Topic 4: Indices and Logarithms

Lecture Notes:

section 3.1 Indices

section 3.2 Logarithms

Jacques Text Book (edition 4):

section 2.3 & 2.4 Indices & Logarithms

INDICES

Any expression written as $\mathbf{a^n}$ is defined as the variable **a** raised to the power of the number **n**

n is called a power, an index or an exponent of **a**

e.g. where n is a positive whole number,

$a^1 = a$
$a^2 = a \times a$
$a^3 = a \times a \times a$
$a^n = a \times a \times a \times a\ldots\ldots n$ times

Indices satisfy the following rules:

1) where n is *positive whole* number

$\mathbf{a^n = a \times a \times a \times a \ldots\ldots} \boldsymbol{n}$ **times**

e.g. $2^3 = 2 \times 2 \times 2 = 8$

2) *<u>Negative</u>* powers…..

$$\mathbf{a^{-n}} = \frac{1}{\boldsymbol{a^n}}$$

e.g. $\mathbf{a^{-2}} = \frac{1}{\mathbf{a}^2}$

e.g. where a = 2

$2^{-1} = \frac{1}{2}$ **or** $2^{-2} = \frac{1}{2 \times 2} = \frac{1}{4}$

3) A ***Zero*** power

$\mathbf{a^0 = 1}$

e.g. $8^0 = 1$

4) A ***Fractional*** power

$$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

e.g. $$9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt{9} = 3$$

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

All indices satisfy the following rules in mathematical applications

Rule 1

$$a^m . a^n = a^{m+n}$$

e.g. $2^2 . 2^3 = 2^5 = 32$

Rule 2

$$\frac{a^m}{a^n} = a^{m-n}$$

e.g. $\frac{2^3}{2^2} = 2^{3-2} = 2^1 = 2$

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note: if m = n,

then $\frac{a^m}{a^n} = a^{m-n} = a^0 = 1$

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note: $\frac{a^m}{a^{-n}} = a^{m-(-n)} = a^{m+n}$

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note: $\frac{a^{-m}}{a^n} = a^{-m-n} = \frac{1}{a^{m+n}}$

Rule 3

$$\mathbf{(a^m)^n = a^{m.n}}$$

e.g. $(2^3)^2 = 2^6 = 64$

Rule 4

$$\mathbf{a^n . b^n = (ab)^n}$$

e.g. $3^2 \times 4^2 = (3\times4)^2 = 12^2 = 144$

Likewise,

$$\frac{\mathbf{a^n}}{\mathbf{b^n}} = \left(\frac{\mathbf{a}}{\mathbf{b}}\right)^{\mathbf{n}} \qquad \text{if } b \neq 0$$

e.g.

$$\frac{6^2}{3^2} = \left(\frac{6}{3}\right)^2 = 2^2 = 4$$

Simplify the following using the above Rules:

1) $b = x^{1/4} \times x^{3/4}$

2) $b = x^{2} \div x^{3/2}$

3) $b = (x^{3/4})^{8}$

4) $b = \dfrac{\mathbf{x}^{2}\,\mathbf{y}^{3}}{\mathbf{x}^{4}\,\mathbf{y}}$

Logarithms

A Logarithm is a mirror image of an index

If $m = b^n$ then $\log_b m = n$

The log of m to base b is n

If $y = x^n$ then $n = \log_x y$

The log of y to the base x is n

e.g.

$1000 = 10^3$ then $3 = \log_{10} 1000$

$0.01 = 10^{-2}$ then $-2 = \log_{10} 0.01$

Evaluate the following:

1) $x = \log_3 9$

the log of m to base b = n then $m = b^n$

the log of 9 to base 3 = x then

⇨ $9 = 3^x$

⇨ $9 = 3 \times 3 = 3^2$

⇨ $x = 2$

2) $x = \log_4 2$

the log of m to base b = n then $m = b^n$

the log of 2 to base 4 = x then

⇨ $2 = 4^x$

⇨ $2 = \sqrt{4} = 4^{1/2}$

⇨ $x = 1/2$

Using Rules of Indices, the following rules of logs apply

1) $\log_b(x \times y) = \log_b x + \log_b y$

eg. $log_{10}(2\times3) = log_{10} 2 + log_{10} 3$

2) $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

eg. $log_{10}\left(\frac{3}{2}\right) = log_{10} 3 - log_{10} 2$

3) $\log_b x^m = m. \log_b x$

e.g. $log_{10} 3^2 = 2\,log_{10} 3$

From the aboverules, it follows that

(1) $\mathbf{\log_b 1 = 0}$

(since => $1 = b^x$, hence x must=0)

e.g. $\log_{10}1=0$

and therefore,

$$\log_b\left(\frac{1}{x}\right) = -\log_b x$$

e.g. $\log_{10}\left({}^1/_3\right) = -\log_{10}3$

(2) $\mathbf{\log_b b = 1}$

(since => $b = b^x$, hence x must = 1)

e.g. $\log_{10} 10 = 1$

(3) $$\log_b\left(\sqrt[n]{x}\right) = \frac{1}{\mathbf{n}}\log_b x$$

A Note of Caution:

- All logs must be to the same base in applying the rules and solving for values

- The most common base for logarithms are logs to the base 10, or logs to the base **e** (e = 2.718281…)

- Logs to the base **e** are called Natural Logarithms

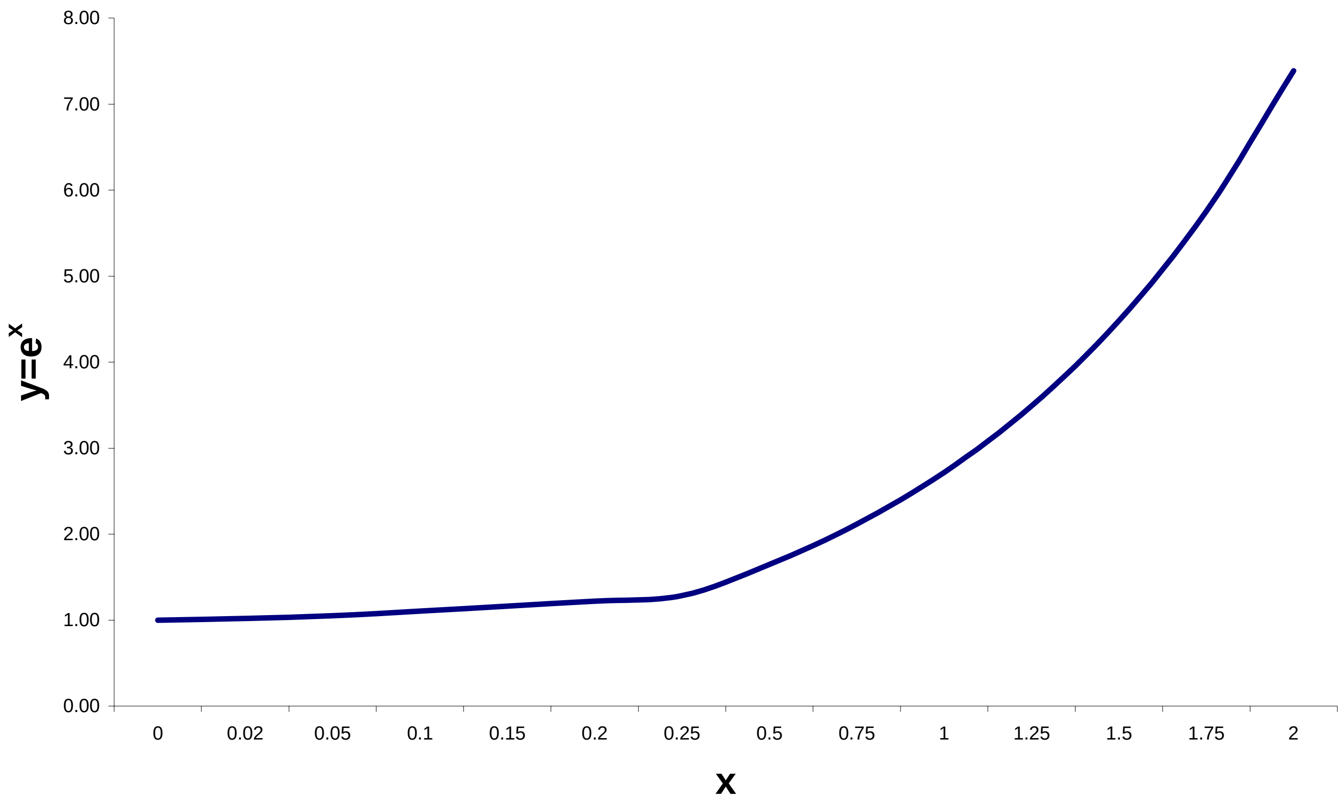
$$\log_e x = \ln x$$

If $y = \exp(x) = e^x$

Then $\log_e y = x$ or $\ln y = x$

Features of y = e^x

- non-linear
- always positive
- as ↑ x get ↑ y and ↑ slope of graph (gets steeper)



Logs can be used to solve algebraic equations where the unknown variable appears as a power

An Example : Find the value of x

$$200(1.1)^x = 20000$$

Simplify

divide across by 200

⇨ $(1.1)^x = 100$

1. **to find x, rewrite equation so that it is no longer a power**

⇨ Take logs of both sides

⇨ $\log(1.1)^x = \log(100)$

⇨ rule 3 => $x.\log(1.1) = \log(100)$

2. Solve for x

$$x = \frac{\mathbf{log}(100)}{\mathbf{log}(1.1)}$$

no matter what base we evaluate the logs, providing the same base is applied both to the top and bottom of the equation

3. Find the value of x by evaluating logs using (for example) base 10

$$x = \frac{\mathbf{log}(100)}{\mathbf{log}(1.1)} = \frac{2}{0.0414} = 48.32$$

4. Check the solution

$200(1.1)^{x} = 20000$

$200(1.1)^{48.32} = 20004$

Another Example: Find the value of x

$5^x = 2(3)^x$

1. **rewrite equation so x is not a power**

⇨ Take logs of both sides

⇨ $\log(5^x) = \log(2\times 3^x)$

⇨ rule 1 => $\log 5^x = \log 2 + \log 3^x$

⇨ rule 3 => $x.\log 5 = \log 2 + x.\log 3$

2. **Solve for x**

$x\,[\log 5 - \log 3] = \log 2$

rule 2 => $x[\log\left(\frac{5}{3}\right)] = \log 2$

$$x = \frac{\log(2)}{\log\left(\frac{5}{3}\right)}$$

3. **Find the value of x by evaluating logs using (for example) base 10**

$$x = \frac{\log(2)}{\log\left(\frac{5}{3}\right)} = \frac{0.30103}{0.2219} = 1.36$$

4. **Check the solution**

$$5^x = 2(3)^x \Rightarrow 5^{1.36} = 2(3)^{1.36} \Rightarrow 8.92$$

An Economics Example 1

$Y = f(K, L) = \quad A K^{\alpha} L^{\beta}$

$Y^{*} = f(\lambda K, \lambda L) = A (\lambda K)^{\alpha} (\lambda L)^{\beta}$

$Y^{*} = A K^{\alpha} L^{\beta} \lambda^{\alpha} \lambda^{\beta} = Y \lambda^{\alpha+\beta}$

$\alpha+\beta = 1$ **Constant Returns to Scale**

$\alpha+\beta > 1$ **Increasing Returns to Scale**

$\alpha+\beta < 1$ **Decreasing Returns to Scale**

Homogeneous of Degree r if:

$f(\lambda X, \lambda Z) = \lambda^{r} f(X, Z) = \lambda^{r} Y$

Homogenous function if by scaling all variables by λ, can write Y in terms of λ^{r}

An Economics Example 2

National Income = £30,000 mill in 1964.
It grows at 4% p.a.
Y = income (units of £10,000 mill)

1964: $Y = 3$
1965: $Y = 3(1.04)$
1966: $Y = 3(1.04)^2$
1984: $Y = 3(1.04)^{20}$

Compute directly using calculator or

Express in terms of logs and solve

1984: $\log Y = \log\{3\times(1.04)^{20}\}$

$$\log Y = \log 3 + \log\{(1.04)^{20}\}$$

$$\log Y = \log 3 + 20.\log(1.04)$$

evaluate to the base 10

$$\log Y = 0.47712 + 20(0.01703)$$

$$\log Y = 0.817788$$

Find the anti-log of the solution:

$$Y = 6.5733$$

In 1984, Y = £65733 mill

Topic 3: Rules of Indices and Logs

Some Practice Questions:

1. Use the rules of indices to simplify each of the following and where possible evaluate:

(i) $\dfrac{3^5.3^2}{3^6}$

(ii) $\dfrac{5^4.6^{-2}}{5^2}$

(iii) $\dfrac{x^6.x^{-2}}{x}$

(iv) $\left(4x^3\right)^2$

(v) $\dfrac{xy^2}{x^2}$

(vi) $\dfrac{15x^6}{3x^4 5x^2}$

2. Solve the following equations:

(i) $\log_4 64 = x$

(ii) $\log_3\left(\frac{1}{27}\right) = x$

(iii) $x = 4\ln 10$

(iv) $5^x = 25$

(v) $4e^x = 100$

(vi) $e^{2x-1} = 100$