Mathematical Model: From Cell Membrane to Simple Neural Network

Sebastian König

Abstract

Researches study our neural system in different levels: from conponents of a single cell, such as membrane, to the connection of two cells (synapsis), and further to the complex network of the whole brain. This summary trying to find a straight pathway of the modeling from sub-cellcular level to simple neural network. Please refer [1] for more detail in.
Keywords: lorentz integral transformation; pionless EFT; bound-state problem

1 Membrane

Membrane has roughly two parts that are vital in the neural activity: phospholipid bilayer and channels (FIG. **??**). phospholipid is permeterbale for water, oil etc., yet impermmitable for ions. This attribution is similar to semi-permeable membrane. Ions carry charges, if the charge densities of the two sides of phospholipid bilayer (intra celluar space and extra-cellcular space) are different, there will be a votage across cell membrane (FIG. **??**). In all, because of the existance of phospholipid bilayer, the cell has a function similar to conductance

$$cV = Q \tag{1}$$

where c is the conductance per unit area, Q is the charge density differences and V is the votage accross the cell. The votage of the extra-cellcular space is set to be zero, therefore V here is also the votage of intra-cellcular space, which is roughly -70mV to -50mV as measured by the experiment.

There are two types of channel: ion channels and metabolic channel. The effect of metabolic channels are more indirect and complex, which is not the center concern of this summary. Ion channel behaviours as the gate of the membrane. It allows only certain types of ions to pass though the membrane, thus produce inward or outward cuurent flow. For example, Na-gated channel is only permitterble to Na^+, when the channel is open (FIG. **??**). One good approximation of the channel is resistence, govern by the Ohm's law

$$I = G_i(V - V_i) \tag{2}$$

where G_i is the conductance which can depends on the votage and time, and V_i is the equivibrilum votage (constant for a certain type of channel) results from the balance of electric force and diffusion. Index i here means ith type of ion channel. Positive I means outward current.

2 Single Cell

Differentiate on both side of equation (1) and use equation (2), we have an equation for single cell

$$c\frac{dV}{dt} = -G_L(V - V_L) - I_e \tag{3}$$

where I_e is the extra-induced current, such as the experimentist inject ions to the cell (negtive I_e). Index L is leakage channel, which is a mathematically equvilent channel of partial function of some types of channel (largely fall into K-selective channel). The rest of effects of channels can be simplified as the following statement:

- If $V > V_{th}$, $V = 50$mV, then quickly fall to V_{res}.

This effect is often introduced by Na-selective channels. Typical value of V_{res} is $-65mV$. There are some choices for V_{th}, but values like $-30mV$ is sufficient to fit a good result. When this statement is triggered, we say the neuron fired, or it made a spike. This single-cell model also be named Integrate and Firing Model. This model contains one resistence (leakage channel), one conductance (membrane), one external current source (I_e), and voltage source (FIG. 3). There are lots more sophiscated models, such as HH model, which replace the statement with more realistic channels as additional terms in equation (3).

Equation 3 can be collected with a more susinct form

$$\tau \frac{dV}{dt} = -V + V_\infty \tag{4}$$

where $\tau = c/G_L$ and $V_\infty = (V_L - I_e)/G_L$. If G_L here is a constant, solution of equation 4 is

$$V = (V_L - V_\infty)e^{-t/\tau} + V_\infty \tag{5}$$

The firing pattern is shown in FIG. **??**

Information is stored in the firing rate of neurons instead of the exact value of votage. Firing rate is defined as the number of spiks in a unit time. The time spacing between two spikes can be calculated from equation (4) by calculate the time that V grows from V_{res} to V_{th}. The inverse of time spacing is firing rate, as show in FIG. **??**. In all, we could conclude the relation with

$$v = F[I_e] \tag{6}$$

where v is the firing rate, and F is some function has the shape in FIG. **??**

3 Synapse

Before moving to neural network, we need to understand how two neurons connect each other by synapse. FIG. **??** shows the biological structure of the synapse. When the pre-neuron is fired, the depolyzation will open the Ca-channel, induces an inward flow of Ca^{2+}. This Ca^{2+} will release the transmitters in the presynapse. Transmitter in the synapse cleft then attarch to particular recepters in the post-synapse, open ion-channel to allow Na^+ flow into the post-synapse. This final lead to the depolyzation of the post-neuron.

The current flow into the post-neuron can be descripted by

$$I_{syn} = -G_{syn}(t)(V - V_{syn}), \tag{7}$$

where V_{syn} is about 0 mV, which is the reversal potential of the post-synapse. Conductance $G_{syn}(t)$ is shown in FIG. **??** If there's no transmitters in the synapse cleft, ion channels in the post-synapse are closed, $G_{syn}(t) = 0$ mV. A spike of the presynapse will suddenly release a large amount of transmitters, therefore largely increase $G_{syn}(t)$. Then the transmitters will gradually either diffuse away or be decomposed by glials, which lead to the slow decay of the conductance. The conductance induced by single spike is often approximated by the exponential function

$$K(t) = G_{max}e^{-t/\tau}. \tag{8}$$

The whole conductance under a train of spikes is

$$G(t) = \sum_i K(t)\delta_{t,t_i} \tag{9}$$

where there is a spike at time t_i. Equation (9) can be simplified with the spike train $S(t) = \sum_i \delta(t - t_i)$

$$G(t) = \int d\tau K(\tau)S(t - \tau) \tag{10}$$

4 Simple Neural Network

Consider a network with only two neurons attarched FIG. **??**. The working status of each neuron is solely stated by its firing rate. So we need to findout the firing-rate relation between neurons. For the single neuron, the only source of external current is from its pre-neuron, i.e. $v = F[I_{syn}]$ from equation (6). Yet I_{syn} followed by equation (7) is still too complex to be pratical. Hence we further throw away the votage term, and treat $G_{syn}(t)$ as votage independent,

$$I_{syn} = \int d\tau K(\tau)S(t - \tau) \tag{11}$$

we could extract a factor w so that the rest of $U(\tau) = K(\tau)/w$ can be normalized,

$$\int d\tau U(\tau) = 1 \tag{12}$$

The convolution of a narrow (correnspond to short time of non-zero $K(\tau)$) window $U(\tau)$ with spike train $S(\tau)$ is exactly the firing rate of the presynapse u. Overall, we have

$$v = F[wu]. \tag{13}$$

This equation reveals the simple relation of the network. Larger w means the two neurons are more strongly connect, only a slow firing rate of the pre-neuron is enough for triggering the post-neuron. This biologically could mean there are more channels in the post-synapse. Different presynapse will attach to different sites of the postsynapse. Furthermore, the post-neual layer can also attarch to each other as shown in FIG. . The overall effect is the summation of all their current.

$$\boldsymbol{v} = F[W\boldsymbol{u} + M\boldsymbol{v}], \tag{14}$$

where W, M are matrices, arrays are written as bold letters. This model is called rate model, which is the fundation of the lots of deep learning architechture.

5 Conclusion

Membrane of the neuron is a conbination of resistor and capacity. The conductances are different under different membrane and more importantly, different to different types of channels. This attribution allows the neuron to fire, and soon go back to the rest potential. A spike from the pre-neuron will triggered a bit depolyzation of the post-neuron. Once there is enough depolyzation, the post-neuron will also make spikes. Despite the complicated procedure for generating a spike, the exact voltage is unimportant. The real information is stored in the frequency of the spikes, which can be described using a single number – firing rate. By carefully simplify the synapse conductance, one could obtain a relation between the firing rate of neurons, thus have a neuron network. This neuron network also lies in the basis current deep network algorithms.

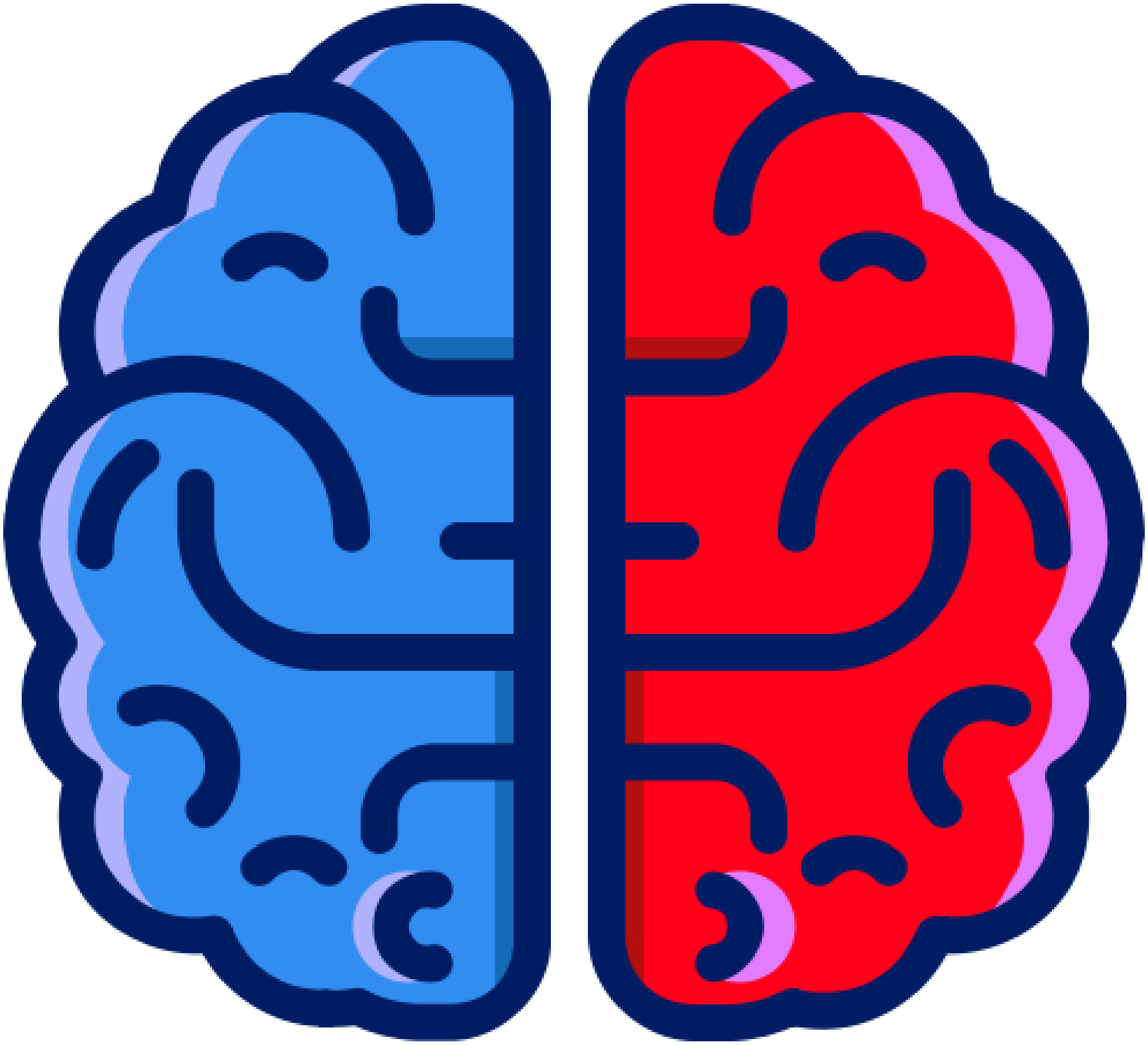

Figure 1: results of problem2_1

References

[1] MIT OpenCourseWare: Introduction to Neural Computation