Simulating phase transition bubble hydrodynamics with general equations of state

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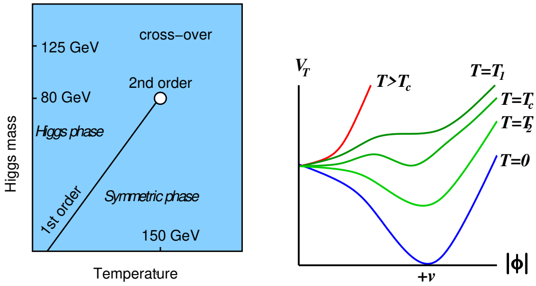
Outline

First-order phase transitions

- If there is a potential barrier between the local and the global minimum, the phase transition is of first order
- The Standard Model Higgs transition is a cross-over, but various extensions convert it to first order
- Klein-Gordon equation with a friction term

$$\Box\phi - V_T'(\phi) = -\eta_T(\phi)u^\mu\partial_\mu\phi \quad (1)$$

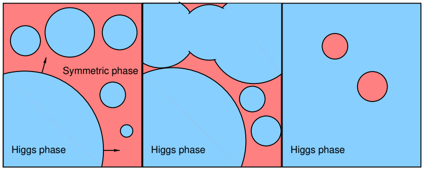
$\rightarrow$ Constant wall speed



Hindmarsh et al., 2021

Bubble nucleation

- Spontaneous tunneling to the new phase $\rightarrow$ bubble nucleation and expansion
- Bubble expansion is governed by relativistic hydrodynamics $\rightarrow$ sound $\rightarrow$ GWs
- "Trying to determine the properties of a fluid in a water kettle based on the sound of boiling"
  - Listening to the sound indirectly through GWs
- Energy-momentum conservation gives rise to
  - Wave equation
  - Bubble wall junction conditions
  - Continuity equations, aka. hydrodynamic equations



Dimensionality of the problem

- Self-similarity
  - Friction results in a constant wall speed v_{wall}
  - As the bubble expands, its shape stays the same
  - $\rightarrow$ Time-independent solution
- Spherical symmetry
- 3+1 dimensional problem reduces to time-independent 1D

Wave equation

- Constant background space-time $\rightarrow$ energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$
- Energy-momentum tensor of an ideal fluid

$$T_f^{\mu\nu} = (e+p)u^\mu u^\nu + p g^{\mu\nu} \tag{2}$$

- For a one-dimensional flow in Cartesian coordinates

$$\partial_t \left[(e+pv^2)\gamma^2\right] + \partial_x \left[(e+p)\gamma^2 v\right] = 0, \tag{3}$$

$$\partial_t \left[(e+p)\gamma^2 v\right] + \partial_x \left[(ev^2+p)\gamma^2\right] = 0 \tag{4}$$

- First-order perturbation $\rightarrow$ wave equation with speed of sound

$$\partial_t^2(\delta e) - \frac{\delta p}{\delta e}\partial_x^2(\delta e) = 0 \qquad c_s^2 \equiv \frac{dp}{de} = \frac{dp/dT}{de/dT} \tag{5}$$

Phase boundary

- Energy-momentum conservation

$$\nabla_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_z T^{zz} = \partial_z T^{z0} = 0 \tag{6}$$

- Inserting ideal fluid $T_f^{\mu\nu} = (e+p)u^\mu u^\nu + pg^{\mu\nu} \to$ junction conditions

$$w_- \tilde{\gamma}_-^2 \tilde{v}_- = w_+ \tilde{\gamma}_+^2 \tilde{v}_+ \tag{7}$$

$$w_- \tilde{\gamma}_-^2 \tilde{v}_-^2 + p_- = w_+ \tilde{\gamma}_+^2 \tilde{v}_+^2 + p_+ \tag{8}$$

- By defining new variables $\theta = \frac{1}{4}(e - 3p)$, $\quad \alpha_+ \equiv \frac{4}{3}\frac{\theta_+(w_+) - \theta_-(w_-)}{w_+}$, $\quad r = \frac{w_+}{w_-}$

$$\tilde{v}_+ = \frac{1}{2(1+\alpha_+)}\left[\left(\frac{1}{3\tilde{v}_-} + \tilde{v}_-\right) \pm \sqrt{\left(\frac{1}{3\tilde{v}_-} - \tilde{v}_-\right)^2 + 4{\alpha_+}^2 + \frac{8}{3}\alpha_+}\right], \tag{9}$$

$$\tilde{v}_- = \frac{1}{2}\left[\left((1+\alpha_+)\tilde{v}_+ + \frac{1-3\alpha_+}{3\tilde{v}_+}\right) \pm \sqrt{\left((1+\alpha_+)\tilde{v}_+ + \frac{1-3\alpha_+}{3\tilde{v}_+}\right)^2 - \frac{4}{3}}\right). \tag{10}$$

Hydrodynamic equations

- Energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$
- Projection $\rightarrow$ hydrodynamic equations away from the phase boundary

$$0 = u_\mu \partial_\nu T^{\mu\nu} = -\partial_\mu (w u^\mu) + u^\mu \partial_\mu p, \tag{11}$$
$$0 = \bar{u}_\mu \partial_\nu T^{\mu\nu} = w \bar{u}^\nu u^\mu \partial_\mu u_\nu + \bar{u}^\mu \partial_\mu p. \tag{12}$$

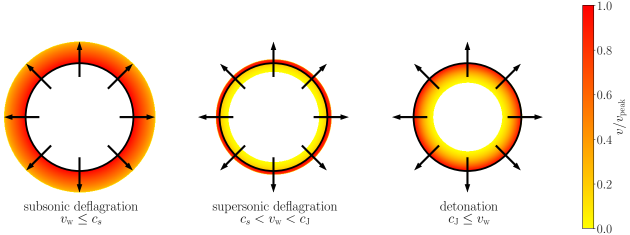
- Using self-similarity $\xi = \frac{r}{t}$ and parametrising

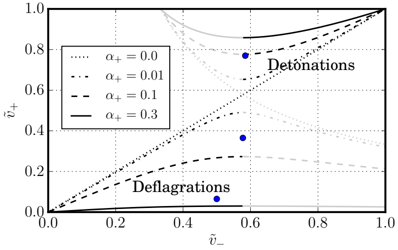
$$\frac{d\xi}{d\tau} = \xi \left[(\xi - v)^2 - c_s^2 (1 - \xi v)^2 \right], \tag{13}$$
$$\frac{dv}{d\tau} = 2 v c_s^2 (1 - v^2)(1 - \xi v), \tag{14}$$
$$\frac{dw}{d\tau} = w \left(1 + \frac{1}{c_s^2} \right) \gamma^2 \mu \frac{dv}{d\tau}. \tag{15}$$

- Note that c_s^2 is computed using user-provided functions

Fluid shells

- Three types of solutions, determined by
  - Wall velocity v_{wall}
  - Transition strength α_n
  - Speed of sound $c_s(T, \phi)$





Black = bubble wall / phase boundary

General equation of state

- Equation of state for an ultrarelativistic plasma with multiple degrees of freedom

$$p(T,\phi) = \frac{\pi^2}{90} g_p(T,\phi) T^4 - V_0(\phi) \tag{16}$$

- The rest can be deduced with thermodynamics
  - Entropy density $s = \frac{dp}{dT}$
  - Enthalpy density $w = Ts$
  - Energy density $e = w - p$
  - Sound speed $c_s = \sqrt{\frac{dp}{de}}$

Simple models

- Bag model: equation of state with constant degrees of freedom

$$g_\pm = \frac{90}{\pi^2} a_\pm \quad \Rightarrow \quad p_\pm = a_\pm T^4 - V_\pm, \quad c_s^2 \equiv \frac{dp}{de} = \frac{1}{3} \tag{17}$$

- Constant sound speed model

$$g_{p\pm} = \frac{90}{\pi^2} a_\pm \left(\frac{T}{T_0} \right)^{\mu_\pm - 4} \tag{18}$$

$$\Rightarrow \quad p_\pm = a_\pm \left(\frac{T}{T_0} \right)^{\mu_\pm - 4} T^4 - V_\pm \approx a_\pm T^{\mu_\pm} - V_\pm \tag{19}$$

$$c_{s\pm}^2 = \frac{1}{\mu_\pm - 1} \tag{20}$$

Equation of state from an arbitrary particle physics model

- Non-constant degrees of freedom $\rightarrow$ varying sound speed $c_s(T, \phi)$
- The equation of state can be constructed from $V_0(\phi)$, $g_p(T, \phi)$ and preferably also $g_e(T, \phi)$ or $g_s(T, \phi)$

$$g_p = 4g_s - 3g_e \tag{21}$$

$$e(T, \phi) = \frac{\pi^2}{30} g_e(T, \phi) T^4 + V_0(\phi) \tag{22}$$

$$p(T, \phi) = \frac{\pi^2}{90} g_p(T, \phi) T^4 - V_0(\phi) \tag{23}$$

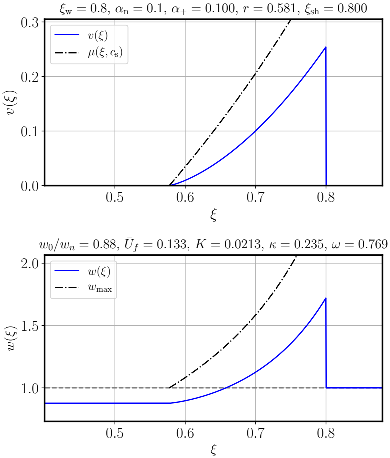
$$s(T, \phi) = \frac{2\pi^2}{45} g_s(T, \phi) T^3 \tag{24}$$

PTtools

- Python-based simulation framework for GW power spectra of first-order phase transitions
  - Based on Numpy, SciPy (solvers & ODE integration) & Numba
- Input
  - Equation of state, either directly as $p(T,\phi)$ (and $e(T,\phi)$ or $s(T,\phi)$), or as degrees of freedom: $g_p(T,\phi)$ (and $g_e(T,\phi)$ or $g_s(T,\phi)$), and potential $V_0(\phi)$
    - Bag model, constant sound speed model and the Standard Model with adjustable phases are already included
    - User provides the model as a Python class. The g_{eff} functions can be arbitrary, e.g. interpolated from data.
  - Bubble wall speed v_{wall}
  - Transition strength parameter α_n
  - Transition rate parameter β
- Output: gravitational wave power spectrum in wavenumber units of mean bubble spacing
- Use case: estimate the likelihood of various Standard Model extensions with future LISA data in the 2030s

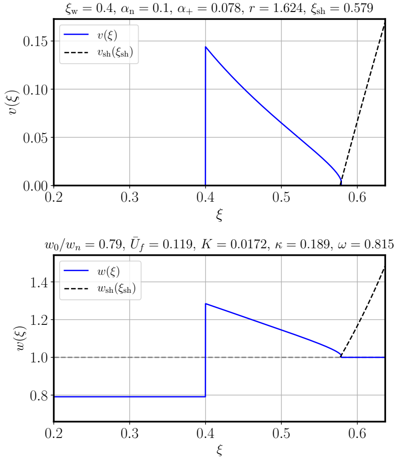
Fluid shell algorithm: detonations

- Solve boundary conditions at the wall for known $w_+ = w_n$, $v_+ = 0$
  - Using user-provided $p(T, \phi) \rightarrow$ numerical solving
- Integrate from the wall to the fixed point at $v = 0$
  - Using $c_s^2(w, \phi)$ based on user-provided functions $\rightarrow$ numerical ODE integration



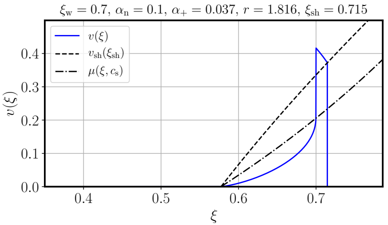
Fluid shell algorithm: deflagrations

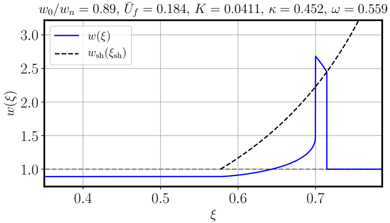
- Guess w_-, the enthalpy behind the wall
- Solve boundary conditions at the wall for w_+, v_+
- Integrate to the shock
  - $v_{sh}(\xi)$ is computed from the boundary conditions $\rightarrow$ has to be found numerically
- Solve boundary conditions at the shock
- Check if $w = w_n$
- If not, change the enthalpy guess and try again



Fluid shell algorithm: hybrids

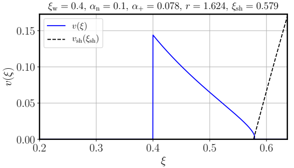
- The same as for subsonic deflagrations, but with an additional integration behind the wall
- Guess w_-, the enthalpy behind the wall
- Solve boundary conditions at the wall for w_+, v_+
- Integrate to the shock
- Solve boundary conditions at the shock
- Check if $w = w_n$
- If not, change the enthalpy guess and try again
- Once the correct enthalpy has been found, integrate from the wall to the fixed point at $v = 0$

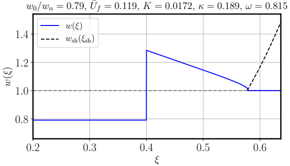


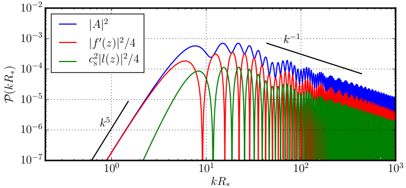


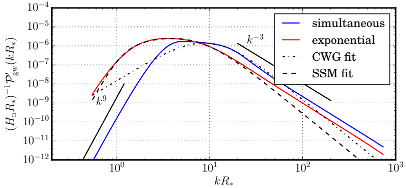
Gravitational wave production

- Sound Shell Model (Hindmarsh, 2018)
- Velocity profile of a single bubble $v_{\mathrm{ip}}(\xi)$ + sine transform
  $\rightarrow$ velocity power spectrum
- Convolution with the nucleation rate function
  $\rightarrow$ Overall velocity power spectrum
- Convolution with a Green's function $\rightarrow$ GW power spectrum



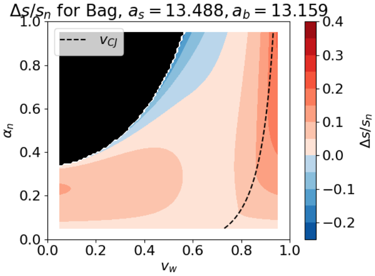


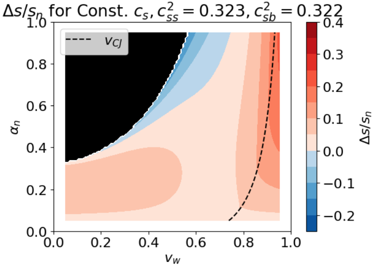




Entropy conservation

- 10 000 bubbles with varying $v_{\text{wall}}, \alpha_n$
- Top-left region ruled out by $\alpha_+ > \frac{1}{3}$
- Unphysical region (blue): decrease in total entropy





Summary

- A phase transition converts energy from the scalar field to kinetic and thermal energy $\rightarrow$ Gravitational wave production
- The phase transition proceeds by bubble nucleation
- The hydrodynamics is based on energy-momentum conservation
- Solving a bubble starts with the equation of state
  - Equation solving & ODE integration $\rightarrow$ fluid velocity profile
- Sound shell model converts the velocity profile to GW power spectrum
  - Sine transform & convolutions $\rightarrow$ GW power spectrum
- PTtools has been updated to support arbitrary equations of state
- PTtools will be published soon

Thank you!

Sources

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- Hindmarsh, Mark, and Mulham Hijazi. Gravitational Waves from First Order Cosmological Phase Transitions in the Sound Shell Model. Journal of Cosmology and Astroparticle Physics 2019, Dec. 2019, doi: 10.1088/1475-7516/2019/12/062.
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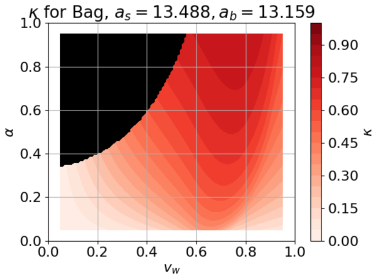
Energy redistribution

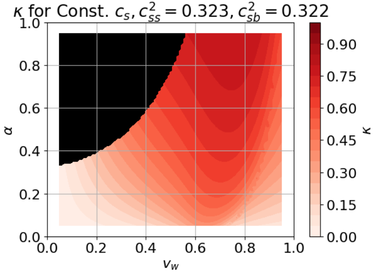
- Efficiency factor κ: ratio of kinetic energy to total energy released

$$\kappa = \frac{e_K}{|\Delta e_\theta|} \tag{25}$$

$$e_K = 4\pi \int_0^{\xi_{\max}} d\xi \xi^2 w \gamma^2 v^2 \tag{26}$$

$$\Delta e_\theta = 4\pi \int_0^{\xi_{\max}} d\xi \xi^2 (\theta - \theta_n) \tag{27}$$





Speedup example with Numba

- Add JIT decorator
- Replace unsupported features with simpler code or split the unsupported parts to another function
- (Restructure the function to make the possible parallelism explicit)

```
@numba.njit(parallel=True)
def sin_transform(t: np.ndarray, f: np.ndarray, freq: np.ndarray) -> np.ndarray:
    integral = np.zeros_like(freq)
    for i in numba.prange(freq.size):
        integrand = f * np.sin(freq[i] * t)
        integral[i] = np.trapz(integrand, t)
    return integral
```