Lecture 10

$$\alpha(\vec{x}) = \vec{y}^T \vec{x} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_D \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_D \end{bmatrix}$$

$$\frac{d\alpha(\vec{x})}{d\vec{x}} = \begin{bmatrix} \frac{d\alpha(\vec{x})}{dx_1} \\ \frac{d\alpha(\vec{x})}{dx_2} \\ \vdots \\ \frac{d\alpha(\vec{x})}{dx_D} \end{bmatrix}$$

$$\vec{y}^T\vec{x} = y_1 x_1 + y_2 x_2 + \cdots + y_D x_D$$

$$\frac{d(\vec{y}^T\vec{x})}{d\vec{x}} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_D \end{bmatrix} = \vec{y}$$

$$\alpha(\vec{x}) = \vec{x}^T A \vec{x} \qquad \vec{x} \in \mathbb{R}^D \qquad A \in \mathbb{R}^{D \times D}$$

$$\frac{d(\vec{x}^T A \vec{x})}{d\vec{x}}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1D} \\ a_{21} & a_{22} & \cdots & a_{2D} \\ \vdots & & \ddots & \\ a_{D1} & & & \end{bmatrix}$$

$$\vec{x}^T A \vec{x} = \sum_{i=1}^{D} \sum_{j=1}^{D} x_i a_{ij} x_j$$

$$\frac{d(\vec{x}^T A \vec{x})}{dx_1} = 2a_{11}x_1 + \sum_{j=2}^{D} a_{1j} x_j + \sum_{i=2}^{D} a_{i1} x_i$$

$$= \sum_j a_{1j} x_j + \sum_i a_{i1} x_i$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1D} \\ a_{21} & a_{22} & \cdots & a_{2D} \\ \vdots & & \ddots & \\ a_{D1} & & & a_{DD} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_D \end{bmatrix} \Rightarrow \begin{bmatrix} (Ax)_1 + (A^T x)_1 \\ (Ax)_2 + (A^T x)_2 \\ \vdots \\ (Ax)_D + (A^T x)_D \end{bmatrix}$$

$$(A + A^T)\vec{x} \Leftarrow$$

$$Tr(A) = \sum_{i=1}^{D} a_{ii}$$

$$Tr(ABCDE)$$

$$= Tr(BCDEA)$$

$$= Tr(CDEAB)$$

$$= Tr(DEABC)$$

Goal: $\vec{x}$ $\Longrightarrow$ classify it as one of $\mathbb{K}$ classes

Firing rate of $\underline{D}$ neurons

which target planned

Train a model to perform this classification

$(\vec{x}^{\,i}, c^i)$ — i^{th} training ex.

$\in \mathbb{R}^{100}$ f.r. of each neuron

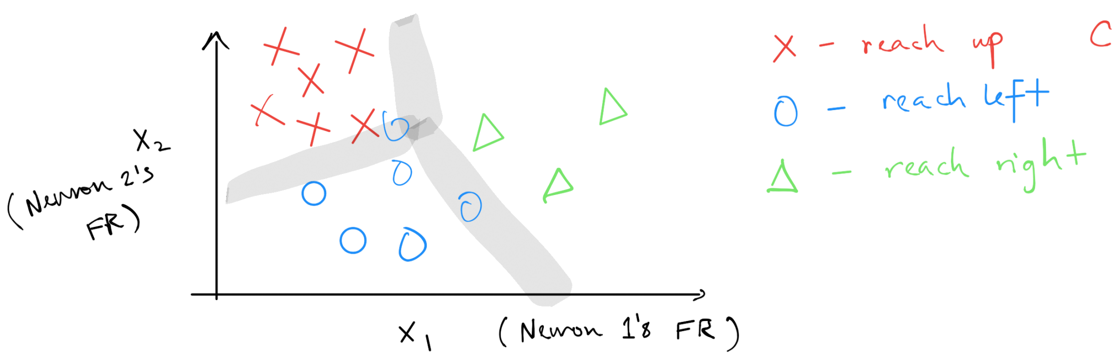
$\Longrightarrow$ right

N

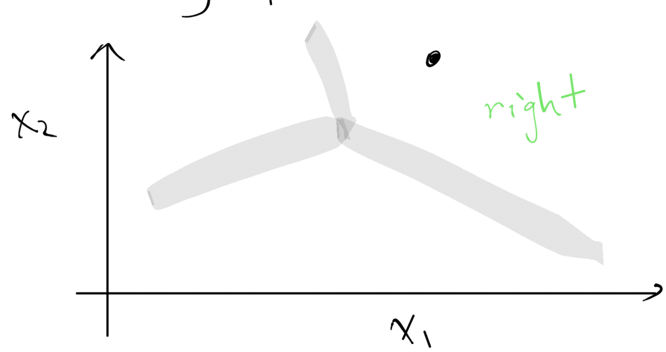
Test: $\vec{x}^{\,j}$

$\Downarrow$ trained model

c^j



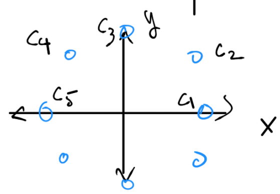
Training phase: learned boundaries



Generative models

Train: class-conditional density : $P(\vec{x} \mid C_k)$

class prior $P(C_k)$



$P(C_k) = \frac{1}{8}$

$P(C_1) = \frac{1}{2}$ $\quad$ $P(C_2) = 0$

Test: $P(C_k \mid \vec{x})$

$$\hat{k} = \arg\max_k P(C_k \mid \vec{x})$$

$$= \arg\max_k \frac{P(\vec{x} \mid C_k)\, P(C_k)}{P(x)}$$

$$= \arg\max_k \frac{P(\vec{x} \mid C_k)\, P(C_k)}{\sum_{k=1}^{K} P(x, C_k)}$$

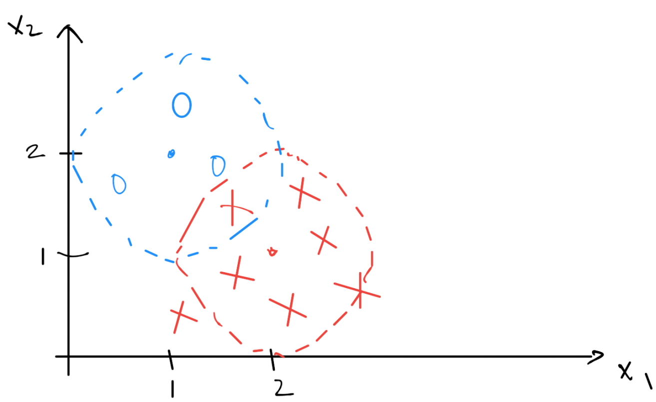
Generative models

$P(C_k)$ $\quad$ $P(\vec{x} \mid C_k)$ $\quad$ $K = 2$

$\vec{x} \in \mathbb{R}^2$

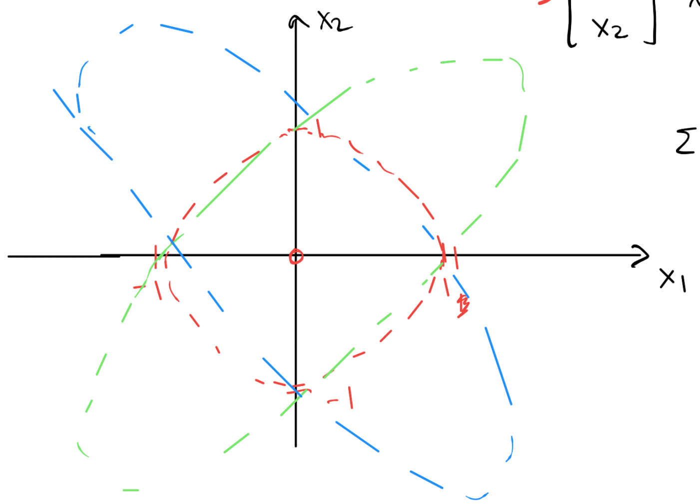
$P(C_1) = 0.7$ $\quad$ $P(\vec{x} \mid C_1) = N\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$P(C_2) = 0.3$ $\quad$ $P(\vec{x} \mid C_2) = N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$



$p(\vec{x} \mid C_k)$ $\quad f$

$p(C_k)$



Σ symmetric

$\Sigma = \Sigma^T$

$\vec{x} \sim N(\mu, \Sigma)$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Sigma = \begin{bmatrix} var(X_1) & cov(X_1, X_2) \\ cov(X_2, X_1) & Var(X_2) \end{bmatrix}$$

$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$$

Max. Likelihood

Ex: two classes, $K = 2$ w/ Gaussian class conditional densities

Training data: $(\vec{x}_n, t_n)$ for $n = 1, \dots, N$

$\in \mathbb{R}^D$, firing rates of D neurons

labels (i.e. reach right/left/etc)

$t_n = 1$ (class C_1, reach left)

$t_n = 0$ (class C_2, reach right)

$t_n \sim \text{Bern}(\pi)$

$Pr(C_1) = Pr(t_n = 1) = \pi$

$Pr(C_2) = Pr(t_n = 0) = 1 - \pi$

Distr. $\vec{x}_n \mid t_n = 1 \sim N(\vec{x}_n \mid \mu_1, \Sigma)$

$\vec{x}_n \mid t_n = 0 \sim N(\vec{x}_n \mid \mu_0, \Sigma)$

$(\vec{x}_1, t_1)$ Reach left $t_1 = 1$

$(\vec{x}_2, t_2)$ Reach left $t_2 = 1$

$(\vec{x}_3, t_3)$ Reach right $t_3 = 0$

$$\theta = \{ \pi, \mu_1, \mu_2, \Sigma \}$$

$$P\left((\vec{x}_1, t_1), (\vec{x}_2, t_2), (\vec{x}_3, t_3) \mid \theta\right)$$

$$= P(\vec{x}_1, t_1 \mid \theta)\, P(\vec{x}_2, t_2 \mid \theta)\, P(\vec{x}_3, t_3 \mid \theta) \cdots$$

$$= \left[\pi\, N(x_1 \mid \mu_1, \Sigma)\right]\left[\pi\, N(x_2 \mid \mu_1, \Sigma)\right]\left[(1-\pi)\, N(x_3 \mid \mu_2, \Sigma)\right] \cdots$$

$$= \prod_{n=1}^{N} \left(\pi\, N(x_n \mid \mu_1, \Sigma)\right)^{t_n} \left((1-\pi)\, N(x_n \mid \mu_2, \Sigma)\right)^{1-t_n}$$

$$= \mathcal{L}$$ "likelihood"

$$\star\star\star \quad \mathcal{L} = \prod_{n=1}^{N} \left(\pi N(x_n | \mu_1, \Sigma)\right)^{t_n} \left((1-\pi) N(x_n | \mu_2, \Sigma)\right)^{1-t_n}$$

$$\boxed{\log \mathcal{L} = \sum_{n=1}^{N} \Big[t_n \log \pi + t_n \log N(x_n | \mu_1, \Sigma) + (1-t_n) \log (1-\pi) + (1-t_n) \log N(x_n | \mu_2, \Sigma) \Big]}$$

$$\log N(x_n | \mu_i, \Sigma) = \log \left[\frac{1}{(2 \cdot 3.14159)^{D/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x_n-\mu_i)^T \Sigma^{-1} (x_n - \mu_i)\right)\right]$$

$$= -\frac{1}{2}(x_n-\mu_i)^T \Sigma^{-1} (x_n-\mu_i) - \frac{D}{2} \log (2 \cdot 3.14159) - \frac{1}{2} \log |\Sigma|$$

$$\theta = \{ \pi, \mu_1, \mu_2, \Sigma \}$$

(i) Find opt π

optimal

$$\frac{\partial \log \mathcal{L}}{\partial \pi} = \sum_{n=1}^{N} \left[t_n \cdot \frac{1}{\pi} - (1-t_n) \frac{1}{1-\pi} \right]$$

$$[=] \ 0$$

$$\Rightarrow (1-\pi) \sum_{n=1}^{N} t_n - \pi \sum_{n=1}^{N} (1-t_n) = 0$$

Define $N_1 = \sum_{n=1}^{N} t_n \Rightarrow$ # trials from class 1

$$N_2 = \sum_{n=1}^{N} (1-t_n)$$

$$N = N_2 + N_1$$

$$\Rightarrow (1-\pi) N_1 - \pi (N - N_1) = 0$$

$$\Rightarrow N_1 \cancel{- N_1 \pi} - \pi N \cancel{+ \pi N_1} = 0$$

$$\Rightarrow \boxed{\pi = \frac{N_1}{N}}$$

$$\frac{\partial \log \mathcal{L}}{\partial \mu_1} = \sum_{n=1}^{N} t_n \left[\frac{\partial}{\partial \mu_1} -\frac{1}{2}(x_n - \mu_1)^T \Sigma^{-1} (x_n - \mu_1) \right]$$

$$= -\frac{1}{2} \sum_{n=1}^{N} t_n \left[\frac{\partial}{\partial \mu_1} \left(x_n^T \Sigma^{-1} x_n - 2\underbrace{\mu_1}_{y}{}^T \underbrace{(\Sigma^{-1} x_n)}_{x} + \mu_1^T \Sigma^{-1} \mu_1 \right) \right]$$

$$\frac{d\, y^T x}{dy} = x$$

$$= -\frac{1}{2} \sum_{n=1}^{N} t_n \left[-2 (\Sigma^{-1} x_n) + (2 \Sigma^{-1}) \mu_1 \right]$$

$$= \sum_{n=1}^{N} t_n \left(\Sigma^{-1} x_n - \Sigma^{-1} \mu_1 \right) [=] 0$$

$$\Rightarrow -\Sigma^{-1} \mu_1 \cdot N_1 + \Sigma^{-1} \left(\sum_{n=1}^{N} t_n x_n \right) = 0$$

$$\Rightarrow N_1 \mu_1 = \sum_{n=1}^{N} t_n x_n \Rightarrow \boxed{\mu_1 = \frac{1}{N_1} \sum_{n \in C_1} x_n}$$

Analogously, $\mu_2 = \dfrac{1}{N_2} \sum_{n \in C_2} x_n$