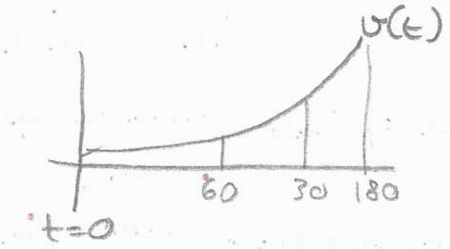
TEMA 4: DERIVACIÓ I INTEGRACIÓ NUMÈRIQUES

Exercici 21.T4) Els temps i les velocitats corresponents a un mòbil vénen donats per la taula: Calculeu valors aprox. de l'acceleració en $t=0$, $t=120$ i $t=300$.

t	0	60	120	180	240	300
v	0,0	0'0824	0'2747	0'6502	1'3851	3'2229

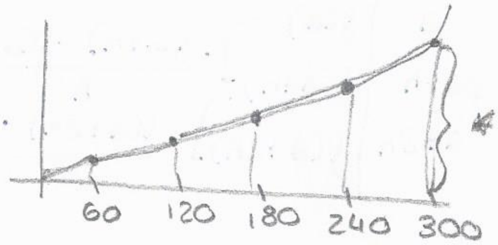


$$v'(t) = a(t) = \frac{dv}{dt} = \lim_{h\to 0} \frac{v(t+h)-v(t)}{h}$$

$t=0$, $h=60$: $a(0) \simeq \dfrac{v(60)-v(0)}{60} = 0'001373$

$a(120) \simeq \dfrac{v(180)-v(60)}{120} = 0'004736$

$a(300) \simeq \dfrac{v(300)-v(240)}{60} = 0'03063$

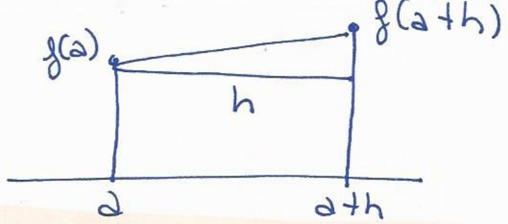


$$* \begin{array}{c|c} a & f(a) \\ a+h & f(a+h) \end{array}$$

$$\left.\begin{array}{c|c} a & f(a) \\ a+h & f(a+h) \end{array}\right\} \frac{f(a+h)-f(a)}{h}$$

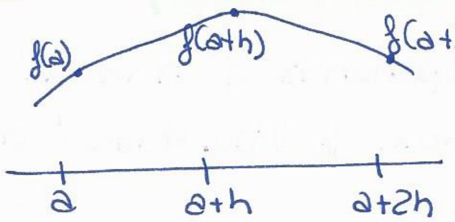
$$p_1(x) = f(a) + \frac{f(a+h)-f(a)}{h}(x-a)$$

$$p_1'(x) = \frac{f(a+h)-f(a)}{h} \simeq f'(a)$$



22.T4 Considereu la taula següent corresponent a la funció $f(x)=xe^x$
Aproximeu els valors de $f'(1.0)$, $f'(1.03)$ i $f'(1.06)$ emprant fórmules progressives, centrades i regressives de 3 punts.

x	1'00	1'01	1'02	1'03	1'04	1'05	1'06
$f(x)$	2'7182	2'7730	2'8286	2'8850	2'9423	3'0005	3'0595
i	0	1	2	3	4	5	6



pol. inter. en $(a,f(a))$, $(a+h,f(a+h))$, $(a+2h,f(a+2h))$
$P_2(x)$

$$\cancel{f'(a) \approx p_2'(a) = \frac{2f(a+h)-2f(a)}{2h} \; f(a+2h)-2f(a+h)+}$$

$$f'(a) \approx p_2'(a) = \frac{2f(a+h)-2f(a)}{2h} - \frac{f(a+2h)-2f(a+h)+f(a)}{2h^2}(\cancel{-h}) = \boxed{\frac{-f(a+2h)+4f(a+h)-3f(a)}{2h}}$$

deriva = $-h$

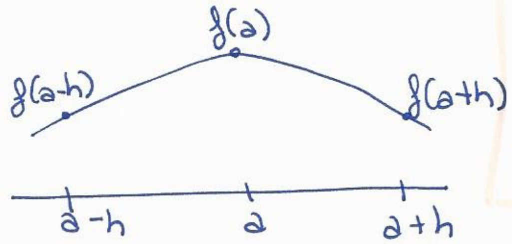
$$p_2(x) = f(a) + \frac{f(a+h)-f(a)}{h}\underbrace{(x-a)}_{\text{deriva}=1} + \frac{f(a+2h)-2f(a+h)+f(a)}{2h^2}\underbrace{(x-a)}_{\text{deriva}=1}(x-a-h)$$

$$\begin{array}{l|l}
a & f(a) \\
a+h & f(a+h) \\
a+2h & f(a+2h)
\end{array}
\quad
\begin{array}{c}
\dfrac{f(a+h)-f(a)}{h} \\[2ex]
\dfrac{f(a+2h)-f(a+h)}{h}
\end{array}
\quad
\boxed{\frac{f(a+2h)-2f(a+h)+f(a)}{2h^2}}$$

formula progressiva

formula centrada

$$\begin{array}{l|l}
a & f(a) \\
a-h & f(a-h) \\
a+h & f(a+h)
\end{array}
\quad
\begin{array}{c}
\dfrac{2f(a)-2f(a-h)}{2h} \\[2ex]
\dfrac{f(a+h)-f(a-h)}{2h}
\end{array}
\quad
\frac{f(a+h)-2f(a)+f(a-h)}{2h^2}$$



$$P_2(x) = f(a) + \frac{f(a)-f(a-h)}{h}(x-a) + \frac{f(a+h)-2f(a)+f(a-h)}{2h^2}(x-a)(x-a+h)$$

$$p_2'(x) = \frac{2f(a)-2f(a-h)}{2h} + \frac{f(a+h)-2f(a)+f(a-h)}{2h^2}\cdot\cancel{h} = \boxed{\frac{f(a+h)-f(a-h)}{2h}}$$

$f'(1'00) \approx \dfrac{-f_2 - 4f_1 - 3f_0}{0'02} \approx 5'47$
$a = 1'00$
$h = 0'01$ $\quad f(a+h) = f_1$ $\quad f(a) = f_0$ $\quad f(a+2h) = f_2$

$f'(1'06) \approx \dfrac{-f_4 + 4f_5 - 3f_0}{0'02} \approx$

$f'(1'03) \approx \dfrac{f_4 - f_2}{0'02} \approx$
$a = 1'03$
$h = 0'01$

formula regresiva

$$P_2'(a) = \frac{f(a) - f(a-2h)}{2h}$$

23.T4 La taula següent es construeix a partir de la funció $f(x)=xe^x$. Aproximeu $f'(2)$ utilitzant diferències progressives, amb dos i tres punts, i fiteu els errors absoluts de les aproximacions obtingudes. Contrasteu les aproximacions amb el valor exacte. $e=2'71828\ldots$

x	$1'8$	$1'9$	$2'0$ (a)	$2'1$ ($a+h$)	$2'2$ ($a+2h$)
$f(x)$	$10'8894$	$12'7032$	$14'7781$	$17'1490$	$19'8550$

$a=2$
$h=0'2$

$f(x)=xe^x$ $\qquad f''(x)=e^x+e^x+xe^x=2e^x+xe^x$

$f'(x)=e^x+xe^x$ $\qquad f'''(x)=2e^x+e^x+xe^3=3e^x+xe^3$

$f'(1)=e+e=2e=5'43656$ $\qquad f'(2)=e^2+2e^2=3e^2=22'167168$

$f(x)\simeq P_1(x)$ interpolador en a y $a+h$.

$$f'(a)\simeq P_1'(a)=\frac{f(a+h)-f(a)}{h}=23'709$$

$f(x)\simeq P_2(x)$ interpolador en $a, a+h, a+2h$

$$f'(a)\simeq P_2'(a)=\frac{-f(a+2h)+4f(a+h)-3f(a)}{2h}=22'035$$

$$f(x)-P_1(x)=\frac{f''(\xi(x))}{2!}(x-a)(x-a+h)$$ (error interpolador)

$$|f'(a)-P_1'(a)|=\left|\frac{f''(\xi(x))}{2!}(-h)\right|=\left|-h\cdot\frac{f''(\xi(x))}{2!}\right|\leq \frac{h}{2}M_2$$

$$f(x)-p_2(x)=\frac{f'''(\xi(x))}{3!}(x-a)(x-a-h)(x-a-2h)$$

$$|f'(a)-p_2'(a)|=\left|\frac{f_3(\xi(x))}{3!}\cdot(2h^2)\right|\leq \frac{h^2}{3}M_3$$

$$M_2=\max_{\substack{x\in[a,a+h]\\ [2,2'1]}}|f''(x)|=2e^{2'1}+2'1\cdot e^{2'1}=4'1e^{2'1}$$ $x=2'1$

$$\boxed{\frac{h}{2}M_2=\frac{0'1}{2}\cdot 4'1e^{2'1}}$$

$$M_3=\max|f'''(x)|=3e^{2'2}+2'2\cdot e^{2'2}=5'2e^{2'2}$$

$$\boxed{\frac{h^2}{3}M_3=\frac{0'01}{3}\cdot 5'2e^{2'2}}$$

(24.T4) Sigui $f(x) = x^2 - e^x + e^{-x}$ es vol aproximar $f'(0'7)$.

a) Treballant amb 8 decimals, useu la fórmula de les diferències finites centrada de primer ordre per aproximar $f'(0'7)$ amb $h = 10^{-i}$ per a $i = 1, 2, \dots, 5$.

b) Si $F(h)$ és la fórmula anterior, es pot veure que

$$F(h) = f'(a) + a_2 h^2 + a_4 h^4 + \dots + a_{2n} h^{2n} + \dots$$

Tenint en compte això i els resultats de l'apartat a), useu el mètode d'extrapolació de Richardson per obtenir una millor aproximació de $f'(0'7)$. Doneu els errors absoluts comparant amb el valor exacte de la derivada.

$f(x) = x^2 - e^x + e^{-x}$

$$f'(\underset{a}{0'7}) \simeq \frac{f(a+h) - f(a-h)}{h}$$

$$\begin{array}{c|c} a & f(a) \\ a-h & f(a-h) \\ a+h & f(a+h) \end{array} \quad \begin{array}{c} \frac{f(a)-f(a-h)}{2h} \\ \frac{f(a+h)-f(a-h)}{2h} \end{array} \quad \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2}$$

$$P_2(x) = f(a) + \frac{f(a) - f(a-h)}{h}(x-a) + \frac{f(a+h) - 2f(a) + f(a-h)}{2h^2}(x-a)(x-a+h)$$

$$f'(a) \simeq P_2'(a) = \frac{f(a+h) - f(a-h)}{2h} \equiv Df(a,h) \equiv F(h) \equiv F_0(h)$$

$$f(x) - P_2(x) = \frac{f'''(\xi(x))}{3!}(x-a)(x-a+h)(x-a-h) \qquad G(x) = (x-a)g(x)$$

$$G'(a) = g(a)$$

$$|f'(a) - P_2'(a)| = \left| -\frac{f'''(\xi(x))}{3!} h^2 \right| \le \frac{M_3}{3!} h^2 \qquad M_3 = \max_{x \in [a-h, a+h]} |f'''(x)|$$

$$F(h) = \frac{f(a+h) - f(a-h)}{2h} = \underset{f'(a)}{f'(a)} + a_1 h^2 + a_2 h^4 + a_3 h^6 = f'(a) + \underbrace{\frac{f'''(a)}{3!}}h^2 + \underbrace{\frac{f^5(a)}{5!}}h^2 \dots$$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f^4(a)}{4!}h^4 + \frac{f^5(a)}{5!}h^5 \dots$$

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{2!}h^2 - \frac{f'''(a)}{3!}h^3 + \frac{f^4(a)}{4!}h^4 - \frac{f^5(a)}{5!}h^5 \dots$$

$+$

$$/ \quad 2f'(a)h \quad / \quad + 2\frac{f'''(a)}{3!}h^3 \quad / \quad + \frac{f^5(a)}{5!}h^5 \dots$$

$q = 10$

$$F_0(h) - f'(a) \simeq a_1 h^2$$

$$f_0(qh) - f'(a) \simeq a_1 q^2 h^2$$

$$F_0(h) - f'(a) \simeq a_1 h^2$$

Si $q = 10$ $\quad f_0(qh) - f'(a) \simeq a_1 q^2 h^2$

$$F_0(qh) - f'(a) \simeq q^2 (F_0(h) - f'(a)) \text{ formula aproxim.}$$

error

$$= q^2 F_0(h) - q^2 f'(a)$$

$$q^2 f'(a) - f'(a) \cong q^2 F_0(h) - F_0(qh)$$

$$(q^2-1) f'(a) \cong (q^2-1) F_0(h) + F_0(h) - F_0(qh)$$

$$f'(a) \simeq F_0(h) + \frac{F_0(h) - F_0(qh)}{q^2-1} = F_1[h, qh]$$

$$f'(0'7) = 2 \cdot 0'7 - e^{0'7} - e^{-0'7} = -1'11033801126 18\ldots$$

$h=$	i	$f(0'7+10^{-i})$	$f(0'7-10^{-i})$	$F_0(10^{-i})$
0'1	1	$f(0'8) = -1'13621196$	$f(0'6) = -0'91330716$	$\frac{f(0'8)-f(0'6)}{0'2} = -1'114524$ ✓
0'01	2	$f(0'71) = -1'03824$	$f(0'69) = -1'01603946$	$\frac{f(0'71)-f(0'69)}{0'02} = -1'11038\ldots$ ✓
0'001	3	$f(0'701) = -1'0282775\ldots$	$f(0'699) = -1'02605682$	$\frac{f(0'701)-f(0'699)}{0'002} = -1'11034$ ✓
0'0001	4	$f(0'7001) =$	$f(0'6999) =$	$\frac{f(0'7001)-f(0'6999)}{0'0002} = -1'11035$ ×
0'00001	5	$f(0'70001) =$	$f(0'69999) =$	$\frac{f(0'70001)-f(0'69999)}{0'00002} = -1'1105$ ×

Quan més petit sigui i més aprop són $f(a+h)$ i $f(a-h)$ i es produeix el mètode de cancelació.

$$F_1[10^{-2}, 10^{-1}] = F_0[10^{-2}] + \frac{F_0[10^{-2}] - F_0[10^{-1}]}{99} = 1'110342$$

extrapolació

taula extrapolació

$$\begin{array}{lll} F_0(q^2h = 10^{-1}) & & \\ & > F_1(10^{-2}, 10^{-1}) & \\ F_0(qh = 10^{-2}) & & > F_2(10^{-3}, 10^{-2}, 10^{-1}) = -1'11033801126 18917\ldots \\ & > F_1(10^{-3}, 10^{-2}) & \\ F_0(h = 10^{-3}) & & \end{array}$$

$$\frac{\Delta}{q^2-1} = \frac{\Delta}{99} \qquad \frac{\Delta}{q^4-1} = \frac{\Delta}{9999}$$

Si $q = 2$ $\quad \frac{\Delta}{q^2-1} = \frac{\Delta}{3}, \quad \frac{\Delta}{q^4-1} = \frac{\Delta}{15}$

25.T4 Es disposa de la taula de la funció $\sqrt{x}$ amb 5 xifres decimals correctes: Utilitzeu aquesta informació per calcular $\int_1^{1'3} \sqrt{x}\,dx$ mitjançant les regles compostes dels trapezis i de Simpson. Constrasteu les aproximacions amb el valor exacte de la integral.

$$I = \int_1^{1'3} \sqrt{x}\,dx = \int_1^{1'3} x^{1/2} = \qquad \frac{1}{2}+1 = \frac{3}{2}$$

$$1'3^{3/2} = (1'3^3)^{1/2} = \sqrt{1'3^3}$$

$$= \left.\frac{x^{1/2+1}}{\frac{1}{2}+1}\right]_1^{1'3} = \left.\frac{2x^{3/2}}{3/\cancel{2}}\right]_1^{1'3} = \frac{2}{3}(1'3^{3/2}-1) = 0'321485$$

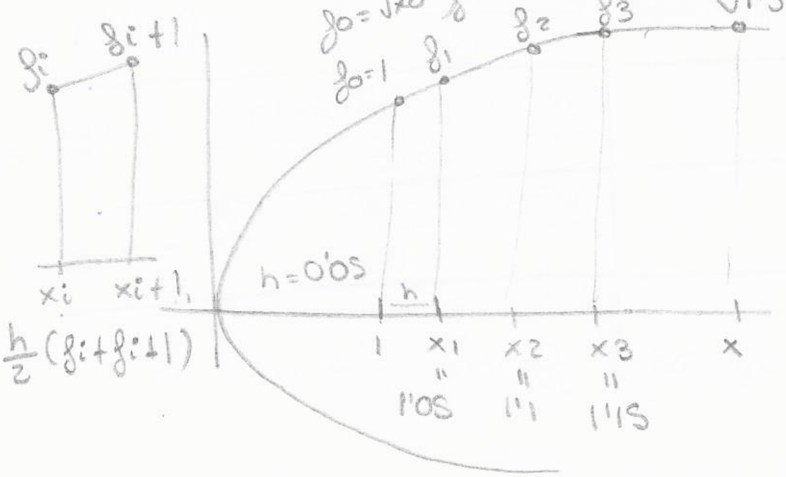
$$F(x) = x^{\alpha}$$
$$F'(x) = \alpha x^{\alpha-1}$$
$$f(x) = x^{\beta}$$
$$\int f(x)\,dx = \frac{x^{\beta+1}}{\beta+1}$$
$$x^{-2} = \frac{1}{x^2}$$
$$x^{\beta} x^{\beta} = x$$
$$\sqrt[3]{4} = 4^{1/3} = 2^{2/3}$$
$$x^{p/q} = (x^p)^{1/q} = \sqrt[q]{x^p}$$

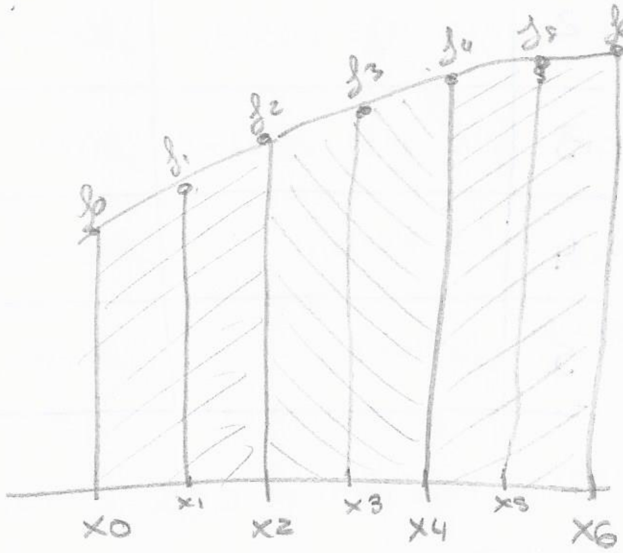
TRAPEZIS $\quad N=2$

$$T(h) = \underbrace{\frac{h}{2}(f_0+f_1)}_{N=1} + \frac{h}{2}(f_1+f_2) + \frac{h}{2}(f_2+f_3) + \dots +$$
$$+ \frac{h}{2}(f_5+f_6) = \frac{h}{2}(f_0+2f_1+2f_2+2f_3+2f_4+$$
$$+2f_5+f_6) = \frac{0'05}{2}(1+2\sqrt{1'05}+2\sqrt{1'1}+\dots+\sqrt{1'3}) = 0'3214725$$

$f_0 = \sqrt{x_0}$
$2f_1 = 2\sqrt{x_1}$

$y = \sqrt{x}$





SIMPSON

$$S(h) = \frac{h}{3}(f_0+4f_1+f_2) + \frac{h}{3}(f_2+4f_3+f_4) + \frac{h}{3}(f_4+4f_5+f_6) =$$
$$= \frac{h}{3}(f_0+4f_1+2f_2+4f_3+2f_4+4f_5+f_6) = 0'32148516\underline{6}$$

parells 2
senars 4

(26.T4) Mitjançant la fórmula composta dels trapezis T(h) volem calcular una aproximació de la integral $\int_{1'8}^{3'4} e^x\,dx$ amb 5 xifres decimals correctes. Quin valor d'h triaríeu?

$$\int_{1'8}^{3'4} e^x\,dx = e^x\Big|_{1'8}^{3'4} = e^{3'4} - e^{1'8} =$$

$$I - T(h) = \frac{b-a}{12} h^2 f^2(\xi) \quad \xi \in [a,b]. \qquad h?$$

$$|I - T(h)| \leq \frac{1}{2} 10^{-5} \qquad \frac{b-a}{12} h^2 M_2 \leq \frac{1}{2} 10^{-5}$$

$$M_2 = \max_{x\in[a,b]} |f''(x)| \qquad \begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \end{aligned} \qquad \max\{\overset{max}{e^{3'4}}, e^{1'8}\} = e^{3'4}$$

$$\Downarrow$$

$$x = 3'4$$

$M_2 = e^{3'4}$

n = nº de troços

$$h^2 \leq \frac{12}{2 \cdot 1'6} \cdot e^{-3'4} \simeq 1'25 \cdot 10^{-6}$$

$$h \leq 1'1 \cdot 10^{-3} \Rightarrow h = 10^{-3}$$

$$h = \frac{b-a}{n} \qquad n = \frac{b-a}{h} = \frac{1'6}{10^{-3}} = 1'6 \cdot 10^{3}$$