Exercise 06 for MA-INF 2201 Computer Vision WS22/23
01.12.2022
Submission on 08.12.2022

1. **Theory Question**
   Prove the following property of k dimensional Gaussian distributions $\text{Norm}_x[\mu, \Sigma]$:
   $$\int \text{Norm}_x[a, A]\text{Norm}_x[b, B]dx = \text{Norm}_a[b, A+B] \int \text{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]dx$$
   where $\Sigma_* = (A^{-1} + B^{-1})^{-1}$.
   (5 points)

2. **Kalman Filtering**
   You need to implement the basic Kalman Filtering algorithm. You observe a set of 2D noisy observations (x_i, y_i) which are the coordinates of the 2D space as shown in Figure 1.

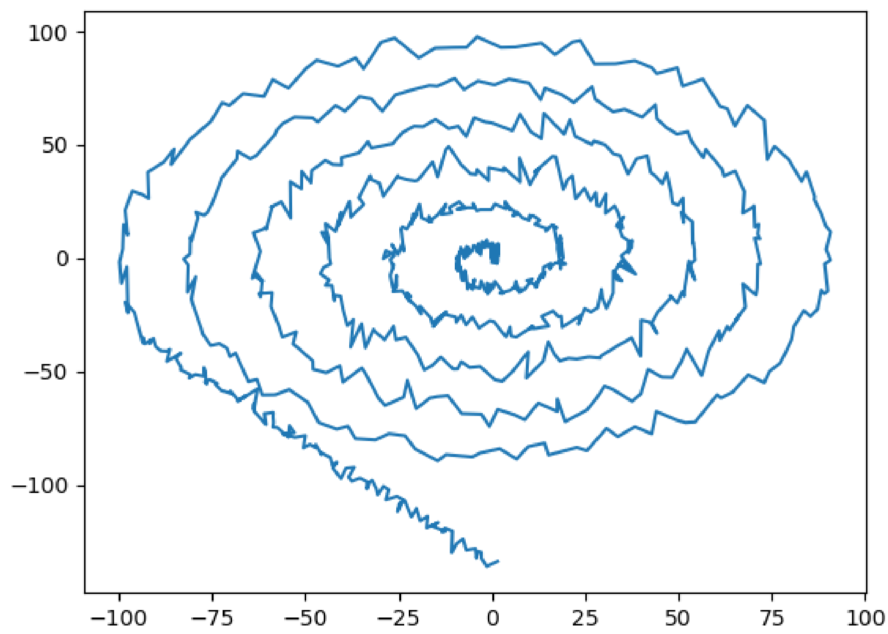

Figure 1: Observations from location of a clockwise rotating object.

State: The state of the object should be the 4D vector (x, y, v_x, v_y) which denote the location and the velocity in each axis.
Initial State: You should consider the initial state of $(-10, -150, 1, -2)$.
Time Evolution Equation: What should be the time evolution equation?
Measurement Equation: What should be the measurement equation?
Code for reading observations is provided. You should write code for performing the kalman filtering. You may use numpy for matrix operations. At the end visualize the filtered output. Use the template `task02.py`.
(5 points)

3. **Background Subtraction using Gaussian Mixture Models**
   In this exercise we want to perform background subtraction for the provided image. The image comes with a rectangular bounding box that contains some skin color pixels (foreground). For this task you are required to implement a Gaussian Mixture Model and the EM algorithm for training. Assume that all covariance matrices are diagonal.

   (a) Implement the function *fit_single_gaussian* which fits a single Gaussian to provided data.
   (1 point)

   (b) GMMs rely on a good initialization. One strategy is to start with a single Gaussian model, split it into two distributions (GMM with two mixtures) and train it using the EM algorithm. For a GMM with four mixtures, both of the previous distributions can be splitted again. Implement the *split* function that doubles the number of components in the current Gaussian mixture model. In particular, generate $2K$ components out of K components as follows:

   - Duplicate the weights λ_k so you have $2K$ weights. Divide by two to ensure $\sum_k \lambda_k = 1$.
   - For each mean μ_k, generate two new means $\mu_{k1} = \mu_k + \epsilon \cdot \sigma_k$ and $\mu_{k2} = \mu_k - \epsilon \cdot \sigma_k$.
   - Duplicate the K diagonal covariance matrices so you have $2K$ diagonal covariance matrices.

   (2 points)

   (c) Implement the EM algorithm to train the GMM.
   (5 points)

   (d) **Background Subtraction** Train a GMM with 4 components (start with a single Gaussian and do 2 component splits) for the background pixels. Using the thresholding approach from the lecture, set every pixel in the image to zero which is above a threshold τ. Display the resulting image.
   (2 points)