第3周作业

1、

序号	随机变量	类型	样本空间
1	50个人中男性个数	离散型	2^{50}个50元组
2	0-1中的一个随机数	连续型	$[0,1]$
3	100个培养皿中某一个的细菌数	离散型	所有培养皿构成的集合
4	某个人的收入	连续型	所有人构成的收入构成的集合
5	随机调查50人对某题评分(1-5分)	离散型	5^{50}个50元组

2、

(1)

$(\Omega, \mathscr{F}, P)$为概率空间，对于随机变量X，设$S_n = \{t \in \Omega, X(t) < -n\}$,则有$S_1 \supset S_2 \supset S_3 \cdots \supset S_n \cdots$
注意到$lim_{n\to\infty} S_n = \varnothing$，因为若其不为$\varnothing$，则$\exists t \in \Omega$，满足$X(t) < n, \forall n > 0$，矛盾

$$\begin{aligned} lim_{x\to-\infty} F(x) &= lim_{n\to\infty} P(x \in S_n) \\ &= P(x \in \varnothing) \\ &= 0 \end{aligned}$$

设$S_n^{'} = \{t \in \Omega, X(t) > n\}$，同样有$S_1 \supset S_2 \supset S_3 \cdots \supset S_n \cdots$，且$lim_{n\to\infty} S_n = \varnothing$

$$\begin{aligned} lim_{x\to+\infty} F(x) &= lim_{n\to\infty} P(x \in (S_n^{'})^C) \\ &= P(x \in (\varnothing)^C) \\ &= P(\Omega) \\ &= 1 \end{aligned}$$

(2)

要证$F(x)$右连续，即$lim_{t\to x^+} F(t) = F(x)$，由海涅定理，只要证对任意从右侧逼近$x$的数列$a_n$，$F(a_n) \to F(x)$。

$$F(x+t) - F(x) = P(x < X \leq x+t) \geq 0$$

因此$F(x)$是一个单调增函数，有界单调数列数列必有极限，因此$F(a_n)$存在且唯一。
因此只要证明存在一个数列$a_n \to x^+, F(a_n) \to F(x)$即可
取$a_n = x + \frac{1}{n}$，$F(a_n) - F(x) = P(x < X \leq x + \frac{1}{n})$，当$n \to \infty, (x, x+\frac{1}{n}] = \varnothing$，因此有：

$$lim_{n\to\infty} F(a_n) - F(x) = 0$$

所以$a_n \to x^+, F(a_n) \to F(x)$,从而$F(x)$右连续

(3)

$$\begin{aligned} P(a \leq x \leq b) &= P(a) + P(a < x \leq b) \\ &= lim_{n\to\infty} P(a - \frac{1}{n} < x \leq a) + F(b) - F(a) \\ &(\text{因为} lim_{n\to\infty}(a - \frac{1}{n}, a] = \{a\}) \\ &= F(a) - lim_{n\to\infty} F(a - \frac{1}{n}) + F(b) - F(a) \\ &= F(b) - lim_{x\to a-} f(x) \end{aligned}$$

4、

(1)

$$\begin{cases} P(\omega_1)+P(\omega_2)+P(\omega_3)=1 \\ P(\omega_1)=P(\omega_2)=P(\omega_3) \end{cases}$$

可得$P(\omega_1)=P(\omega_2)=P(\omega_3)=\frac{1}{3}$
X,Y均满足以下分布：

X/Y	1	2	3
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(2)

$X+Y$	3	4	5
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$Y-X$	-2	1
P	$\frac{1}{3}$	$\frac{2}{3}$

4、

$$\begin{aligned} Var(X) &= \sum_i (x_i-E(X))^2 P_i \\ &= \sum_i (x_i^2-2E(X)x_i+E^2(X))P_i \\ &= \sum_i x_i^2 P_i - 2E(X)\times\sum_i x_i P_i + E^2(X)\sum_i P_i \\ &= E(X^2)-2E^2(X)+E^2(X) \\ &= E(X^2)-E^2(X) \end{aligned}$$

定义是一样的，中学方差$\sigma^2=\frac{1}{n}\sum_i(x_i-u)^2$是将每个数据的概率视为$\frac{1}{n}$的结果

5、

(1)

X	1	2	3	$\cdots$	$a+1$
P	$\frac{b}{a+b}$	$\frac{a}{a+b}\times\frac{b}{a+b-1}$	$\frac{a}{a+b}\times\frac{a-1}{a+b-1}\times\frac{b}{a+b-2}$	$\cdots$	$\frac{a!b!}{(a+b)!}$

(2)

记$p=\frac{b}{a+b}$

X	1	2	3	$\cdots$	n	$\cdots$
P	p	$p(1-p)$	$p(1-p)^2$	$\cdots$	$p(1-p)^{n-1}$	$\cdots$

$$\begin{aligned} E(X) &= p+2p(1-p)+3p(1-p)^2+\cdots+np(1-p)^{n-1}+\cdots \\ &= p(1+2(1-p)+3(1-p)^2+\cdots+n(1-p)^{n-1}+\cdots) \end{aligned}$$

记$S=1+2(1-p)+3(1-p)^2+\cdots+n(1-p)^{n-1}+\cdots$
$(1-p)S=(1-p)+2(1-p)^2+3(1-p)^3+\cdots+n(1-p)^n+\cdots$

$$pS = 1 + (1-p) + (1-p)^2 + \cdots = \frac{1}{p}$$
$$S = \frac{1}{p^2}$$
$$E(X) = pS = p \times \frac{1}{p^2} = \frac{1}{p} = \frac{a+b}{b}$$

6、

存在
X的分布：

X	0	1	2	3	$\cdots$	98	10^8
P	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\cdots$	$\frac{1}{100}$	$\frac{1}{100}$

Y的分布：

Y	1	2	3	4	$\cdots$	99	100
P	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{100}$	$\cdots$	$\frac{1}{100}$	$\frac{1}{100}$

$E(X) = \frac{1}{100}(0 + 1 + \cdots + 98) + 10^6 = 48.51 + 10^6 \approx 10^6$

$E(Y) = \frac{1}{100}(0 + 1 + \cdots + 100) = 50.5$

$\frac{E(X)}{E(Y)} \approx 20000$

7、

(1)

第5题中已有相同的分析

(2)

$E(X) = \frac{1}{p}$

$$E(X^2) = p(1 + 4(1-p) + 9(1-p)^2 + \cdots + n^2(1-p)^{n-1} + \cdots)$$

记$S = 1 + 4(1-p) + 9(1-p)^2 + \cdots + n^2(1-p)^{n-1} + \cdots$

$(1-p)S = (1-p) + 4(1-p)^2 + \cdots + n^2(1-p)^n + \cdots$
$pS = 1 + 3(1-p) + 5(1-p)^2 + \cdots$
$(1-p)pS = (1-p) + 3(1-p)^2 + 5(1-p)^3 + \cdots$
$p^2S = 1 + 2(1-p) + 2(1-p)^2 + \cdots = 1 + 2(1-p)(1 + (1-p) + \cdots)$
$= 1 + 2 \times \frac{1-p}{p} = \frac{2-p}{p}$
$E(X^2) = pS = \frac{2-p}{p^2}$

$$\begin{aligned} Var(X) &= E(X^2) - E^2(X) \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} \\ &= \frac{1-p}{p^2} \end{aligned}$$

8、

设有X人通过某电商平台购买商品，$X \sim B(25, 0.6)$
记$p = 0.6$

(1)

$$
\begin{aligned}
P(X \ge 15) &= \sum_{k=15}^{25} P(X=k) = \sum_{k=15}^{25} C_{25}^{k} p^k (1-p)^{n-k} \\
&\approx 0.5858
\end{aligned}
$$

(2)

$$
\begin{aligned}
P(X > 20) &= \sum_{k=21}^{25} P(X=k) = \sum_{k=21}^{25} C_{25}^{k} p^k (1-p)^{n-k} \\
&\approx 0.0095
\end{aligned}
$$

(3)

$$
\begin{aligned}
P(X < 10) &= \sum_{k=0}^{9} P(X=k) = \sum_{k=0}^{9} C_{25}^{k} p^k (1-p)^{n-k} \\
&\approx 0.0132
\end{aligned}
$$

9、

对n次二项分布：

$$
P(X=k) = C_n^k p^k (1-p)^{n-k}
$$

$$
\begin{aligned}
E(X) &= \sum_k k C_n^k p^k (1-p)^{n-k} \\
&= \sum_k n C_{n-1}^{k-1} p^k (1-p)^{n-k} \\
&= np \sum_k C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} \\
&= np(p+1-p)^{n-1} \\
&= np
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - E^2(X) \\
&= \sum_k k^2 C_n^k p^k (1-p)^{n-k} - E^2(X) \\
&= \sum_k nk C_{n-1}^{k-1} p^k (1-p)^{n-k} - E^2(X) \\
&= np \sum_k (k-1+1) C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} - E^2(X) \\
&= np(1 + \sum_k (k-1) C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k}) - E^2(X) \\
&= np(1 + (n-1)p \sum_k C_{n-2}^{k-2} p^{k-2} (1-p)^{n-k}) - n^2p^2 \\
&= np(1+(n-1)p) - n^2p^2 \\
&= np(1-p)
\end{aligned}
$$

10、

(1)

$$
P(X=m) = \frac{C_M^m C_{N-M}^{n-m}}{C_N^n}
$$

(2)

$$
\begin{aligned}
E(X) &= \sum_m mP(X=m) \\
&= \sum_m \frac{mC_M^m C_{N-M}^{n-m}}{C_N^n} \\
&= \sum_m \frac{MC_{M-1}^{m-1} C_{N-M}^{n-m}}{C_N^n} \\
&= \frac{M}{C_N^n} \sum_m C_{M-1}^{m-1} C_{N-M}^{n-m} \\
&= \frac{M}{C_N^n} C_{N-1}^{n-1} \\
&= M \frac{n!(N-n)!}{N!} \frac{(N-1)!}{(n-1)!(N-n)!} \\
&= \frac{nM}{N}
\end{aligned}
$$

如果认为捕上来的鱼m条为期望数

$$
\begin{aligned}
m &= \frac{nM}{N} \\
N &= \frac{nM}{m}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\text{记} a_N &= \frac{C_M^m C_{N-M}^{n-m}}{C_N^n} \\
\frac{a_{N+1}}{a_N} &= \frac{C_{N-M+1}^{n-m} C_N^n}{C_{N+1}^n C_{N-M}^{n-m}} \\
&= \frac{(N-M+1)!}{(n-m)!(N-M+1-(n-m))!} \frac{(n-m)!(N-M-(n-m))!}{(N-M)!} \frac{n!(N+1-n)! \times N!}{(N+1)! \times n!(N-n)!} \\
&= \frac{N-M+1}{N-M+1-(n-m)} \frac{N+1-n}{N+1}
\end{aligned}
$$

$$
\begin{aligned}
\text{令} \frac{N-M+1}{N-M+1-(n-m)} \frac{N+1-n}{N+1} &> 1 \\
\frac{N+1-n}{N+1} &> \frac{N-M+1-(n-m)}{N-M+1} \\
1 - \frac{n}{N+1} &> 1 - \frac{n-m}{N-M+1} \\
\frac{n-m}{N-M+1} &> \frac{n}{N+1} \\
(n-m)(N+1) &> n(N-M+1) \\
-mN - m + nM &> 0 \\
N < \frac{nM-m}{m} &= \frac{nM}{m} - 1
\end{aligned}
$$

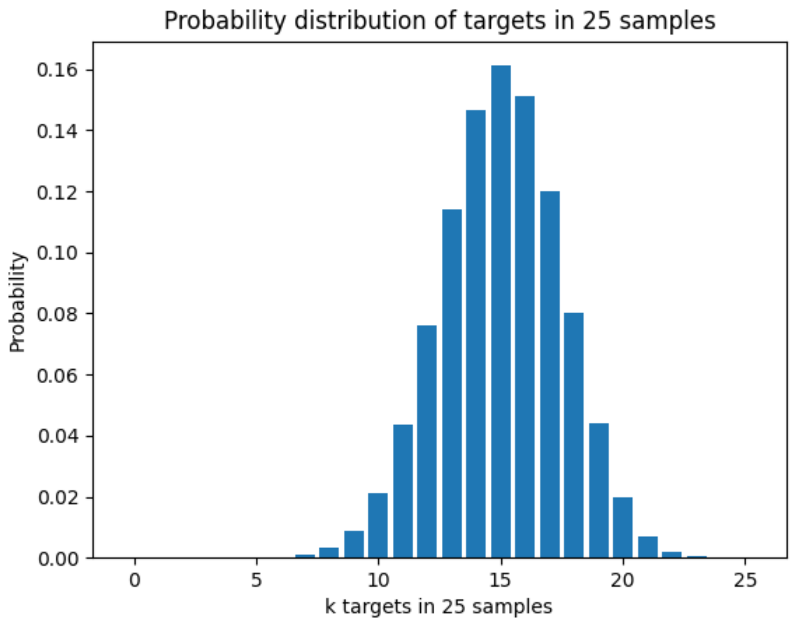
记$L = \lfloor \frac{nM}{m} \rfloor$，则有$max(a_N) = a_L$，这与(2)中的估计值相同

(4)

当n足够大时，该随机变量近似服从二项分布$B(n,p), p = \frac{M}{N}$
这种极端情况下，$B(n,p)$的期望与之前推算的期望一致

11、

(1)



由图可知，$x = 15$有最大概率

(2)

通过计算得到$E(X) = 15$，这与最大概率对应的x的大小相等

(3)

通过计算得到$Var(X) = 6$

(6)

介于$u \pm 2\sigma$的概率约为93.6%

```python
from scipy.stats import binom
import matplotlib.pyplot as plt
import numpy as np

n = 25  # n为试验次数
p = 0.6  # p为成功的概率
ps = []
u = 0
var = 0

for i in range(0, 26):
    ps.append(binom.pmf(i, n, p))
    u += ps[i] * i
plt.bar(range(0, 26), ps)
plt.title("Probability distribution of targets in 25 samples")
plt.xlabel("k targets in 25 samples")
plt.ylabel("Probability")
plt.savefig("1.png")

for i in range(0, 26):
    var += ps[i] * ((i - u) ** 2)
sd = np.sqrt(var)
sum_p = 0

# print(u - 2 * sd, u + 2 * sd)
for i in range(0, 26):
    if(u - 2 * sd <= i and i <= u + 2 * sd):
        # print(i)
        sum_p += ps[i]

print(f"E(X)={u}")
print(f"Var(X)={var}")
print(f"Probability in (u-2{chr(963)}, u+2{chr(963)})={sum_p}")
```