Note 5.10

$$\log L = \sum \Big[t_n \log N(\vec{x_n} | \mu_1, \Sigma) + t_n \log \pi + (1-t_n) \log N(\vec{x_n} | \mu_2, \Sigma)$$

$$+ (1-t_n) \log (1-\pi) \Big]$$

$$\frac{\partial \log L}{\partial \Sigma} = \frac{\partial}{\partial \Sigma} \Big[\sum_{n=1}^{N} t_n \Big(-\frac{1}{2} \underbrace{(x_n-\mu_1)^T \Sigma^{-1} (x_n-\mu_1)}_{Tr(\cdot)} - \frac{1}{2} \log|\Sigma| \Big) +$$

$$(1-t_n) \Big(-\frac{1}{2} \underbrace{(x_n-\mu_2)^T \Sigma^{-1} (x_n-\mu_2)}_{Tr(\cdot)} - \frac{1}{2} \log|\Sigma| \Big)$$

$$\boxed{\begin{aligned} & Tr\left(\Sigma^{-1}(x_n-\mu_2)(x_n-\mu_2)^T\right) \\ & \frac{\partial}{\partial \Sigma} Tr(\Sigma^{-1} A) = -\Sigma^{-T} A^T \Sigma^{-T} = -\Sigma^{-1} A^T \Sigma^{-1} \\ & \frac{\partial}{\partial \Sigma} \log|\Sigma| = \Sigma^{-T} = \Sigma^{-1} \end{aligned}}$$

$$= \sum_{n=1}^{N} \Big\{ t_n \Big[-\frac{1}{2} \cdot -\Sigma^{-1} (x_n-\mu_1)(x_n-\mu_1)^T \Sigma^{-1} - \frac{1}{2} \Sigma^{-1} \Big] +$$

$$(1-t_n) \Big[-\frac{1}{2} \cdot -\Sigma^{-1} (x_n-\mu_2)(x_n-\mu_2)^T \Sigma^{-1} - \frac{1}{2} \Sigma^{-1} \Big] \Big\}$$

$$= 0$$

$$\Rightarrow -N\Sigma + \sum_{n \in C_1} (x_n-\mu_1)(x_n-\mu_1)^T + \sum_{n \in C_2} (x_n-\mu_2)(x_n-\mu_2)^T$$

$$\Sigma = \frac{1}{N} \Big[\sum_{n \in C_1} (x_n-\mu_1)(x_n-\mu_1)^T + \sum_{n \in C_2} (x_n-\mu_2)(x_n-\mu_2)^T \Big]$$

$$S_i = \frac{1}{N_i} \sum_{n \in C_i} (x_n - \mu_i)(x_n - \mu_i)^T$$

$$\Sigma = \frac{N_1}{N} S_1 + \frac{N_2}{N} S_2$$

Hint $\log L$ s.t $\sum \pi_i = 1$

Lagrangian: $\log L - \lambda\left(\sum \pi_i - 1\right)$

$$\hat{k} = \operatorname{argmax} P(C_k | \vec{x})$$

$$= \underset{k}{\operatorname{argmax}} \frac{P(C_k)\, P(\vec{x} | C_k)}{P(X)}$$

$$= \operatorname{argmax}\left(\log P(C_k) + \log P(\vec{x} | C_k)\right)$$