Theory for Grid Cell

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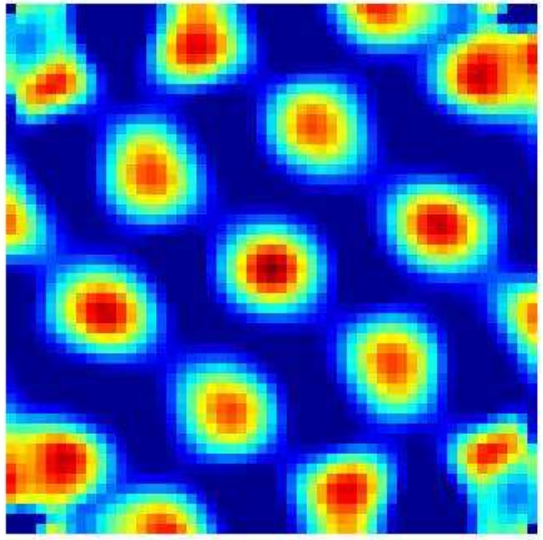


Image source: Wiki – Grid Cell

0. Navigation

No individual butterfly completes the entire round trip. Four generations are involved in the annual cycle.

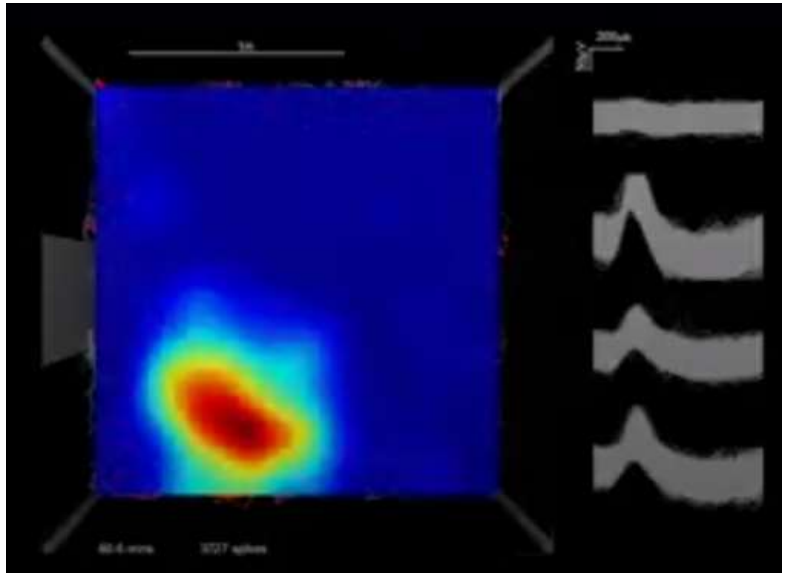
How exactly animals navigate?

Image source: wiki

0. Cells

There are several cell types about navigation

Place Cell: 4:11
https://www.youtube.com/watch?v=km4203tZXnY

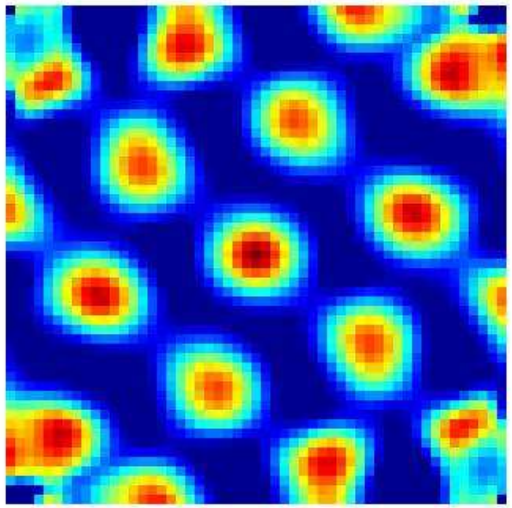


0. Cells

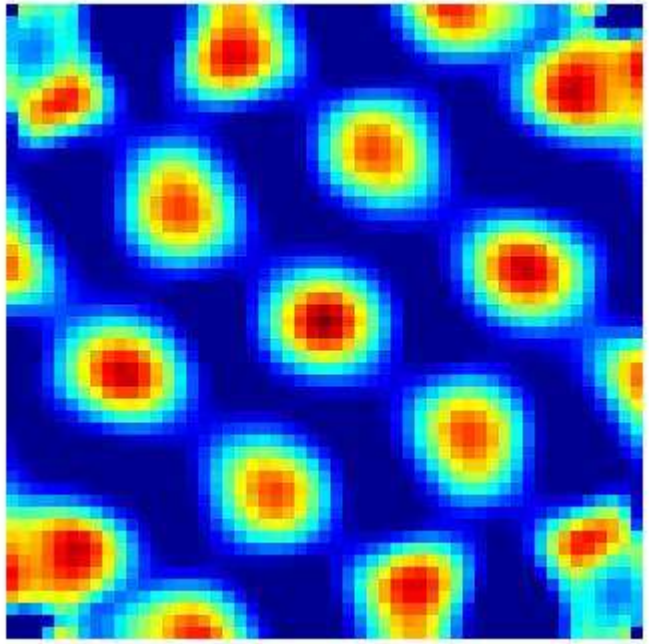
There are several cell types about navigation

Grid Cell: 1:55
https://www.youtube.com/watch?v=CQPswbIuCkk

Autocorrelation



Hexagonal pattern!



Why Grid?

Image source: Wiki – Grid Cell

Content

1. Path way 1: Mechanistic Model
2. Path way 2: Normative model
    1. 1-layer model – grid cell for encoder
    2. RNN model – grid cell for navigation
3. Unified Theory
4. Conclusion and Perspective

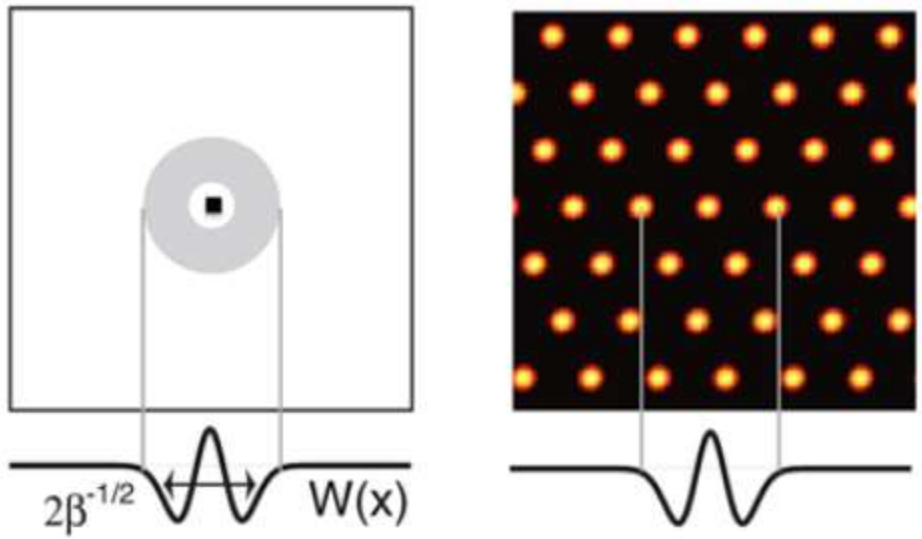
1. Mechanistic Model

$$\tau \frac{ds_i}{dt} + s_i = f\left[\sum_j W_{ij} s_j + B_i\right]$$

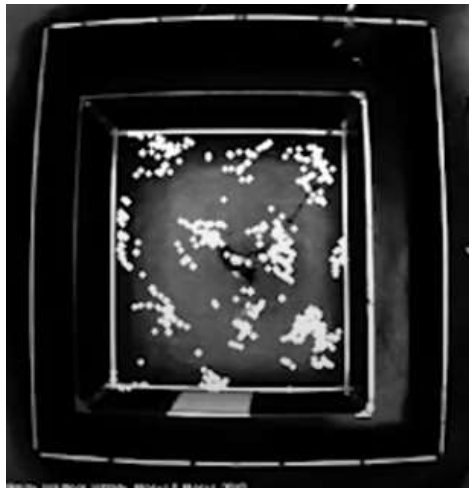
W = ?

[1] Burak, Y., & Fiete, I. R. (2009). 10.1371/journal.pcbi.1000291

1. Mechanistic Model



?



Firing rate of all Neurons in one location

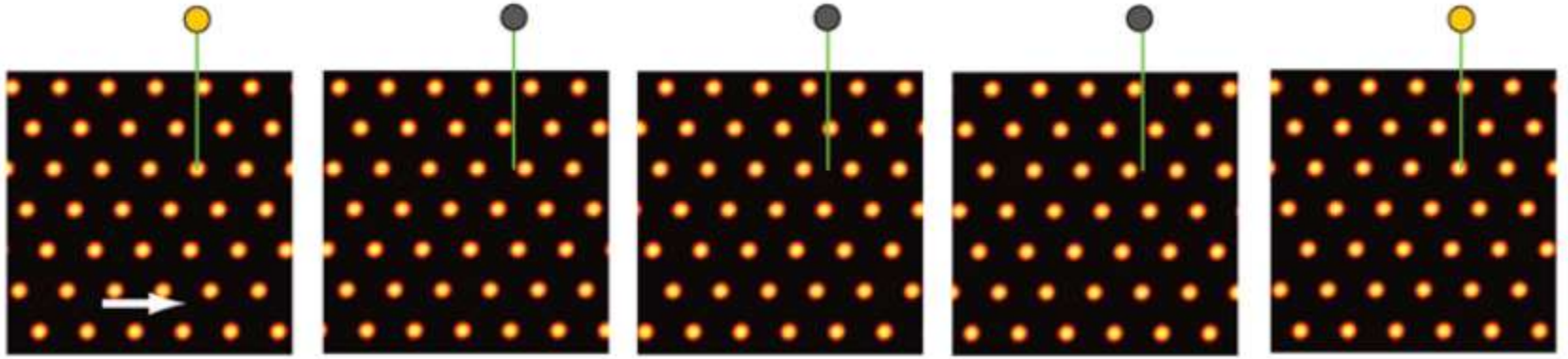
Firing rate of one neuron in all location

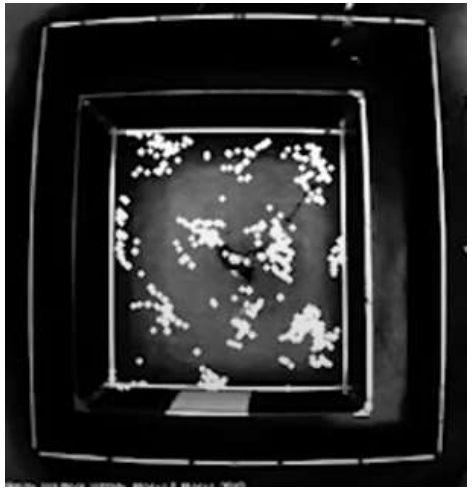
$$\tau \frac{ds_i}{dt} + s_i = f\left[\sum_j W_{ij} s_j + B_i\right]$$

[1] Burak, Y., & Fiete, I. R. (2009). 10.1371/journal.pcbi.1000291

1. Mechanistic Model

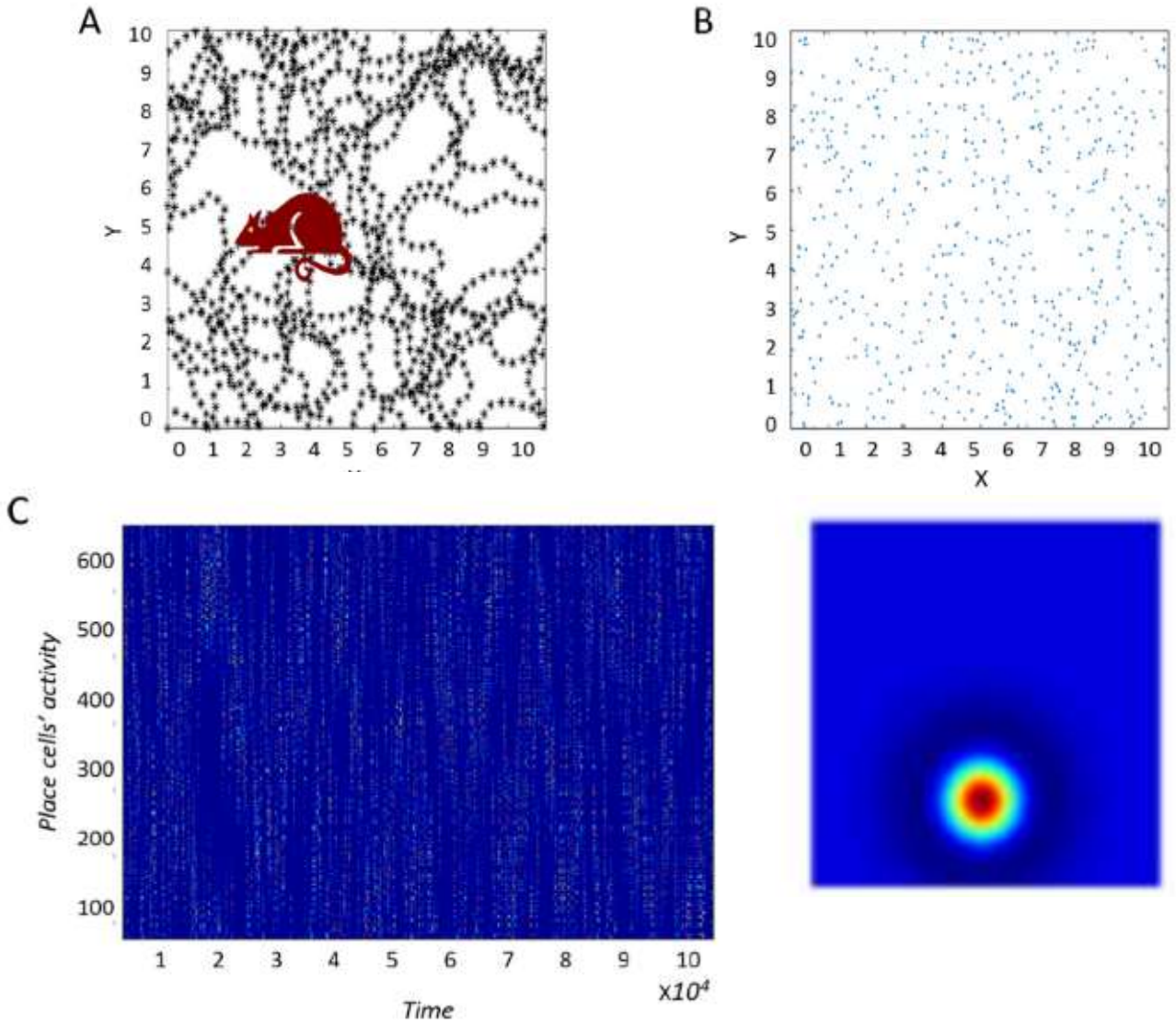
Mechanistic model tells us how to obtain the hexagonal pattern, but not why





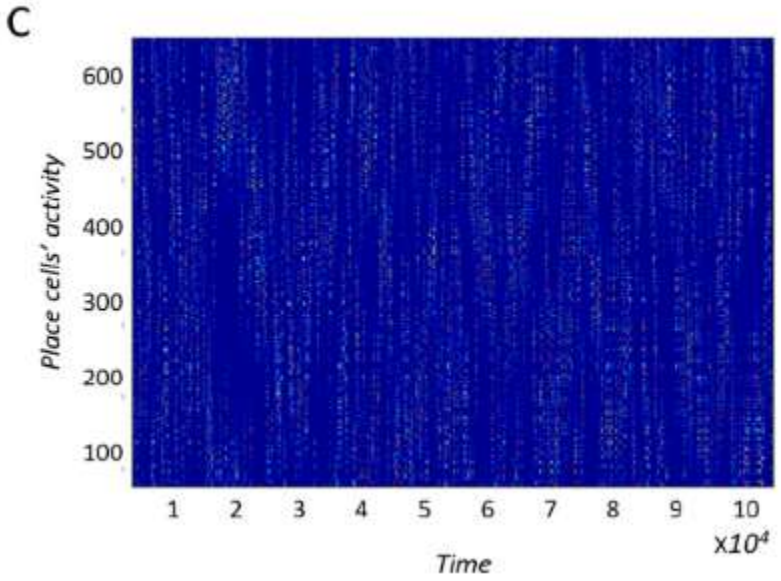
[1] Burak, Y., & Fiete, I. R. (2009). 10.1371/journal.pcbi.1000291

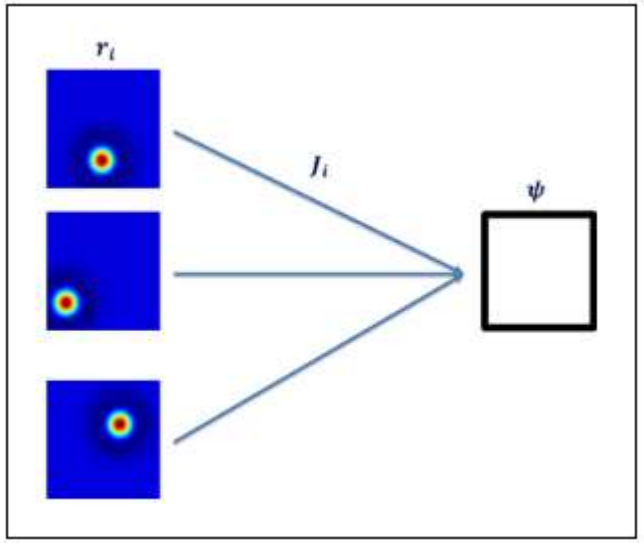
2. Normative model -- 1-layer model



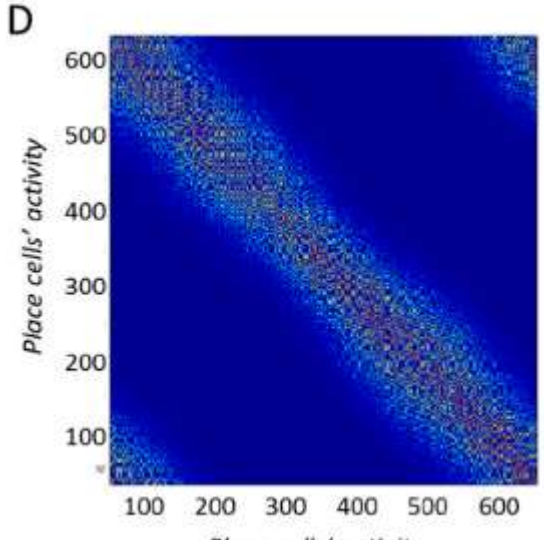
Dordek, Y., Soudry, D., Meir, R., & Derdikman, D. (2016). 10.7554/eLife.10094

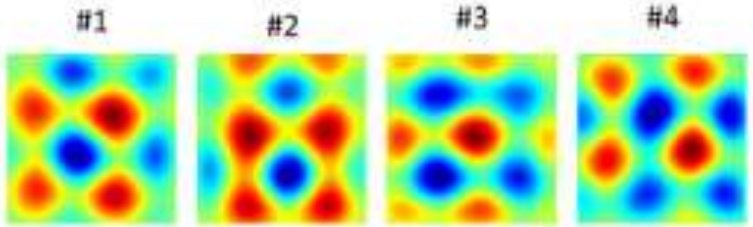
2. Normative model -- 1-layer model





Hebb Rule





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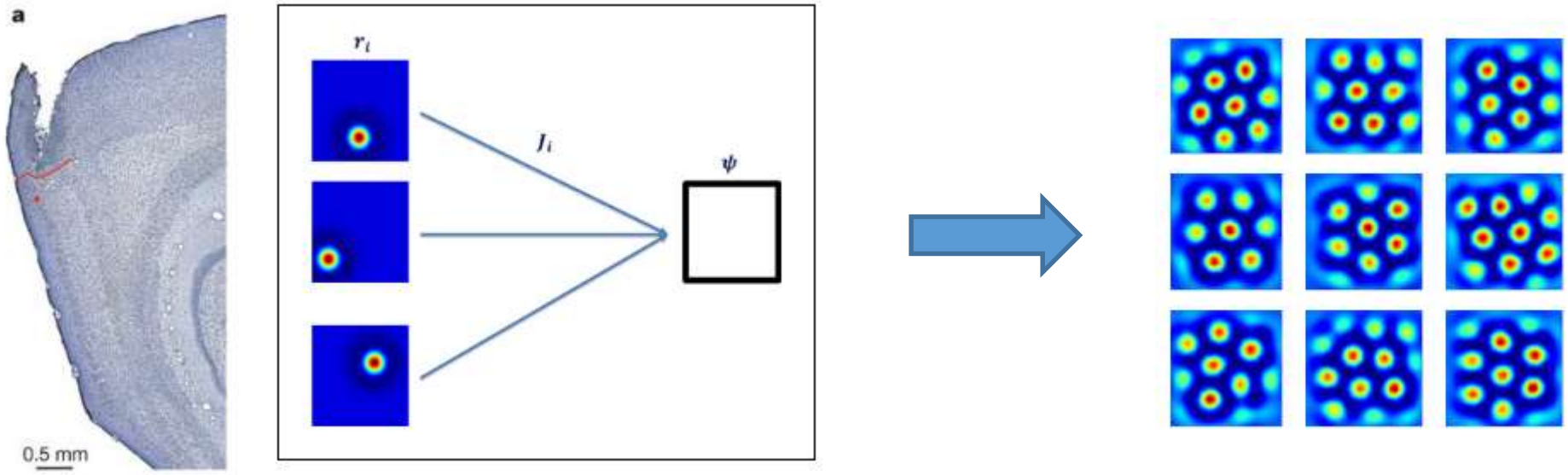
2. Normative model -- 1-layer model

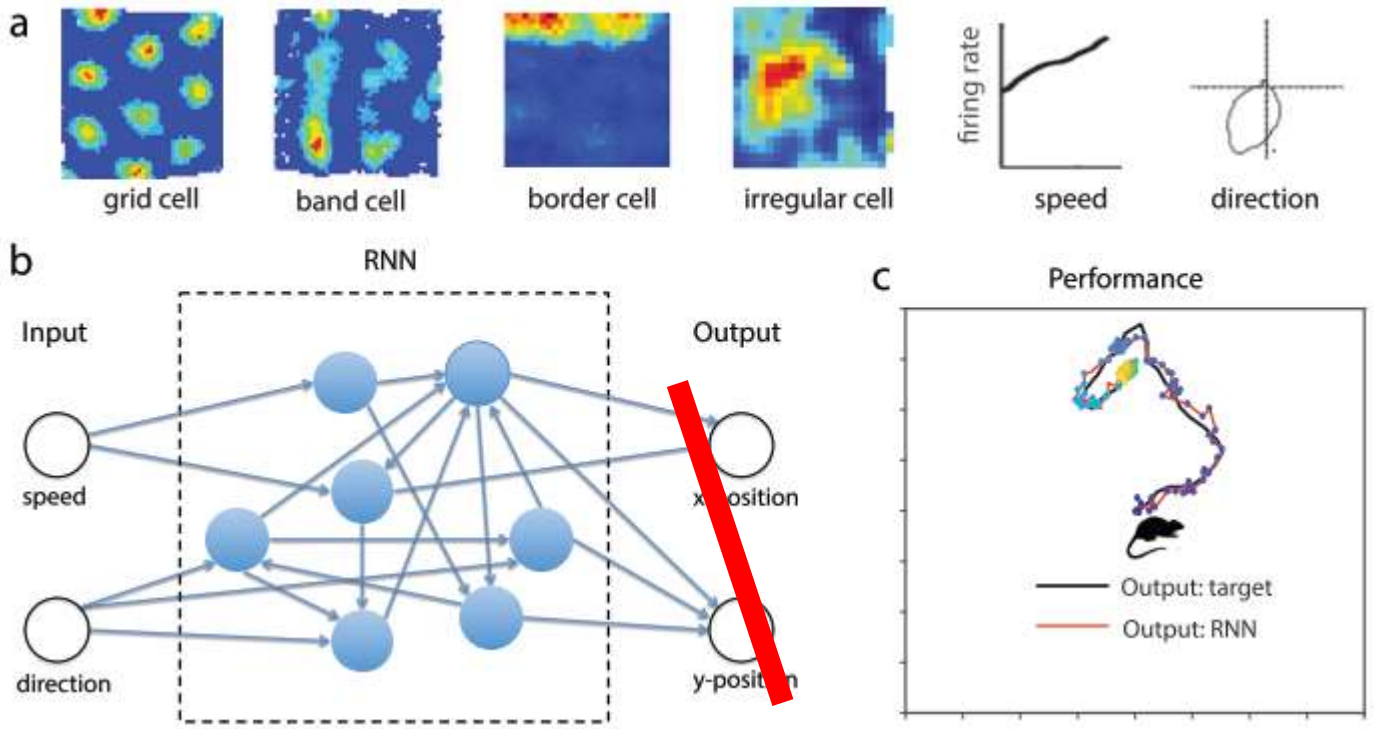


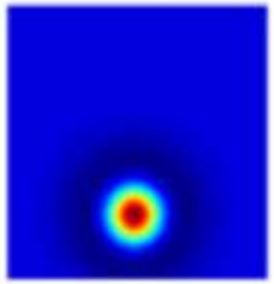
Near hippocampus →Most synapse are exitary

Grid pattern is used for encode the place cell information

Dordek, Y., Soudry, D., Meir, R., & Derdikman, D. (2016). 10.7554/eLife.10094

2. Normative model -- RNN model

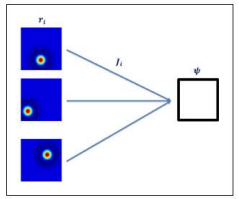
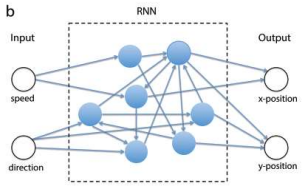




Grid cell is the hidden neurons while learning where am I.

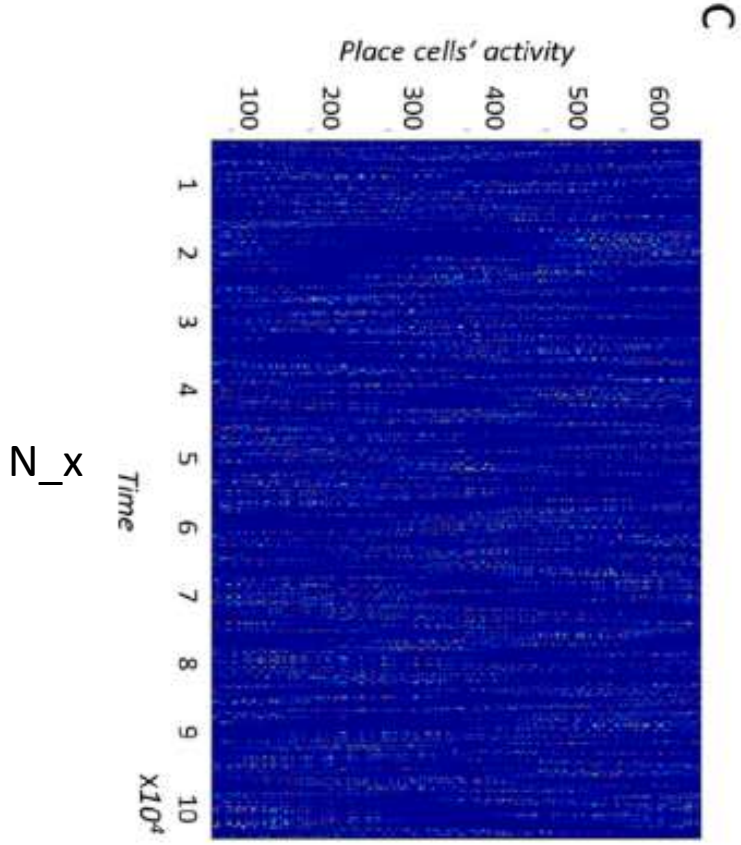
Cueva, C. J., & Wei, X. X. (2018)

2. Conclusion of Normative model

Network	Input	Output	Why Grid
1-layer network 	Place cell information	Encoded feature	Grid cell is used for efficient encoding the place cell information
RNN 	Velocity	Place cell information	Grid cell is the hidden units when learn the position from velocity

3. Unified Theory

$$\mathcal{E}(G, W) = \|P - \hat{P}\|_F^2, \text{ where } \hat{P} = GW.$$



$$P \in \mathbb{R}^{n_x \times n_P}$$

$$G \in \mathbb{R}^{n_x \times n_G}$$

$$W \in \bar{\mathbb{R}}^{n_G \times n_P}$$

Column of G – Receptive field of a grid cell

Minimize the loss function – The columns of G has hexagonal pattern

Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

3. Unified Theory

$$\mathcal{E}(G, W) = \|P - \hat{P}\|_F^2, \text{ where } \hat{P} = GW.$$

G's columns are orthonormal.

$$\underset{W}{\operatorname{argmin}}\, \mathcal{E}(G, W) = (G^T G)^{-1} G^T P.$$

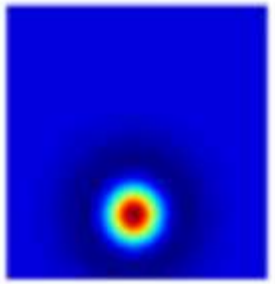
$$\mathcal{L} = \operatorname{Tr}\left[G^T \Sigma G - \lambda(G^T G - I)\right], \quad \Sigma = P P^T$$

Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

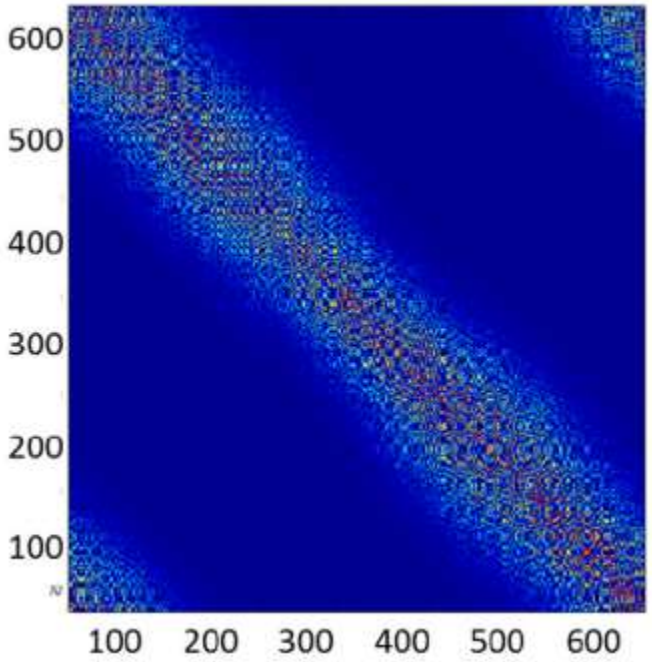
3. Unified Theory

$$\mathcal{L} = \mathrm{Tr}\left[G^T \Sigma G - \lambda(G^T G - I)\right], \ \Sigma = PP^T$$

Rotational Symmetry



Place cell is uniformly distributed in the whole space

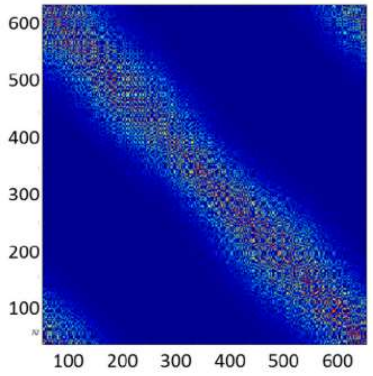


Circulate Matrix

G is the linear combination of eigenvectors (with max eigenvalue) of Σ

Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

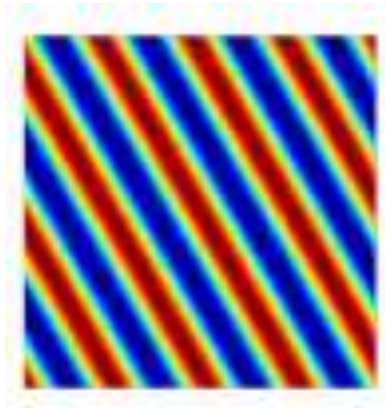
3. Unified Theory



N_x

G(x)

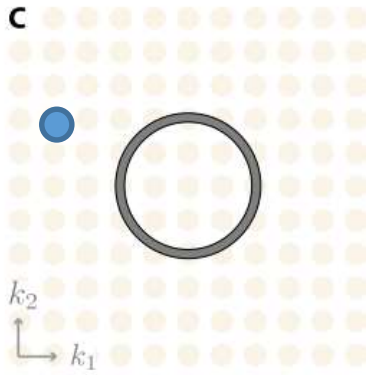
Go back to 2D



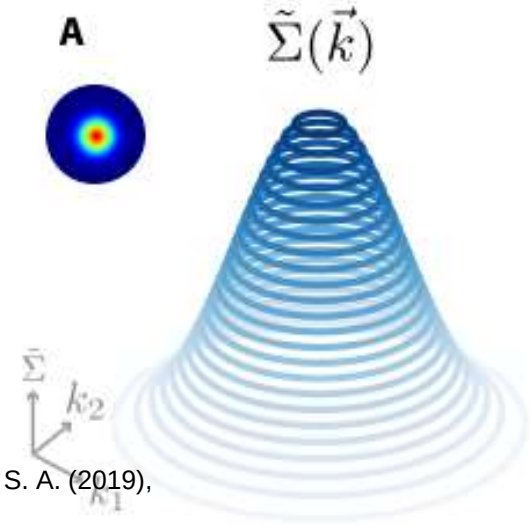
G(x)

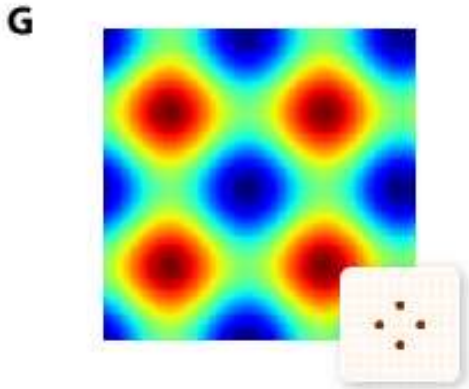
DFT

because of the rotational invariant of the place cell



G(k)





Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

3. Unified Theory

$$\mathcal{L} = \mathrm{Tr}\left[G^T \Sigma G - \lambda(G^T G - I)\right],$$

$$\mathcal{L} = g^T \Sigma g + \lambda(1 - g^T g) + \sigma(g),$$

$$\mathcal{L}[g(\vec{x})] = \iint_{\vec{x},\vec{x}'} g(\vec{x})\Sigma(\vec{x} - \vec{x}')g(\vec{x}') + \lambda\left(1 - \int_{\vec{x}} g^2(\vec{x})\right) + \int_{\vec{x}} \sigma(g(\vec{x})).$$

Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

3. Unified Theory

$$\mathcal{L}[g(\vec{x})] = \iint_{\vec{x},\vec{x}'} g(\vec{x})\Sigma(\vec{x}-\vec{x}')g(\vec{x}') + \lambda\left(1 - \int_{\vec{x}} g^2(\vec{x})\right) + \int_{\vec{x}} \sigma(g(\vec{x})).$$

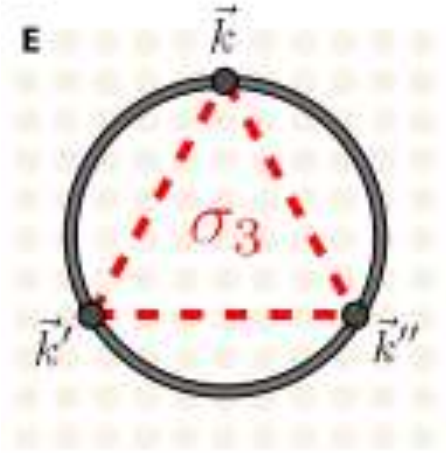
$$\sigma(g) \approx \sigma_0 + \sigma_1 g + \sigma_2 g^2 + \ \sigma_3 g^3$$

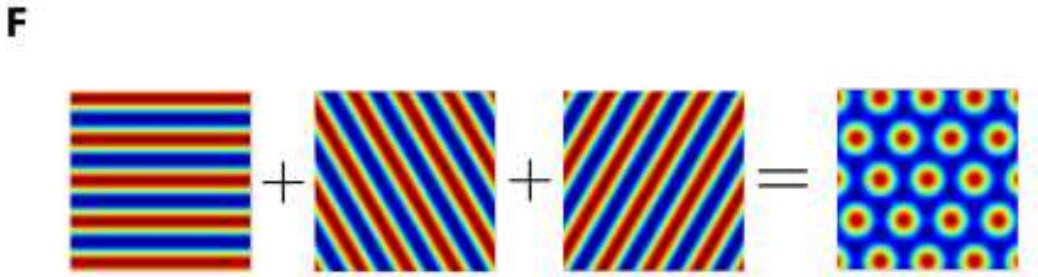
$$\tilde{\mathcal{L}}[\tilde{g}(\vec{k})] \approx \int_{\vec{k}} |\tilde{g}(\vec{k})|^2 \tilde{\Sigma}(\vec{k}) + \tilde{\lambda}\left(1 - \int_{\vec{k}} |\tilde{g}(\vec{k})|^2\right)$$
$$+ [\sigma_0 + \sigma_1 \tilde{g}(\vec{0}) + \sigma_2 \int_{\vec{k}} |\tilde{g}(\vec{k})|^2 + \sigma_3 \iiint_{\vec{k},\vec{k}',\vec{k}''} \tilde{g}(\vec{k})\tilde{g}(\vec{k}')\tilde{g}(\vec{k}'')\delta(\vec{k}+\vec{k}'+\vec{k}'')]. \quad (8)$$

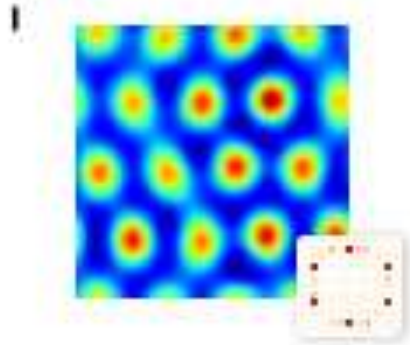
Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

3. Unified Theory

$$\mathcal{L}_{\text{int}} = \iiint_{\vec{k},\vec{k}',\vec{k}''} \tilde{g}(\vec{k})\tilde{g}(\vec{k}')\tilde{g}(\vec{k}'')\delta(\vec{k}+\vec{k}'+\vec{k}'').$$





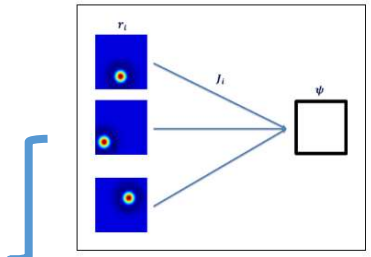


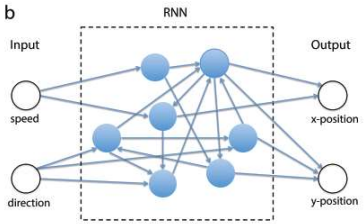
Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

3. Unified Theory

Unifying normative and mechanistic models of grid cells

$$\mathcal{E}(G, W) = \|P - \hat{P}\|_F^2,$$





$$\mathcal{L} = g^T \Sigma g + \lambda(1 - g^T g) + \sigma(g),$$

$$\frac{d}{dt}g = \begin{cases} -\lambda g + \Sigma g + \mu & g > 0 \\ -\lambda g + \sigma\left[\Sigma g + \mu\right] & g = 0 \end{cases}$$

$$\tau \frac{ds_i}{dt} + s_i = f\left[\sum_j W_{ij} s_j + B_i\right]$$

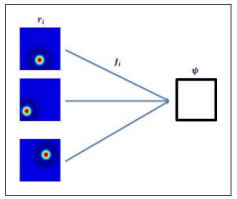
Sorscher, B., Mel, G. C., Ganguli, S., & Ocko, S. A. (2019), (NeurIPS), 1–18.

4. Conclusion

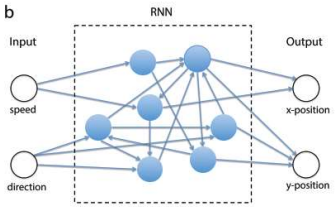
3 models + unified theory

1. Mechanistic model, we know how to construct the wiring of the neurons to produce the hexagonal pattern

1. 1-layer model, hexagonal pattern occurs as the weight of an encoder. Grid cell is used for encoding the place cell



1. RNN, grid cell is the hidden unit when the network learn the position of the velocity

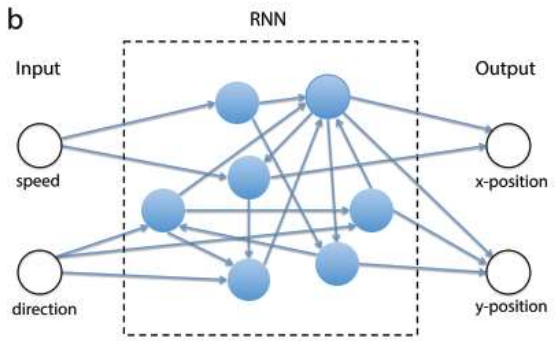


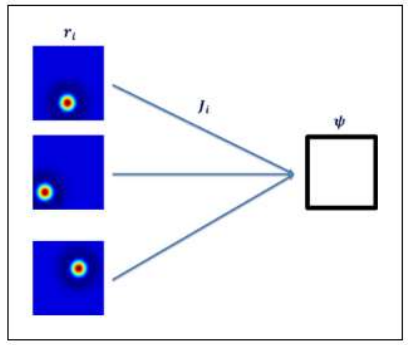
1. Unified theory. All the above model are different aspects of minimizing loss function. Hexagonal pattern is the linear combination of the plane waves

$$\mathcal{E}(G, W) = \|P - \hat{P}\|_F^2,$$

5. Discussion

1. What is the biological meaning of the grid cell?





a. Grid cell occurs as unit in the network. This has very clear biological meaning.
b. The place cell is inside the hippocampus, grid cell is near the hippocampus.

2. What can we learn from the methodology of this research?
a. Architecture, Learning rule and Objective function! Use Machine learning to interpret the biological system
b. Interpret (at least partially) the Machine learning using Math!

Thanks!