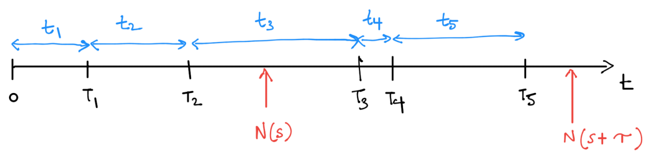
Lecture 8

Poisson Process Summary (Homogeneous) PP w/ rate λ



1. What are the distributions of:
   (a) t_1 ? $\sim \exp(\lambda)$
   (b) t_4 ? $\sim \exp(\lambda)$
   (c) T_1 ? $\sim \exp(\lambda)$
   (d) T_4 ? $\sim$ Erlang $(n=4, \lambda)$
   (e) $t_4 \mid t_1$? $\sim \exp(\lambda)$
   (f) $N(s+\tau)$? $\sim$ Poisson $(\lambda(s+\tau))$

2. Are the following variables independent?
   (a) $t_2 \perp\!\!\!\perp t_3$? Yes.
   (b) $T_2 \perp\!\!\!\perp T_3$? No.
   (c) $N(s+\tau) \perp\!\!\!\perp N(s)$? No.
   (d) $N(s+\tau) - N(s) \perp\!\!\!\perp N(s)$? Yes.

3. What is the Fano Factor of $N(s)$?

$$FF = \frac{\text{var}}{\text{mean}} = \frac{\lambda s}{\lambda s} = 1$$

4. What is the Fano Factor of t_3?

$$FF = \frac{1/\lambda^2}{1/\lambda} = 1/\lambda$$

<u>Inhomogeneous Poisson Process</u>

Key difference: $\lambda(r)$ is now a fn. of time.

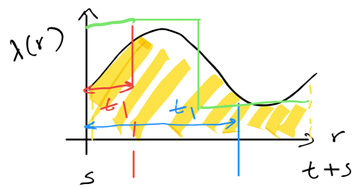
Defn: $\{N(s),\ s \geq 0\}$ is an inhomogeneous Poisson Process with rate $\lambda(r)$ if:

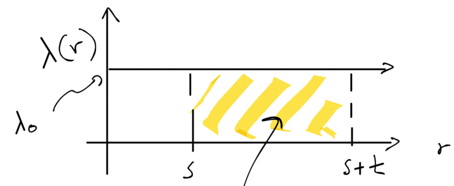
(1) $N(0) = 0$

(2) $N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^{t+s} \lambda(r)\, dr\right)$

(3) $N(s)$ has independent Increments

Question: how do we think of this in terms of a Bernoulli process?





$Pr\left(t_i > s \mid \lambda(r),\ t_{i-1} = t\right)$

area $= \lambda_0 t$

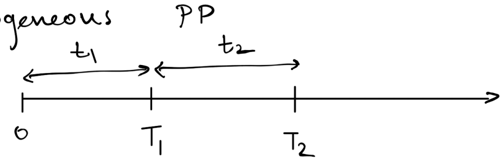
$Pr(t_1 > t + s \mid t_1 > t) = Pr(t_1 > s)$ Poisson $(\lambda_0 t)$

1. What is the distr. of the ISI's ?

   Yakovlev et al. (arXiv, 2005)

2. Are the ISI's memoryless?
3. Dist. of spike times?
4. Are the ISI's independent?

Inhomogeneous PP



$$\mu(t) = \int_0^t \lambda(r)\, dr$$

$$Pr(t_1 > t) = Pr\left(N(t) = 0\right)$$

$$= \frac{(\mu(t))^0}{0!} e^{-\mu(t)}$$

$$= e^{-\mu(t)}$$

$$f_{t_1}(t) = -\frac{d}{dt} Pr(t_1 > t)$$

$$= -e^{-\mu(t)}\left[-\lambda(t)\right]$$

$$= \lambda(t)\, e^{-\mu(t)}$$

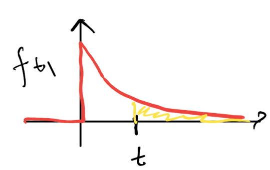
$$f_{t_1}(t) = \lambda e^{-\lambda t} \quad t \geq 0$$

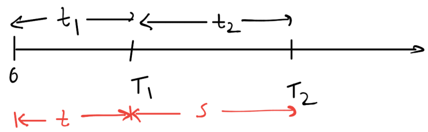
$$F_{t_1}(t) = Pr(t_1 < t)$$

$$= 1 - e^{-\lambda t}$$

$$Pr(t_1 > t) = 1 - Pr(t_1 < t)$$

$$= e^{-\lambda t}$$





$Pr(N(s+t) = 1)$

$$Pr(t_2 > s \mid t_1 = t) = Pr(N(s+t) - N(t) = 0)$$

$$= e^{-\int_t^{s+t} \lambda(r)\, dr}$$

$$= e^{-(\mu(s+t) - \mu(t))}$$

$$f_{t_2|t_1}(s|t) = -\frac{d}{ds} Pr(t_2 > s \mid t_1 = t)$$

$$= \lambda(s+t)\, e^{-(\mu(s+t) - \mu(t))}$$

$$\Rightarrow \quad t_1 \not\perp t_2$$

Joint ISI's

$$f_{t_1, t_2}(t, s) = f_{t_2|t_1}(s|t)\, f_{t_1}(t)$$

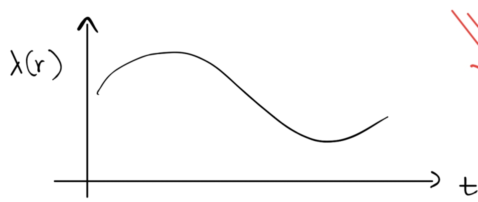
$$= \lambda(t)\, \lambda(t+s)\, e^{-\mu(s+t)}$$

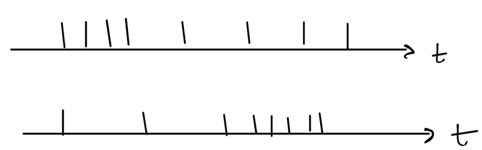
$$f_{T_1, T_2}(v_1, v_2) = \lambda(v_1)\, \lambda(v_2)\, e^{-\mu(v_2)}$$

$\Downarrow$ Homogeneous PP

$$\lambda(r) = \lambda_0 \ \forall r$$

$$f_{T_1, T_2}(v_1, v_2) = \lambda_0^2\, e^{-\mu(v_2)}$$





Generate PP's — Homogeneous PP w/ rate λ

Method 1 — generate over length τ

- Generate a lot of $t_i \sim \exp(\lambda)$
  "exprnd"
- Gen. $t_1 \sim \exp(\lambda) \longrightarrow T_1 = t_1$
  $t_2 \sim \exp(\lambda) \longrightarrow T_2 = t_1 + t_2$
  $\vdots$
- Spike times are $T_n = \sum_{i=1}^{n} t_i$
- If $T_n > \tau$, stop.

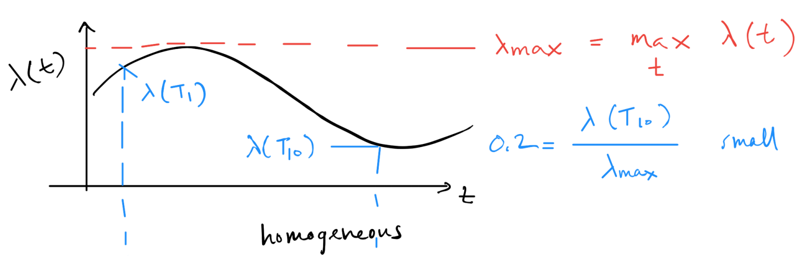
Method 2 — window τ, rate λ

- $N \sim \text{Poiss}(\lambda \tau) \Leftarrow$ # of spikes in my window τ

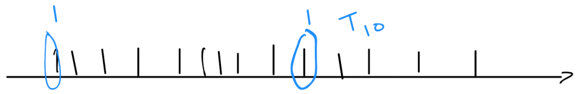
  "poissrnd"

- $T_1, \ldots, T_N \sim \text{Uniform}[0, \tau]$ iid

Gen. Inhomogeneous PP



1. Generate a PP with rate λ_{max}.



N spikes

2. For each spike $n = 1, \ldots, N$

Draw Unif$[0, 1]$ r.v. (u)

u vs. $\dfrac{\lambda(T_n)}{\lambda_{max}}$

If $u > \dfrac{\lambda(T_n)}{\lambda_{max}}$ reject.

$u \leq \dfrac{\lambda(T_n)}{\lambda_{max}}$ keep