CFD HW 4

Cameron Cummins (csc3323)

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1 Introduction

Consider the following ODE:

$$u''(x) = f(x) \text{ for } x \in [0, 1] \tag{1}$$

$$f(x) = (4\pi)^2 \cos(4\pi x) \tag{2}$$

We define the number of interior grid points over $x \in (0, 1)$ (not including the boundaries) as N and the grid spacing h as $h = \frac{1}{N+1}$. We thus define our interior coordinates as $x_i = ih$ for $i = 1, ..., N$. The numerical solution at x_i is u_i which is an approximation to the exact solution $\tilde{u}_i$.

2 Taylor Expansion Coefficient (20 Points)

We start with the second order, center approximation to the second order derivative, where we define the constant coefficient C:

$$h^2 u''(x_i) = u(x_{i+1}) - 2u(x_i) + u(x_{i-1}) + Ch^4 u^{(4)}(x_i) + \text{e.h.o.t.} \tag{3}$$

Here, **h.o.t.** stands for "higher order terms" as the approximation includes infinitely additional higher order terms. In this instance however, they are only evenly-ordered, so we rename this to "evenly higher order terms" or **e.h.o.t.**. Using this equation, we can preform a Taylor expansion of u'' centered at x_i with coefficients a, b, c:

$$h^2 u''(x_i) = au(x_{i+1}) + bu(x_i) + cu(x_{i-1}) + \text{h.o.t.} \tag{4}$$

We can preform Taylor expansions of u centered at x_i and evaluated at x_{i+1} and x_{i-1} to approximate the fourth order term. Recall that $x_i = ih$ so $x_{i+1} = x_i + h$ and $x_{i-1} = x_i - h$.

$$u(x_{i+1}) = u(x_i) + hu'(x_i) + \frac{h^2}{2} u''(x_i) + \frac{h^3}{6} u'''(x_i) + \text{h.o.t.} \tag{5}$$

$$u(x_{i-1}) = u(x_i) - hu'(x_i) + \frac{h^2}{2} u''(x_i) - \frac{h^3}{6} u'''(x_i) + \text{h.o.t.} \tag{6}$$

We can then substitute these expansions into equation 4.

$$h^2u''(x_i) = (a+b+c)u(x_i) + (a-c)hu'(x_i) + (a+b)\frac{h^2}{2}u''(x_i) + \text{h.o.t.} \quad (7)$$

We can define the coefficients using equation 3, where $a = 1$, $b = -2$, and $c = 1$. This zeroes out the oddly ordered terms, leaving only evenly ordered terms. Here, since we lose some terms from the zeros, the fourth term is displayed separated from **e.h.o.t.**.

$$h^2u''(x_i) = h^2u''(x_i) + \frac{h^4}{12}u^{(4)}(x_i) + \text{e.h.o.t.} \quad (8)$$

$$h^2u''(x_i) = u(x_{i+1}) - 2u(x_i) + u(x_{i-1}) + \frac{h^4}{12}u^{(4)}(x_i) + \text{e.h.o.t.} \quad (9)$$

Comparing equation 9 to equation 3, we can conclude that $C = \frac{1}{12}$.

3 Numerical Solution with 10 Points (20 Points)

We can compute the numerical solution by solving the system of central finite difference equations for N interior points within $x \in (0, 1)$ and with the boundary conditions $u(0) = 0$ and $u(1) = 2$. The result is compared against the exact solution ($N = 10^4$) in figure 1.

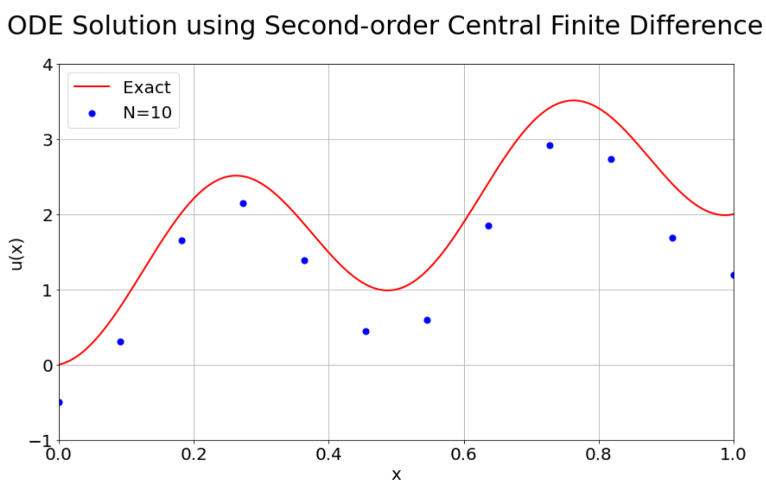

Figure 1: Plot of numerical approximation versus the exact solution ($N = 10^4$) to the ODE.

i	x	u(x)
1	0.0	-0.499
2	0.091	0.308
3	0.182	1.656
4	0.273	2.15
5	0.364	1.392
6	0.455	0.448
7	0.545	0.602
8	0.636	1.853
9	0.727	2.919
10	0.818	2.733

Figure 2: Points plotted in figure 1.

4 Numerical Solution with 10^4 Points (20 Points)

We plot the numerical solution and exact solution in figure 3 to make sure that the coding implantation is sound.

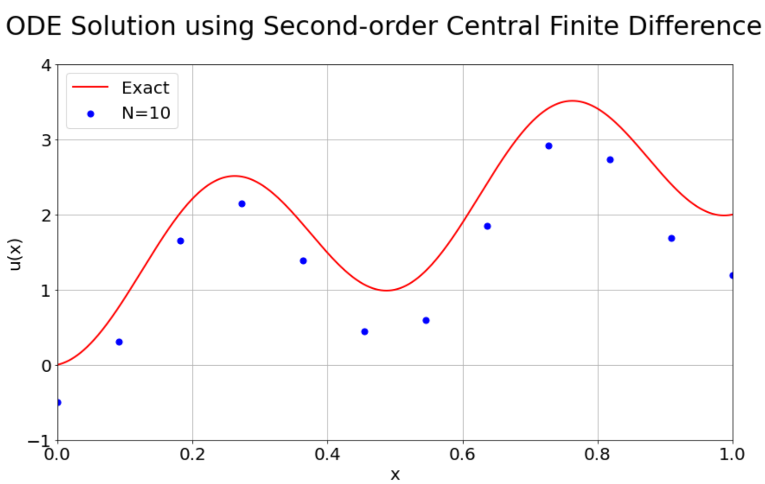

Figure 3: Plot of numerical approximation ($N = 10^4$) versus the exact solution ($N = 10^4$) to the ODE.

We can then calculate the error associated with lower values of N. In figure 4, we investigate the following definitions of error:

$$E = \left[\sum_{i=1}^{N} (u_i - \tilde{u}_i)^2 \right]^{\frac{1}{2}} \tag{10}$$

$$e = \frac{1}{N} \left[\sum_{i=1}^{N} (u_i - \tilde{u}_i)^2 \right]^{\frac{1}{2}} \tag{11}$$

The results are featured in figure 4.

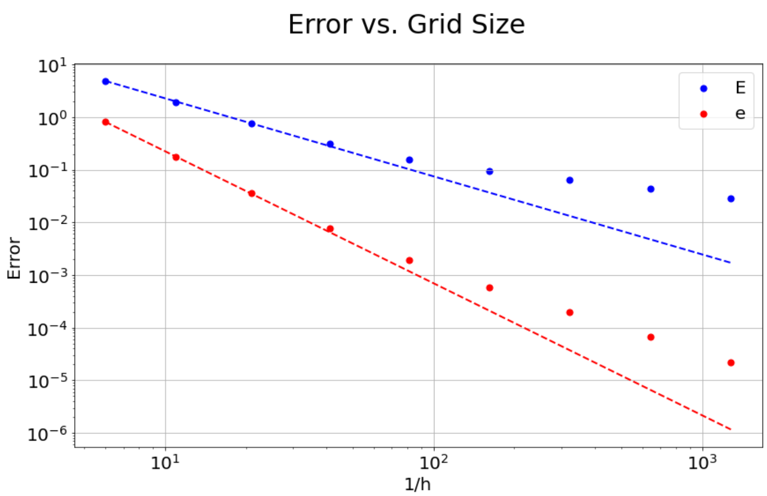

Figure 4: Plot of error associated with different grid sizes N and fitted trend lines.

The function βh^{α} was fitted to the two errors, producing the following approximations:

$$E \approx 69.10h^{1.48} \tag{12}$$

$$e \approx 111.53h^{2.65} \tag{13}$$

We can thus conclude that error decreases with larger computational grid sizes N and thus smaller values of h.

5 Taylor Expansion Coefficients Part 2 (20 Points)

Here, we will approximate $u'(x)$ with a one-sided second order finite difference formula:

$$hu'(x_i) = au_i + bu_{i+1} + cu_{i+2} + O(h^3) \tag{14}$$

We can then solve for the coefficients a, b, and c, through Taylor expansions:

$$u_{i+1} = u(x_i + h) = u_i + hu'(x_i) + \frac{h^2}{2}u''(x_i) + O(h^3) \tag{15}$$

$$u_{i+2} = u(x_i + 2h) = u_i + 2hu'(x_i) + \frac{4h^2}{2}h^2u''(x_i) + O(h^3) \tag{16}$$

Put into vector-matrix form:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{17}$$

Taking the inverse of the matrix allows us to solve for the coefficients:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 2 \\ -\frac{1}{2} \end{bmatrix} \tag{18}$$

6 New Boundary Conditions (20 Points)

Here, I tried and failed to implement the first derivative into the approximation and seemed to approach the pattern of the exact solution, but failed overall.

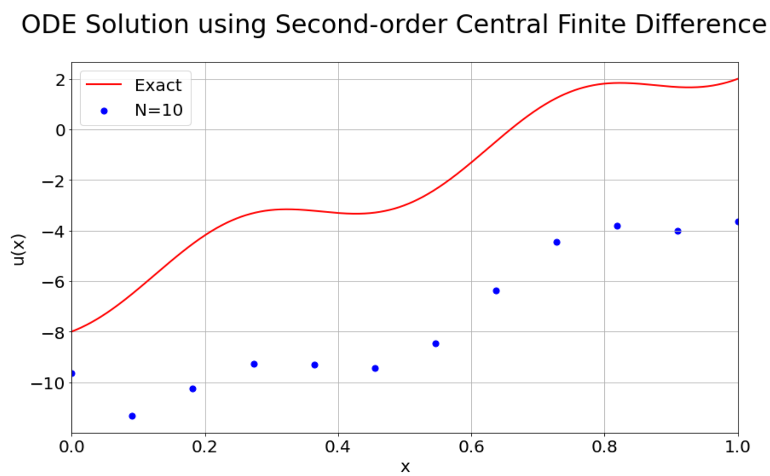

Figure 5: Failed attempt at using the first derivative approximation in the numerical solution.

Here are the points

i	x	u(x)
1	0.0	-9.652 2
0.091	-11.333 3	0.182
-10.25 4	0.273	-9.28 5
0.364	-9.315 6	0.455
-9.455 7	0.545	-8.468 8
0.636	-6.375 9	0.727
-4.464 10	0.818	-3.805 height

Figure 6: Points plotted in figure 5.