Mathematical Modeling: From Membrame to Simple Neural Network

Sebastian König

Abstract

Theoretical neural scientist study our neural system in different level: from conponents of a single cell, such as memberame, synaps, to the complex network of the whole brain. This summary trying to find a straight pathway from the modeling of sub-cellcular level to simple neural network. Please refer more detail in []

Keywords: lorentz integral transformation; pionless EFT; bound-state problem

1 Membrame

Membrame can be roughly decomposed into two parts that are vital in the neural activity: double-layer and channels. Double layer is permeterbale for water, oil etc., yet impermmitable for ions, which is similar with bantoumo. If the charge densities of the two sides of double layer (intra celluar space and extra-cellcular space) are different, there will be a votage across cell memberame. In all, because the existance of the double-layer, the cell has a function similar to conductance

$$cV = Q \tag{1}$$

where c is the conductance per unit area, Q is the charge density differences and V is the votage accross the cell. The votage of the extra-cellcular space is set to be zero, therefore V here is also the votage of intra-cellcular space, which has realistic value -64mV as measured by the experiment.

There are two types of channel: ion channels and metabolic channel. The effect of metabolic channels are more indirect and complex which is not the center concern of this summary. Ion channels are the gates of the memberane, in which certain types of ions can pass though with a volecity. By defination, the movement of the ions is the current. One good approximation of the channels are resistence, govern by the Ohm's law

$$I = G_i(V - V_i) \tag{2}$$

where G_i is the conductance which can depends on the votage and time, and V_i is the equivibrilum votage (it is also a constant) results from the balance of electric force and diffusion []. Index i here means ith type of ion channel. Positive I means outward current.

2 Single Cell

Differentiate both side of equation (1) and use equation2, we have an equation for single cell

$$c\frac{dV}{dt} = -G_L(V - V_L) - I_e \tag{3}$$

where I_e is the extra-induced current, such as the experimentist inject ions to the cell. We use index L to descript the overall effect of some types of chennels (largely fall into K-selective channel). The rest effect of channel can be concluded as the following statement:

- If $V > V_{th}$, $V = 50$mV, then quickly fall to V_{res}.

This effect is often induced by Na-selective channels. Typical value of V_{res} is $-65mV$. There are some choice for V_{th}, but values like $-30mV$ is sufficient to fit a good result. When the statement is triggered, we say the neuron fired, or it made a spike. This single-cell model also be named Integrate and Firing Model. There are lots more sophiscated models, such as HH model, which replace the statement with more realistic channels as additional terms in equation (3).

Equation 3 can be collected with a more susinct form

$$\tau \frac{dV}{dt} = -V + V_\infty \tag{4}$$

where $\tau = c/G_L$ and $V_\infty = (V_L - I_e)/G_L$, where G_L here is a constant. Solution of equation 4 is

$$V = (V_L - V_\infty)e^{-t/\tau} + V_\infty \tag{5}$$

The firing pattern is shown in FIG. **??**

It is in consensus that the information is stored in the firing rate of the neuron instead of the exact value of votage. Firing rate is defined as the number of spiks in a unit time. The time spacing between two spikes can be calculated from equation (4) by calculate the time that V grows from V_{res} to V_{th}. The inverse of time spacing is firing rate, as show in FIG. **??**. In all, we could conclude the relation with

$$v = F[I_e] \tag{6}$$

where v is the firing rate, and F is some function has the shape in FIG.

3 Synapse

Before moving to neural network, we need to understand how two neurons connect each other by synapse. FIG. **??** shows the biological structure of the synapse. When the pre-neuron is fired, the depolyzation will open the Ca-channel, induces a inward flow of Ca^{2+}. This Ca^{2+} release the transmitters in the presynapse. Transmitter in the synapse cleft then attarch to particular recepters in the post-synapse, open ion-channel to allow Na+ flow into the post-synapse. This final lead to the depolyzation of the post-neuron.

The current flow into the post-neuron can be descripted by

$$I_{syn} = -G_{syn}(t)(V - V_{syn}), \tag{7}$$

where V_{syn} is about 0 mV, which is the reversal potential of the post-synapse. Conductance $G_{syn}(t)$ is shown in FIG. If there's no transmitters in the synapse cleft, ion channels in the post-synapse are closed, $G_{syn}(t) = 0$ mV. A spike of the presynapse will suddenly release a large amount of transmitters, therefore largely increase $G_{syn}(t)$. Then the transmitters will graduatly either diffuse away or be decomposed by glials, which lead to the slow decay of the conductance. The conductance induced by single spike is often approximated by the exponential function

$$K(t) = G_{max}e^{-t/\tau}. \tag{8}$$

The whole conductance under a train of spikes is

$$G(t) = \sum_i K(t)\delta_{t,t_i} \tag{9}$$

where there is a spike at time t_i. Equation (9) can be simplified with the spike train $S(t) = \sum_i \delta(t - t_i)$

$$G(t) = \int d\tau K(\tau)S(t - \tau) \tag{10}$$

4 Simple Neural Network

Consider a network with only two neurons attarched FIG. **??**. The working status of each neuron is solely stated by its firing rate. So we need to findout the firing-rate relation between neurons. For the single neuron, the only source of external current is from its pre-neuron, i.e. $v = F[I_{syn}]$ from equation (**??**). This equation is still too complex to be pratical. Hence we further throw away the votage term, and treat $G_{syn}(t)$ as votage independent in equation (**??**)

$$I_{syn} = \int d\tau K(\tau) S(t - \tau) \tag{11}$$

we could stripe a factor w so that the rest of $U(\tau) = K(\tau)/w$ can be normalized,

$$\int d\tau U(\tau) = 1 \tag{12}$$

The convolution of a narrow (correnspond to short time of non-zero $K(\tau)$) window $U(\tau)$ with spike train $S(\tau)$ is exactly the firing rate of the presynapse u. Overall, we have

$$v = F[wu]. \tag{13}$$

This equation reveals the simple relation of the network. Larger w means the spike of this pre-neuron induce larger current. This often means there are more channels in the post-synapse. Different presynapse will attach to different sites of the postsynapse. Furthermore, the post-neual layer can also attach to each other as shown in FIG. . The overall effect is the summation of all their current.

$$\boldsymbol{v} = F[W\boldsymbol{u} + M\boldsymbol{v}], \tag{14}$$

where W, M are matrices, arrays are written as bold letters. This model is called rate model, which is the fundation of the lots of deep learning architechture.

5 Time Dependence

6 Theoretical formulation

7 Results

8 Appendix

References

MIT OpenCourseWare: Introduction to Neural Computation, and text