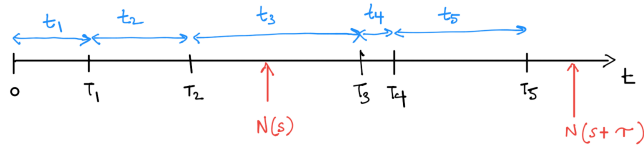


Lecture 8

Poisson Process Summary (Homogeneous) PP w/ rate λ



- What are the distributions of:
 - (a) t_1 ? $\sim \exp(\lambda)$
 - (b) t_4 ? $\sim \exp(\lambda)$
 - (c) T_1 ? $\sim \exp(\lambda)$
 - (d) T_4 ? $\sim \text{Erlang}(n=4, \lambda)$
 - (e) $t_4 | t_1$? $\sim \exp(\lambda)$
 - (f) $N(s+\tau)$? $\sim \text{Poisson}(\lambda(s+\tau))$
- Are the following variables independent?
 - (a) $t_2 \perp t_3$? **Yes.**
 - (b) $T_2 \perp T_3$? **No.**
 - (c) $N(s+\tau) \perp N(s)$? **No.**
 - (d) $N(s+\tau) - N(s) \perp N(s)$? **Yes.**
- What is the Fano Factor of $N(s)$?

$$FF = \frac{\text{var}}{\text{mean}} = \frac{\lambda s}{\lambda s} = 1$$

- What is the Fano Factor of t_3 ?

$$FF = \frac{1/\lambda^2}{1/\lambda} = 1/\lambda$$

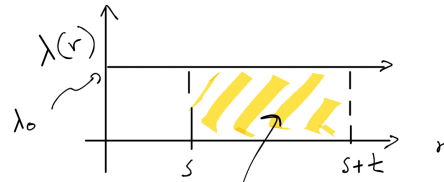
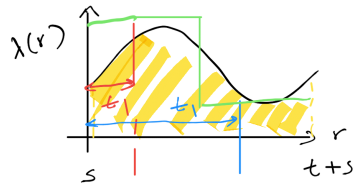
Inhomogeneous Poisson Process

Key difference: $\lambda(r)$ is now a fn. of time.

Defn: $\{N(s), s \geq 0\}$ is an inhomogeneous Poisson Process with rate $\lambda(r)$ if:

- (1) $N(0) = 0$
- (2) $N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^{t+s} \lambda(r) dr\right)$
- (3) $N(s)$ has independent increments

Question: how do we think of this in terms of a Bernoulli process?



$$\Pr(t_i > s \mid \lambda(r), t_{i-1} = t)$$

$$\text{area} = \lambda_0 t$$

$$\Pr(t_1 > t+s \mid t_1 > t) = \Pr(t_1 > s) \text{ Poisson}(\lambda_0 t)$$

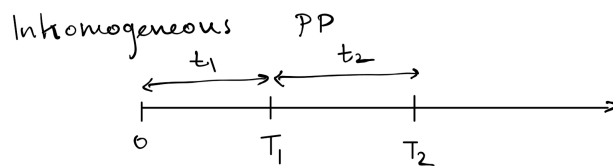
1. What is the distr. of the ISI's?

2. Are the ISI's memoryless?

3. Distr. of spike times?

4. Are the ISI's independent?

↓ Yakovlev et al.
(arXiv, 2005)



$$\mu(t) = \int_0^t \lambda(r) dr$$

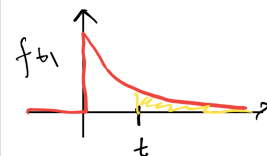
$$\begin{aligned} \Pr(t_1 > t) &= \Pr(N(t) = 0) \\ &= \frac{(\mu(t))^0}{0!} e^{-\mu(t)} \\ &= e^{-\mu(t)} \end{aligned}$$

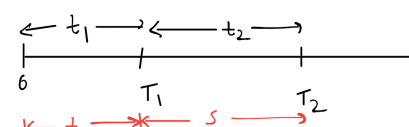
$$\begin{aligned} f_{t_1}(t) &= -\frac{d}{dt} \Pr(t_1 > t) \\ &= -e^{-\mu(t)} [-\lambda(t)] \\ &= \lambda(t) e^{-\mu(t)} \end{aligned}$$

$$f_{t_1}(t) = \lambda e^{-\lambda t} \quad t \geq 0$$

$$\begin{aligned} F_{t_1}(t) &= \Pr(t_1 < t) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

$$\begin{aligned} \Pr(t_1 > t) &= 1 - \Pr(t_1 < t) \\ &= e^{-\lambda t} \end{aligned}$$





$$\Pr(N(s+t) = 1)$$

$$\Pr(t_2 > s \mid t_1 = t) = \Pr(N(s+t) - N(t) = 0)$$

$$= e^{-\int_t^{s+t} \lambda(r) dr}$$

$$= e^{-(\mu(s+t) - \mu(t))}$$

$$f_{t_2|t_1}(s|t) = -\frac{d}{ds} \Pr(t_2 > s \mid t_1 = t)$$

$$= \lambda(s+t) e^{-(\mu(s+t) - \mu(t))}$$

$$\Rightarrow t_1 \not\propto t_2$$

Joint 1s1's

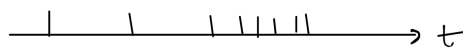
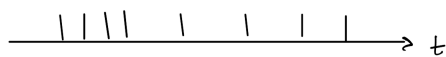
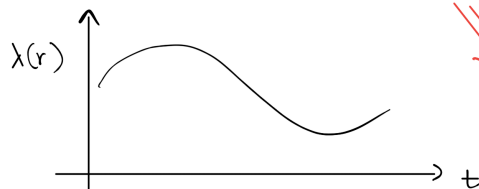
$$f_{t_1, t_2}(t, s) = f_{t_2|t_1}(s|t) f_{t_1}(t)$$

$$= \lambda(t) \lambda(t+s) e^{-\mu(s+t)}$$

$$f_{\tau_1, \tau_2}(v_1, v_2) = \lambda(v_1) \lambda(v_2) e^{-\mu(v_2)}$$

Homogeneous PP
 $\lambda(r) = \lambda_0 \forall r$

$$f_{\tau_1, \tau_2}(v_1, v_2) = \lambda_0^2 e^{-\mu(v_2)}$$



Generate PP's - Homogeneous PP w/ rate λ

Method 1 - generate over length τ

- Generate a lot of $t_i \sim \exp(\lambda)$
"exprrnd"
- Gen. $t_1 \sim \exp(\lambda) \rightarrow T_1 = t_1$
 $t_2 \sim \exp(\lambda) \rightarrow T_2 = t_1 + t_2$
 \vdots
- Spike times are $T_n = \sum_{i=1}^n t_i$
- If $T_n > \tau$, stop.

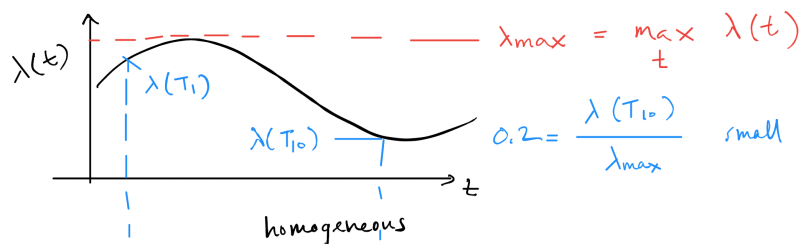
Method 2 - window τ , rate λ

- $N \sim \text{Poiss}(\lambda\tau) \Leftarrow$ # of spikes in my window τ

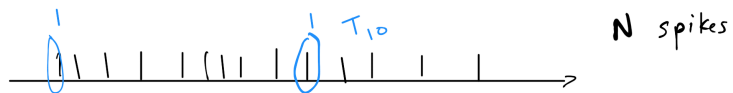
"poissrnd"

- $T_1, \dots, T_N \sim \text{Uniform}[0, \tau]$ iid

Gen. Inhomogeneous PP



1. Generate a $\hat{P}P$ with rate λ_{\max} .



2. For each spike $n=1, \dots, N$

Draw $\text{Unif}[0,1]$ r.v.

u vs. $\frac{\lambda(T_n)}{\lambda_{\max}}$

(u)

If $u > \frac{\lambda(T_n)}{\lambda_{\max}}$ reject.

$u \leq \frac{\lambda(T_n)}{\lambda_{\max}}$ keep

