EE239AS.2, Lecture 7, 25 Apr 2017

$T \sim \exp(\lambda)$ [Exponential distribution]

Key property: memoryless, i.e. $\Pr(T > t+s \mid T > t) = \Pr(T > s)$

$$\Downarrow$$

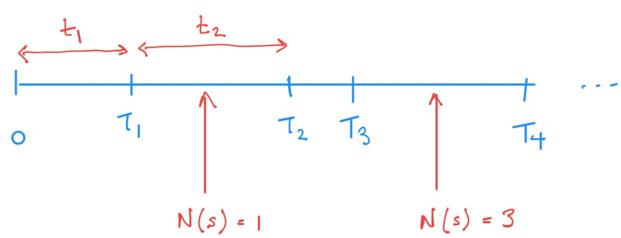
Poisson Process

$t_1, t_2, \ldots \sim \exp(\lambda)$ iid.

$T_n = t_1 + t_2 + \ldots + t_n$ for $n \geq 1$ ($T_0 = 0$)

$N(s) = \max\{n : T_n \leq s\}$

$N(s)$ is a Poisson Process



$$Pr(N(s)=n) = Pr(T_n \leq s,\ T_{n+1} > s)$$

$$= Pr(T_{n+1} > s \mid T_n \leq s)\ Pr(T_n \leq s)$$

Let's consider times are discrete

$$= Pr(T_{n+1} > s \mid T_n = s)\ Pr(T_n = s)$$

$$+ Pr(T_{n+1} > s \mid T_n = s-1)\ Pr(T_n = s-1)$$

$$+ \quad \vdots$$

$$+ Pr(T_{n+1} > s \mid T_n = 0)\ Pr(T_n = 0)$$

$$= \sum_{t=0}^{s} Pr(T_{n+1} > s \mid T_n = t)\ Pr(T_n = t)$$

$$= \int_0^s Pr(T_{n+1} > s \mid T_n = t)\ f_{T_n}(t)\, dt$$

$$Pr(N(s)=n) = \int_0^s Pr(T_{n+1} > s \mid T_n = t)\ f_{T_n}(t)\, dt$$

$$= \int_0^s Pr(T_n + t_{n+1} > s \mid T_n = t) \quad \cdots$$

$$= \int_0^s Pr(t_{n+1} > s - t)\ f_{T_n}(t)\, dt$$

$$e^{-\lambda(s-t)} \qquad ???$$

$$T_n = t_1 + t_2 + \cdots + t_n$$

$$f_{T_n} = f_{t_1} * f_{t_2} * \cdots * f_{t_n}$$

$$\mathcal{F}(f_{T_n}) = \mathcal{F}(f_{t_1})\,\mathcal{F}(f_{t_2})\cdots\mathcal{F}(f_{t_n})$$

$$= \prod_{i=1}^{n}\mathcal{F}(f_{t_i}) = \prod_{i=1}^{n}\mathcal{F}\left(\lambda e^{-\lambda t}u(t)\right)$$

$$= \left[\mathcal{F}\left(\lambda e^{-\lambda t}u(t)\right)\right]^n$$

$$e^{-\lambda t}u(t) \rightarrow \frac{1}{\lambda + j\omega}$$

$$= \left[\lambda\cdot\frac{1}{\lambda+j\omega}\right]^n$$

$$t^n e^{-\lambda t}u(t) \rightarrow \frac{n!}{(\lambda+j\omega)^{n+1}}$$

$$= \left(\frac{\lambda}{\lambda+j\omega}\right)^n$$

$$\mathcal{F}^{-1}\left[\left(\frac{\lambda}{\lambda+j\omega}\right)^n\right] = \frac{\lambda^n}{(n-1)!}\mathcal{F}^{-1}\left[\frac{(n-1)!}{(\lambda+j\omega)^n}\right]$$

$$f_{T_n} = \frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t}u(t)$$

$$Pr(N(s)=n) = \int_0^s e^{-\lambda(s-t)}\,\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}\,dt$$

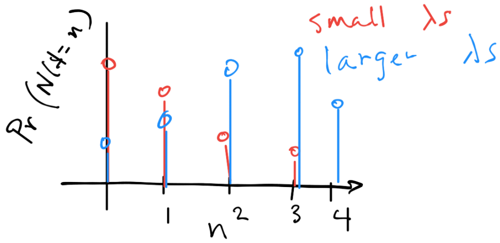
$$= \frac{\lambda^n}{(n-1)!}e^{-\lambda s}\int_0^s t^{n-1}\,dt$$

$$= \frac{\lambda^n}{(n-1)!}e^{-\lambda s}\left[\frac{t^n}{n}\right]_0^s$$

$$= \frac{\lambda^n}{n!}e^{-\lambda s}s^n$$

$$= \frac{(\lambda s)^n}{n!}e^{-\lambda s}$$

$$Pr\{N(s)=n\} = \checkmark \qquad N(s) \sim \text{Poisson}(\lambda s)$$



$$\mathbb{E}[N(s)] = \lambda s$$

$$\mathbb{E}[N(s)] = \sum_{n=0}^{\infty} n \Pr(N(s) = n)$$

$$= \sum_{n=0}^{\infty} n e^{-\lambda s} \frac{(\lambda s)^n}{n!}$$

$$= \sum_{n=1}^{\infty} n e^{-\lambda s} \frac{(\lambda s)^n}{n!}$$

$$= \sum_{n=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^n}{(n-1)!}$$

$$= \lambda s \cancel{\sum_{n=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!}} \quad 1$$

$$\mathbb{E}\left(N(s)\,(N(s)-1)\right) = \sum_{n=0}^{\infty} n(n-1)\, P(N(s)=n)$$

$$= \sum_{n=2}^{\infty} n(n-1)\, e^{-\lambda s}\, \frac{(\lambda s)^n}{n!}$$

$$= (\lambda s)^2 \underbrace{\sum_{n=2}^{\infty} e^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!}}_{1}$$

$$= (\lambda s)^2$$

$$\mathrm{var}(N(s)) = \mathbb{E}\left(N(s)^2\right) - \left[\mathbb{E}\left(N(s)\right)\right]^2$$

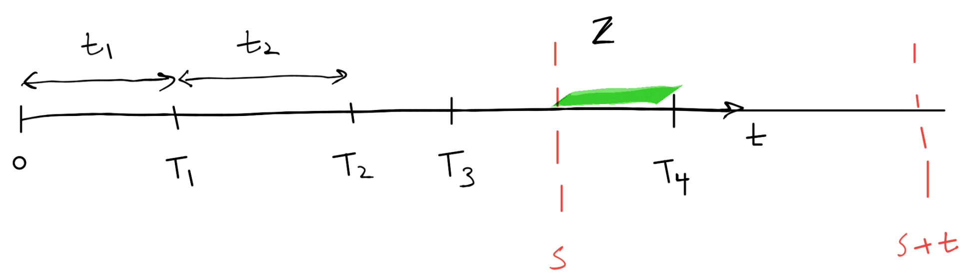
$$= \mathbb{E}\left(N(s)\left(N(s)-1\right)\right) + \mathbb{E}\left(N(s)\right) - \left(\mathbb{E}\left[N(s)\right]\right)^2$$

$$= (\lambda s)^2 + \lambda s - (\lambda s)^2$$

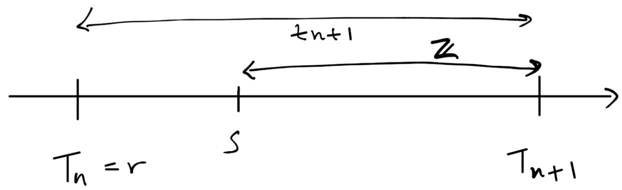
$$= \lambda s$$

$$\text{Fano Factor} = \frac{\text{Var}}{\text{Mean}} = 1$$

<u>Prop. 2</u> $N(t+s) - N(s)$, $t \geq 0$, is a rate λ PP & is $\perp\!\!\!\perp$ of $N(r)$, $0 \leq r \leq s$.



$$N(s+t) - N(s) \sim \text{Poiss}(\lambda t)$$



$$\Pr\{ Z > z \mid N(s) = n,\ T_n = r \}$$

$$= \Pr\{ t_{n+1} > z + s - r \mid \cancel{T_n = r},\ t_{n+1} > s - r \}$$

$$= \Pr\{ t_{n+1} > z + s - r \mid t_{n+1} > s - r \}$$

$$= \Pr\{ t_{n+1} > z \}$$

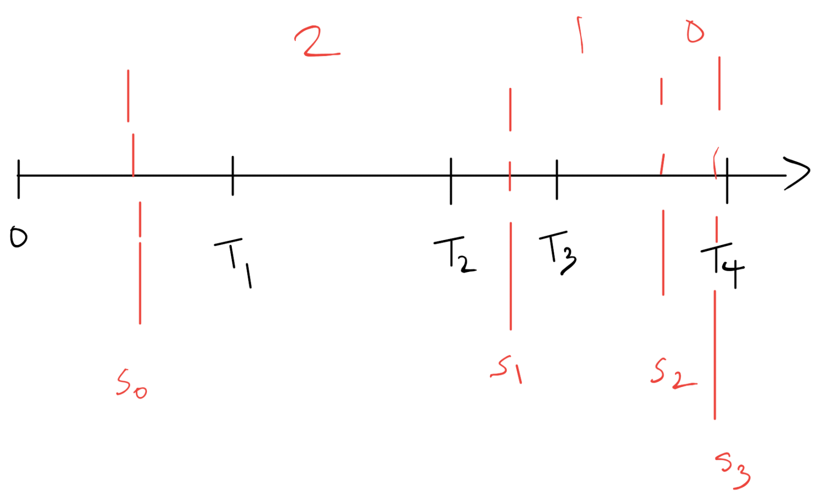
$$= e^{-\lambda z}$$

Prop. 3 : $N(t)$ has indep. incr.

If $s_0 < s_1 < \cdots < s_n$, then

$N(s_1) - N(s_0)$, $N(s_2) - N(s_1)$, ...

are independent



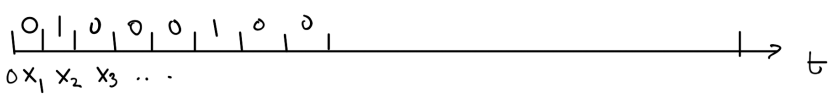
If $\{N(s), s \geq 0\}$ is a PP w/ rate λ, then

(i) $N(0) = 0$

(ii) $N(t+s) - N(s) \sim \text{Poiss}(\lambda t)$

(iii) $N(s)$ has indep. increments.

Bernoulli Process



n discrete time steps

p coin coming up heads (spike)

i^{th} time step

$$X_i \sim \text{Bern}(p)$$

$$X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

S_n : # of spikes up to & including the n^{th} time step

$$S_n = \sum_{i=1}^{n} X_i$$

$S_n \sim \text{Binom}(n, p)$

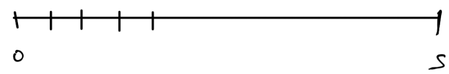
$$\Pr(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}[S_n] = np \quad \Rightarrow$$

As $n \to \infty$, $p \to 0$, Bern Process $\to$ Poiss Process

$$np = \lambda s$$

$$p = \frac{\lambda s}{n}$$



$$P(S_n = k) \xrightarrow[p = \frac{\lambda s}{n}]{n \to \infty} \text{Poiss}(\lambda s)$$

$$\lim_{n\to\infty} P(S_n = k) = \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda s}{n}\right)^k \left(1 - \frac{\lambda s}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(\lambda s)^k}{k!} \frac{\left(1 - \frac{\lambda s}{n}\right)^n}{\left(1 - \frac{\lambda s}{n}\right)^k}$$

$$= \lim_{n \to \infty} \frac{n^k + O(n^{k-1})}{n^k} \frac{(\lambda s)^k}{k!} \frac{\left(1 - \frac{\lambda s}{n}\right)^n}{\left(1 - \frac{\lambda s}{n}\right)^k}$$

$$= 1 \cdot \frac{(\lambda s)^k}{k!} \frac{e^{-\lambda s}}{1}$$

$$\left.\begin{array}{l} PP \quad \lambda \\ \qquad\ \ \delta \end{array}\right\} \quad N(\delta) \sim \text{Poiss}(\lambda\delta)$$

$$Pr(0 \text{ spikes in } [t, t+\delta]) = \frac{e^{-\lambda\delta}(\lambda d)^0}{0!}$$

$$= e^{-\lambda\delta} = 1 - \lambda\delta + o(\delta^2)$$

$$Pr(1 \text{ spike in } [t, t+\delta]) = e^{-\lambda\delta}(\lambda\delta) = \lambda\delta + o(\delta^2)$$

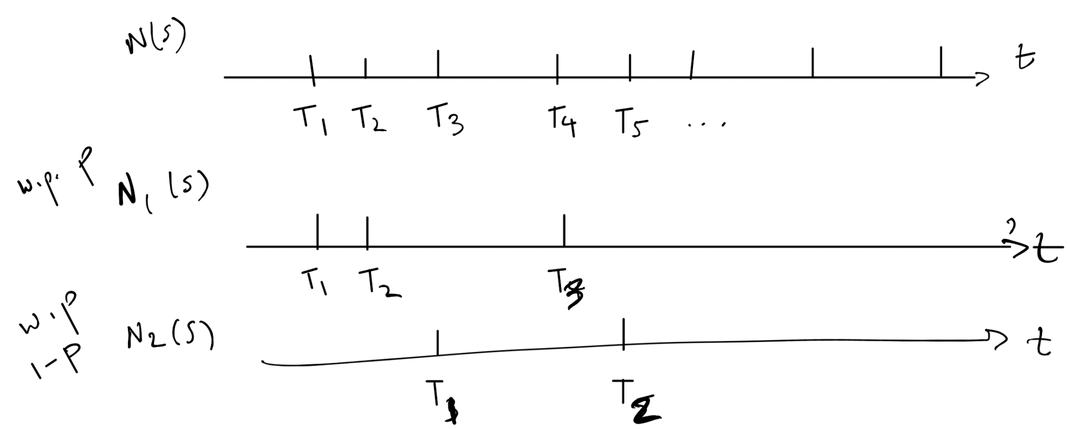
$$Pr(> 1 \text{ spike in } [t, t+\delta]) = O(\delta^2)$$

If δ is small, $O(\delta^2) \to 0$

δ : spike happens w.p. $\lambda\delta$

No spikes happen w.p. $1 - \lambda\delta$

Thinning $N(s)$ rate λ



$N_1(s)$ is a PP w/ rate λp

$N_2(s)$ is a PP w/ rate $\lambda(1-p)$

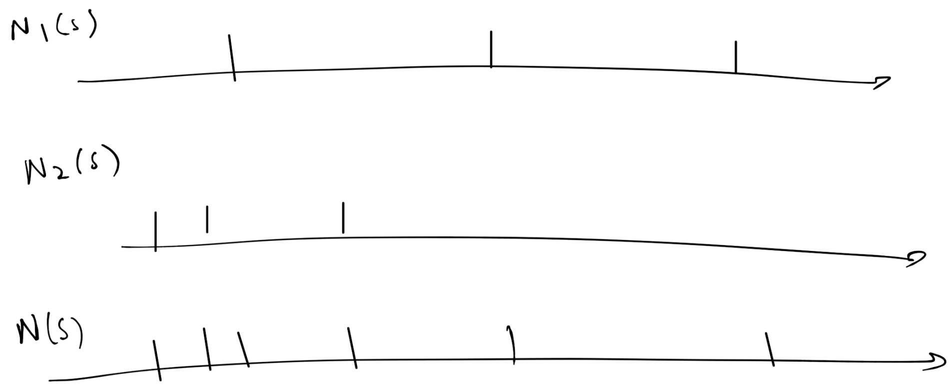
$Pr(N_1(s) = n)$

$\sim \text{Poiss}(\lambda p s)$

Superposition

$$\left.\begin{array}{ll} N_1(s) & \text{w/ rate } \lambda_1 \\ N_2(s) & \text{w/ rate } \lambda_2 \end{array}\right\} \quad \perp\!\!\!\perp$$

$$N(s) = N_1(s) + N_2(s) \qquad \text{w/ rate } \lambda_1 + \lambda_2$$



Inhomogeneous PP

Define: $\{N(s),\ s \geq 0\}$ is an inhomogeneous PP w/ rate $\lambda(r)$ if

(i) $N(0) = 0$

(ii) $N(t+s) - N(s) \sim \text{Poiss}\left(\int_s^{t+s} \lambda(r)\, dr \right)$

(iii) $N(s)$ has indep. incr.