Exeter Math Club Competition
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Chapter 1

EMC^2 2016 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Compute the value of $2 + 20 + 201 + 2016$.

2. Gleb is making a doll, whose prototype is a cube with side length 5 centimeters. If the density of the toy is 4 grams per cubic centimeter, compute its mass in grams.

3. Find the sum of 20% of 16 and 16% of 20.

4. How many times does Akmal need to roll a standard six-sided die in order to guarantee that two of the rolled values sum to an even number?

5. During a period of one month, there are ten days without rain and twenty days without snow. What is the positive difference between the number of rainy days and the number of snowy days?

6. Joanna has a fully charged phone. After using it for 30 minutes, she notices that 20 percent of the battery has been consumed. Assuming a constant battery consumption rate, for how many additional minutes can she use the phone until 20 percent of the battery remains?

7. In a square $ABCD$, points P, Q, R, and S are chosen on sides AB, BC, CD, and DA respectively, such that $AP = 2PB$, $BQ = 2QC$, $CR = 2RD$, and $DS = 2SA$. What fraction of square $ABCD$ is contained within square $PQRS$?

8. The sum of the reciprocals of two not necessarily distinct positive integers is 1. Compute the sum of these two positive integers.

9. In a room of government officials, two-thirds of the men are standing and 8 women are standing. There are twice as many standing men as standing women and twice as many women in total as men in total. Find the total number of government officials in the room.

10. A string of lowercase English letters is called *pseudo-Japanese* if it begins with a consonant and alternates between consonants and vowels. (Here the letter "y" is considered neither a consonant nor vowel.) How many 4-letter pseudo-Japanese strings are there?

11. In a wooden box, there are 2 identical black balls, 2 identical grey balls, and 1 white ball. Yuka randomly draws two balls in succession without replacement. What is the probability that the first ball is strictly darker than the second one?

12. Compute the real number x for which
$$(x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2.$$

13. Let ABC be an isosceles right triangle with $\angle C = 90^\circ$ and $AB = 2$. Let D, E, and F be points outside ABC in the same plane such that the triangles DBC, AEC, and ABF are isosceles right triangles with hypotenuses BC, AC, and AB, respectively. Find the area of triangle DEF.

14. Salma is thinking of a six-digit positive integer n divisible by 90. If the sum of the digits of n is divisible by 5, find n.

15. Kiady ate a total of 100 bananas over five days. On the $(i + 1)$-th day $(1 \leq i \leq 4)$, he ate i more bananas than he did on the i-th day. How many bananas did he eat on the fifth day?

16. In a unit equilateral triangle ABC, points D, E, and F are chosen on sides BC, CA, and AB, respectively. If lines DE, EF, and FD are perpendicular to CA, AB, and BC, respectively, compute the area of triangle DEF.

17. Carlos rolls three standard six-sided dice. What is the probability that the product of the three numbers on the top faces has units digit 5?

18. Find the positive integer n for which $n^{n^n} = 3^{3^{82}}$.

19. John folds a rope in half five times then cuts the folded rope with four knife cuts, leaving five stacks of rope segments. How many pieces of rope does he now have?

20. An integer $n > 1$ is *conglomerate* if all positive integers less than n and relatively prime to n are not composite. For example, 3 is conglomerate since 1 and 2 are not composite. Find the sum of all conglomerate integers less than or equal to 200.

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. A right triangle has a hypotenuse of length 25 and a leg of length 16. Compute the length of the other leg of this triangle.

2. Tanya has a circular necklace with 5 evenly-spaced beads, each colored red or blue. Find the number of distinct necklaces in which no two red beads are adjacent. If a necklace can be transformed into another necklace through a series of rotations and reflections, then the two necklaces are considered to be the same.

3. Find the sum of the digits in the decimal representation of $10^{2016} - 2016$.

4. Let x be a real number satisfying
$$x^1 \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 = 8^7.$$
Compute the value of x^7.

5. What is the smallest possible perimeter of an acute, scalene triangle with integer side lengths?

6. Call a sequence $a_1, a_2, a_3, \ldots, a_n$ *mountainous* if there exists an index t between 1 and n inclusive such that
$$a_1 \le a_2 \le \cdots \le a_t \quad \text{and} \quad a_t \ge a_{t+1} \ge \cdots \ge a_n.$$
In how many ways can Bishal arrange the ten numbers $1, 1, 2, 2, 3, 3, 4, 4, 5$, and 5 into a mountainous sequence? (Two possible mountainous sequences are $1, 1, 2, 3, 4, 4, 5, 5, 3, 2$ and $5, 5, 4, 4, 3, 3, 2, 2, 1, 1$.)

7. Find the sum of the areas of all (non self-intersecting) quadrilaterals whose vertices are the four points $(-3, -6), (7, -1), (-2, 9)$, and $(0, 0)$.

8. Mohammed Zhang's favorite function is $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8}$. Find the minumum possible value of $f(x)$ over all real numbers x.

9. A segment AB with length 1 lies on a plane. Find the area of the set of points P in the plane for which $\angle APB$ is the second smallest angle in triangle ABP.

10. A binary string is a *dipalindrome* if it can be produced by writing two non-empty palindromic strings one after the other. For example, 10100100 is a dipalindrome because both 101 and 00100 are palindromes. How many binary strings of length 18 are both palindromes and dipalindromes?

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

1. Lisa is playing the piano at a tempo of 80 beats per minute. If four beats make one measure of her rhythm, how many seconds are in one measure?

2. Compute the smallest integer $n > 1$ whose base-2 and base-3 representations both do not contain the digit 0.

3. In a room of 24 people, $5/6$ of the people are old, and $5/8$ of the people are male. At least how many people are both old and male?

4. Juan chooses a random even integer from 1 to 15 inclusive, and Gina chooses a random odd integer from 1 to 15 inclusive. What is the probability that Juan's number is larger than Gina's number? (They choose all possible integers with equal probability.)

5. Set S consists of all positive integers less than or equal to 2016. Let A be the subset of S consisting of all multiples of 6. Let B be the subset of S consisting of all multiples of 7. Compute the ratio of the number of positive integers in A but not B to the number of integers in B but not A.

6. Three peas form a unit equilateral triangle on a flat table. Sebastian moves one of the peas a distance d along the table to form a right triangle. Determine the minimum possible value of d.

7. Oumar is four times as old as Marta. In m years, Oumar will be three times as old as Marta will be. In another n years after that, Oumar will be twice as old as Marta will be. Compute the ratio m/n.

8. Compute the area of the smallest square in which one can inscribe two non-overlapping equilateral triangles with side length 1.

9. Teemu, Marcus, and Sander are signing documents. If they all work together, they would finish in 6 hours. If only Teemu and Sander work together, the work would be finished in 8 hours. If only Marcus and Sander work together, the work would be finished in 10 hours. How many hours would Sander take to finish signing if he worked alone?

10. Triangle ABC has a right angle at B. A circle centered at B with radius BA intersects side AC at a point D different from A. Given that $AD = 20$ and $DC = 16$, find the length of BA.

11. A regular hexagon H with side length 20 is divided completely into equilateral triangles with side length 1. How many regular hexagons with sides parallel to the sides of H are formed by lines in the grid?

12. In convex pentagon $PEARL$, quadrilateral $PERL$ is a trapezoid with side PL parallel to side ER. The areas of triangle ERA, triangle LAP, and trapezoid $PERL$ are all equal. Compute the ratio $\dfrac{PL}{ER}$.

13. Let m and n be positive integers with $m < n$. The first two digits after the decimal point in the decimal representation of the fraction $\frac{m}{n}$ are 74. What is the smallest possible value of n?

14. Define functions
$$f(x,y) = \frac{x+y}{2} - \sqrt{xy} \quad \text{and} \quad g(x,y) = \frac{x+y}{2} + \sqrt{xy}.$$
Compute
$$g\left(g\left(f\left(1,3\right),\ f\left(5,7\right)\right),\ g\left(f\left(3,5\right),\ f\left(7,9\right)\right)\right).$$

15. Natalia plants two gardens in a 5×5 grid of points. Each garden is the interior of a rectangle with vertices on grid points and sides parallel to the sides of the grid. How many unordered pairs of two non-overlapping rectangles can Nataliia choose as gardens? (The two rectangles may share an edge or part of an edge but should not share an interior point.)

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [5] Suppose that gold satisfies the relation $p = v + v^2$, where p is the price and v is the volume. How many pieces of gold with volume 1 can be bought for the price of a piece with volume 2?

2. [5] Find the smallest prime number with each digit greater or equal to 8.

3. [5] What fraction of regular hexagon $ZUMING$ is covered by both quadrilateral $ZUMI$ and quadrilateral $MING$?

1.4.2 Round 2

4. [7] The two smallest positive integers expressible as the sum of two (not necessarily positive) perfect cubes are $1 = 1^3 + 0^3$ and $2 = 1^3 + 1^3$. Find the next smallest positive integer expressible in this form.

5. [7] In how many ways can the numbers $1, 2, 3$, and 4 be written in a row such that no two adjacent numbers differ by exactly 1?

6. [7] A real number is placed in each cell of a grid with 3 rows and 4 columns. The average of the numbers in each column is 2016, and the average of the numbers in each row is a constant x. Compute x.

1.4.3 Round 3

7. [9] Fardin is walking from his home to his office at a speed of 1 meter per second, expecting to arrive exactly on time. When he is halfway there, he realizes that he forgot to bring his pocketwatch, so he runs back to his house at 2 meters per second. If he now decides to travel from his home to his office at x meters per second, find the minimum x that will allow him to be on time.

8. [9] In triangle ABC, the angle bisector of $\angle B$ intersects the perpendicular bisector of AB at point P on segment AC. Given that $\angle C = 60^\circ$, determine the measure of $\angle CPB$ in degrees.

9. [9] Katie colors each of the cells of a 6×6 grid either black or white. From top to bottom, the number of black squares in each row are 1, 2, 3, 4, 5, and 6, respectively. From left to right, the number of black squares in each column are 6, 5, 4, 3, 2, and 1, respectively. In how many ways could Katie have colored the grid?

1.4.4 Round 4

10. [11] Lily stands at the origin of a number line. Each second, she either moves 2 units to the right or 1 unit to the left. At how many different places could she be after 2016 seconds?

11. [11] There are 125 politicians standing in a row. Each either always tells the truth or always lies. Furthermore, each politician (except the leftmost politician) claims that at least half of the people to his left always lie. Find the number of politicians that always lie.

12. [11] Two concentric circles with radii 2 and 5 are drawn on the plane. What is the side length of the largest square whose area is contained entirely by the region between the two circles?

1.4.5 Round 5

13. [13] Initially, the three numbers 20, 201, and 2016 are written on a blackboard. Each minute, Zhuo selects two of the numbers on the board and adds 1 to each. Find the minimum n for which Zhuo can make all three numbers equal to n.

14. [13] Call a three-letter string *rearrangeable* if, when the first letter is moved to the end, the resulting string comes later alphabetically than the original string. For example, AAA and BAA are not rearrangeable, while ABB is rearrangeable. How many three-letters strings with (not necessarily distinct) uppercase letters are rearrangeable?

15. [13] Triangle ABC is an isosceles right triangle with $\angle C = 90^\circ$ and $AC = 1$. Points D, E, and F are chosen on sides BC, CA, and AB, respectively, such that AEF, BFD, CDE, and DEF are isosceles right triangles. Find the sum of all distinct possible lengths of segment DE.

1.4.6 Round 6

16. [15] Let p, q, and r be prime numbers such that $pqr = 17(p + q + r)$. Find the value of the product pqr.

17. [15] A cylindrical cup containing some water is tilted 45 degrees from the vertical. The point on the surface of the water closest to the bottom of the cup is 6 units away. The point on the surface of the water farthest from the bottom of the cup is 10 units away. Compute the volume of the water in the cup.

18. [15] Each dot in an equilateral triangular grid with 63 rows and $2016 = \frac{1}{2} \cdot 63 \cdot 64$ dots is colored black or white. Every unit equilateral triangle with three dots has the property that exactly one of its vertices is colored black. Find all possible values of the number of black dots in the grid.

1.4.7 Round 7

19. [18] Tomasz starts with the number 2. Each minute, he either adds 2 to his number, subtracts 2 from his number, multiplies his number by 2, or divides his number by 2. Find the minimum number of minutes he will need in order to make his number equal 2016.

20. [18] The edges of a regular octahedron $ABCDEF$ are painted with 3 distinct colors such that no two edges with the same color lie on the same face. In how many ways can the octahedron be painted? Colorings are considered different under rotation or reflection.

21. [18] Jacob is trapped inside an equilateral triangle ABC and must visit each edge of triangle ABC at least once. (Visiting an edge means reaching a point on the edge.) His distances to sides AB, BC, and CA are currently 3, 4, and 5, respectively. If he does not need to return to his starting point, compute the least possible distance that Jacob must travel.

1.4.8 Round 8

22. [22] Four integers a, b, c, and d with $a \le b \le c \le d$ satisfy the property that the product of any two of them is equal to the sum of the other two. Given that the four numbers are not all equal, determine the 4-tuple (a, b, c, d).

23. [22] In equilateral triangle ABC, points D, E, and F lie on sides BC, CA, and AB, respectively, such that $BD = 4$ and $CD = 5$. If DEF is an isosceles right triangle with right angle at D, compute $EA + FA$.

24. [22] On each edge of a regular tetrahedron, four points that separate the edge into five equal segments are marked. There are sixteen planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these sixteen planes, how many new tetrahedrons are produced?

Chapter 2

EMC^2 2016 Solutions

2.1 Speed Test Solutions

1. Compute the value of $2 + 20 + 201 + 2016$.

   Solution. The answer is $\boxed{2239}$.

   We compute $2 + 20 + 201 + 2016 = 2239$.

2. Gleb is making a doll, whose prototype is a cube with side length 5 centimeters. If the density of the toy is 4 grams per cubic centimeter, compute its mass in grams.

   Solution. The answer is $\boxed{500}$.

   The volume of the doll is $5^3 = 125$ cubic centimeters, so the mass is $4 \cdot 125 = 500$ grams.

3. Find the sum of 20% of 16 and 16% of 20.

   Solution. The answer is $\boxed{\frac{32}{5}}$.

   We compute
   $$20\% \cdot 16 + 16\% \cdot 20 = 320\% + 320\% = 640\% = \frac{32}{5}.$$

4. How many times does Akmal need to roll a standard six-sided die in order to guarantee that two of the rolled values sum to an even number?

   Solution. The answer is $\boxed{3}$.

   If Akmal rolls twice, then the numbers can be 1 and 2 and their sum will be odd. However when he rolls three times, by Pigeonhole Principle two of them will have the same parity and their sum will be even. Therefore the least number of rolls needed is 4.

5. During a period of one month, there are ten days without rain and twenty days without snow. What is the positive difference between the number of rainy days and the number of snowy days?

   Solution. The answer is $\boxed{10}$.

   Suppose a month has n days. Then ten days without rain means $n - 10$ days with rain, and twenty days without snow means $n - 20$ days with snow. Therefore the difference is $(n - 10) - (n - 20) = (-10) + 20 = 10$ days.

6. Joanna has a fully charged phone. After using it for 30 minutes, she notices that 20 percent of the battery has been consumed. Assuming a constant battery consumption rate, for how many additional minutes can she use the phone until 20 percent of the battery remains?

   Solution. The answer is $\boxed{90}$.

   To go from 20 percent gone (or 80 percent remaining), to 20 percent remaining, Joanna needs to use 60 percent of the battery. Since using 20 percent takes 30 minutes, using 60 percent will take $\frac{30}{20} \cdot 60 = 90$ minutes.

7. In a square $ABCD$, points P, Q, R, and S are chosen on sides AB, BC, CD, and DA respectively, such that $AP = 2PB$, $BQ = 2QC$, $CR = 2RD$, and $DS = 2SA$. What fraction of square $ABCD$ is contained within square $PQRS$?

   Solution. The answer is $\boxed{\frac{5}{9}}$.

   Suppose the side length of the square $ABCD$ is 1. Then it is easy to see that the area of triangle SAP,PBQ,QCR,RDS are all $\frac{1}{3} \cdot \frac{2}{3} \div 2 = \frac{1}{9}$, so the area of the quadrilateral $PQRS$ is $1 - 4 \cdot \frac{1}{9} = \frac{5}{9}$, and that is also the desired fraction since the whole area of $ABCD$ is 1.

8. The sum of the reciprocals of two not necessarily distinct positive integers is 1. Compute the sum of these two positive integers.

   Solution. The answer is $\boxed{4}$.

   Suppose the two integers are a, b. Then $\frac{1}{a} + \frac{1}{b} = 1$, so neither a or b can be 1. Hence $\frac{1}{a} + \frac{1}{b} \le \frac{1}{2} + \frac{1}{2} = 1$, with the equality case of $a = b = 2$. Thus the answer is $a + b = 4$.

9. In a room of government officials, two-thirds of the men are standing and 8 women are standing. There are twice as many standing men as standing women and twice as many women in total as men in total. Find the total number of government officials in the room.

   Solution. The answer is $\boxed{72}$.

   There are $8 \cdot 2 = 16$ men standing, and thus $\frac{16}{2/3} = 24$ men in the room. Then we deduce there are $24 \cdot 2 = 48$ women, so the room contains $24 + 48 = 72$ government officials in total.

10. A string of lowercase English letters is called *pseudo-Japanese* if it begins with a consonant and alternates between consonants and vowels. (Here the letter "y" is considered neither a consonant nor vowel.) How many 4-letter pseudo-Japanese strings are there?

    Solution. The answer is $\boxed{10000}$.

    There are 5 vowels and $26 - 1 - 5 = 20$ consonants, so there are $20 \cdot 5 \cdot 20 \cdot 5 = 10000$ possible pseudo-Japanese strings.

11. In a wooden box, there are 2 identical black balls, 2 identical grey balls, and 1 white ball. Yuka randomly draws two balls in succession without replacement. What is the probability that the first ball is strictly darker than the second one?

    Solution. The answer is $\boxed{\frac{2}{5}}$.

    There are 2 pairs out of $\binom{5}{2} = 10$ pairs of balls that have the same color, so the probability of getting two balls of the same color is $\frac{1}{5}$. Since the probability of the first ball being darker than the second one is equal that of the first ball being lighter, both probabilities are $(1 - \frac{1}{5}) \div 2 = \frac{2}{5}$.

12. Compute the real number x for which

$$(x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2.$$

Solution. The answer is $\boxed{-\frac{7}{2}}$.

Expanding both sides, we see that

$$x^2+2x+1+x^2+4x+4+x^2+6x+9=x^2+8x+16+x^2+10x+25+x^2+12x+36.$$

This simplifies to $18x=-63$, or $x=-\frac{7}{2}$.

13. Let ABC be an isosceles right triangle with $\angle C=90^\circ$ and $AB=2$. Let D, E, and F be points outside ABC in the same plane such that the triangles DBC, AEC, and ABF are isosceles right triangles with hypotenuses BC, AC, and AB, respectively. Find the area of triangle DEF.

Solution. The answer is $\boxed{2}$.

Notice that $ACBF$ is a square with diagonal length 2 and $AEDB$ is a rectangle with length 2 and width 1. Therefore triangle DEF has base $DE=2$ and height $CF=2$, and its area is $\frac{2\cdot 2}{2}=2$.

14. Salma is thinking of a six-digit positive integer n divisible by 90. If the sum of the digits of n is divisible by 5, find n.

Solution. The answer is $\boxed{999990}$.

Since n is a multiple of 90, it ends with zero and the sum of digits is a multiple of 9. Since the sum of digits is also a multiple of 5, the sum of digits is a multiple of 45. Since there are at most five non-zero digits, each of them have to be a 9, so $n=999990$.

15. Kiady ate a total of 100 bananas over five days. On the $(i+1)$-th day $(1\le i\le 4)$, he ate i more bananas than he did on the i-th day. How many bananas did he eat on the fifth day?

Solution. The answer is $\boxed{26}$.

Suppose Kiady ate x bananas on the first day, so he ate $(x+1)$ bananas on the second days, $(x+3)$ on the third day, $(x+6)$ on fourth and $(x+10)$ on fifth. Solving $x+(x+1)+(x+3)+(x+6)+(x+10)=100$ gives $x=16$, so on fifth day he ate $x+10=26$ bananas.

16. In a unit equilateral triangle ABC, points D, E, and F are chosen on sides BC, CA, and AB, respectively. If lines DE, EF, and FD are perpendicular to CA, AB, and BC, respectively, compute the area of triangle DEF.

Solution. The answer is $\boxed{\frac{\sqrt{3}}{12}}$.

Notice that triangles DEC, EFA, FDB are all 30-60-90 triangles, so $EA=2AF, FB=2BD, DC=2CE$. Then $AE=BF=CD=\frac{2}{3}$ and $AF=BD=CE=\frac{1}{3}$, and thus $DE=EF=FD=\frac{\sqrt{3}}{3}$. Since the area of an equilateral triangle with side length a has area $\frac{\sqrt{3}}{4}a^2$, the area of $\triangle DEF$ is equal to $\frac{\sqrt{3}}{4}(\frac{\sqrt{3}}{3})^2=\frac{\sqrt{3}}{12}$.

17. Carlos rolls three standard six-sided dice. What is the probability that the product of the three numbers on the top faces has units digit 5?

Solution. The answer is $\boxed{\frac{19}{216}}$.

The unit digit of a product is a 5 if and only if all of the rolls are odd and at least one of them is a 5. There are $3^3 = 27$ ways to roll three odd numbers, and $2^3 = 8$ ways to roll three odd numbers *without* a 5, so there are $27 - 8 = 19$ ways in total, and the probability is $\frac{19}{6^3} = \frac{19}{216}$.

18. Find the positive integer n for which $n^{n^n} = 3^{3^{82}}$.

Solution. The answer is $\boxed{27}$.

First note that n must be a power of 3, so let $n = 3^k$. Now we have $n^{n^n} = (3^k)^{(3^k)^{3^k}} = 3^{k\cdot 3^{k\cdot 3^k}} = 3^{3^{82}}$, so $k \cdot 3^{k\cdot 3^k} = 3^{82}$, and k must be a power of 3 as well. Notice that when $k = 3$, the LHS of the equation becomes $3 \cdot 3^{3\cdot 27} = 3^{82}$, which is equal to the RHS. Therefore $n = 3^3 = 27$.

19. John folds a rope in half five times then cuts the folded rope with four knife cuts, leaving five stacks of rope segments. How many pieces of rope does he now have?

Solution. The answer is $\boxed{129}$.

A rope folded in half five times has $2^5 = 32$ layers, and cutting into five equal parts requires $5 - 1 = 4$ cuts for each layer, or $4 \cdot 32 = 128$ cuts in total on the original rope. Since each cut creates one more piece, there will be $128 + 1 = 129$ disjoint pieces at the end.

20. An integer $n > 1$ is *conglomerate* if all positive integers less than n and relatively prime to n are not composite. For example, 3 is conglomerate since 1 and 2 are not composite. Find the sum of all conglomerate integers less than or equal to 200.

Solution. The answer is $\boxed{107}$.

If p is the smallest prime not dividing n, then p^2 is the smallest composite number that is relatively prime to n, which means that n is very composite if and only if $n < p^2$. Now we base our cases on p.

If $p = 2$, then $n < 4$ and n is odd. This gives $n = 3$.

If $p = 3$, then $n < 9$, n is even but not divisible by 3. This gives $n = 2, 4, 8$.

If $p = 5$, then $n < 25$ and n is a multiple of $2 \cdot 3$. This gives $n = 6, 12, 18, 24$.

If $p = 7$, then $n < 49$ and n is a multiple of $2 \cdot 3 \cdot 5$. This gives $n = 30$.

If $p > 7$, then n has to be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210 > 200$, which yields no results inside the bounds.

Therefore the sum of all possibilities is $3 + 2 + 4 + 8 + 6 + 12 + 18 + 24 + 30 = 107$.

2.2 Accuracy Test Solutions

1. A right triangle has a hypotenuse of length 25 and a leg of length 16. Compute the length of the other leg of this triangle.

   Solution. The answer is $\boxed{3\sqrt{41}}$.

   By the Pythagorean Theorem we have $16^2+x^2 = 25^2$, so the length of the other leg is $x = \sqrt{25^2 - 16^2} = \sqrt{(25-16)(25+16)} = 3\sqrt{41}$.

2. Tanya has a circular necklace with 5 evenly-spaced beads, each colored red or blue. Find the number of distinct necklaces in which no two red beads are adjacent. If a necklace can be transformed into another necklace through a series of rotations and reflections, then the two necklaces are considered to be the same.

   Solution. The answer is $\boxed{3}$.

   If there are no red beads there is one way (all blue). If there is one red bead then there is only one way as well, since all other ways can be rotated to make the red bead placed at the top. If there are two red beads then there is also only one way which separates the blue beads into groups of 1 and 2. Adding up all cases, we have $1 + 1 + 1 = 3$ distinct configurations.

3. Find the sum of the digits in the decimal representation of $10^{2016} - 2016$.

   Solution. The answer is $\boxed{18136}$.

   $10^{2016} - 2016$ is less than 10^{2016} so it has 2016 digits. It's easy to see that $N = \overline{999\cdots997984}$, where there are 2012 9's preceding the last four digits. Therefore the sum of digits is $2012 \cdot 9+7+9+8+4 = 18136$.

4. Let x be a real number satisfying
   $$x^1 \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 = 8^7.$$
   Compute the value of x^7.

   Solution. The answer is $\boxed{128}$.

   The LHS of the equation is equal to x^{21}, so $x^7 = \sqrt[3]{x^{21}} = \sqrt[3]{8^7} = (\sqrt[3]{8})^7 = 2^7 = 128$.

5. What is the smallest possible perimeter of an acute, scalene triangle with integer side lengths?

   Solution. The answer is $\boxed{15}$.

   When the longest side is 3, the shorter side must be at most 1 and 2, and the triangle is degenerate. When the longest side is 4, the shorter side must be at most 2 and 3, but since $2^2 + 3^2 < 4^2$ the triangle is obtuse. When the longest side is 5, the shorter side must be at most 3 and 4, but this triangle is a right triangle. When the longest side is 6, the shorter sides can be 4 and 5, and since $4^2 + 5^2 > 6^2$ the triangle is acute and its perimeter is $4 + 5 + 6 = 15$. If the sides are $3, 5, 6$ then it is obtuse since $3^2 + 5^2 < 6^2$. If the longest side is 7 or larger, the perimeter will be at least $7 + (7 + 1) = 15$ so we cannot improve the minimum. Therefore the shortest perimeter is 15.

6. Call a sequence $a_1, a_2, a_3, \ldots, a_n$ *mountainous* if there exists an index t between 1 and n inclusive such that
$$a_1 \le a_2 \le \cdots \le a_t \quad \text{and} \quad a_t \ge a_{t+1} \ge \cdots \ge a_n.$$
In how many ways can Bishal arrange the ten numbers $1, 1, 2, 2, 3, 3, 4, 4, 5,$ and 5 into a mountainous sequence? (Two possible mountainous sequences are $1, 1, 2, 3, 4, 4, 5, 5, 3, 2$ and $5, 5, 4, 4, 3, 3, 2, 2, 1, 1$.)

   Solution. The answer is $\boxed{81}$.

   The two 5's have to be together in the sequence. Now we put 4's around them: there are three ways, corresponding to two 4's on the left, one 4 and no 4's. The same logic applies to 3's, 2's and 1's. So in total there are $3^4 = 81$ possible sequences.

7. Find the sum of the areas of all (non self-intersecting) quadrilaterals whose vertices are the four points $(-3, -6), (7, -1), (-2, 9)$, and $(0, 0)$.

   Solution. The answer is $\boxed{145}$.

   Let $A = (-3, -6), B = (7, -1), C = (-2, 9), O = (0, 0)$. There are three possible quadrilaterals: $ABCO, BCAO, CABO$, and each of the triangles ABO, BCO, CAO are contained in exactly two of the quadrilaterals, so the sum of the areas is equal to twice the area of triangle ABC. By bounding the triangle with a rectangle and subtracting the three extra triangles from it, we can find the area of triangle ABC to be $10 \cdot 15 - \frac{1}{2}(10 \cdot 5 + 1 \cdot 15 + 9 \cdot 10) = 150 - \frac{1}{2}(155) = \frac{145}{2}$, so the sum of the areas is equal to 145.

8. Mohammed Zhang's favorite function is $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8}$. Find the minumum possible value of $f(x)$ over all real numbers x.

   Solution. The answer is $\boxed{5}$.

   Observe that $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8} = \sqrt{(x-2)^2 + (-1)^2} + \sqrt{(x+2)^2 + 2^2}$. If we let $A = (2, 1), B = (-2, -2), P = (x, 0)$, then $f(x) = AP + BP \le AB = \sqrt{(2+2)^2 + (1+2)^2} = 5$ by triangle inequality. The equality can be achieved when P lies on segment AB, so the minimum is indeed 5.

9. A segment AB with length 1 lies on a plane. Find the area of the set of points P in the plane for which $\angle APB$ is the second smallest angle in triangle ABP.

   Solution. The answer is $\boxed{\sqrt{3} + \frac{2}{3}\pi}$.

   Since the largest angle in a triangle faces the longest side, and the smallest angle faces the shortest side, the original condition is equivalent to AB being neither the longest nor the shortest side in the triangle ABP, which means either $AP > 1, BP < 1$, or $AP < 1, BP > 1$. The locus (the set of all such points) is therefore the region inside exactly one of the unit circles drawn around A and B. Since the area in common is equal to $2(2 \cdot \frac{\pi}{6} - \frac{\sqrt{3}}{4}) = \frac{2}{3}\pi - \frac{\sqrt{3}}{2}$, the area of the desired regions is $2\pi - 2(\frac{2}{3}\pi - \frac{\sqrt{3}}{2}) = \sqrt{3} + \frac{2}{3}\pi$.

10. A binary string is a *dipalindrome* if it can be produced by writing two non-empty palindromic strings one after the other. For example, 10100100 is a dipalindrome because both 101 and 00100 are palindromes. How many binary strings of length 18 are both palindromes and dipalindromes?

Solution. The answer is $\boxed{36}$.

We make the claim that each dipalindrome must be comprised of at least two identical palindromes written back-to-back, and that one can only split it into two palindromes by separating between two of these copies.

Proof: For a string S, define S' as the reversed version of S. A palindrome is therefore defined as a string P that satisfies $P = P'$.

Now we prove our claim using induction. When the dipalindrome has length 2 the claim is true. Now suppose all dipalindromes with length less than or equal to k satisfy this property. Consider a dipalindrome Q with length $k+1$, which can be expressed as $S+T$, where S, T are both palindromes. If S and T has the same length, then $S = T' = T = S'$ and we are done (if we can split it in another way we can simply consider S and T as having different length). Otherwise WLOG assume T is longer than S, then since Q itself is a palindrome, T must be able to be expressed as $U + S' = U + S$ for some string U. But also since $Q = Q'$, $S + U + S = S' + U' + S' = S + U' + S$, so $U' = U$ and it is a palindrome. This means that T itself must be a dipalindrome. By induction hypothesis T is comprised of some number of identical palindromes P, and that S must contain a smaller number of the same P. Therefore, $Q = S + T$ must contain only identical copies of P as well and we proved our inductive step and thus proved our claim.

Now we need to find the number of dipalindromes using this property. When the string has length 18, it can be split into 2 identical palindromes of length 9, 3 palindromes of length 6, 6 palindromes of length 3, 9 palindromes of length 2, 18 palindromes of length 1. Here we need to use Principle of Inclusion-Exclusion for the case for 6 palindromes since it's accounted for twice in the 2-palindrome and 3-palindrome case. Other cases will be properly taken care of after this consideration. Making use of the fact that there are $2^{\lceil \frac{n}{2} \rceil}$ binary palindromes with length n, the answer will be

$$2^{\lceil \frac{9}{2} \rceil} + 2^{\lceil \frac{6}{2} \rceil} - 2^{\lceil \frac{3}{2} \rceil} = 32 + 8 - 4 = 36.$$

2.3 Team Test Solutions

1. Lisa is playing the piano at a tempo of 80 beats per minute. If four beats make one measure of her rhythm, how many seconds are in one measure?

   Solution. The answer is $\boxed{3}$.

   Lisa plays the piano at $\frac{80}{4} = 20$ measures per minute, or 60 seconds, so each measure takes $\frac{60}{20} = 3$ seconds.

2. Compute the smallest integer $n > 1$ whose base-2 and base-3 representations both do not contain the digit 0.

   Solution. The answer is $\boxed{7}$.

   Since base 2 uses digits 0 and 1 only, the base-2 representation of n must contain only 1s. So we list $11_2 = 3, 111_2 = 7, 1111_2 = 15, \cdots$ and express each of them in base-3. We see that $3 = 10_3$ doesn't work but $7 = 21_3$ does, so the smallest integer is $n = 7$.

3. In a room of 24 people, $5/6$ of the people are old, and $5/8$ of the people are male. At least how many people are both old and male?

   Solution. The answer is $\boxed{11}$.

   There are $24 \cdot \frac{5}{6} = 20$ people older than 20 years old, and $24 \cdot \frac{5}{8} = 15$ people who are male. By the Principle of Inclusion-Exclusion, there are at least $20 + 15 - 24 = 11$ people who satisfy both properties.

4. Juan chooses a random even integer from 1 to 15 inclusive, and Gina chooses a random odd integer from 1 to 15 inclusive. What is the probability that Juan's number is larger than Gina's number? (They choose all possible integers with equal probability.)

   Solution. The answer is $\boxed{\frac{1}{2}}$.

   Suppose Juan and Gina chooses J and G, respectively. Since each ordered pair (J, G) for which $J > G$ can be paired with a pair $(16 - J, 16 - G)$ for which $16 - J < 16 - G$, the probability of Juan's number exceeding Gina's is equal to the probability of it being smaller than Gina's. Because it's impossible for $J = G$, the desired probability is $\frac{1}{2}$.

5. Set S consists of all positive integers less than or equal to 2016. Let A be the subset of S consisting of all multiples of 6. Let B be the subset of S consisting of all multiples of 7. Compute the ratio of the number of positive integers in A but not B to the number of integers in B but not A.

   Solution. The answer is $\boxed{\frac{6}{5}}$.

   Since 2016 is a multiple of both 6 and 7, we see that $|A| = \frac{1}{6}(2016), |B| = \frac{1}{7}(2016)$, and $|A \cap B| = \frac{1}{42}(2016)$. So the number of positive integers that are in A but not in B is equal to $(\frac{1}{6} - \frac{1}{42})(2016) = \frac{6}{42}(2016)$ and the number of integers that are in B but not in A is equal to $(\frac{1}{7} - \frac{1}{42})(2016) = \frac{5}{42}(2016)$. Therefore the ratio is $\frac{6/42}{5/42} = \frac{6}{5}$.

6. Three peas form a unit equilateral triangle on a flat table. Sebastian moves one of the peas a distance d along the table to form a right triangle. Determine the minimum possible value of d.

   Solution. The answer is $\boxed{\frac{\sqrt{3}-1}{2}}$.

   Suppose that the three peas are P, E, A respectively, and the one that Sebastian moved was P (from P to P'). Suppose the midpoint of EA is M. If $\angle P' = 90^\circ$, then the minimal distance is equal to $PM - P'M = \frac{\sqrt{3}-1}{2}$. If $\angle E = 90^\circ$, then the minimal distance is equal to $EM = \frac{1}{2}$. It's easy to see that the first one is smaller, so $d_{min} = \frac{\sqrt{3}-1}{2}$.

7. Oumar is four times as old as Marta. In m years, Oumar will be three times as old as Marta will be. In another n years after that, Oumar will be twice as old as Marta will be. Compute the ratio m/n.

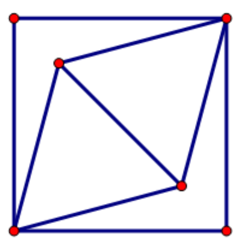
   Solution. The answer is $\boxed{\frac{1}{3}}$.

   Suppose that Marta is x years old and Oumar is $4x$ years old. So $4x+m = 3(x+m)$ and $4x+m+n = 2(x+m+n)$. Solving the first equation gives $m = \frac{1}{2}x$, and plugging that into the second gives $n = \frac{3}{2}x$. Therefore $\frac{m}{n} = \frac{1/2}{3/2} = \frac{1}{3}$.

8. Compute the area of the smallest square in which one can inscribe two non-overlapping equilateral triangles with side length 1.

   Solution. The answer is $\boxed{\frac{3}{2}}$.

   No matter how one arranges the two equilateral triangles, the distance between the two farthest vertices must be at least $2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$, achieved when the two triangles share an edge. Since the square has to contain both of these vertices, the diagonal must be at least $\sqrt{3}$ long, so the area is at least $\frac{1}{2} \cdot (\sqrt{3})^2 = \frac{3}{2}$. We can easily see that this squares works, by putting the two triangles with an overlapping edge along with the diagonal.



9. Teemu, Marcus, and Sander are signing documents. If they all work together, they would finish in 6 hours. If only Teemu and Sander work together, the work would be finished in 8 hours. If only Marcus and Sander work together, the work would be finished in 10 hours. How many hours would Sander take to finish signing if he worked alone?

   Solution. The answer is $\boxed{\frac{120}{7}}$.

   Suppose the whole job of signing is 1 unit, and the rates of Teemu, Marcus, and Sander are T, M, S respectively. We can see that $T+M+S = \frac{1}{6}, T+S = \frac{1}{8}, M+S = \frac{1}{10}$, so $S = (T+S)+(M+S)-(T+M+S) = \frac{1}{10} + \frac{1}{8} - \frac{1}{6} = \frac{7}{120}$. Therefore, it takes $\frac{1}{S} = \frac{120}{7}$ hours for Sander alone to sign all documents.

10. Triangle ABC has a right angle at B. A circle centered at B with radius BA intersects side AC at a point D different from A. Given that $AD = 20$ and $DC = 16$, find the length of BA.

Solution. The answer is $\boxed{6\sqrt{10}}$.

Since $BA = BD$, if H is the midpoint of AD, then BH is perpendicular to AD. From this we can compute $AC = 20 + 16 = 36, AH = 20 \div 2 = 10$. Since ABC is a right triangle, ABC is similar to AHB, so $\frac{AB}{AH} = \frac{AC}{AB}$, or $BA = \sqrt{AC \cdot AH} = \sqrt{360} = 6\sqrt{10}$.

11. A regular hexagon H with side length 20 is divided completely into equilateral triangles with side length 1. How many regular hexagons with sides parallel to the sides of H are formed by lines in the grid?

Solution. The answer is $\boxed{8000}$.

Define H_n as the number of vertices in a regular hexagon that has side length n. So $H_0 = 1, H_1 = 7, H_2 = 19, \cdots\ H_n = (n+1)^3 - n^3$ for all n. When the side length of a sub-hexagon of H is 1, the center of this sub-hexagon can be any vertex except for the ones on the boundary, so there are H_{19} possibilities, similarly when the side length is 2 there are H_{18} possibilities, $\cdots$, and when the side length is 20 there are H_0 possibilities. Therefore in total the number of sub-hexagons is

$$H_{19} + H_{18} + \cdots + H_0 = (20^3 - 19^3) + (19^3 - 18^3) + \cdots + (1^3 - 0^3) = 20^3 = 8000.$$

12. In convex pentagon $PEARL$, quadrilateral $PERL$ is a trapezoid with side PL parallel to side ER. The areas of triangle ERA, triangle LAP, and trapezoid $PERL$ are all equal. Compute the ratio $\dfrac{PL}{ER}$.

Solution. The answer is $\boxed{\frac{\sqrt{5}-1}{2}}$.

Suppose $PL = x, RE = y$, and the heights of trapezoid $REPL$ and triangle EAR (on RE) are u, v respectively. So we have $yv = (x+y)u = x(u+v)$, which gives $yu = xv$, and $\frac{u}{v} = \frac{x}{y} = \frac{y}{x+y}$. If $\frac{x}{y} = k$ then $\frac{y}{x+y} = \frac{1}{k+1} = k$, and since $k > 0$ we can solve the quadratic equation $k^2 + k = 1$ and get $k = \frac{\sqrt{5}-1}{2}$. Since k is equal to $\frac{ER}{OA}$, we get our desired ratio.

13. Let m and n be positive integers with $m < n$. The first two digits after the decimal point in the decimal representation of the fraction $\frac{m}{n}$ are 74. What is the smallest possible value of n?

Solution. The answer is $\boxed{27}$.

The original condition is equivalent to $0.74 \leq \frac{m}{n} < 0.75$, so we have that $\frac{1}{100} = 0.01 \geq \frac{3}{4} - \frac{m}{n} > 0$. Since $\frac{3}{4} - \frac{m}{n} = \frac{3n-4m}{4n}$, we need $4n \geq 100$ or $n \geq 25$. If $n = 25$ then we need $3 \cdot 25 - 4m = 1$ for some integer m, which is impossible. Similar situation happen with $n = 26$, but when $n = 27$, $3 \cdot 27 - 4m = 1$ has a valid solution $m = 20$, and indeed $\frac{20}{27} = 0.\overline{740}$ works. So the smallest possible value for n is 27.

14. Define functions

$$f(x,y) = \frac{x+y}{2} - \sqrt{xy} \quad \text{and} \quad g(x,y) = \frac{x+y}{2} + \sqrt{xy}.$$

Compute

$$g\left(g\left(f\left(1,3\right),\ f\left(5,7\right)\right),\ g\left(f\left(3,5\right),\ f\left(7,9\right)\right)\right).$$

Solution. The answer is $\boxed{\frac{1}{2}}$.

Notice that $f(x,y) = (\sqrt{x}-\sqrt{y})^2/2,\ g(x,y) = (\sqrt{x}+\sqrt{y})^2/2$. Therefore

$$g(f(a,b), f(c,d)) = \frac{1}{2}\left(\sqrt{\frac{(\sqrt{a}-\sqrt{b})^2}{2}} + \sqrt{\frac{(\sqrt{c}-\sqrt{d})^2}{2}}\right)^2 = \frac{1}{4}\left(\left|\sqrt{b}-\sqrt{a}\right| + \left|\sqrt{d}-\sqrt{c}\right|\right)^2$$

for all positive real numbers a, b, c, d. Plugging this result into the expression, we obtain

$$g\left(\frac{1}{4}\left(\sqrt{3}-\sqrt{1}+\sqrt{7}-\sqrt{5}\right)^2, \frac{1}{4}\left(\sqrt{5}-\sqrt{3}+\sqrt{9}-\sqrt{7}\right)^2\right)$$
$$= \frac{1}{2}\left(\frac{(\sqrt{3}-\sqrt{1}+\sqrt{7}-\sqrt{5}) + (\sqrt{5}-\sqrt{3}+\sqrt{9}-\sqrt{7})}{2}\right)^2 = \frac{(\sqrt{9}-\sqrt{1})^2}{8} = \frac{1}{2}.$$

15. Natalia plants two gardens in a 5×5 grid of points. Each garden is the interior of a rectangle with vertices on grid points and sides parallel to the sides of the grid. How many unordered pairs of two non-overlapping rectangles can Nataliia choose as gardens? (The two rectangles may share an edge or part of an edge but should not share an interior point.)

Solution. The answer is $\boxed{2550}$.

First assume the two rectangles are distinct and call them P, Q. Label the 5 horizontal lines 0,1,2,3,4 from top to bottom and 5 vertical lines 0,1,2,3,4 from left to right. Let P_{h1}, P_{h2}, Q_{h1}, Q_{h2}, P_{v1}, P_{v2}, Q_{v1}, Q_{v2} be the number corresponding to the boundaries of each rectangle such that $P_{h1} < P_{h2}, Q_{h1} < Q_{h2}, P_{v1} < P_{v2}, Q_{v1} < Q_{v2}$ (h stand for horizontal, and v stand for vertical). Notice that a rectangle is uniquely determined by the four lines that bounds the rectangle. There are $\binom{5}{2} = 10$ ways to choose the boundaries for one rectangle from one dimension, so there are $\binom{5}{2}^2 = 100$ ways to choose a rectangle and therefore $100^2 = 10000$ ways to choose an *ordered* pair of (possibly overlapping) rectangles.

By complementary counting we only need to count the number of pairs of rectangles that do intersect. Now notice that the two rectangles intersect if and only if the intervals $[P_{h1}, P_{h2}]$ and $[Q_{h1}, Q_{h2}]$ intersect and so do $[P_{v1}, P_{v2}]$ and $[Q_{v1}, Q_{v2}]$. Here we focus on one dimension, say, horizontal. There are $10^2 = 100$ ways to choose two intervals on this dimension, and again by complementary counting we only need count the number of intervals that do *not* intersect. The four ends of the interval have to use either 4 or 3 distinct numbers, and for each subset of 4 or 3 numbers there are two ways to designate them to the two intervals such that they don't intersect, except for possibly at a common point. For example, if the numbers chosen are 0, 2, 3, 4, then the two possibilities are $(P_{h1}, P_{h2}, Q_{h1}, Q_{h2}) = (0,2,3,4)$ or $(3,4,0,2)$; if the numbers are 0, 1, 3 only, then the two possibilities are $(P_{h1}, P_{h2}, Q_{h1}, Q_{h2}) = (0,1,1,3)$ or $(1,3,0,1)$. So there will be $2(\binom{5}{4} + \binom{5}{3}) = 30$ ordered pairs of non-intersecting intervals and $100 - 30 = 70$ pairs of intersecting intervals.

This implies that there are $70^2 = 4900$ ways to choose two overlapping (possibly identical) rectangles in the grid, and $10000 - 4900 = 5100$ ways that they don't overlap.

At this point, remember that the problem asks for *unordered* pairs, and each pair so far is counted exactly twice. Therefore, we divide 5100 by 2 and get 2550 as the answer.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [5] Suppose that gold satisfies the relation $p = v + v^2$, where p is the price and v is the volume. How many pieces of gold with volume 1 can be bought for the price of a piece with volume 2?

   Solution. The answer is $\boxed{3}$.

   The price of the gold with volume 2 has price $2 + 2^2 = 6$, and the price of the gold with volume 1 has price $1 + 1^2 = 2$, so one can buy $\frac{6}{2} = 3$ pieces of gold.

2. [5] Find the smallest prime number with each digit greater or equal to 8.

   Solution. The answer is $\boxed{89}$.

   The problem statement is equivalent to finding the smallest prime comprising of digit 8 and 9 only. Since the number has to be odd the unit digit has to be 9. 9 itself is obviously not a prime, but the second smallest candidate, 89, is. Therefore the answer is 89.

3. [5] What fraction of regular hexagon $ZUMING$ is covered by both quadrilateral $ZUMI$ and quadrilateral $MING$?

   Solution. The answer is $\boxed{\frac{1}{6}}$.

   Suppose that the center of the hexagon is O. Since ZI and MG both goes through O, the intersection of the two quadrilaterals is the triangle IMO, which is $\frac{1}{6}$ of the entire area.

2.4.2 Round 2

4. [7] The two smallest positive integers expressible as the sum of two (not necessarily positive) perfect cubes are $1 = 1^3 + 0^3$ and $2 = 1^3 + 1^3$. Find the next smallest positive integer expressible in this form.

   Solution. The answer is $\boxed{7}$.

   Notice that $7 = 2^3 + (-1)^3$, and no smaller number greater than 2 can be expressed in this form since the smallest difference of two nonnegative perfect cubes that is greater than 2 is $2^3 - 1 = 7$. Hence 7 is the smallest such number after 1 and 2.

5. [7] In how many ways can the numbers $1, 2, 3$, and 4 be written in a row such that no two adjacent numbers differ by exactly 1?

Solution. The answer is $\boxed{2}$.

Note that 2 and 3 can each have only one neighbor, so they cannot be in the middle. Hence the middle two can be 14 or 41, and the only valid ways are 3142 and 2413, for 2 ways in total.

6. [7] A real number is placed in each cell of a grid with 3 rows and 4 columns. The average of the numbers in each column is 2016, and the average of the numbers in each row is a constant x. Compute x.

Solution. The answer is $\boxed{2016}$.

The average of the numbers in each column is 2016, so the average of all numbers is 2016. Similarly the average of all numbers is also x since the average of each row is x. Thus $x = 2016$.

2.4.3 Round 3

7. [9] Fardin is walking from his home to his office at a speed of 1 meter per second, expecting to arrive exactly on time. When he is halfway there, he realizes that he forgot to bring his pocketwatch, so he runs back to his house at 2 meters per second. If he now decides to travel from his home to his office at x meters per second, find the minimum x that will allow him to be on time.

Solution. The answer is $\boxed{4}$.

When he remembers his pocketwatch, Fardin has already used up half of his total time, and when he ran back home he travelled half the entire distance at double the speed, so he used another quarter of his time. When he departs again from his home, he has only $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ of the time left, so he needs to go 4 times as fast as he started, or 4 meters per second.

8. [9] In triangle ABC, the angle bisector of $\angle B$ intersects the perpendicular bisector of AB at point P on segment AC. Given that $\angle C = 60^\circ$, determine the measure of $\angle CPB$ in degrees.

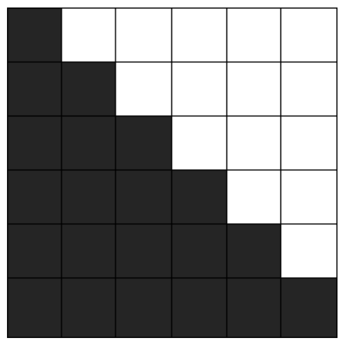
Solution. The answer is $\boxed{80^\circ}$.

From the problem statement we see that $AP = BP$ and $\angle A = \angle ABP = \angle CBP = x$, and $\angle CPB = \angle A + \angle ABP = 2x$. Since $x + 2x + 60^\circ = 180^\circ$, $x = 40^\circ$ and $\angle CPB = 80^\circ$.

9. [9] Katie colors each of the cells of a 6×6 grid either black or white. From top to bottom, the number of black squares in each row are 1, 2, 3, 4, 5, and 6, respectively. From left to right, the number of black squares in each column are 6, 5, 4, 3, 2, and 1, respectively. In how many ways could Katie have colored the grid?

Solution. The answer is $\boxed{1}$.

Since both the last row and first column have 6 black squares, we see that all squares on that row and column are black. Then since both the first row and last column have 1 black square, and the first square of the first row and last square of the last column are already black, all other squares on that row must be white. Similarly we can apply our argument to the row/column with 5, 2, 4, 3 black squares in that order, and find out that there is only one way Katie could have colored the grid:



2.4.4 Round 4

10. [11] Lily stands at the origin of a number line. Each second, she either moves 2 units to the right or 1 unit to the left. At how many different places could she be after 2016 seconds?

Solution. The answer is $\boxed{2017}$.

Suppose Lily moved right for x seconds and left for $(2016 - x)$ seconds, so at the end she is on $x - 2(2016 - x) = 3x - 4032$. Since x is an integer between 0 and 2016, there are 2017 possibilities for x and thus 2017 possibilities for the final position.

11. [11] There are 125 politicians standing in a row. Each either always tells the truth or always lies. Furthermore, each politician (except the leftmost politician) claims that at least half of the people to his left always lie. Find the number of politicians that always lie.

Solution. The answer is $\boxed{62}$.

If the leftmost person is a liar, then the second person is telling the truth and so is the third person, then the fourth person is a liar, fifth person is a truth-teller, sixth person is a liar, etc. So there will be $\frac{125-3}{2} + 1 = 62$ liars in this case.

If the leftmost person is telling the truth, then the second person is a liar, third person is telling the truth, and the rest is identical to the first case. So there will also be 62 liars in this case.

Since there can only be 62 liars in either case, it is the only possibility and therefore our answer.

12. [11] Two concentric circles with radii 2 and 5 are drawn on the plane. What is the side length of the largest square whose area is contained entirely by the region between the two circles?

Solution. The answer is $\boxed{\frac{14}{5}}$.

The largest square $ABCD$ must satisfy that a side AB must be the chord of the larger circle and the opposite side CD must be tangent to the smaller circle. Let the midpoint of AB, CD be M, N respectively, then O, M, N should be collinear, and N is the point of tangency to the smaller circle. If the side length of $ABCD$ is x, then by expressing the length of OA using Pythagorean Theorem, we have

$$\left(\frac{x}{2}\right)^2 + (2+x)^2 = 5^2.$$

This equation has two solutions: $x = -6$ and $x = \frac{14}{5}$. Since x is positive, we disregard the negative solution and obtain $\frac{14}{5}$ as the desired side length.

2.4.5 Round 5

13. [13] Initially, the three numbers 20, 201, and 2016 are written on a blackboard. Each minute, Zhuo selects two of the numbers on the board and adds 1 to each. Find the minimum n for which Zhuo can make all three numbers equal to n.

Solution. The answer is $\boxed{2197}$.

Suppose that Zhuo adds the 20 and 201 sides x times, 20 and 2016 sides y times, and 201 and 2016 sides z times, then we have

$$x + y = n - 20, x + z = n - 201, y + z = n - 2016.$$

This implies that $n - 20, n - 201, n - 2016$ are three sides of a (possibly degenrate) triangle, which means $(n - 20) \geq (n - 201) + (n - 2016)$, or $n \geq 201 + 2016 - 20 = 2197$. This minimum is established when $x = 1996, y = 181, z = 0$.

14. [13] Call a three-letter string *rearrangeable* if, when the first letter is moved to the end, the resulting string comes later alphabetically than the original string. For example, AAA and BAA are not rearrangeable, while ABB is rearrangeable. How many three-letters strings with (not necessarily distinct) uppercase letters are rearrangeable?

Solution. The answer is $\boxed{8775}$.

Suppose the word is $\overline{\alpha\beta\gamma}$, where α, β, γ represent an uppercase letter. We then treat the alphabet as base-26 integers, and therefore the problem condition is then equivalent to $\overline{\alpha\beta\gamma} < \overline{\beta\gamma\alpha}$. This requires (a) $\alpha < \beta$ or (b) $\alpha = \beta$ and $\beta < \gamma$. In case (a), there are $\binom{26}{2}$ ways to choose two different letters for α and β and 26 ways to choose a letter for γ. In case (b), there are $\binom{26}{2}$ ways to choose two different letters for β and ω and α will be determined by β. Therefore there are $\binom{26}{2}\cdot(26+1) = 13\cdot 25\cdot 27 = 8775$ possible strings.

Alternatively, since the probability that the new string comes strictly later is equal to that the new string comes strictly earlier, we only need to discount the case where the two strings are equal, which requires all three letters to be the same. Therefore there are $\frac{26^3-26}{2} = 8775$ possible strings.

15. [13] Triangle ABC is an isosceles right triangle with $\angle C = 90°$ and $AC = 1$. Points D, E, and F are chosen on sides BC, CA, and AB, respectively, such that AEF, BFD, CDE, and DEF are isosceles right triangles. Find the sum of all distinct possible lengths of segment DE.

Solution. The answer is $\boxed{\frac{5}{6}\sqrt{2}}$.

Since C is a right angle, it's clear that $CD = CE$. Now we separate into cases.

Case 1: $DF = EF$. This means that $\angle DFE = 90°$, and $CDFE$ is a square. It's not difficult to see that in this case AEF and BDF are both right isosceles triangles, and $DF = \frac{1}{2}$. Therefore $DE = \frac{\sqrt{2}}{2}$.

Case 2: $DF \neq EF$. WLOG assume then that $DF = DE$. Then we need $AE = EF, DF = FB$. Since $DE = \sqrt{2}CD = \sqrt{2}CE$, and $AE = EF = \sqrt{2}DE = \sqrt{2}DF = DB$, we have $AE = 2CE, BD = 2CD$, so $CE = CD = \frac{1}{3}$. Therefore $DE = \frac{\sqrt{2}}{3}$.

Adding up the two cases, we get $(\frac{1}{2} + \frac{1}{3})\sqrt{2} = \frac{5}{6}\sqrt{2}$ as the answer.

2.4.6 Round 6

16. [15] Let p, q, and r be prime numbers such that $pqr = 17(p + q + r)$. Find the value of the product pqr.

Solution. The answer is $\boxed{646}$.

Since pqr is a multiple of 17, at least one of p, q, r is 17. Without loss of generality, let $r = 17$, so we have $pq = p+q+17$, or $(p-1)(q-1) = 18$. Assuming $p \leq q$, we have that $(p-1, q-1) = (1, 18), (2, 9)$ or $(3, 6)$, corresponding to $(p, q) = (2, 19), (3, 10)$ or $(4, 7)$. Since 10 and 4 are not prime, p, q, r has to be a permutation of $2, 19, 17$, and therefore their product is $2 \cdot 19 \cdot 17 = 646$.

17. [15] A cylindrical cup containing some water is tilted 45 degrees from the vertical. The point on the surface of the water closest to the bottom of the cup is 6 units away. The point on the surface of the water farthest from the bottom of the cup is 10 units away. Compute the volume of the water in the cup.

    Solution. The answer is $\boxed{32\pi}$.

    Since the cup is tilted exactly 45 degrees, we see that the diameter of the cup is equal to the difference between the highest point and lowest point from the bottom. which is $10 - 6 = 4$ inches, so the diameter is $\frac{4}{2} = 2$ inches. Now notice that if we replace the water with ice (with equal volume), then we can place another identical copy of the ice block on top of the current one and create a cylinder of height $10 + 6 = 16$ inches. Since the entire cylinder has volume $2^2 \cdot 16 \cdot \pi = 64\pi$ cubic inches, the volume of water originally would be $64\pi \div 2 = 32\pi$ cubic inches.

18. [15] Each dot in an equilateral triangular grid with 63 rows and $2016 = \frac{1}{2} \cdot 63 \cdot 64$ dots is colored black or white. Every unit equilateral triangle with three dots has the property that exactly one of its vertices is colored black. Find all possible values of the number of black dots in the grid.

    Solution. The answer is $\boxed{672}$.

    Notice that knowing the configuration of a row (except for the first one with only one dot) uniquely determines the next row. If the single dot on the first row is black, then both dots on the second row must be white. On the third row the dots must be white, black, white in that order, and on fourth black, white, white, black, fifth white, white, black, white, white, etc. In general the number of black dots on each row will be $1, 0, 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, \cdots, 19, 18, 19, 20, 19, 20, 21, 20, 21$, where each number repeats itself three times (except for 0 and 21) on the row with the same parity as itself, so the sum will be $3 \cdot \frac{(1+20)\cdot 20}{2} + 2 \cdot 21 = 672$.

    If the dot on the first row is white, then we can assume (without loss of generality) that the left dot on the second row is black. By a similar logic we get that the number of black dots on each row is $0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \cdots, 20, 20, 20, 21, 21$, where each number between 1 and 20 inclusive repeats itself three times. It's not difficult to see that this yields the same sum. Therefore there will be 672 black dots in the grid.

    Note: In general, the number of black dots will be either $\lfloor \frac{N}{3} \rfloor$ or $\lceil \frac{N}{3} \rceil$, where N is the number of dots in the triangular grid. A proof follows a similar line as the solution.

2.4.7 Round 7

19. [18] Tomasz starts with the number 2. Each minute, he either adds 2 to his number, subtracts 2 from his number, multiplies his number by 2, or divides his number by 2. Find the minimum number of minutes he will need in order to make his number equal 2016.

Solution. The answer is $\boxed{11}$.

Let a_n be the number Tomasz has after n minutes. When $n = 11$, we can make $a_0 = 2, a_1 = 4, a_2 = 8, a_3 = 16, a_4 = 32, a_5 = 64, a_6 = 128, a_7 = 126, a_8 = 252, a_9 = 504, a_{10} = 1008, a_{11} = 2016$. Now we show that $n = 10$ is impossible. In fact, since $a_1 \le 4$, if $a_n \neq a_{n-1} \cdot 2$ for any $n > 2$, then $a_{10} \le (4+2) \cdot 2^8 = 1536 < 2016$, but otherwise $a_{10} = 2^{10} = 2048 > 2016$, which means that it is impossible for us to achieve 2016. This means that the minimum number of minutes is indeed 11.

20. [18] The edges of a regular octahedron $ABCDEF$ are painted with 3 distinct colors such that no two edges with the same color lie on the same face. In how many ways can the octahedron be painted? Colorings are considered different under rotation or reflection.

Solution. The answer is $\boxed{24}$.

WLOG let the faces be $ABC, ACD, ADE, AEB, FBC, FCD, FDE, FEB$. Also let the colors be red, green and blue. Consider the coloring of edges AB, AC, AD, AE, which, by Pigeonhole Principle, includes two edges of the same color. However, no two edges of the same color can be the same color. This leaves us with two cases.

Case 1: Only two colors are used in these four edges. Then AB, AD takes on a color and AC, AE takes on another color. There are $3 \cdot 2 = 6$ ways to choose the two colors and WLOG suppose AB is red and AC is green. This forces BC, CD, DE, EB to be all blue, and therefore FB, FC, FD, FE alternates red and green. There are two ways to color the those four edges, so there are $6 \cdot 2 = 12$ ways in this case.

Case 2: All three colors are used in these four edges. Then either AB, AD or AC, AE takes the same color. There are two ways to choose which pair, three ways to choose the color that the pair takes, and two ways to choose the color for the remaining two edges. WLOG both AB and AD takes blue, AC takes red and AE takes green. Then BC, CD, DE, EB have to be green, green, red, red respectively. Then we have FB, FD having to be blue, and FC, FE are red and green respectively. Since all the remaining colors are uniquely determined by the first four edges, there are $2 \cdot 3 \cdot 2 = 12$ ways in this case.

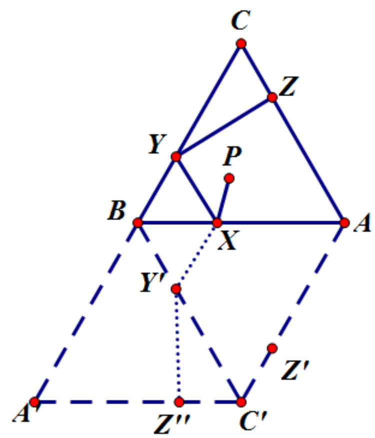
Adding up the two cases, we have $12 + 12 = 24$ ways in total.

21. [18] Jacob is trapped inside an equilateral triangle ABC and must visit each edge of triangle ABC at least once. (Visiting an edge means reaching a point on the edge.) His distances to sides AB, BC, and CA are currently 3, 4, and 5, respectively. If he does not need to return to his starting point, compute the least possible distance that Jacob must travel.

Solution. The answer is $\boxed{15}$.

Suppose that Jacob starts at point P. Then since the area of triangle ABC is equal to $\frac{1}{2}(3a+4a+5a) = \frac{1}{2}(ha)$, where a is the side length of the triangle and h is its height, so $h = 3 + 4 + 5 = 12$.

First assume that Jacob visits AB first at point X, then BC, CA at Y, Z respectively. Reflect C, Y, Z across line AB to get C', Y', Z', and then reflect Z' across BC' to get Z''. It is not difficult to see that $PX + XY + YZ = AX + XY' + Y'Z'' \le PZ''$ (with equality achieved when P, X, Y', Z'' are all collinear and Z'' lies on the reflection line AC' over BC', which is a line that is parallel to AB that is h apart from it. Since P is 3 away from AB, it is $3 + h = 15$ away from the line Z'' is on, so $PZ'' \ge 15$, with equality case achieved when PZ'' is perpendicular to AB. Similarly we can do the case for all possible sequences of visits and get $3 + h, 4 + h, 5 + h$ as the respective lower bounds. Therefore the minimum is $3 + h = 15$ and it is not difficult to achieve this bound when we reflect the points back onto the sides of ABC.



2.4.8 Round 8

22. [22] Four integers a, b, c, and d with $a \leq b \leq c \leq d$ satisfy the property that the product of any two of them is equal to the sum of the other two. Given that the four numbers are not all equal, determine the 4-tuple (a, b, c, d).

Solution. The answer is $\boxed{(-1, -1, -1, 2)}$.

We can in fact solve the problem in real numbers. Let $S = a + b + c + d$. From the problem condition we see that $ab + cd = (c + d) + (a + b) = S$. Similarly $ac + bd = ad + bc = S$. Notice that $(a + b)(c + d) = (ac + bd) + (ad + bc) = 2S$, and similarly $(a + c)(b + d) = (a + d)(b + c) = 2S$. By Vieta's Theorem we get that all three pairs $(a + b, c + d), (a + c, b + d), (a + d, b + c)$ are solutions of the equation $x^2 - Sx + 2S = 0$, so all three pairs must be equal to (u, v) or (v, u) for some real number u, v. Since there are at most two distinct values in $\{a + b, a + c, a + d\}$, by Pigeonhole Principle at least two of b, c, d are the same. WLOG let $b = c$. Then by making the same observation on a, b, d gives that either there are two pairs of equal numbers among the four three of the four numbers are equal. If it's the first possibility then a, b, c, d is a permutation of p, p, q, q for some distinct reals p, q, but it's not difficult to see that $\{p + p, q + q\} \neq \{p + q, p + q\}$. This leaves the second possibility (where the numbers is a permutation of p, p, p, q for $p \neq q$), which satisfy the desired property.

Now it remains to solve the following system of equations:

$$p + p = pq, p + q = p^2.$$

The first equation gives $p = 0$ or $q = 2$. If $p = 0$ then $q = 0$, which contradicts the requirement that $p \neq q$. If $q = 2$, then $p + 2 = p^2$ has two solutions -1 or 2, but since $p \neq q = 2$ we are forced to have $p = -1$, and by putting the numbers in nondecreasing order we get $(-1, -1, -1, 2)$ as the one and only possible quadruple.

23. [22] In equilateral triangle ABC, points D, E, and F lie on sides BC, CA, and AB, respectively, such that $BD = 4$ and $CD = 5$. If DEF is an isosceles right triangle with right angle at D, compute $EA + FA$.

Solution. The answer is $\boxed{27 - 9\sqrt{3}}$.

The side length of the triangle is $4 + 5 = 9$. Rotate triangle CDE by 90 degrees to $C'DF$, where $CD = C'D = 5$. $C'D$ and BD are perpendicular. Draw rectangle $XFYD$ such that X, Y lie on $C'D$ and BD respectively. Suppose $BF = p, CE = C'F = q$, then

$$BD = BY + FX = \frac{1}{2}p + \frac{\sqrt{3}}{2}q,$$

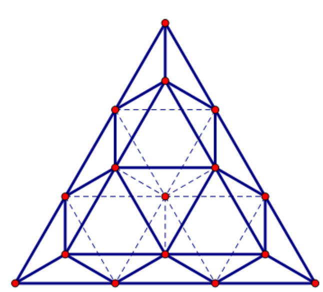
$$C'D = FY + C'X = \frac{\sqrt{3}}{2}p + \frac{1}{2}q.$$

Adding up the two equations, we have $BD + C'D = 9 = \frac{\sqrt{3}+1}{2}(p + q)$, so $p + q = 9 \div \frac{\sqrt{3}+1}{2} = 9\sqrt{3} - 9$, and thus $EA + FA = (9 - p) + (9 - q) = 18 - (p + q) = 27 - 9\sqrt{3}$.

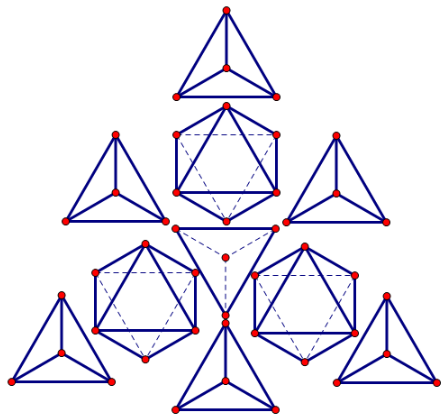
24. [22] On each edge of a regular tetrahedron, four points that separate the edge into five equal segments are marked. There are sixteen planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these sixteen planes, how many new tetrahedrons are produced?

Solution. The answer is $\boxed{45}$.

Consider the four cuts in the same direction, which cuts the tetrahedron into 4 frustums of different size and a unit regular tetrahedron. Now put the remaining four frustums separately such that one angle is pointing up and order them in increasing size. The i-th smallest frustum has a smaller base being an equilateral triangle with side length i and a larger base being an equilateral triangle with side length $i + 1$. Now perform the cuts in the remaining directions on each frustum. Below is an example for the cuts on second smallest frustum.



After that, on the smaller base there are $\frac{i(i-1)}{2}$ downward-pointing triangles and each of them faces an intersection on the larger base, meaning that each correspond to a tetrahedron; on the larger base there are $\frac{(i+1)(i+2)}{2}$ upward-pointing triangles, each facing an intersection and therefore corresponding to a tetrahedron as well. The $\frac{i(i+1)}{2}$ upward-pointing triangles on the smaller base and downward-pointing triangles on the larger base correspond to each other and each pair correspond to a regular *octahedron*. For example, when we separate all the pieces in the second frustum, we should get the following pieces: (in their relative position in order to aid visualization)



Finally, by summing $\frac{i(i-1)}{2}$ and $\frac{(i+1)(i+2)}{2}$ over $i = 0, 1, 2, 3, 4$, we get that there are $(0 + 0 + 1 + 3 + 6) + (1 + 3 + 6 + 10 + 15) = 45$ tetrahedrons (and $0 + 1 + 3 + 6 + 10 = 20$ octahedrons).