})^2$$
$$= 64(3) - 4(15)$$
$$= 192 - 60$$
$$= \mathbf{132}$$

19) Simplify the following by rationalising the denominators. Leave your answers in simplified form.

(a) $\frac{1}{\sqrt{2}+1}$

Solution

$$\frac{1}{\sqrt{2}+1} = \left[\frac{1}{\sqrt{2}+1}\right] \times \left[\frac{\sqrt{2}-1}{\sqrt{2}-1}\right]$$
$$= \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}$$
$$= \frac{\sqrt{2}-1}{(\sqrt{2})^2 - 1^2}$$
$$= \frac{\sqrt{2}-1}{2-1}$$
$$= \mathbf{\sqrt{2}-1}$$

(b) $\frac{5}{2-\sqrt{3}}$

Solution

$$\frac{5}{2-\sqrt{3}} = \left[\frac{5}{2-\sqrt{3}}\right] \times \left[\frac{2+\sqrt{3}}{2+\sqrt{3}}\right]$$
$$= \frac{5(2+\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$
$$= \frac{5(2+\sqrt{3})}{2^2 - (\sqrt{3})^2}$$
$$= \frac{5(2+\sqrt{3})}{4-3}$$
$$= 5(2+\sqrt{3})$$
$$= \mathbf{10+5\sqrt{3}}$$

(c) $\frac{12}{\sqrt{15}-3}$

Solution

$$\frac{12}{\sqrt{15}-3} = \left[\frac{12}{\sqrt{15}-3}\right] \times \left[\frac{\sqrt{15}+3}{\sqrt{15}+3}\right]$$
$$= \frac{12(\sqrt{15}+3)}{(\sqrt{15}-3)(\sqrt{15}+3)}$$
$$= \frac{12(\sqrt{15}+3)}{(\sqrt{15})^2 - 3^2}$$
$$= \frac{12(\sqrt{15}+3)}{15-9}$$
$$= \frac{12(\sqrt{15}+3)}{6}$$
$$= 2(\sqrt{15}+3)$$

$$= \mathbf{6 + 2\sqrt{15}}$$

(d) $\frac{1}{5\sqrt{7}+4\sqrt{11}}$

Solution

$$\frac{1}{5\sqrt{7}+4\sqrt{11}} = \left[\frac{1}{5\sqrt{7}+4\sqrt{11}}\right] \times \left[\frac{5\sqrt{7}-4\sqrt{11}}{5\sqrt{7}-4\sqrt{11}}\right]$$

$$= \frac{5\sqrt{7}-4\sqrt{11}}{(5\sqrt{7}+4\sqrt{11})(5\sqrt{7}-4\sqrt{11})}$$

$$= \frac{5\sqrt{7}-4\sqrt{11}}{(5\sqrt{7})^2-(4\sqrt{11})^2}$$

$$= \frac{5\sqrt{7}-4\sqrt{11}}{175-176}$$

$$= \frac{5\sqrt{7}-4\sqrt{11}}{-1}$$

$$= \mathbf{4\sqrt{11} - 5\sqrt{7}}$$

(e) $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{5}-\sqrt{2}}$

Solution

$$\frac{\sqrt{6}-\sqrt{3}}{\sqrt{5}-\sqrt{2}} = \left[\frac{\sqrt{6}-\sqrt{3}}{\sqrt{5}-\sqrt{2}}\right] \times \left[\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}\right]$$

$$= \frac{(\sqrt{6}-\sqrt{3})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$

$$= \frac{\sqrt{6\times5}+\sqrt{6\times2}-\sqrt{3\times5}-\sqrt{3\times2}}{(\sqrt{5})^2-(\sqrt{2})^2}$$

$$= \frac{\sqrt{30}+\sqrt{12}-\sqrt{15}-\sqrt{6}}{5-2}$$

$$= \frac{\sqrt{30}+\sqrt{4\times3}-\sqrt{15}-\sqrt{6}}{3}$$

$$= \mathbf{\frac{1}{3}(\sqrt{30}+2\sqrt{3}-\sqrt{15}-\sqrt{6})}$$

(f) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

Solution

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} = \left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right] \times \left[\frac{\sqrt{3}+1}{\sqrt{3}+1}\right]$$

$$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{(\sqrt{3})^2+1^2+2(1)(\sqrt{3})}{(\sqrt{3})^2-1^2}$$

$$= \frac{3+1+2\sqrt{3}}{3-1}$$

$$= \frac{4+2\sqrt{3}}{2}$$

$$= \mathbf{2+\sqrt{3}}$$

(g) $\frac{3+\sqrt{5}}{3-\sqrt{5}}$

Solution

$$\frac{3+\sqrt{5}}{3-\sqrt{5}} = \left[\frac{3+\sqrt{5}}{3-\sqrt{5}}\right] \times \left[\frac{3+\sqrt{5}}{3+\sqrt{5}}\right]$$

$$= \frac{(3+\sqrt{5})^2}{(3-\sqrt{5})(3+\sqrt{5})}$$

$$= \frac{3^2+(\sqrt{5})^2+2(3)(\sqrt{5})}{3^2-(\sqrt{5})^2}$$

$$= \frac{9+5+6\sqrt{5}}{9-5}$$

$$= \frac{14+6\sqrt{5}}{4}$$

$$= \mathbf{\frac{1}{2}(7+3\sqrt{5})}$$

(h) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

Solution

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \left[\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right] \times \left[\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right]$$

$$= \frac{(\sqrt{3}+\sqrt{2})^2}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}$$

$$= \frac{(\sqrt{3})^2+(\sqrt{2})^2+2(\sqrt{3\times2})}{(\sqrt{3})^2-(\sqrt{2})^2}$$

$$= \frac{3+2+2\sqrt{6}}{3-2}$$

$$= \mathbf{5+2\sqrt{6}}$$

(i) $\frac{\sqrt{2}+5\sqrt{18}}{\sqrt{2}-5\sqrt{18}}$

Solution

$$\frac{\sqrt{2}+5\sqrt{18}}{\sqrt{2}-5\sqrt{18}}=\left[\frac{\sqrt{2}+5\sqrt{18}}{\sqrt{2}-5\sqrt{18}}\right]\times\left[\frac{\sqrt{2}+5\sqrt{18}}{\sqrt{2}+5\sqrt{18}}\right]$$

$$=\frac{(\sqrt{2}+5\sqrt{18})(\sqrt{2}+5\sqrt{18})}{(\sqrt{2}-5\sqrt{18})(\sqrt{2}+5\sqrt{18})}$$

$$=\frac{(\sqrt{2})^2+(5\sqrt{18})^2+2(\sqrt{2})(5\sqrt{18})}{(\sqrt{2})^2-(5\sqrt{18})^2}$$

$$=\frac{2+(25\times 18)+2(5)(\sqrt{2\times 18})}{2-450}$$

$$=\frac{2+450+10\sqrt{36}}{2-450}$$

$$=\frac{452+10(6)}{-448}$$

$$=\frac{512}{-448}$$

$$=-\frac{\mathbf{8}}{\mathbf{7}}$$

NOTE

It is obvious from the last few worked examples that dividing a surd by its conjugate can either result in a rational or irrational number. This is in contrary to when they are multiplied together, which always results in a rational answer.

Section 9: Finding Roots of Surds

Hint

In this section, we will use the following property of surds, i.e. equality of surds.

Given that

$$\boxed{\boldsymbol{a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2}}}$$

then

$$\boxed{\boldsymbol{a_1 = a_2 \text{ and } b_1 = b_2}}$$

20) Find the square roots of each of the following surds.

(a) $11+2\sqrt{28}$

Solution

Let the square root of $\mathbf{11+2\sqrt{27}}$ be $\sqrt{\boldsymbol{a}}+\sqrt{\boldsymbol{b}}$ for which $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}$. Therefore

$$11+2\sqrt{28}=\left(\sqrt{a}+\sqrt{b}\right)^2$$

$$=a+b+2\sqrt{ab}$$

Comparing the two sides of the above equation, we have

$$a+b=11 \qquad ------\textbf{(i)}$$

and

$$2\sqrt{ab}=2\sqrt{28}$$

Divide both sides by 2

$$\sqrt{ab}=\sqrt{28}$$

Square both sides

$$ab=28 \qquad ------\textbf{(ii)}$$

From (i)

$$b=11-a \qquad ------\textbf{(iii)}$$

Substitute equation (iii) in equation (ii),

$$a(11-a)=28$$

$$11a-a^2=28$$

$$a^2-11a+28=0$$

$$(a-4)(a-7)=0$$

Therefore, either

$$a-4=0$$

$$a=4$$

or

$$a-7=0$$

$$a=7$$

Now we need to find the corresponding values for b, thus

when $a=4$, from (iii)

$$b=11-4=7$$

and when $a=7$, from (iii)

$$b = 11 - 7 = 4$$

Taking the square root of $\mathbf{11 + 2\sqrt{27}}$ to be $\sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}}$ for which $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$ implies

$$\sqrt{a} + \sqrt{b} = \sqrt{7} + \sqrt{4}$$

and

$$\sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}} = \sqrt{\mathbf{4}} + \sqrt{\mathbf{7}}$$

These are the same.

$$\therefore \sqrt{\left(\mathbf{11 + 2\sqrt{28}}\right)} = \sqrt{\mathbf{4}} + \sqrt{\mathbf{7}}$$

(b) $32 - 6\sqrt{15}$

Solution

Let the square root of the $\mathbf{32 - 6\sqrt{15}}$ be $\sqrt{\boldsymbol{x}} - \sqrt{\boldsymbol{y}}$ for which $\boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}$. Therefore,

$$32 - 6\sqrt{15} = \left(\sqrt{x} - \sqrt{y}\right)^2$$

$$= x + y - 2\sqrt{xy}$$

Comparing the two sides of the above equation, we have

$$x + y = 32 \qquad - - - - - \textbf{(i)}$$

and

$$-2\sqrt{xy} = -6\sqrt{15}$$

Divide both sides by -2

$$\sqrt{xy} = 3\sqrt{15}$$

Square both sides

$$xy = 135 \qquad - - - - - - \textbf{(ii)}$$

From (i)

$$y = 32 - x \qquad - - - - - - \textbf{(iii)}$$

Substitute equation (iii) in equation (ii),

$$x(32 - x) = 135$$

$$32x - x^2 = 135$$

$$x^2 - 32x + 135 = 0$$

$$(x - 27)(x - 5) = 0$$

Therefore, either

$$x - 27 = 0$$

$$x = 27$$

or

$$x - 5 = 0$$

$$x = 5$$

Now we need to find the corresponding values for y, thus

when $x = 27$, from (iii)

$$y = 32 - 27 = 5$$

and when $x = 5$, from (iii)

$$y = 32 - 5 = 27$$

Taking the square root of $\mathbf{32 - 6\sqrt{15}}$ to be $\sqrt{\boldsymbol{x}} - \sqrt{\boldsymbol{y}}$ for which $\boldsymbol{x}, \ \boldsymbol{y} \in \mathbb{R}$ implies

$$\sqrt{x} - \sqrt{y} = \sqrt{27} - \sqrt{5}$$

$$= \mathbf{3\sqrt{3} - 5}$$

and

$$\sqrt{a} + \sqrt{b} = \sqrt{5} - \sqrt{27}$$

$$= \sqrt{\mathbf{5}} - \mathbf{3\sqrt{3}}$$

$$\therefore \sqrt{\left(\mathbf{32 - 6\sqrt{15}}\right)} = \pm\left(\mathbf{3\sqrt{3}} - \sqrt{\mathbf{5}}\right)$$

Section 10: Advanced Simplification

21) Express $\frac{3\sqrt{2}+5\sqrt{6}}{3\sqrt{2}-5\sqrt{6}}$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers. State the values of a, b and c.

Solution

$$\frac{3\sqrt{2} + 5\sqrt{6}}{3\sqrt{2} - 5\sqrt{6}} = \left[\frac{3\sqrt{2} + 5\sqrt{6}}{3\sqrt{2} - 5\sqrt{6}}\right] \times \left[\frac{3\sqrt{2} + 5\sqrt{6}}{3\sqrt{2} + 5\sqrt{6}}\right]$$

$$= \frac{\left(3\sqrt{2} + 5\sqrt{6}\right)\left(3\sqrt{2} + 5\sqrt{6}\right)}{\left(3\sqrt{2} - 5\sqrt{6}\right)\left(3\sqrt{2} + 5\sqrt{6}\right)}$$

$$= \frac{\left[\left(3\sqrt{2}\right)^2\right] + \left[\left(5\sqrt{6}\right)^2\right] + \left[2\left(3\sqrt{2}\right)\left(5\sqrt{6}\right)\right]}{\left(3\sqrt{2}\right)^2 - \left(5\sqrt{6}\right)^2}$$

$$= \frac{[9 \times 2] + [25 \times 6] + \left[2(3 \times 5)\left(\sqrt{2 \times 6}\right)\right]}{(9 \times 2) - (25 \times 6)}$$

$$= \frac{18 + 150 + 30\sqrt{12}}{18 - 150}$$

$$= \frac{168 + 60\sqrt{3}}{-132}$$

$$= -\frac{168}{132} - \frac{60\sqrt{3}}{132}$$

$$= -\frac{\mathbf{14}}{\mathbf{11}} - \frac{\mathbf{5}}{\mathbf{11}}\sqrt{\mathbf{3}}$$

$$\therefore \boldsymbol{a} = -\frac{\mathbf{14}}{\mathbf{11}}, \boldsymbol{b} = -\frac{\mathbf{5}}{\mathbf{11}} \textbf{ and } \boldsymbol{c} = \mathbf{3}$$

22) Evaluate

$$\frac{5 + \sqrt{3}}{\sqrt{5} - 2} \div \frac{\sqrt{5} + 2}{\sqrt{2}}$$

Solution

$$\frac{5 + \sqrt{3}}{\sqrt{5} - 2} \div \frac{\sqrt{5} + 2}{\sqrt{2}} = \left[\frac{5 + \sqrt{3}}{\sqrt{5} - 2}\right] \times \left[\frac{\sqrt{2}}{\sqrt{5} + 2}\right]$$

$$= \frac{\sqrt{2}(5 + \sqrt{3})}{(\sqrt{5} - 2)(\sqrt{5} + 2)}$$

$$= \frac{(5\sqrt{2}) + (\sqrt{2} \times \sqrt{3})}{(\sqrt{5})^2 - 2^2}$$

$$= \frac{5\sqrt{2} + \sqrt{6}}{5 - 4}$$

$$= \mathbf{5\sqrt{2} + \sqrt{6}}$$

23) Given that $\beta = 8 + 3\sqrt{7}$, find $\beta + \frac{1}{\beta}$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers.

Solution

Given that

$$\beta = 8 + 3\sqrt{7}$$

then

$$\frac{1}{\beta} = \frac{1}{8 + 3\sqrt{7}}$$

$$= \left[\frac{1}{8 + 3\sqrt{7}}\right] \times \left[\frac{8 - 3\sqrt{7}}{8 - 3\sqrt{7}}\right]$$

$$= \frac{8 - 3\sqrt{7}}{(8 + 3\sqrt{7})(8 - 3\sqrt{7})}$$

$$= \frac{8 - 3\sqrt{7}}{(8)^2 - (3\sqrt{7})^2}$$

$$= \frac{8 - 3\sqrt{7}}{64 - 63}$$

$$= 8 - 3\sqrt{7}$$

Hence

$$\beta + \frac{1}{\beta} = (8 + 3\sqrt{7}) + (8 - 3\sqrt{7})$$

$$= 8 + 8 = 16$$

$$\therefore \boldsymbol{\beta} + \frac{\mathbf{1}}{\boldsymbol{\beta}} = \mathbf{16}$$

24) Find the values of a and b for which

$$\frac{5}{2\sqrt{2} + 3} - \sqrt{2} = a + b\sqrt{2}$$

Solution

LHS

$$\frac{5}{2\sqrt{2} + 3} - \sqrt{2} = \frac{5 - \sqrt{2}(2\sqrt{2} + 3)}{2\sqrt{2} + 3}$$

$$= \frac{5 - 2(\sqrt{2})^2 - 3\sqrt{2}}{2\sqrt{2} + 3}$$

$$= \frac{5 - 4 - 3\sqrt{2}}{2\sqrt{2} + 3}$$

It is time to rationalise the above surdic expression

$$= \frac{1 - 3\sqrt{2}}{2\sqrt{2} + 3} = \left[\frac{1 - 3\sqrt{2}}{2\sqrt{2} + 3}\right] \times \left[\frac{2\sqrt{2} - 3}{2\sqrt{2} - 3}\right]$$

$$= \frac{(1 - 3\sqrt{2})(2\sqrt{2} - 3)}{(2\sqrt{2} + 3)(2\sqrt{2} - 3)}$$

$$= \frac{2\sqrt{2} - 3 - 6(\sqrt{2})^2 + 9\sqrt{2}}{(2\sqrt{2})^2 - 3^2}$$

$$= \frac{2\sqrt{2} + 9\sqrt{2} - 3 - 12}{8 - 9}$$

$$= -(11\sqrt{2} - 15)$$

$$= 15 - 11\sqrt{2}$$

Therefore,

$$\frac{5}{2\sqrt{2} + 3} - \sqrt{2} = a + b\sqrt{2}$$

implies that

$$15 - 11\sqrt{2} = a + b\sqrt{2}$$

Comparing both sides, we have

$$\boldsymbol{a = 15, b = -11}$$

NOTE

In this question, we may alternatively simplify $\frac{5}{2\sqrt{2}+3}$ in the first place, which gives $15 - 10\sqrt{2}$, and then subtract $\sqrt{2}$.

Section 11: Some Application of Surds (Geometry & Trigonometry)

25) Find the length of the line joining $A\ (-5, -2)$ and $B\ (-3, 4)$ in a Cartesian plane, leaving the answer in surd form.

Solution

The length of AB is given by

$$\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where (x_1, y_1) and (x_2, y_2) are the two points A and B respectively connecting the line AB. For this case

$$(x_1, y_1) = (-5, -2)$$

and

$$(x_2, y_2) = (-3, 4)$$

$$\therefore \overline{AB} = \sqrt{(-3+5)^2 + (4+2)^2} = \sqrt{(2)^2 + (6)^2}$$
$$= \sqrt{4 + 36} = \sqrt{40}$$
$$= \sqrt{4 \times 10} = \sqrt{40} \times \sqrt{10}$$
$$= 2 \times \sqrt{10}$$
$$\boldsymbol{= 2\sqrt{10}}$$

26) Given that $\tan 45° = 1$ and $\tan 60° = \sqrt{3}$, find the value of tan 15, leaving the answer in simplified surd form .

Solution

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

This implies

$$\tan 15° = \tan(60° - 45°)$$

Thus

$$A = 65° \text{ and } B = 45°$$

Hence

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\Rightarrow$

$$\tan 15° = \tan(60° - 45°)$$
$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$$
$$= \frac{\sqrt{3} - 1}{1 + (\sqrt{3} \times 1)} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$
$$= \left[\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right] \times \left[\frac{\sqrt{3} - 1}{\sqrt{3} - 1}\right]$$
$$= \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$
$$= \frac{(\sqrt{3})^2 - 2(\sqrt{3})(1) + 1^2}{(\sqrt{3})^2 - 1^2}$$
$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$
$$= \frac{4 - 2\sqrt{3}}{2} = \frac{2(2 - \sqrt{3})}{2}$$
$$\boldsymbol{= 2 - \sqrt{3}}$$

27) If $\sin\theta = \frac{2}{3}$ without using a calculator, find the value of $\tan\theta$, leaving the answer in simplified surd form.

Solution

Given that

$$\sin\theta = \frac{2}{3}$$

We know that

$$\sin^2\theta + \cos^2\theta = 1$$

which implies

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

substitute for $\sin\theta$

$$\therefore \cos\theta = \sqrt{1-\left(\frac{2}{3}\right)^2} = \sqrt{1-\frac{4}{9}}$$

$$= \sqrt{\frac{9-4}{9}} = \sqrt{\frac{5}{9}}$$

$$= \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}$$

But

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Now substitute for sine and cosine of θ

$$= \frac{2/3}{\sqrt{5}/3}$$

$$= \left[\frac{2}{3}\right] \times \left[\frac{3}{\sqrt{5}}\right] = \frac{2}{\sqrt{5}}$$

$$= \left[\frac{2}{\sqrt{5}}\right] \times \left[\frac{\sqrt{5}}{\sqrt{5}}\right]$$

$$= \frac{2\sqrt{5}}{\left(\sqrt{5}\right)^2}$$

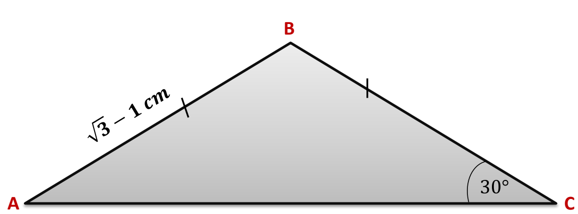
$$= \boldsymbol{\frac{2}{5}\sqrt{5}}$$

28) In a triangle ABC, $\overline{AB} = \overline{BC} = \sqrt{3}-1\ cm$ and $\angle ACB = 30°$, without using a calculator find the length of AC.

Note that

$\sin 30° = \cos 60° = \frac{1}{2}$ and $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$

Solution



Using cosine rule, we have

$$b^2 = a^2 + c^2 - 2ac\cos B$$

where

$$\overline{AB} = c = \sqrt{3}-1\ cm$$
$$\overline{BC} = a = \sqrt{3}-1\ cm$$
$$\overline{AC} = b$$

Also,

$$\angle BAC = \angle ACB = 30°$$
$$\because \overline{AB} = \overline{BC}$$
$$\therefore \hat{B} = 180° - (\angle BAC + \angle ACB)$$
$$= 180° - (30° + 30°)$$
$$= 120°$$

We can now find the length AC as

$$b^2 = \left(\sqrt{3}-1\right)^2 + \left(\sqrt{3}-1\right)^2$$
$$- 2\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)\cos 120°$$
$$= 2\left(\sqrt{3}-1\right)^2 - 2\left(\sqrt{3}-1\right)^2\cos(-60°)$$
$$= 2\left[\left(\sqrt{3}\right)^2 + (-1)^2 - 2\sqrt{3}\right]$$
$$- 2\left(\left(\sqrt{3}\right)^2 + (-1)^2 - 2\sqrt{3}\right)\left(-\frac{1}{2}\right)$$
$$= 2\left[3+1-2\sqrt{3}\right] + \left(3+1-2\sqrt{3}\right)$$
$$= 2\left[4-2\sqrt{3}\right] + \left(4-2\sqrt{3}\right)$$
$$= 3\left[4-2\sqrt{3}\right]$$
$$= 12 - 6\sqrt{3}$$

$$\therefore b = \sqrt{\left(12-6\sqrt{3}\right)}$$

Let the square root of the $\mathbf{12 - 6\sqrt{3}}$ be $\sqrt{\boldsymbol{x}} - \sqrt{\boldsymbol{y}}$ for which $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$. Therefore,

$$12 - 6\sqrt{3} = \left(\sqrt{x} - \sqrt{y}\right)^2$$
$$= x + y - 2\sqrt{xy}$$

Comparing the two sides of the above equation, we have

$$x + y = 12 \qquad -----\ (\mathbf{i})$$

and

$$-2\sqrt{xy} = -6\sqrt{3}$$

Divide both sides by -2

$$\sqrt{xy} = 3\sqrt{3}$$

Square both sides

$$xy = 27 \qquad ------ \textbf{(ii)}$$

From (i)

$$y = 12 - x \qquad ------ \textbf{(iii)}$$

Substitute equation (iii) in equation (ii),

$$x(12 - x) = 27$$

$$12x - x^2 = 27$$

$$x^2 - 12x + 27 = 0$$

$$(x - 9)(x - 3) = 0$$

Therefore, either

$$x - 9 = 0$$

$$x = 9$$

or

$$x - 3 = 0$$

$$x = 3$$

We now need to find the corresponding values for y, thus

when $x = 9$, from (iii)

$$y = 12 - 9 = 3$$

and when $x = 3$, from (iii)

$$y = 12 - 3 = 9$$

Hence, the square root of the $12 - 6\sqrt{3}$ are

$$\sqrt{x} - \sqrt{y} = \sqrt{9} - \sqrt{3}$$

$$= 3 - \sqrt{3}$$

and

$$\sqrt{a} + \sqrt{b} = \sqrt{3} - \sqrt{9}$$

$$= \sqrt{3} - 3$$

$$\therefore \sqrt{\mathbf{(12 - 6\sqrt{3})}} = \pm\mathbf{(3 - \sqrt{3})}$$

Since the length AC can only be a positive value, the only answer here is

$$\boldsymbol{\overline{AC} = 3 - \sqrt{3}\ cm}$$

$$\boldsymbol{\because = \sqrt{3} - 3\ \ is\ negative}$$

END OF WORKED EXAMPLES

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