Exeter Math Club Competition
January 21, 2017

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- **Judges** Zuming Feng, Greg Spanier

Chapter 1

EMC^2 2017 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Ben was trying to solve for x in the equation $6 + x = 1$. Unfortunately, he was reading upside-down and misread the equation as $1 = x + 9$. What is the positive difference between Ben's answer and the correct answer?

2. Anjali and Meili each have a chocolate bar shaped like a rectangular box. Meili's bar is four times as long as Anjali's, while Anjali's is three times as wide and twice as thick as Meili's. What is the ratio of the volume of Anjali's chocolate to the volume of Meili's chocolate?

3. For any two nonnegative integers m, n, not both zero, define $m?n = m^n + n^m$. Compute the value of $((2?0)?1)?7$.

4. Eliza is making an in-scale model of the Phillips Exeter Academy library, and her prototype is a cube with side length 6 inches. The real library is shaped like a cube with side length 120 feet, and it contains an entrance chamber in the front. If the chamber in Eliza's model is 0.8 inches wide, how wide is the real chamber, in feet?

5. One day, Isaac begins sailing from Marseille to New York City. On the exact same day, Evan begins sailing from New York City to Marseille along the exact same route as Isaac. If Marseille and New York are exactly 3000 miles apart, and Evan sails exactly 40 miles per day, how many miles must Isaac sail each day to meet Evan's ship in 30 days?

6. The conversion from Celsius temperature C to Fahrenheit temperature F is:
$$F = 1.8C + 32.$$
If the lowest temperature at Exeter one day was 20°F, and the next day the lowest temperature was 5°C higher, what would be the lowest temperature that day, in degrees Fahrenheit?

7. In a school, 60% of the students are boys and 40% are girls. Given that 40% of the boys like math and 50% of the people who like math are girls, what percentage of girls like math?

8. Adam and Victor go to an ice cream shop. There are four sizes available (kiddie, small, medium, large) and seventeen different flavors, including three that contain chocolate. If Victor insists on getting a size at least as large as Adam's, and Adam refuses to eat anything with chocolate, how many different ways are there for the two of them to order ice cream?

9. There are 10 (not necessarily distinct) positive integers with arithmetic mean 10. Determine the maximum possible range of the integers. (The range is defined to be the nonnegative difference between the largest and smallest number within a list of numbers.)

10. Find the sum of all distinct prime factors of $11! - 10! + 9!$.

11. Inside regular hexagon $ZUMING$, construct square $FENG$. What fraction of the area of the hexagon is occupied by rectangle $FUME$?

12. How many ordered pairs (x, y) of nonnegative integers satisfy the equation $4^x \cdot 8^y = 16^{10}$?

13. In triangle ABC with $BC = 5$, $CA = 13$, and $AB = 12$, Points E and F are chosen on sides AC and AB, respectively, such that $EF \parallel BC$. Given that triangle AEF and trapezoid $EFBC$ have the same perimeter, find the length of EF.

14. Find the number of two-digit positive integers with exactly 6 positive divisors. (Note that 1 and n are both counted among the divisors of a number n.)

15. How many ways are there to put two identical red marbles, two identical green marbles, and two identical blue marbles in a row such that no red marble is next to a green marble?

16. Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

17. The tiny island nation of Konistan is a cone with height twelve meters and base radius nine meters, with the base of the cone at sea level. If the sea level rises four meters, what is the surface area of Konistan that is still above water, in square meters?

18. Nicky likes to doodle. On a convex octagon, he starts from a random vertex and doodles a path, which consists of seven line segments between vertices. At each step, he chooses a vertex randomly among all unvisited vertices to visit, such that the path goes through all eight vertices and does not visit the same vertex twice. What is the probability that this path does not cross itself?

19. In a right-angled trapezoid $ABCD$, $\angle B = \angle C = 90°$, $AB = 20, CD = 17$, and $BC = 37$. A line perpendicular to DA intersects segment BC and DA at P and Q respectively and separates the trapezoid into two quadrilaterals with equal area. Determine the length of BP.

20. A sequence of integers a_i is defined by $a_1 = 1$ and $a_{i+1} = 3i - 2a_i$ for all integers $i \geq 1$. Given that $a_{15} = 5476$, compute the sum $a_1 + a_2 + a_3 + \cdots + a_{15}$.

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. Chris goes to Matt's Hamburger Shop to buy a hamburger. Each hamburger must contain exactly one bread, one lettuce, one cheese, one protein, and at least one condiment. There are two kinds of bread, two kinds of lettuce, three kinds of cheese, three kinds of protein, and six different condiments: ketchup, mayo, mustard, dill pickles, jalapeños, and Matt's Magical Sunshine Sauce. How many different hamburgers can Chris make?

2. The degree measures of the interior angles in convex pentagon $NICKY$ are all integers and form an increasing arithmetic sequence in some order. What is the smallest possible degree measure of the pentagon's smallest angle?

3. Daniel thinks of a two-digit positive integer x. He swaps its two digits and gets a number y that is less than x. If 5 divides $x - y$ and 7 divides $x + y$, find all possible two-digit numbers Daniel could have in mind.

4. At the Lio Orympics, a target in archery consists of ten concentric circles. The radii of the circles are $1, 2, 3, \ldots, 9$, and 10 respectively. Hitting the innermost circle scores the archer 10 points, the next ring is worth 9 points, the next ring is worth 8 points, all the way to the outermost ring, which is worth 1 point. If a beginner archer has an equal probability of hitting any point on the target and never misses the target, what is the probability that his total score after making two shots is even?

5. Let $F(x) = x^2 + 2x - 35$ and $G(x) = x^2 + 10x + 14$. Find all distinct real roots of $F(G(x)) = 0$.

6. One day while driving, Ivan noticed a curious property on his car's digital clock. The sum of the digits of the current hour equaled the sum of the digits of the current minute. (Ivan's car clock shows 24-hour time; that is, the hour ranges from 0 to 23, and the minute ranges from 0 to 59.) For how many possible times of the day could Ivan have observed this property?

7. Qi Qi has a set Q of all lattice points in the coordinate plane whose x- and y-coordinates are between 1 and 7 inclusive. She wishes to color 7 points of the set blue and the rest white so that each row or column contains exactly 1 blue point and no blue point lies on or below the line $x + y = 5$. In how many ways can she color the points?

8. A piece of paper is in the shape of an equilateral triangle ABC with side length 12. Points A_B and B_A lie on segment AB, such that $AA_B = 3$, and $BB_A = 3$. Define points B_C and C_B on segment BC and points C_A and A_C on segment CA similarly. Point A_1 is the intersection of A_CB_C and A_BC_B. Define B_1 and C_1 similarly. The three rhombi — $AA_BA_1A_C, BB_CB_1B_A, CC_AC_1C_B$ — are cut from triangle ABC, and the paper is folded along segments A_1B_1, B_1C_1, C_1A_1, to form a tray without a top. What is the volume of this tray?

9. Define $\{x\}$ as the fractional part of x. Let S be the set of points (x, y) in the Cartesian coordinate plane such that $x + \{x\} \leq y$, $x \geq 0$, and $y \leq 100$. Find the area of S.

10. Nicky likes dolls. He has 10 toy chairs in a row, and he wants to put some indistinguishable dolls on some of these chairs. (A chair can hold only one doll.) He doesn't want his dolls to get lonely, so he wants each doll sitting on a chair to be adjacent to at least one other doll. How many ways are there for him to put any number (possibly none) of dolls on the chairs? Two ways are considered distinct if and only if there is a chair that has a doll in one way but does not have one in the other.

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

1. Compute $2017 + 7201 + 1720 + 172$.

2. A number is called *downhill* if its digits are distinct and in descending order. (For example, 653 and 8762 are downhill numbers, but 97721 is not.) What is the smallest downhill number greater than 86432?

3. Each vertex of a unit cube is sliced off by a planar cut passing through the midpoints of the three edges containing that vertex. What is the ratio of the number of edges to the number of faces of the resulting solid?

4. In a square with side length 5, the four points that divide each side into five equal segments are marked. Including the vertices, there are 20 marked points in total on the boundary of the square. A pair of distinct points A and B are chosen randomly among the 20 points. Compute the probability that $AB = 5$.

5. A positive two-digit integer is one less than five times the sum of its digits. Find the sum of all possible such integers.

6. Let
$$f(x) = 5^{4^{3^{2^x}}}.$$
Determine the greatest possible value of L such that $f(x) > L$ for all real numbers x.

7. If $\overline{AAAA}+\overline{BB} = \overline{ABCD}$ for some distinct base-10 digits A, B, C, D that are consecutive in some order, determine the value of $\overline{ABCD}$. (The notation $\overline{ABCD}$ refers to the four-digit integer with thousands digit A, hundreds digit B, tens digit C, and units digit D.)

8. A regular tetrahedron and a cube share an inscribed sphere. What is the ratio of the volume of the tetrahedron to the volume of the cube?

9. Define $\lfloor x \rfloor$ as the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$ as the fractional part of x. If $\lfloor x^2 \rfloor = 2\lfloor x \rfloor$ and $\{x^2\} = \frac{1}{2}\{x\}$, determine all possible values of x.

10. Find the largest integer $N > 1$ such that it is impossible to divide an equilateral triangle of side length 1 into N smaller equilateral triangles (of possibly different sizes).

11. Let f and g be two quadratic polynomials. Suppose that f has zeroes 2 and 7, g has zeroes 1 and 8, and $f - g$ has zeroes 4 and 5. What is the product of the zeroes of the polynomial $f + g$?

12. In square $PQRS$, points A, B, C, D, E, and F are chosen on segments PQ, QR, PR, RS, SP, and PR, respectively, such that $ABCDEF$ is a regular hexagon. Find the ratio of the area of $ABCDEF$ to the area of $PQRS$.

13. For positive integers m and n, define $f(m, n)$ to be the number of ways to distribute m identical candies to n distinct children so that the number of candies that any two children receive differ by at most 1. Find the number of positive integers n satisfying the equation $f(2017, n) = f(7102, n)$.

14. Suppose that real numbers x and y satisfy the equation

$$x^4 + 2x^2y^2 + y^4 - 2x^2 + 32xy - 2y^2 + 49 = 0.$$

Find the maximum possible value of $\frac{y}{x}$.

15. A point P lies inside equilateral triangle ABC. Let A', B', C' be the feet of the perpendiculars from P to BC, AC, AB, respectively. Suppose that $PA = 13, PB = 14$, and $PC = 15$. Find the area of $A'B'C'$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. **[5]** If $2m = 200cm$ and $m \neq 0$, find c.

2. **[5]** A right triangle has two sides of lengths 3 and 4. Find the smallest possible length of the third side.

3. **[5]** Given that $20(x + 17) = 17(x + 20)$, determine the value of x.

1.4.2 Round 2

4. **[7]** According to the Egyptian Metropolitan Culinary Community, food service is delayed on $\frac{2}{3}$ of flights departing from Cairo airport. On average, if flights with delayed food service have twice as many passengers per flight as those without, what is the probability that a passenger departing from Cairo airport experiences delayed food service?

5. **[7]** In a positive geometric sequence $\{a_n\}$, $a_1 = 9, a_9 = 25$. Find the integer k such that $a_k = 15$.

6. **[7]** In the Delicate, Elegant, and Exotic Music Organization, pianist Hans is selling two types of flowers with different prices (per flower): magnolias and myosotis. His friend Alice originally plans to buy a bunch containing 50% more magnolias than myosotis for \$50, but then she realizes that if she buys 50% less magnolias and 50% more myosotis than her original plan, she would still need to pay the same amount of money. If instead she buys 50% more magnolias and 50% less myosotis than her original plan, then how much, in dollars, would she need to pay?

1.4.3 Round 3

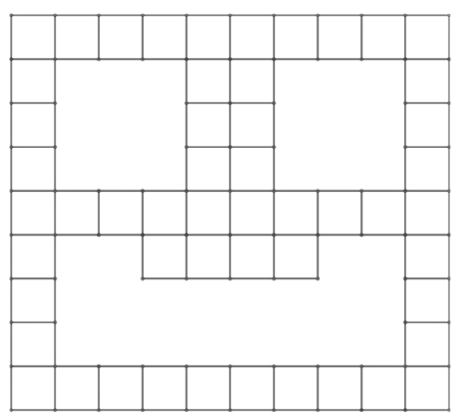
7. **[9]** In square $ABCD$, point P lies on side AB such that $AP = 3, BP = 7$. Points Q, R, S lie on sides BC, CD, DA respectively such that $PQ = PR = PS = AB$. Find the area of quadrilateral $PQRS$.

8. **[9]** Kristy is thinking of a number $n < 10^4$ and she says that 143 is one of its divisors. What is the smallest number greater than 143 that could divide n?

9. **[9]** A positive integer n is called *special* if the product of the n smallest prime numbers is divisible by the sum of the n smallest prime numbers. Find the sum of the three smallest special numbers.

1.4.4 Round 4

10. **[11]** In the diagram below, all adjacent points connected with a segment are unit distance apart. Find the number of squares whose vertices are among the points in the diagram and whose sides coincide with the drawn segments.



11. **[11]** Geyang tells Junze that he is thinking of a positive integer. Geyang gives Junze the following clues:

    - My number has three distinct odd digits.
    - It is divisible by each of its three digits, as well as their sum.

    What is the sum of all possible values of Geyang's number?

12. **[11]** Regular octagon $ABCDEFGH$ has center O and side length 2. A circle passes through A, B, and O. What is the area of the part of the circle that lies outside of the octagon?

1.4.5 Round 5

13. **[13]** Kelvin Amphibian, a not-frog who lives on the coordinate plane, likes jumping around. Each step, he jumps either to the spot that is 1 unit to the right and 2 units up, or the spot that is 2 units to the right and 1 unit up, from his current location. He chooses randomly among these two choices with equal probability. He starts at the origin and jumps for a long time. What is the probability that he lands on $(10, 8)$ at some time in his journey?

14. **[13]** Points A, B, C, and D are randomly chosen on the circumference of a unit circle. What is the probability that line segments AB and CD intersect inside the circle?

15. **[13]** Let $P(x)$ be a quadratic polynomial with two consecutive integer roots. If it is also known that $\frac{P(2017)}{P(2016)} = \frac{2016}{2017}$, find the larger root of $P(x)$.

1.4.6 Round 6

16. **[15]** Let S_n be the sum of reciprocals of the integers between 1 and n inclusive. Find a triple (a, b, c) of positive integers such that $S_{2017} \cdot S_{2017} - S_{2016} \cdot S_{2018} = \frac{S_a + S_b}{c}$.

17. **[15]** Suppose that m and n are both positive integers. Alec has m standard 6-sided dice, each labelled 1 to 6 inclusive on the sides, while James has n standard 12-sided dice, each labelled 1 to 12 inclusive on the sides. They decide to play a game with their dice. They each toss all their dice simultaneously and then compute the sum of the numbers that come up on their dice. Whoever has a higher sum wins (if the sums are equal, they tie). Given that both players have an equal chance of winning, determine the minimum possible value of mn.

18. **[15]** Overlapping rectangles $ABCD$ and $BEDF$ are congruent to each other and both have area 1. Given that A, C, E, F are the vertices of a square, find the area of the square.

1.4.7 Round 7

19. **[18]** Find the number of solutions to the equation

$$|||\dots|||||x|+1|-2|+3|-4|+\cdots-98|+99|-100|=0.$$

20. **[18]** A *split* of a positive integer in base 10 is the separation of the integer into two nonnegative integers, allowing leading zeroes. For example, 2017 can be split into 2 and 017 (or 17), 20 and 17, or 201 and 7. A split is called *squarish* if both integers are nonzero perfect squares. 49 and 169 are the two smallest perfect squares that have a squarish split (4 and 9, 16 and 9 respectively). Determine all other perfect squares less than 2017 with at least one squarish split.

21. **[18]** Polynomial $f(x) = 2x^3 + 7x^2 - 3x + 5$ has zeroes a, b, and c. Cubic polynomial $g(x)$ with x^3-coefficient 1 has zeroes a^2, b^2, and c^2. Find the sum of coefficients of $g(x)$.

1.4.8 Round 8

22. **[22]** Two congruent circles, ω_1 and ω_2, intersect at points A and B. The centers of ω_1 and ω_2 are O_1 and O_2 respectively. The arc AB of ω_1 that lies inside ω_2 is trisected by points P and Q, with the points lying in the order A, P, Q, B. Similarly, the arc AB of ω_2 that lies inside ω_1 is trisected by points R and S, with the points lying in the order A, R, S, B. Given that $PQ = 1$ and $PR = \sqrt{2}$, find the measure of $\angle AO_1B$ in degrees.

23. **[22]** How many ordered triples of (a, b, c) of integers between -10 and 10 inclusive satisfy the equation $-abc = (a + b)(b + c)(c + a)$?

24. **[22]** For positive integers n and b where $b > 1$, define $s_b(n)$ as the sum of digits in the base-b representation of n. A positive integer p is said to *dominate* another positive integer q if for all positive integers n, $s_p(n)$ is greater than or equal to $s_q(n)$. Find the number of ordered pairs (p, q) of *distinct* positive integers between 2 and 100 inclusive such that p dominates q.

Chapter 2

EMC^2 2017 Solutions

2.1 Speed Test Solutions

1. Ben was trying to solve for x in the equation $6 + x = 1$. Unfortunately, he was reading upside-down and misread the equation as $1 = x + 9$. What is the positive difference between Ben's answer and the correct answer?

   Solution. The answer is $\boxed{3}$.

   $x = -5$ and $x = -8$ are the solutions to these two equations. The positive difference is 3.

2. Anjali and Meili each have a chocolate bar shaped like a rectangular box. Meili's bar is four times as long as Anjali's, while Anjali's is three times as wide and twice as thick as Meili's. What is the ratio of the volume of Anjali's chocolate to the volume of Meili's chocolate?

   Solution. The answer is $\boxed{\frac{3}{2}}$.

   The volume of the chocolate bar is given by length · width · thickness. If Meili's bar has dimensions l, w, t, Anjali's bar must have dimensions $\frac{1}{4}l, 3w, 2t$, which yields a volume of $\frac{3}{2}lwt$. The answer is $\frac{3}{2}$.

3. For any two nonnegative integers m, n, not both zero, define $m?n = m^n + n^m$. Compute the value of $((2?0)?1)?7$.

   Solution. The answer is $\boxed{177}$.

   We compute
   $$((2?0)?1)?7 = ((2^0 + 0^2)?1)?7 = (1?1)?7 = (1^1 + 1^1)?7 = 2?7 = 2^7 + 7^2 = 128 + 49 = 177.$$

4. Eliza is making an in-scale model of the Phillips Exeter Academy library, and her prototype is a cube with side length 6 inches. The real library is shaped like a cube with side length 120 feet, and it contains an entrance chamber in the front. If the chamber in Eliza's model is 0.8 inches wide, how wide is the real chamber, in feet?

   Solution. The answer is $\boxed{16}$.

   120 feet is equivalent to $120 \cdot 12 = 1440$ inches. Therefore, the actual library is $\frac{1440}{6} = 240$ times bigger than the model. The size of the actual chamber is thus $240 \cdot 0.8 = 192$ inches, or $\frac{192}{12} = 16$ feet. Alternatively, since the width of the chamber of the model is $\frac{0.8}{6} = \frac{2}{15}$ of the side length of the model, the width of the real chamber is also $\frac{2}{15}$ of the real side length, which is $\frac{2}{15} \cdot 120 = 16$ feet.

5. One day, Isaac begins sailing from Marseille to New York City. On the exact same day, Evan begins sailing from New York City to Marseille along the exact same route as Isaac. If Marseille and New York are exactly 3000 miles apart, and Evan sails exactly 40 miles per day, how many miles must Isaac sail each day to meet Evan's ship in 30 days?

   Solution. The answer is $\boxed{60}$.

   Isaac and Evan, together, must sail the distance of exactly 3000 miles in 30 days. This means that, together, they must sail $\frac{3000}{30} = 100$ miles every day. Evan sails 40 miles a day, so Isaac must sail $100 - 40 = 60$ miles a day.

6. The conversion from Celsius temperature C to Fahrenheit temperature F is:

$$F = 1.8C + 32.$$

If the lowest temperature at Exeter one day was 20°F, and the next day the lowest temperature was 5°C higher, what would be the lowest temperature that day, in degrees Fahrenheit?

Solution. The answer is $\boxed{29}$.

From the given equation, we notice that a x degree Celsius increase in temperature is equivalent to a $1.8 \cdot x$ degree Fahrenheit increase in temperature. Therefore, the increase in temperature between these two days is $1.8 \cdot 5 = 9$ degrees Fahrenheit. The lowest temperature on the second day is $20+9 = 29$ Fahrenheit.

7. In a school, 60% of the students are boys and 40% are girls. Given that 40% of the boys like math and 50% of the people who like math are girls, what percentage of girls like math?

Solution. The answer is $\boxed{60\%}$.

Because the number of boys who like math is equal to the number of girls who like math, $0.6 \cdot 0.4 = 0.4 \cdot x$, where x is the fraction of girls who like math. Solving gives $x = 0.6$, so the answer is 60%.

8. Adam and Victor go to an ice cream shop. There are four sizes available (kiddie, small, medium, large) and seventeen different flavors, including three that contain chocolate. If Victor insists on getting a size at least as large as Adam's, and Adam refuses to eat anything with chocolate, how many different ways are there for the two of them to order ice cream?

Solution. The answer is $\boxed{2380}$.

Consider size and flavor separately. For size, if Adam chooses kiddie/small/medium/large, Victor has 4/3/2/1 choices in order to get a size at least as large as Adam's. In total, there are 10 ways for Adam and Victor to pick sizes. For flavor, Victor can pick any of the 17 flavors, and Adam can pick any of the 14 non-chocolate flavors. Therefore, there is a total of $10 \cdot 17 \cdot 14 = 2380$ different ways to order ice cream.

9. There are 10 (not necessarily distinct) positive integers with arithmetic mean 10. Determine the maximum possible range of the integers. (The range is defined to be the nonnegative difference between the largest and smallest number within a list of numbers.)

Solution. The answer is $\boxed{90}$.

Because the average of 10 numbers is 10, the sum of these 10 numbers must be 100. In order to maximize the range, we need to maximize the largest number and minimize the smallest number. Since all 10 numbers are positive integers, the largest integer is at most $100 - 9 \cdot 1 = 91$, and the smallest integer is at least 1, so the maximal range is $91 - 1 = 90$. This maximum is achieved when nine of the numbers are 1 and the tenth is 91.

10. Find the sum of all distinct prime factors of $11! - 10! + 9!$.

Solution. The answer is $\boxed{118}$.

We compute

$$11! - 10! + 9! = (11 \cdot 10 - 10 + 1) \cdot 9! = (110 - 10 + 1) \cdot 9! = 101 \cdot 9!$$

Note that 101 is prime. The distinct prime factors of this number are: 2, 3, 5, 7, 101. The sum is 118.

11. Inside regular hexagon $ZUMING$, construct square $FENG$. What fraction of the area of the hexagon is occupied by rectangle $FUME$?

    Solution. The answer is $\boxed{\frac{6-2\sqrt{3}}{9}}$.

    Suppose that the side length of the hexagon is 1, then the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. The length of UN is $\sqrt{3}$, so the length of FU is $\sqrt{3} - 1$, and the area of the rectangle of $FUME$ is also $\sqrt{3} - 1$. Finally, we compute the fraction $\frac{\sqrt{3}-1}{3\sqrt{3}/2} = \frac{6-2\sqrt{3}}{9}$.

12. How many ordered pairs (x, y) of nonnegative integers satisfy the equation $4^x \cdot 8^y = 16^{10}$?

    Solution. The answer is $\boxed{7}$.

    We can first rewrite the equation into $2^{2x} \cdot 2^{3y} = 2^{40}$, or $2^{2x+3y} = 2^{40}$. We need the exponents to equal each other, so $2x + 3y = 40$. The solution with the largest possible value of x should be obvious: $x = 20, y = 0$. Notice that decreasing x by 3 and increasing y by 2 will always give another solution, and by doing this we can generate all solutions: $(x, y) = (20, 0), (17, 2), (14, 4), (11, 6), (8, 8), (5, 10), (2, 12)$, for a total of 7.

13. In triangle ABC with $BC = 5$, $CA = 13$, and $AB = 12$, Points E and F are chosen on sides AC and AB, respectively, such that $EF \parallel BC$. Given that triangle AEF and trapezoid $EFBC$ have the same perimeter, find the length of EF.

    Solution. The answer is $\boxed{3}$.

    Notice that $\triangle ABC$ and $\triangle AFE$ are similar. Therefore, $\frac{AF}{AB} = \frac{AE}{AC} = \frac{EF}{BC}$.
    Because the perimeters of triangle AEF and trapezoid $EFBC$ share the segment EF, the perimeters being equal is equivalent to $AE + AF = FB + BC + CE$. However, the two sides of this equation together make up the perimeter of triangle ABC. Therefore, our equation is equivalent to $AE + AF = \frac{1}{2}(AB + AC + BC) = 15$.
    Now we use our earlier similarity. Because $\frac{AF}{12} = \frac{AE}{13} = \frac{EF}{5}$, $AF = \frac{12}{5}EF$ and $AE = \frac{13}{5}EF$. Plugging back into the previous equation, we get $5EF = 15$, or $EF = 3$.

14. Find the number of two-digit positive integers with exactly 6 positive divisors. (Note that 1 and n are both counted among the divisors of a number n.)

    Solution. The answer is $\boxed{16}$.

    If the prime factorization of n is ${p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$, it has $(e_1 + 1)(e_2 + 1)...(e_k + 1)$ divisors. In order for a number to have exactly 6 divisors, it must be in the form of either ${p_1}^5$ or ${p_1}^2{p_2}^1$. The first case only gives 32. The second case can be done by casework on p_1. If $p_1 = 2$, $p_2 = 3, 5, 7, 11, 13, 17, 19, 23$. If $p_1 = 3$, $p_2 = 2, 5, 7, 11$. If $p_1 = 5$, $p_2 = 2, 3$. If $p_1 = 7$, $p_2 = 2$. This gives a total of 16 two-digit integers.

15. How many ways are there to put two identical red marbles, two identical green marbles, and two identical blue marbles in a row such that no red marble is next to a green marble?

    Solution. The answer is $\boxed{12}$.

    Consider the positions of the two blue marbles (denoted by B). Note that we cannot have a block of three or more consecutive non-blue marbles in a row. Let P, Q, R, S denote the other four marbles.

    - $BPQBRS$. P and Q must be the same color, and so must be R and S. Thus there are 2 ways in this case.
    - $PBQBRS$. R and S must be the same color, and so must be P and Q. Thus there are 2 ways in this case.
    - $PBQRBS$. Q and R must be the same color, and so must be P and S. Thus there are 2 ways in this case.
    - $PQBBRS$. P and Q must be the same color, and so must be R and S. Thus there are 2 ways in this case.
    - $PQBRBS$. This is the identical case as the second case, and thus there are 2 ways in this case.
    - $PQBRSB$. This is the identical case as the first case, and thus there are 2 ways in this case.

    Adding up all cases, we have $2 + 2 + 2 + 2 + 2 + 2 = 12$ ways in total.

16. Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

    Solution. The answer is $\boxed{\text{Monday}}$.

    Let the Monday when both Yannick and Vinjai submit 1 problem be day 1, then:

    On day 2, Yannick submits 9 problems, Vinjai submits 2 problems.

    On day 3, Yannick submits 17 problems, Vinjai submits 4 problems.

    On day 4, Yannick submits 25 problems, Vinjai submits 8 problems.

    On day 5, Yannick submits 33 problems, Vinjai submits 16 problems.

    On day 6, Yannick submits 41 problems, Vinjai submits 32 problems.

    On day 7, Yannick submits 49 problems, Vinjai submits 64 problems. At this point Yannick have submitted $((1 + 49) \cdot 7)/2 = 175$ problems, and Vinjai have submitted $2^7 - 1 = 127$ problems.

    On day 8, Yannick submits 57 problems, Vinjai submits 128 problems. At this point Yannick have submitted $175 + 57 = 232$ problems, and Vinjai have submited $127 + 128 = 255$ problems.

    Therefore, day 8 is the first day that Vinjai have submitted more problems than Yannick in total. Since day 1 is Monday, day 8 is Monday as well.

17. The tiny island nation of Konistan is a cone with height twelve meters and base radius nine meters, with the base of the cone at sea level. If the sea level rises four meters, what is the surface area of Konistan that is still above water, in square meters?

Solution. The answer is $\boxed{60\pi}$.

We first find the radius of the cone at the risen sea level, which is four meters above the base. Let this radius be x. Because the part of Konistan above the risen sea level is similar in shape to the entire Konistan, we can write the equation $\frac{x}{12-4} = \frac{9}{12}$, or $x = 6$. Then, we need to find the lateral area of the cone with height 8 and base radius 6. By the Pythagorean theorem, the lateral height of the cone is 10. We can use lateralSA $= \pi \cdot \text{radius} \cdot \text{lateral height}$, which comes out to $\pi \cdot 6 \cdot 10 = 60\pi$.

18. Nicky likes to doodle. On a convex octagon, he starts from a random vertex and doodles a path, which consists of seven line segments between vertices. At each step, he chooses a vertex randomly among all unvisited vertices to visit, such that the path goes through all eight vertices and does not visit the same vertex twice. What is the probability that this path does not cross itself?

Solution. The answer is $\boxed{\frac{4}{315}}$.

We first count the number of paths that don't cross itself. The key observation is as follows: if at any time, Nicky draws a line with unvisited vertices on both sides of the line, then the path has to cross itself.

Nicky can start at any vertex, and for every move after that except for the very last move, Nicky only has two choices for his next vertex, in order to not have unvisited vertices on both sides of his next line. In the very last move, he has no choices. Therefore, there is a total of $8 \cdot 2^6$ paths that don't self intersect.

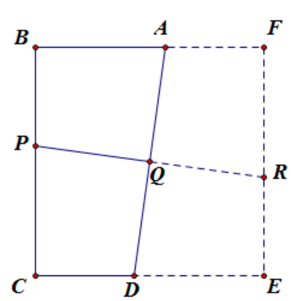
Then we count the total number of paths that Nicky can draw. He can start anywhere, and for each move afterwards, choose any of the yet unvisited vertices. There is a total of 8! total paths.

The answer is $\frac{8 \cdot 2^6}{8!} = \frac{2^6}{7!} = \frac{64}{5040} = \frac{4}{315}$.

19. In a right-angled trapezoid $ABCD$, $\angle B = \angle C = 90^\circ$, $AB = 20, CD = 17$, and $BC = 37$. A line perpendicular to DA intersects segment BC and DA at P and Q respectively and separates the trapezoid into two quadrilaterals with equal area. Determine the length of BP.

Solution. The answer is $\boxed{17}$.

Construct a second copy $ADEF$ of trapezoid $ABCD$ such that $BCEF$ is a square with side length of 37, and let R be the intersection of line PQ with segment EF. When Q is the midpoint of AD, by symmetry we can see that all four quadrilaterals $ABPQ, CDQP, DERQ, FAQR$ are congruent and therefore have the same area. If Q is closer to A, then $ABPQ$ would have a smaller area and $CDQP$ would have a larger area, and vice versa. Therefore, Q is the midpoint of AD and $BP = CD = 17$.



20. A sequence of integers a_i is defined by $a_1 = 1$ and $a_{i+1} = 3i - 2a_i$ for all integers $i \geq 1$. Given that $a_{15} = 5476$, compute the sum $a_1 + a_2 + a_3 + \cdots + a_{15}$.

    Solution. The answer is $\boxed{3756}$.

    Notice that $2a_i + a_{i+1} = 3i$ for all $i \geq 1$, and so we have

$$\begin{aligned} 3(a_1 + a_2 + \cdots + a_{15}) &= a_1 + (2a_1 + a_2) + (2a_2 + a_3) + \cdots + (2a_{14} + a_{15}) + 2a_{15} \\ &= 1 + 3(1 + 2 + 3 + \cdots + 14) + 2 \cdot 5476 = 10953 + 3 \cdot 105 = 11268 \end{aligned}$$

    Therefore $a_1 + a_2 + \cdots + a_{15} = \frac{11268}{3} = 3756$.

2.2 Accuracy Test Solutions

1. Chris goes to Matt's Hamburger Shop to buy a hamburger. Each hamburger must contain exactly one bread, one lettuce, one cheese, one protein, and at least one condiment. There are two kinds of bread, two kinds of lettuce, three kinds of cheese, three kinds of protein, and six different condiments: ketchup, mayo, mustard, dill pickles, jalapeños, and Matt's Magical Sunshine Sauce. How many different hamburgers can Chris make?

   Solution. The answer is $\boxed{2268}$.

   We first consider the number of ways to choose the set of condiments. If there are no restrictions on condiments, then there are 2 ways to determine whether each condiment is chosen or not, for $2^6 = 64$ ways. But since we cannot choose no condiments, we exclude one to get 63 ways to choose the set of condiments. Now we consider the other elements of the hamburger and multiply each of them to get the number of ways in total: $2 \cdot 2 \cdot 3 \cdot 3 \cdot 63 = 2268$ ways.

2. The degree measures of the interior angles in convex pentagon $NICKY$ are all integers and form an increasing arithmetic sequence in some order. What is the smallest possible degree measure of the pentagon's smallest angle?

   Solution. The answer is $\boxed{38^\circ}$.

   Suppose that the angles of the pentagon are $a - 2d, a - d, a, a + d, a + 2d$ degrees respectively for some positive integer a and d. Notice that $(a - 2d) + (a - d) + a + (a + d) + (a + 2d) = 5a = (5 - 2) \cdot 180 = 540 \Rightarrow a = 108$, so we need to minimize $108 - 2d$ while $108 + 2d < 180$. The inequality implies $d < 36$, or $d \leq 35 \Rightarrow 108 - 2d \geq 38$. Therefore, the smallest possible measure of the smallest angle is 38 degrees.

3. Daniel thinks of a two-digit positive integer x. He swaps its two digits and gets a number y that is less than x. If 5 divides $x - y$ and 7 divides $x + y$, find all possible two-digit numbers Daniel could have in mind.

   Solution. The answer is $\boxed{61}$.

   Let the two digit integer x be $10a + b$. From the given information, $10a + b - (10b + a) = 9(a - b)$ is divisible by 5, and $10a + b + (10b + a) = 11(a + b)$ is divisible by 7. Because $\gcd(5, 9) = \gcd(11, 7) = 1$, we must have that $a - b$ is divisible by 5 and $a + b$ is divisible by 7. In order for a, b to have integer solutions, $a - b$ and $a + b$ must have the same parity, and $a + b \geq a - b$, so either $a - b = 5, a + b = 7$, or $a - b = 10, a + b = 14$. Only the first pair gives a valid solution of 61.

4. At the Lio Orympics, a target in archery consists of ten concentric circles. The radii of the circles are $1, 2, 3, \ldots, 9$, and 10 respectively. Hitting the innermost circle scores the archer 10 points, the next ring is worth 9 points, the next ring is worth 8 points, all the way to the outermost ring, which is worth 1 point. If a beginner archer has an equal probability of hitting any point on the target and never misses the target, what is the probability that his total score after making two shots is even?

   Solution. The answer is $\boxed{\frac{101}{200}}$.

   We first find the probability that a single shot has an odd score. This is equivalent to the probability of hitting an area with score $1, 3, 5, 7, 9$, or the ratio of the area of the odd-score rings to the total area. The area of a ring with outer radius a and inner radius b is $\pi(a^2-b^2)$, and we can use this formula to calculate the sum of areas of the odd-score rings as $(10^2-9^2+8^2-7^2+\cdot+2^2-1^2)\pi = (10+9+8+\cdot+1)\pi = 55\pi$. Therefore, the probability of a single shot being odd is $\frac{55\pi}{100\pi} = \frac{11}{20}$.

For the sum of two shots to be even, they must be either both even or both odd. The probability of this happening is $(\frac{11}{20})^2 + (\frac{9}{20})^2 = \frac{121+81}{400} = \frac{101}{200}$

5. Let $F(x) = x^2 + 2x - 35$ and $G(x) = x^2 + 10x + 14$. Find all distinct real roots of $F(G(x)) = 0$.

   Solution. The answer is $\boxed{-1, -3, -7, -9}$.

   $F(x) = (x+7)(x-5)$, which means that in order for $F(G(x)) = 0$, we need $G(x) = -7$ or 5. $G(x) = -7$ factors into $(x+3)(x+7)$, and $G(x) = 5$ factors into $(x+1)(x+9)$. The roots are $-1, -3, -7, -9$.

6. One day while driving, Ivan noticed a curious property on his car's digital clock. The sum of the digits of the current hour equaled the sum of the digits of the current minute. (Ivan's car clock shows 24-hour time; that is, the hour ranges from 0 to 23, and the minute ranges from 0 to 59.) For how many possible times of the day could Ivan have observed this property?

   Solution. The answer is $\boxed{112}$.

   We consider cases based on the sum of digits in both hour and minute.

Sum	Hour	Min	# of ways
0	1	1	1
1	2	2	4
2	3	3	9
3	3	4	12
4	3	5	15
5	3	6	18
6	2	6	12
7	2	6	12
8	2	6	12
9	2	6	12
10	1	5	5

   Summing all the totals in the right-hand column yields 112.

7. Qi Qi has a set Q of all lattice points in the coordinate plane whose x- and y-coordinates are between 1 and 7 inclusive. She wishes to color 7 points of the set blue and the rest white so that each row or column contains exactly 1 blue point and no blue point lies on or below the line $x + y = 5$. In how many ways can she color the points?

   Solution. The answer is $\boxed{486}$.

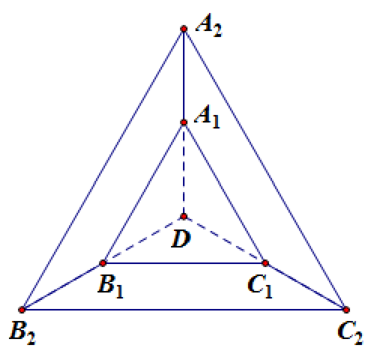
   Qi Qi has to choose one of the three points $(1,5), (1,6), (1,7)$ to be colored blue, and once this point is chosen, one of the $(2,4), (2,5), (2,6), (2,7)$ cannot be blue anymore, so she still has 3 choices. We can continue this process for the first five columns, each with 3 possibilities, and then for the sixth and seventh column there are $2 \cdot 1 = 2$ ways to choose a blue point for each. Therefore, there are $3^5 \cdot 2 = 486$ ways to color the grid.

8. A piece of paper is in the shape of an equilateral triangle ABC with side length 12. Points A_B and B_A lie on segment AB, such that $AA_B = 3$, and $BB_A = 3$. Define points B_C and C_B on segment BC and points C_A and A_C on segment CA similarly. Point A_1 is the intersection of A_CB_C and A_BC_B. Define B_1 and C_1 similarly. The three rhombi — $AA_BA_1A_C, BB_CB_1B_A, CC_AC_1C_B$ — are cut from triangle ABC, and the paper is folded along segments A_1B_1, B_1C_1, C_1A_1, to form a tray without a top. What is the volume of this tray?

   Solution. The answer is $\boxed{\frac{63}{4}\sqrt{2}}$.

   By labeling equal segments, one can see that the top of the tray is in the shape of an equilateral triangle $A_2B_2C_2$ with side length 6, and the bottom of the tray $A_1B_1C_1$ is in the shape of an equilateral triangle with side length 3. The sides of the tray are trapezoids with angles 60 and 120 degrees. Imagine extending rays A_2A_1, B_2B_1, C_2C_1. The three rays will concur at some point D. This "completes" the tray into a regular tetrahedron, because all faces are equilateral triangles. In other words, the tray may be viewed as a regular tetrahedron with side length 6, with one of its corners cut off half way from the tip to the base.

   Now we calculate the volume of the whole tetrahedron. Drop the altitude from D to triangle ABC, and let the foot of perpendicular be E. ABE is a 30-30-120 triangle, so $AE = 2\sqrt{3}$. By the pythagorean theorem, $DE = \sqrt{36-12} = \sqrt{24} = 2\sqrt{6}$. The volume of the tetrahedron is $\frac{1}{3} \cdot \text{base} \cdot \text{height} = \frac{1}{3} \cdot 9\sqrt{3} \cdot 2\sqrt{6} = 18\sqrt{2}$. The cut-off tip is simply a smaller version of this tetrahedron, and since its side length is half of the large tetrahedron, its volume is one eighth of the large tetrahedron. The remaining tray has volume that is seven-eighth of the large tetrahedron, or $\frac{7}{8} \cdot 18\sqrt{2} = \frac{63}{4}\sqrt{2}$.



9. Define $\{x\}$ as the fractional part of x. Let S be the set of points (x, y) in the Cartesian coordinate plane such that $x + \{x\} \leq y$, $x \geq 0$, and $y \leq 100$. Find the area of S.

   Solution. The answer is $\boxed{\frac{19801}{4}}$.

   Define $f(x) = x + \{x\}$. Divide the region S up into vertical strips, each containing the range $a \leq x < a+1$ for some nonnegative integer a. Within each strip, $f(x) = x + (x-a) = 2x - a$. When $0 \leq a \leq 98$, the region of S within each strip is a trapezoid with two bases $100 - a$ and $98 - a$ and the height 1, which has area $99 - a$. When $a = 99$, the region of S within that strip is a right triangle with base 1 and height $\frac{1}{2}$, which has area $\frac{1}{4}$. When $a > 99$, thre are no points in S that lie in the strip. Therefore, the area of S is equal to $99 + 98 + 97 + \cdots + 2 + 1 + \frac{1}{4} = \frac{99(1+99)}{2} + \frac{1}{4} = 4950.25$ or $\frac{19801}{4}$.

10. Nicky likes dolls. He has 10 toy chairs in a row, and he wants to put some indistinguishable dolls on some of these chairs. (A chair can hold only one doll.) He doesn't want his dolls to get lonely, so he wants each doll sitting on a chair to be adjacent to at least one other doll. How many ways are there for him to put any number (possibly none) of dolls on the chairs? Two ways are considered distinct if and only if there is a chair that has a doll in one way but does not have one in the other.

    Solution. The answer is $\boxed{200}$.

    We do casework on the number of "groups" of dolls there are. We'll count the number of seatings for each case by selecting the beginning and ending chair for each group.

    There is one way for there to be no dolls.

    If there is only one group of dolls, we only need to choose two distinct chairs. There are $\binom{10}{2} = 45$ seatings.

    If there are two groups, we need to choose four distinct chairs, with at least one empty space between the second and third chair. We can solve this problem by choosing four chairs out of nine chairs, then "adding" an unchosen chairs between the second and third. Following this logic, there are $\binom{9}{4} = 126$ seatings.

    If there are three groups, we continue using the logic of the previous cases and pick six chairs out of eight chairs. There are $\binom{8}{6} = 28$ seatings.

    It is not possible for there to be four or more groups of dolls.

    Overall, there are $1 + 45 + 126 + 28 = 200$ seatings.

    Alternatively, define A_n to be the number of ways to place dolls on n chairs in a row such that no doll is alone. Then, By considering whether there is a doll in the first chair or not, and if so, how many dolls are there next to each other, we see that $A_n = A_{n-1} + (A_{n-3} + A_{n-4} + \cdots + A_1 + A_0 + A_0)$ for $n \geq 3$. (A_{n-1} represents the case where there is no dolls on the first chair, A_{n-3} represents two dolls in the first two chairs, A_{n-4} represents three dolls in the first three chairs, etc. Note that we need to include the empty chair after the leading group, and there are two A_0 at the end of the expression because one is the case of the first $n-1$ chairs being occupied and the other is the case of all n chairs being occupied.)

    Starting with $A_0 = 1, A_1 = 1$, and $A_2 = 2$, we get $A_3 = 4, A_4 = 7, A_5 = 12, A_6 = 21, A_7 = 37, A_8 = 65, A_9 = 114$, and finally $A_{10} = 200$.

2.3 Team Test Solutions

1. Compute $2017 + 7201 + 1720 + 172$.

   Solution. The answer is $\boxed{11110}$.

   Notice that digits $0, 1, 2$, and 7 appear on each place once each, so the sum is just $7777+2222+1111 = 11110$.

2. A number is called *downhill* if its digits are distinct and in descending order. (For example, 653 and 8762 are downhill numbers, but 97721 is not.) What is the smallest downhill number greater than 86432?

   Solution. The answer is $\boxed{86510}$.

   We cannot only alter the last two digits of the number since 432 is the largest downhill number with hundreds digit 4, so the next downhill number must have hundreds digit 5. When the hundreds digit is 5, the last two digits must be at least 10. Indeed 86510 is a downhill number, so that is the answer.

3. Each vertex of a unit cube is sliced off by a planar cut passing through the midpoints of the three edges containing that vertex. What is the ratio of the number of edges to the number of faces of the resulting solid?

   Solution. The answer is $\boxed{\frac{12}{7}}$.

   After the cuts are made, each edge belongs to exactly one of the six original faces, and each of these six faces has four of these edges, so there are $6 \cdot 4 = 24$ edges. As for faces, since each cut creates exactly one new face, and there are eight cuts, so we have $6 + 8 = 14$ faces. Finally, we compute the ratio $\frac{24}{14} = \frac{12}{7}$.

4. In a square with side length 5, the four points that divide each side into five equal segments are marked. Including the vertices, there are 20 marked points in total on the boundary of the square. A pair of distinct points A and B are chosen randomly among the 20 points. Compute the probability that $AB = 5$.

   Solution. The answer is $\boxed{\frac{2}{19}}$.

   There are two main ways to get $AB = 5$: one is to have segment AB parallel to one of the edges, and one is to have segment AB cut the square into a pentagon and a $3-4-5$ triangle. There are $6 \cdot 2 = 12$ (unordered) pairs of points that satisfy the first condition, and $2 \cdot 4 = 8$ pairs that satisfy the second, for $12 + 8 = 20$ pairs in total. Since there are $\binom{20}{2} = 190$ possible pairs in total, the probability is $\frac{20}{190} = \frac{2}{19}$.

5. A positive two-digit integer is one less than five times the sum of its digits. Find the sum of all possible such integers.

   Solution. The answer is $\boxed{113}$.

   It is easy to see that this integer must leave a remainder of 4 when divided by 5, which means that the last digit must be 4 or 9. If the two-digit number is $\overline{a4}$, then we have $10a + 4 = 5(a + 4) - 1$, which gives $a = 3$. If the number is $\overline{a9}$, then we have $10a + 9 = 5(a + 9) - 1$, which gives $a = 7$. Therefore the two possible numbers are 34 and 79, and their sum is 113.

6. Let
$$f(x) = 5^{4^{3^{2^x}}}.$$
Determine the greatest possible value of L such that $f(x) > L$ for all real numbers x.

Solution. The answer is $\boxed{625}$.

Since $2^x > 0$ for all real number x, we have $3^{2^x} > 3^0 = 1$, $4^{3^{2^x}} > 4^1 = 4$, and finally $5^{4^{3^{2^x}}} > 5^4 = 625$. When x is very small (very negative), 2^x can get arbitrarily close to 0, and similarly all the expression on the left-hand side of each inequality can be arbitrarily close to the right-hand side. Therefore $L_{\max} = 625$.

7. If $\overline{AAAA}+\overline{BB} = \overline{ABCD}$ for some distinct base-10 digits A, B, C, D that are consecutive in some order, determine the value of $\overline{ABCD}$. (The notation $\overline{ABCD}$ refers to the four-digit integer with thousands digit A, hundreds digit B, tens digit C, and units digit D.)

Solution. The answer is $\boxed{7865}$.

Clearly $B = A + 1$ because exactly one carry-over happens on the tens digit, and consequently $\overline{CD} < \overline{AA}$. In particular, since $A + B > 10$, we see that a carry-over also occurs on the ones digit, we have $D = A + B - 10 < A$ and $C = D + 1$. Therefore, the four digits satisfy $D < C < A < B$. Using the fact that $D = A + B - 10 = A + (A + 1) - 10 = 2A - 9$ and $D = A - 2$, we find that $A = 7$, and therefore $\overline{ABCD} = 7865$. It is not difficult to see that $7777 + 88 = 7865$ is a valid solution.

8. A regular tetrahedron and a cube share an inscribed sphere. What is the ratio of the volume of the tetrahedron to the volume of the cube?

Solution. The answer is $\boxed{\sqrt{3}}$.

Suppose that the side length of the tetrahedron $ABCD$ is a, and let the center of triangle BCD be H. Then the distance from B to H is $\frac{1}{\sqrt{3}}a$, and thus the distance from A to H is $\sqrt{\frac{2}{3}}a = \frac{\sqrt{6}}{3}a$. Since the center of the inscribed sphere lies three-fourth the way from A to H, the radius of the inscribed sphere is $\frac{\sqrt{6}}{12}a$. Consequently, the side length of the cube is $\frac{\sqrt{6}}{6}a$. The volume of the tetrahedron is $\frac{1}{3}(\frac{\sqrt{6}}{3}a \cdot (\frac{\sqrt{3}}{4}a^2)) = \frac{\sqrt{2}}{12}a^3$, and the volume of the cube is $(\frac{\sqrt{6}}{6}a)^3 = \frac{\sqrt{6}}{36}a^3$. Therefore, the ratio of the two volumes is $\frac{\sqrt{2}/12}{\sqrt{6}/36} = \sqrt{3}$.

9. Define $\lfloor x \rfloor$ as the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$ as the fractional part of x. If $\lfloor x^2 \rfloor = 2\lfloor x \rfloor$ and $\{x^2\} = \frac{1}{2}\{x\}$, determine all possible values of x.

Solution. The answer is $\boxed{0, \frac{1}{2}, \frac{3}{2}, 2}$.

Since $\lfloor x^2 \rfloor \geq 0$, we have $\lfloor x \rfloor \geq 0$. On the other hand, if $\lfloor x \rfloor > 2$, we have $\lfloor x^2 \rfloor \geq \lfloor x \rfloor^2 > 2\lfloor x \rfloor$, so $\lfloor x \rfloor = 0, 1$ or 2.

When $\lfloor x \rfloor = 0$, we have $\lfloor x^2 \rfloor = 0$, and $x^2 = \frac{1}{2}x$, which gives $x = 0$ or $\frac{1}{2}$.

When $\lfloor x \rfloor = 1$, we have $\lfloor x^2 \rfloor = 2$, and $\{x^2\} = x^2 - 2 = \frac{1}{2}(x-1) = \frac{1}{2}\{x\}$, which gives $x = \frac{3}{2}$ or -1 (the second solution is discarded).

When $\lfloor x \rfloor = 2$, we have $\lfloor x^2 \rfloor = 2$, and $\{x^2\} = x^2 - 4 = \frac{1}{2}(x-2) = \frac{1}{2}\{x\}$, which gives $x = 2$ or $-\frac{3}{2}$ (the second solution is discarded). Therefore, all possible values of x are $0, \frac{1}{2}, \frac{3}{2}$ or 2.

10. Find the largest integer $N > 1$ such that it is impossible to divide an equilateral triangle of side length 1 into N smaller equilateral triangles (of possibly different sizes).

Solution. The answer is $\boxed{5}$.

We first show that it is impossible to divide an equilateral triangle into five smaller equilateral triangles. First of all, each vertex of the original triangle must belong to different smaller triangles, otherwise the triangle that contains two of these vertices will cover the entire triangle. If we cut away these three triangles, then we would either have a hexagon, a pentagon, an isosceles trapezoid, or an equilateral triangle. However, if two equilateral triangles combined form a convex figure, they must form a rhombus. This means that the remaining region cannot be divided into two equilateral triangles.

Now we show that for all $N > 5$, it is possible to do such a division. First, note that all even numbers $2n$ with $n \geq 2$ are possible: Cut out a triangle with side length $\frac{n-1}{n}$, and then divide the remaining trapezoid into $2n-1$ triangles with side length $\frac{1}{n}$. Second, given a valid division, we can always create exactly 3 more triangles by dividing a current one into four smaller congruent triangles. This covers all the odd numbers greater than or equal to 7. Therefore, whenever $N > 5$, we can always find a division with N triangles.

11. Let f and g be two quadratic polynomials. Suppose that f has zeroes 2 and 7, g has zeroes 1 and 8, and $f - g$ has zeroes 4 and 5. What is the product of the zeroes of the polynomial $f + g$?

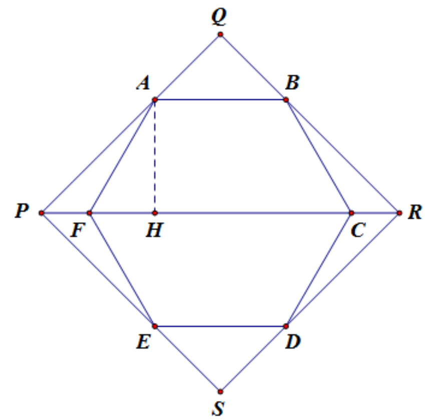
Solution. The answer is $\boxed{12}$.

Note that we can express $f = p(x-2)(x-7)$ and $g = q(x-1)(x-8)$ where p and q are two nonzero constants. The problem condition tells us that $f = g$ when $x = 4$ or $x = 5$, and plugging in one of them gives $-6p = -12q$, or $p = 2q$. Without loss of generality, assume that $q = 1$ and $p = 2$, then $f + g = 3x^2 - 27x + 36$, and by Vieta's Theorem the product of the two roots is $\frac{36}{3} = 12$.

12. In square $PQRS$, points A, B, C, D, E, and F are chosen on segments PQ, QR, PR, RS, SP, and PR, respectively, such that $ABCDEF$ is a regular hexagon. Find the ratio of the area of $ABCDEF$ to the area of $PQRS$.

Solution. The answer is $\boxed{3\sqrt{3} - \frac{9}{2}}$.

Notice that the entire diagram is symmetric about both diagonals PR and QS. Suppose that the side length of the hexagon is a, then the area of the hexagon is $6(\frac{\sqrt{3}}{4})a^2 = \frac{3\sqrt{3}}{2}a^2$. Let AH be perpendicular to CF at H, then $AH = \frac{\sqrt{3}}{2}a$ and $AP = \sqrt{2}AH = \frac{\sqrt{6}}{2}a$. Moreover, we have $AQ = \frac{\sqrt{2}}{2}a$, so $PQ = \frac{\sqrt{6}+\sqrt{2}}{2}a$, and the area of the square is $PQ^2 = (\sqrt{3}+2)a^2$. Therefore the desired ratio is $\frac{3\sqrt{3}/2}{\sqrt{3}+2} = 3\sqrt{3} - \frac{9}{2}$.



13. For positive integers m and n, define $f(m,n)$ to be the number of ways to distribute m identical candies to n distinct children so that the number of candies that any two children receive differ by at most 1. Find the number of positive integers n satisfying the equation $f(2017,n) = f(7102,n)$.

Solution. The answer is $\boxed{15}$.

If $m = nq + r$ where $0 \le r < n$, then we need to first give each child q candies, and then give the rest to r distinct children. Therefore $f(m,n) = \binom{n}{r}$. Since $\binom{n}{x} = \binom{n}{y}$ if and only if $x = y$ or $x + y = n$, we see that either $2017 \equiv 7102 \pmod{n}$ or $2017 + 7102 \equiv 0 \pmod{n}$. This means that n either divides $7102 - 2017 = 5085 = 3^2 \cdot 5 \cdot 113$, or $7102 + 2017 = 9119 = 11 \cdot 829$. There are $3 \cdot 2 \cdot 2 = 12$ factors of 5085, and $2 \cdot 2 = 4$ factors of 9119, but we counted $n = 1$ twice, so there are $12 + 4 - 1 = 15$ possible values of n.

14. Suppose that real numbers x and y satisfy the equation

$$x^4 + 2x^2y^2 + y^4 - 2x^2 + 32xy - 2y^2 + 49 = 0.$$

Find the maximum possible value of $\frac{y}{x}$.

Solution. The answer is $\boxed{\frac{-4+\sqrt{7}}{3}}$.

We may rewrite the left-hand side into

$$\begin{aligned}(x^2+y^2+7)^2 - 16x^2 + 32xy - 16y^2 &= (x^2+y^2+7)^2 - (4x-4y)^2 \\ &= (x^2 - 4x + y^2 + 4y + 7)(x^2 + 4x + y^2 - 4y + 7) \\ &= ((x-2)^2 + (y+2)^2 - 1)((x+2)^2 + (y-2)^2 - 1)\end{aligned}$$

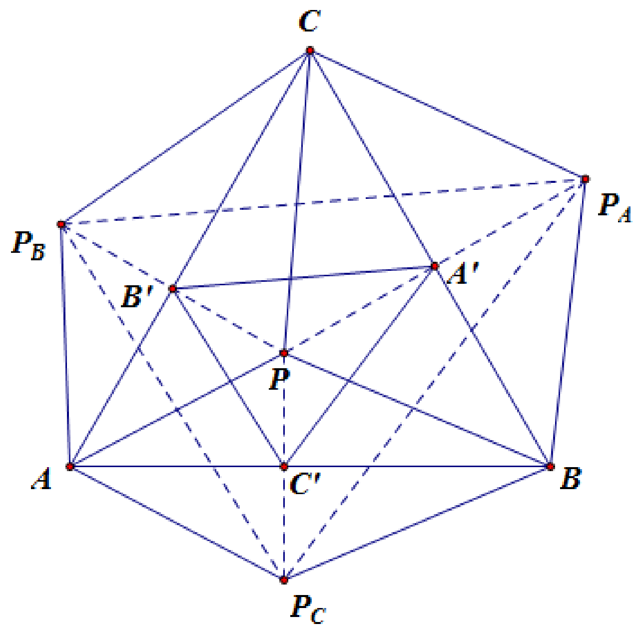
Graphically speaking, the equation is equivalent to two unit circles centered at $(2,-2)$ and $(-2,2)$ respectively, and we aim to maximize the slope through the origin and a point on either circle. Since the two circles are symmetric about the origin, we only need to consider one of them.

WLOG we assume that $x^2 - 4x + y^2 + 4y + 7 = 0$. Set $y = kx$, and we have $(k^2+1)x^2 + 4(k-1)x + 7 = 0$. This equation has solutions if and only if $(4(k-1))^2 - 4(7)(k^2+1) = 4(-3k^2 - 8k - 3) \ge 0$, or $3k^2 + 8k + 3 \le 0$. Solving this inequality, we get $\frac{-4-\sqrt{7}}{3} \le k \le \frac{-4+\sqrt{7}}{3}$, so the maximum value of k is $\frac{-4+\sqrt{7}}{3}$.

15. A point P lies inside equilateral triangle ABC. Let A', B', C' be the feet of the perpendiculars from P to BC, AC, AB, respectively. Suppose that $PA = 13, PB = 14$, and $PC = 15$. Find the area of $A'B'C'$.

    Solution. The answer is $\boxed{63}$.

    Reflect P across sides BC, CA, AB to get P_A, P_B, P_C respectively.



Notice that $CP_A = CP_B = CP = 15$ and $\angle P_ACP_B = 2\angle ACB = 120°$, which implies $P_AP_B = 15\sqrt{3}$. Since $A'B'$ is the midline of the triangle P_APP_B, we have $A'B' = \frac{1}{2}P_AP_B = 7.5\sqrt{3}$. Similarly, we get $B'C' = 6.5\sqrt{3}$ and $C'A' = 7\sqrt{3}$. We can then use Heron's formula:

$$[A'B'C'] = \sqrt{s'(s'-a')(s'-b')(s'-c')} = \sqrt{(10.5\sqrt{3}) \cdot (4\sqrt{3}) \cdot (3.5\sqrt{3}) \cdot (3\sqrt{3})} = 63.$$

2.4 Guts Test Solutions

2.4.1 Round 1

1. **[5]** If $2m = 200cm$ and $m \neq 0$, find c.

   Solution. The answer is $\boxed{\frac{1}{100}}$.

   We may divide both sides by $2m$ to get $1 = 100c$, which means $c = \frac{1}{100}$.

2. **[5]** A right triangle has two sides of lengths 3 and 4. Find the smallest possible length of the third side.

   Solution. The answer is $\boxed{\sqrt{7}}$.

   If the third side is the hypotenuse, then the length is $\sqrt{3^2 + 4^2} = 5$. If the third side is not the hypotenuse, then the length is $\sqrt{4^2 - 3^2} = \sqrt{7}$. Clearly the second one is smaller, so that is our answer.

3. **[5]** Given that $20(x + 17) = 17(x + 20)$, determine the value of x.

   Solution. The answer is $\boxed{0}$.

   By expanding both sides, we get $20x + 20 \cdot 17 = 17x + 17 \cdot 20$, which means $20x = 17x$. Therefore the only solution is $x = 0$.

2.4.2 Round 2

4. **[7]** According to the Egyptian Metropolitan Culinary Community, food service is delayed on $\frac{2}{3}$ of flights departing from Cairo airport. On average, if flights with delayed food service have twice as many passengers per flight as those without, what is the probability that a passenger departing from Cairo airport experiences delayed food service?

   Solution. The answer is $\boxed{\frac{4}{5}}$.

   Suppose that there are $3f$ flights, then $2f$ of them have delayed food service. If the flightss with normal food service have p passengers per flight, then in total there are $(2f)(2p) + fp = 5fp$ departing passengers, among which $4fp$ of them experiences delayed food service. Therefore the probability is $\frac{4}{5}$.

5. **[7]** In a positive geometric sequence $\{a_n\}$, $a_1 = 9, a_9 = 25$. Find the integer k such that $a_k = 15$.

   Solution. The answer is $\boxed{5}$.

   Notice that 15 is the geometric mean of 9 and 25, which means that in a geometric sequence 15 should be exactly half-way between 9 and 25. Therefore $k = \frac{1+9}{2} = 5$.

6. **[7]** In the Delicate, Elegant, and Exotic Music Organization, pianist Hans is selling two types of flowers with different prices (per flower): magnolias and myosotis. His friend Alice originally plans to buy a bunch containing 50% more magnolias than myosotis for \$50, but then she realizes that if she buys

50% less magnolias and 50% more myosotis than her original plan, she would still need to pay the same amount of money. If instead she buys 50% more magnolias and 50% less myosotis than her original plan, then how much, in dollars, would she need to pay?

Solution. The answer is $\boxed{50}$.

Suppose that the total price of magnolias in Alice's original plan is M dollars, and the total price of myosotis is N dollars. Then we have $M + N = \frac{1}{2}M + \frac{3}{2}N = 50$, which implies $M = N$, and thus $\frac{3}{2}M + \frac{1}{2}N = M + N = 50$ dollars as well.

2.4.3 Round 3

7. **[9]** In square $ABCD$, point P lies on side AB such that $AP = 3, BP = 7$. Points Q, R, S lie on sides BC, CD, DA respectively such that $PQ = PR = PS = AB$. Find the area of quadrilateral $PQRS$.

Solution. The answer is $\boxed{50}$.

Since $PR = AB$, we see that triangle PQR has half the area of rectangle $PBCR$, and triangle PRS has half the area of rectangle $PRCD$. Therefore, the quadrilateral $PQRS$ has half the area of square $ABCD$, which has side length $3 + 7 = 10$. So the area of $PQRS$ is $\frac{1}{2}(10^2) = 50$.

8. **[9]** Kristy is thinking of a number $n < 10^4$ and she says that 143 is one of its divisors. What is the smallest number greater than 143 that could divide n?

Solution. The answer is $\boxed{154}$.

Notice that if the next divisor of n is $143 + k$, then $\text{lcm}(143, 143 + k)$ divides n. Since $143 = 11 \cdot 13$ we see that $\text{lcm}(143, 143 + k) = 143(143 + k) > 143^2 > 10^4$ if k is divisible by neither 11 nor 13, so k has to be at least 11. When $k = 11$, we see that $\text{lcm}(143, 154) = 11 \cdot 13 \cdot 14 = 2002 < 10^4$, which means that $n = 2002$ could satisfy this condition. Therefore the smallest possible number is 154.

9. **[9]** A positive integer n is called *special* if the product of the n smallest prime numbers is divisible by the sum of the n smallest prime numbers. Find the sum of the three smallest special numbers.

Solution. The answer is $\boxed{12}$.

We proceed by trying small cases:

When $n = 1$, 2 divides 2, so 1 is special.

When $n = 2$, $2 + 3 = 5$ does not divide $2 \cdot 3$, so 2 is not special.

When $n = 3$, $2 + 3 + 5 = 10$ divides $2 \cdot 3 \cdot 5$, so 3 is special.

When $n = 4$, $2 + 3 + 5 + 7 = 17$ does not divide $2 \cdot 3 \cdot 5 \cdot 7$, so 4 is not special.

When $n = 5$, $2 + 3 + 5 + 7 + 11 = 28$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, so 5 is not special.

When $n = 6$, $2 + 3 + 5 + 7 + 11 + 13 = 41$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, so 6 is not special.

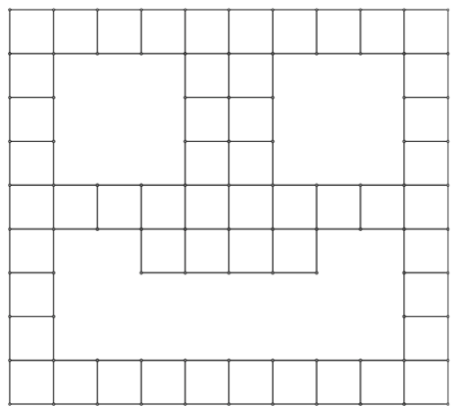
When $n = 7$, $2+3+5+7+11+13+17 = 58$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, so 7 is not special.

When $n = 8$, $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 77$ divides $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so 8 is special.

So the three smallest special numbers are 1, 3, and 8, and their sum is $1 + 3 + 8 = 12$.

2.4.4 Round 4

10. **[11]** In the diagram below, all adjacent points connected with a segment are unit distance apart. Find the number of squares whose vertices are among the points in the diagram and whose sides coincide with the drawn segments.



Solution. The answer is $\boxed{77}$.

There are 52 squares with side length 1, 7 squares with side length 2, 2 squares with side length 3, 8 squares with side length 4, 4 squares with side length 5, 2 squares with side length 8, and 2 squares with side length 9. for $52 + 7 + 2 + 8 + 4 + 2 + 2 = 77$ squares in total.

11. **[11]** Geyang tells Junze that he is thinking of a positive integer. Geyang gives Junze the following clues:

- My number has three distinct odd digits.
- It is divisible by each of its three digits, as well as their sum.

What is the sum of all possible values of Geyang's number?

Solution. The answer is $\boxed{1185}$.

We consider two cases:

Case 1: One of the digits is 5. Then since the number must be a multiple of 5, 5 must be on the units digit. Suppose that neither 3 nor 9 is in this number, then the number must be 175 or 715, but the first one is not divisible by $1 + 7 + 5 = 13$ and the second one is not divisible by 7. Therefore, one of the digits must be 3 or 9 and therefore the sum of digits must be a multiple of 3, which means that the other digit must be 1 or 7. When a digit is 3, the other can be 1 or 7. We see that 135, 315, and 735 all satisfy the requirements, but 375 is not divisible by 7. When a digit is 9, the other one can be neither 1 nor 7 because neither $1 + 9 + 5 = 15$ nor $7 + 9 + 5 = 24$ is divisible by 9.

Case 2: None of the digits is 5. Then we see that one of the digits must be 3 or 9, which means that the sum of digits is a multiple of 9. Since 1 and 7 are both congruent to 1 modulo 3, and we must use at least one of them, it is impossible to choose three numbers out of $\{1, 3, 7, 9\}$ whose sum is divisible by 3.

Therefore, all possible values of Geyang's number are 135, 315, and 735, and their sum is 1185.

12. **[11]** Regular octagon $ABCDEFGH$ has center O and side length 2. A circle passes through A, B, and O. What is the area of the part of the circle that lies outside of the octagon?

Solution. The answer is $\boxed{\frac{\pi}{2} - 1}$.

Since $\angle AOB = 45°$, it follows that arc AB in the circumcircle of ABO has measure $90°$, which also means that the radius of this circle is $\sqrt{2}$. The area outside the octagon is a quarter-circle minus a right isosceles triangle, or $\frac{1}{4}\pi(\sqrt{2})^2 - \frac{1}{2}(\sqrt{2})^2 = \frac{\pi}{2} - 1$.

2.4.5 Round 5

13. **[13]** Kelvin Amphibian, a not-frog who lives on the coordinate plane, likes jumping around. Each step, he jumps either to the spot that is 1 unit to the right and 2 units up, or the spot that is 2 units to the right and 1 unit up, from his current location. He chooses randomly among these two choices with equal probability. He starts at the origin and jumps for a long time. What is the probability that he lands on $(10, 8)$ at some time in his journey?

Solution. The answer is $\boxed{\frac{15}{64}}$.

The sum of coordinates of Kelvin must increase by 3 each time, so if he reaches $(10, 8)$ he must reach it after exactly $\frac{10+8}{3} = 6$ jumps. Suppose that among these 6 jumps, x of them are 1 unit right and 2 units up, and y of them are 2 units right and 1 unit up. Then we have $x+2y = 10$ and $2x+y = 8$, which gives $x = 4, y = 2$. The number of ways to get to $(10, 8)$ is therefore $\binom{6}{2} = 15$, out of 2^6 ways to make the 6 jumps, so the probability is $\frac{15}{64}$.

14. **[13]** Points A, B, C, and D are randomly chosen on the circumference of a unit circle. What is the probability that line segments AB and CD intersect inside the circle?

Solution. The answer is $\boxed{\frac{1}{3}}$.

Suppose that we first choose four random points on the circle without assigning the labels. If A is assigned to one of the points, then AB and CD intersects if and only if B is assigned to the one point that is not adjacent to A, which happens with $\frac{1}{3}$ possibility.

15. **[13]** Let $P(x)$ be a quadratic polynomial with two consecutive integer roots. If it is also known that $\frac{P(2017)}{P(2016)} = \frac{2016}{2017}$, find the larger root of $P(x)$.

Solution. The answer is $\boxed{6050}$.

Suppose that the two roots of $P(x)$ are a and $a + 1$, then we have $P(x) = k(x - a)(x - a - 1)$ for some nonzero constant k. The left-hand side of the given equation can then be expressed as $\frac{k(2017-a)(2016-a)}{k(2016-a)(2015-a)} = \frac{2017-a}{2015-a}$, which implies $2016(2015 - a) = 2017(2017 - a)$, and so $a = 2017 \cdot 2017 - 2016 \cdot 2015 = 6049$. The larger root is $a + 1 = 6050$.

2.4.6 Round 6

16. **[15]** Let S_n be the sum of reciprocals of the integers between 1 and n inclusive. Find a triple (a, b, c) of positive integers such that $S_{2017} \cdot S_{2017} - S_{2016} \cdot S_{2018} = \frac{S_a+S_b}{c}$.

Solution. The answer is $\boxed{(2017, 1, 4070306) \text{ or } (1, 2017, 4070306)}$.

We can rewrite the left hand side as

$$S_{2017} \cdot S_{2017} - (S_{2017} - \frac{1}{2017})(S_{2017} + \frac{1}{2018}) = \frac{S_{2017}}{2017} - \frac{S_{2017}}{2018} + \frac{1}{2017 \cdot 2018} = \frac{S_{2017} + 1}{4070306}.$$

Therefore one possible triple is $(2017, 1, 4070306)$ since $S_1 = 1$. We can also swap a and b to get $(1, 2017, 4070306)$.

17. **[15]** Suppose that m and n are both positive integers. Alec has m standard 6-sided dice, each labelled 1 to 6 inclusive on the sides, while James has n standard 12-sided dice, each labelled 1 to 12 inclusive on the sides. They decide to play a game with their dice. They each toss all their dice simultaneously and then compute the sum of the numbers that come up on their dice. Whoever has a higher sum wins (if the sums are equal, they tie). Given that both players have an equal chance of winning, determine the minimum possible value of mn.

Solution. The answer is $\boxed{91}$.

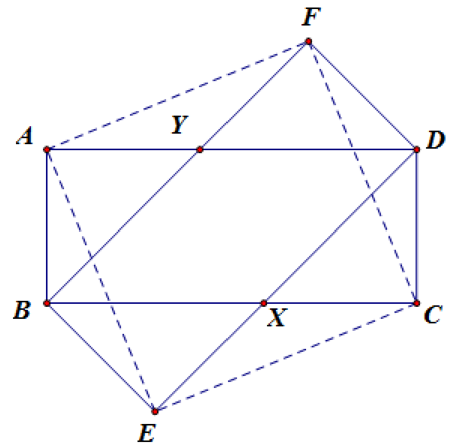
Notice that on average Alec's sum is $(\frac{1+2+3+4+5+6}{6})m = \frac{7}{2}m$ and James' sum is $(\frac{1+2+3+\cdots+12}{12})n = \frac{13}{2}n$, and the probability distribution for each person is symmetric about his average value. This means that it is equally likely for Alec to get x and $7m - x$ for his sum, and equally likely for James to get y and $13n - y$ for his sum.

If Alec and James have an equal chance of winning, we should be able to pair up the cases where Alec wins with the cases where James wins. This can happen if and only if they have the same average value, because only then can we have a one-to-one correspondence between (x, y) and $(7m - x, 13n - y)$, with either both of them representing a tie or one representing an Alec victory and the other representing a James victory. Therefore $\frac{7}{2}m = \frac{13}{2}n$, or $m : n = 13 : 7$, and the minimum value of mn is 91, achieved when $m = 13, n = 7$.

18. **[15]** Overlapping rectangles $ABCD$ and $BEDF$ are congruent to each other and both have area 1. Given that A, C, E, F are the vertices of a square, find the area of the square.

Solution. The answer is $\boxed{\sqrt{2}}$.

Without loss of generality assume that $AB < BC$, BC and DE intersects at X, and AD and BF intersects at Y. Notice that $AB = BE$, $EX = XC$ (by symmetry), and $\angle ABE = \angle EXC$, so if $AE = EC$, then triangles ABE and EXC are congruent. This means that $BE = EX$, $\angle ABE = 135°$, and $BX = \sqrt{2}EX = \sqrt{2}XC$. Therefore the overlapping area $BXYD$ is equal to $\frac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$. Since the area of the square is equal to that of the concave octagon $ABEXCDFY$, which is $1+1-(2-\sqrt{2}) = \sqrt{2}$. Our answer is $\sqrt{2}$.



2.4.7 Round 7

19. **[18]** Find the number of solutions to the equation

$$|||\dots|||||x|+1|-2|+3|-4|+\dots-98|+99|-100|=0.$$

Solution. The answer is $\boxed{51}$.

To simplify notation, we define $A_0 = |x|, A_1 = ||x|+1|, A_2 = |||x|+1|-2|, \dots, A_{100} = ||||\dots|||||x|+1|-2|+3|-4|+\dots|+99|-100|$.

Notice that when n is odd, $A_n = |A_{n-1}+n| = A_{n-1}+n$, and then $A_{n+1} = |A_n-(n+1)| = |A_{n-1}-1|$. So in general, we have $A_{2k} = |A_{2(k-1)}-1|$.

We now work backwards from $A_{100} = 0$: the equation implies $A_{98} = 1, A_{96} = 0$ or $2, A_{94} = 1$ or $3, A_{92} = 0$ or 2 or $4, A_{90} = 1$ or 3 or $5, \dots$.

Therefore, we can follow this pattern and get $A_0 = 0$ or 2 or 4 or $\dots$ or 50. When $A_0 = 0$, there is 1 solution, otherwise there are 2 possible values for each A_0, so there are $1+2\cdot 25 = 51$ solutions in total.

20. **[18]** A *split* of a positive integer in base 10 is the separation of the integer into two nonnegative integers, allowing leading zeroes. For example, 2017 can be split into 2 and 017 (or 17), 20 and 17, or 201 and 7. A split is called *squarish* if both integers are nonzero perfect squares. 49 and 169 are the two smallest perfect squares that have a squarish split (4 and 9, 16 and 9 respectively). Determine all other perfect squares less than 2017 with at least one squarish split.

Solution. The answer is $\boxed{361, 1225, 1444, 1681}$.

Let's start by considering three-digit numbers greater than 169. Since the first digit or the first two digits must be a perfect square, the possibilities are: $\overline{1AB}, \overline{25A}, \overline{36A}, \overline{4AB}, \overline{49A}, \overline{64A}, \overline{81A}, \overline{9AB}$. It is not difficult to check that among all these options, $361 = 19^2$ is the only one that works.

Now we consider the four-digit numbers whose thousand digit is 1. If the last three digits form a perfect square m^2, and the four-digit number is n^2, then we have $n^2 - m^2 = 1000$, or $(n-m)(n+m) =$

$1000 = 2^3 \cdot 5^3$. Keeping in mind that $n-m$ and $n+m$ have the same parity, the only solution with $m^2 < 1000$ is $(n-m, n+m) = (20, 50) \Rightarrow (n, m) = (35, 15) \Rightarrow n^2 = 1225$.

If the first two digits are 16, then similarly we need $n^2 - m^2 = 1600$, since $m^2 < 100$, the only solution is $(n-m, n+m) = (32, 50) \Rightarrow (n, m) = (41, 9) \Rightarrow n^2 = 1681$.

If the first three digits are a perfect square, then the only options are $\overline{121A}, \overline{144A}, \overline{169A}, \overline{196A}$. Among these possibilities, only $1444 = 38^2$ is a perfect square with the last digit being a perfect square as well.

There are also no perfect squares between 2000 and 2016 inclusive, so all perfect squares greater than 169 with a squarish split are 361, 1225, 1444, and 1681.

21. **[18]** Polynomial $f(x) = 2x^3 + 7x^2 - 3x + 5$ has zeroes a, b, and c. Cubic polynomial $g(x)$ with x^3-coefficient 1 has zeroes a^2, b^2, and c^2. Find the sum of coefficients of $g(x)$.

Solution. The answer is $\boxed{-\frac{143}{4}}$.

Note that we can write $f(x) = 2(x-a)(x-b)(x-c)$, and $g(x) = (x-a^2)(x-b^2)(x-c^2)$. The sum of coefficients of $g(x)$ is equal to $g(1)$, which can be rewritten as

$$\begin{aligned} g(1) &= (1-a^2)(1-b^2)(1-c^2) \\ &= (1-a)(1-b)(1-c)(1+a)(1+b)(1+c) \\ &= \frac{2(1-a)(1-b)(1-c)}{2} \cdot \frac{-2(-1-a)(-1-b)(-1-c)}{2} \\ &= -\frac{f(1)f(-1)}{4}. \end{aligned}$$

The last expression can be computed simply by plugging in $x = 1$ and $x = -1$ into $f(x)$, and we get $f(1) = 2 + 7 - 3 + 5 = 11$ and $f(-1) = -2 + 7 + 3 + 5 = 13$, so the sum of coefficients of $g(x)$ is $-\frac{11 \cdot 13}{4} = -\frac{143}{4}$.

2.4.8 Round 8

22. **[22]** Two congruent circles, ω_1 and ω_2, intersect at points A and B. The centers of ω_1 and ω_2 are O_1 and O_2 respectively. The arc AB of ω_1 that lies inside ω_2 is trisected by points P and Q, with the points lying in the order A, P, Q, B. Similarly, the arc AB of ω_2 that lies inside ω_1 is trisected by points R and S, with the points lying in the order A, R, S, B. Given that $PQ = 1$ and $PR = \sqrt{2}$, find the measure of $\angle AO_1B$ in degrees.

Solution. The answer is $\boxed{135}$.

Since P, Q trisects arc AB, $AP = PQ = QB = 1$, and by symmetry $AR = RS = SB = 1$. Because $PR = \sqrt{2}$, we can conclude that APQ is a right isosceles triangle with right angle at A. Note that by symmetry $PQSR$ is a rectangle, which means $\angle RPQ = 90^\circ$ and $\angle APQ = \angle RPQ + \angle APR = 90^\circ + 45^\circ = 135^\circ$. Then, we get $\angle APO_1 = \frac{1}{2}\angle APQ = 67.5^\circ$, $\angle AO_1P = 180^\circ - 2 \cdot 67.5^\circ = 45^\circ$, and finally $\angle AO_1B = 3\angle AO_1P = 135^\circ$.

23. **[22]** How many ordered triples of (a, b, c) of integers between -10 and 10 inclusive satisfy the equation $-abc = (a+b)(b+c)(c+a)$?

Solution. The answer is $\boxed{433}$.

We can expand and rewrite the equation as

$$a^2b + b^2c + c^2a + a^2c + b^2a + c^2b + 3abc = 0,$$

which factors to

$$(a + b + c)(ab + bc + ca) = 0.$$

Now we need either $a + b + c = 0$ or $ab + bc + ca = 0$. If one of a, b, c is zero (WLOG assume $c = 0$), then either $a = -b$ or $ab = 0$. The former case gives all the cases $(x, -x, 0)$ and its permutations. If $x = 0$, there is 1 possible solution, and otherwise we have $10 \cdot 6 = 60$ solutions. The latter gives all the cases $(x, 0, 0)$ and its permutations. Since $x = 0$ has been accounted before, we only need to consider the case with $x \neq 0$, which gives $20 \cdot 3 = 60$ solutions.

If none of a, b, c are zero, then if $a+b+c = 0$, exactly one of them will have different signs as the other two. WLOG assume that c is the only negative one, so $-c = a + b$. By caseworking on the value of c between -2 and -10 inclusive, we see that there are $1 + 2 + \cdots + 9 = 45$ solutions for a and b. Since there are 3 ways to choose the variable with the different sign and 2 ways to choose the sign, there are $3 \cdot 2 \cdot 45 = 270$ solutions in this case.

If instead $ab+bc+ca = 0$, then by dividing both sides by abc, we get $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. Again, exactly one of the three have different signs as the other two, and WLOG assume that c is the only negative one, so $-\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. (Note that this implies $|c| < a$ and $|c| < b$, so there will be no overlap with the case $-c = a + b$.) The only possible solutions for (a, b, c) are $(2, 2, -1), (4, 4, -2), (3, 6, -2), (6, 3, -2), (6, 6, -3), (8, 8, -4)$, and $(10, 10, -5)$, for 7 solutions in total. Similarly, because there are $3 \cdot 2 = 6$ ways to choose the "special" variable and its sign, there are $6 \cdot 7 = 42$ solutions in this case.

Adding up all cases, we have $1 + 60 + 60 + 270 + 42 = 433$ triples in total.

24. **[22]** For positive integers n and b where $b > 1$, define $s_b(n)$ as the sum of digits in the base-b representation of n. A positive integer p is said to *dominate* another positive integer q if for all positive integers n, $s_p(n)$ is greater than or equal to $s_q(n)$. Find the number of ordered pairs (p, q) of *distinct* positive integers between 2 and 100 inclusive such that p dominates q.

Solution. The answer is $\boxed{16}$.

We claim that p dominates q if and only if p is a power of q. In other words, there exists a positive integer k such that $p = q^k$.

If p dominates q, then by plugging in $n = p = \overline{10}_p$, we have $1 = s_p(p) \geq s_q(p) > 0$, so $s_q(p) = 1$, which happens if and only if the base-q representation of p is $\overline{100\ldots00}$, or p is a power of q.

Now suppose that $p = q^k$ for some positive integer k. Notice that for any base-q representation of a number n, we can break the string into several k-digit blocks from right to left (possibly with leading zeroes), convert each k-digit block to a single digit in base p, and put the digits back together to get the base-p representation of n. Consider a k-digit block $D = \overline{d_{k-1}d_{k-2}\cdots d_0}$ in base q, and its base-p equivalent $D' = \overline{d'}$. Since

$$s_p(D') = D' = D = d_{k-1} \cdot q^{k-1} + d_{k-2} \cdot q^{k-2} + \cdots + d_0 \cdot q^0 \geq d_{k-1} + d_{k-2} + \cdots + d_0 = s_q(D),$$

the sum of digits in each block in base q is no greater than the corresponding digit in base p, which means that $s_p(n) \geq s_q(n)$ for all n.

It remains to find the number of pairs (p, q) between 2 and 100 such that p is a power of q. Since $100 \geq p \geq q^2$, we only need to consider the cases where $q \leq 10$.

If $q = 2$, $p = 4, 8, 16, 32, 64$, for 5 pairs. If $q = 3$, $p = 9, 27, 81$, for 3 pairs. If $q = 4$, $p = 16, 64$, for 2 pairs. If $q = 5, 6, 7, 8, 9, 10$, $p = q^2$ for each case, for 6 pairs.

Therefore we have $5 + 3 + 2 + 6 = 16$ pairs in total.